If we now consider functions of $\hat{\mathbf{r}}_{\mathrm{I}}$, $\hat{\mathbf{r}}_{\mathrm{II}}$, $\hat{\mathbf{P}}_{\mathrm{I}}$, and $\hat{\mathbf{P}}_{\mathrm{II}}$ suggested from classical analogs, symmetric with respect to interchange of I and II, which can be expressed as power series of terms of which Eq. (A14) is an example, the matrix elements in the two-body subspace can be written as

$$\langle \mathbf{r}_1\mathbf{r}_2|\hat{V}(\hat{\mathbf{r}}_{\mathrm{I}}\hat{\mathbf{r}}_{\mathrm{II}}, \hat{\mathbf{P}}_{\mathrm{I}}, \hat{\mathbf{P}}_{\mathrm{II}})|\mathbf{r}_3\mathbf{r}_4\rangle = V(\mathbf{r}_1\mathbf{r}_2 - i\hbar\nabla_1, -i\hbar\nabla_2)\langle \mathbf{r}_1\mathbf{r}_2|\mathbf{r}_3\mathbf{r}_4\rangle. \tag{A16}$$

It should be noted that the classical functions that suggest the form of the operator function $\hat{V}(\hat{\mathbf{r}}_{\mathrm{I}}, \hat{\mathbf{r}}_{\mathrm{II}}, \hat{\mathbf{P}}_{\mathrm{I}}, \hat{\mathbf{P}}_{\mathrm{II}})$ are required to be symmetric with respect to interchange of I and II in the positions coordinates alone and in the momenta coordinates alone, which insures the overall symmetry with respect to an exchange of I and II.

[1]It has been noted in a previous article in this Journal [S. T. Epstein, Am. J. Phys. **44**, 484 (1976)] that a discussion of second quantized operators is not standard textbook material.

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[13]Reference 9, pp. 12–20.
[14]Reference 2, Vol. I, pp. 306–309.

Classical electrodynamics in a universe with two space dimensions

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Equations for classical electrodynamics in a universe with two space dimensions may be derived directly from those in three space dimensions by beginning with the $3+1$ space-time formulation. In $2+1$ space-time the electromagnetic field tensor has three independent components corresponding to two components of the electric field and one component of the magnetic field. Several applications of the general equations are discussed.

Recently, Dewdney[1] has discussed physics in a universe with two space dimensions. While the task of solving problems for such a universe may be largely recreational, their solution may serve a more practical purpose by suggesting technological innovations with applications in the three-dimensional world.

In addition, the study of physics in a universe with one or two space dimensions can provide a set of pedagogical exercises that may be useful to both instructors and students by elucidating some of the features of the usual physics of our universe with three space dimensions. In particular, it may be possible to obtain insight into the structure of classical electrodynamics by considering the electrodynamic equations and a number of familiar problems in a universe with only one or two space dimensions.[2]

The purpose of this note is to formulate the equations of classical electrodynamics in two-dimensional space. The Maxwell equations and other relations that are required are obtained by assuming that the usual $3+1$-dimensional space-time formulation of electrodynamics may be reduced to a $2+1$-dimensional space-time formulation by setting $z\equiv 0$. All derivatives with respect to z also vanish. The equations of classical electrodynamics in one-dimensional space are obtained similarly in the Appendix.

Following the formulation in $3+1$-dimensional space-time, the electromagnetic field tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1}$$

where A_μ are a set of potentials, ∂_μ are partial derivatives, and the indices have the values $\mu = 0,1,2$.

As in $3+1$ dimensions $F_{\mu\nu}$ is antisymmetric. Thus there are $(3\times 3-3)/2 = 3$ independent components of $F_{\mu\nu}$. These correspond to an electric field with two components and a magnetic field with one component.

The Maxwell equations are

$$\partial_\nu F_{\mu\nu} = 2\pi J_\mu, \tag{2}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0, \tag{3}$$

where the three components of J_μ are two currents and one density. In component form Eq. (1) is

$$E_x = \partial A_x/\partial t - \partial\phi/\partial x, \tag{4a}$$

$$E_y = \partial A_y/\partial t - \partial\phi/\partial y, \tag{4b}$$

$$B = \partial A_y/\partial x - \partial A_x/\partial y. \tag{4c}$$

The Maxwell equations in component form are

$$\partial E_x/\partial y - \partial E_y/\partial x = (1/c)\partial B/\partial t, \tag{5}$$

$$\partial E_x/\partial x + \partial E_y/\partial y = 2\pi\rho, \tag{6}$$

$$\partial B/\partial x = -(1/c)\partial E_y/\partial t - (2\pi/c)J_y, \tag{7a}$$

$$\partial B/\partial y = (1/c)\partial E_x/\partial t + (2\pi/c)J_x. \tag{7b}$$

One may write down immediately a variety of other relations that hold for two-dimensional electrodynamics. The equation of continuity,

$$\partial_\mu J_\mu = 0, \quad (8)$$

becomes in component form

$$\partial J_x/\partial x + \partial J_y/\partial y + \partial \rho/\partial t = 0. \quad (9)$$

The potentials satisfy wave equations

$$\Box^2 A_\mu = -(2\pi/c)J_\mu, \quad (10)$$

where $\Box^2 = \nabla^2 - (1/c^2)\partial^2/\partial t^2$.

In the absence of charges and currents the fields satisfy free wave equations:

$$\Box^2 F_{\mu\nu} = 0. \quad (11)$$

Hence there are electromagnetic waves in two-dimensional space that travel at a speed c.

The Lorentz condition is

$$\partial_\mu A_\mu = 0. \quad (12)$$

One may define an energy–momentum stress tensor

$$T_{\mu\nu} = (1/2\pi)[F_{\mu\sigma}F_{\sigma\nu} + (1/3)\delta_{\mu\nu}F_{\lambda\rho}F_{\lambda\rho}]. \quad (13)$$

This tensor is symmetric and therefore has six independent components. These are

$$T_{01} = BE_y/2\pi, \quad (14a)$$

$$T_{02} = -BE_x/2\pi, \quad (14b)$$

$$T_{00} = (E_x^2 + E_y^2 + B^2)/4\pi, \quad (14c)$$

$$T_{12} = E_xE_y/2\pi, \quad (14d)$$

$$T_{11} = (E_x^2 - E_y^2)/2\pi, \quad (14e)$$

$$T_{22} = T_{11}. \quad (14f)$$

The divergence of $T_{\mu\nu}$ equals the Lorentz force density

$$\partial_\nu T_{\mu\nu} = f_\mu. \quad (15)$$

In component form Eq. (15) includes a statement of conservation of total energy:

$$\frac{\partial u}{\partial t} + \frac{c}{2\pi}\frac{\partial(BE_y)}{\partial x} + \frac{\partial(BE_x)}{\partial y} = -(E_xJ_x + E_yJ_y), \quad (16)$$

where the energy density is given by

$$u = (E_x^2 + E_y^2 + B^2)/4\pi. \quad (17)$$

Equation (15) also contains a statement of conservation of total linear momentum. The field linear momentum density is proportional to the "Poynting vector" given by

$$S_x = (c/2\pi)BE_y, \quad (18a)$$

$$S_y = (c/2\pi)BE_x. \quad (18b)$$

The angular momentum density, which is a scalar quantity in two-dimensional space, is given by

$$\mathscr{L} = -(1/2\pi)(xE_x + yE_y)B. \quad (19)$$

The motion of charged particles in electric and magnetic fields is obtained from the equation

$$m_0 dU_\mu/d\tau = (q/c)F_{\mu\nu}U_\nu, \quad (20)$$

where m_0 is the rest mass of the particle, U_μ is the "three-velocity," and τ is the proper time. In component form Eq. (20) yields the Lorentz force

$$F_x = q(E_x + v_yB/c), \quad (21a)$$

$$F_y = q(E_y - v_xB/c), \quad (21b)$$

and the energy equation

$$d\epsilon/dt = q(E_xv_x + E_yv_y), \quad (22)$$

where $\epsilon = m_0c^2\gamma$ is the energy of the particle and $\gamma = (1 - v^2/c^2)^{-1/2}$.

If $E_x = E_y = 0$ and $B = \text{const}$ in Eq. (21) the charge moves in a circle with a frequency $\omega = qB/m_0\gamma c$.

A velocity selector may be built by sending a beam of charged particles through "crossed" electric and magnetic fields. There is no deflection if

$$v/c = E/B. \quad (23)$$

An amusing problem that resembles the motion of a charge moving in the field of a magnetic monopole may be solved completely. The three-dimensional motion of a charge in a magnetic field has been discussed by Lapidus and Pietenpol.[3] A two-dimensional monopole field would be expected to have a $1/r$ dependence. Thus one may compute the orbit of a charge in a scalar magnetic field, $B = g/r$, where g is the "monopole" strength. The equations of motion are

$$m\dot{v}_x = (q/c)Bv_y, \quad (24a)$$

$$m\dot{v}_y = (q/c)Bv_x. \quad (24b)$$

It is clear that circular orbits satisfy Eq. (24), since in this case $B(r)$ is a constant along the orbit. The frequency is given by $\omega = qB/mc = (qg/mc)/r$. Hence, for any r the speed is a constant

$$v = v_0 = qg/mc. \quad (25)$$

This result may also be obtained by multiplying Eq. (24a) by v_x, Eq. (24b) by v_y, and adding. Since $v_x = v\cos\theta$, $v_y = v\sin\theta$, Eq. (24) becomes

$$\dot{\theta} = -v_0/r. \quad (26)$$

The angular momentum of the charge is given by

$$l = mr^2\dot{\theta} = -mv_0r, \quad (27)$$

which is not constant.

The general motion may be obtained as follows. In polar coordinates Eq. (24) may be written as

$$\ddot{r} - r\dot{\theta}^2 = v_0\dot{\theta}, \quad (28a)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = -v_0\dot{r}/r. \quad (28b)$$

Equation (28b) may be rewritten as

$$d(r^2\dot{\theta})/dt = -v_0\dot{r}, \quad (29)$$

which integrates to Eq. (27).

Substituting Eq. (27) into Eq. (28b) one obtains

$$\ddot{r} = 0. \quad (30)$$

Integrating Eq. (30),

$$r = r_i + v_it, \quad (31)$$

where r_i and v_i are the initial position and velocity. Substituting into Eq. (26) and integrating

$$\theta = -v_0/v_i \ln(1 + v_it/r_i). \quad (32)$$

Thus the orbit is given by

$$r = r_i \exp(-v_i\theta/v_0). \quad (33)$$

The general motion is a spiral that is outward or inward depending upon the sign of v_i. If $v_i = 0$, the radial coordinate is a constant and θ varies linearly with t. Then the orbit is a circle with fixed speed. The period of the orbit to proportional to the radius, i.e.,

$$T = 2\pi r/v_0. \quad (34)$$

It is interesting to note that the period of a charge moving in a circular orbit in a two-dimensional "Coulomb" field satisfies a relation similar to Eq. (34). In two dimensions the force between two charges is inverse linear. Hence,

$$qQ/r = mv^2/r. \tag{35}$$

Then

$$v^2 = qQ/m \tag{36}$$

is a constant for this problem also. Thus

$$T = 2\pi(m/qQ)^{1/2}r. \tag{37}$$

If the periods in Eqs. (34) and (37) are equal, $v = v_0$ and

$$g^2 = (Q/q)mc^2. \tag{38}$$

In two dimensions both q^2 and g^2 have the units of energy.

One may also speculate on the nature of electrodynamics in a universe with four or more space dimensions. If one again assumes that the convariant equations hold in a 4 + 1-dimensional universe, one defines five potentials. The electromagnetic field tensor then has ten components. There are six homogeneous and five inhomogeneous Maxwell equations, and the energy–momentum stress tensor has fifteen components.

But, what are the fields? In a 1 + 1-dimensional universe (see Appendix) there is no magnetic field and the electric field has one component. In a 2 + 1-dimensional universe the magnetic field has one component and the electric field has two components. In a 3 + 1-dimensional universe the magnetic field has three components and the electric field has three components. In a 4 + 1-dimensional universe the magnetic field has six components and the electric field has four components. In an $n + 1$-dimensional universe the magnetic field has $n(n-1)/2$ components and the electric field has n components.

APPENDIX: CLASSICAL ELECTRODYNAMICS IN A UNIVERSE WITH ONE SPACE DIMENSION

The equations for classical electrodynamics in a universe with one space dimension are obtained here by reducing the usual 3 + 1-dimensional equations to a set of 1 + 1-dimensional equations. In this formulation $F_{\mu\nu}$ has $(2\times 2 - 2)/2 = 1$ component. Hence the electromagnetic field is

$$F_{01} = \partial A/\partial t - \partial\phi/\partial x = E. \tag{A1}$$

The Maxwell equations are

$$\partial E/\partial x = \rho, \tag{A2}$$

$$\partial E/\partial t = -J. \tag{A3}$$

The equation of continuity is

$$\partial J/\partial x + \partial\rho/\partial t = 0. \tag{A4}$$

The potentials satisfy wave equations

$$\Box^2 A_\mu = -(1/c)J_\mu \tag{A5}$$

with $\mu = 0,1$ and $\Box^2 = \partial^2/\partial x^2 - \partial^2/\partial t^2$.

The field E also satisfies a wave equation

$$\Box^2 E = 0. \tag{A6}$$

The Lorentz condition is

$$\partial A/\partial x + \partial\phi/\partial t = 0. \tag{A7}$$

The electromagnetic energy–momentum stress tensor has three independent components:

$$T_{00} = T_{11} = E^2/2, \tag{A8a}$$

$$T_{01} = T_{10} = 0. \tag{A8b}$$

Conservation of total energy is written as

$$\partial u/\partial t = -EJ \tag{A9}$$

with

$$u = E^2/2. \tag{A10}$$

Thus

$$\dot{E} = -J, \tag{A11}$$

as given in Eq. (A3).

The Lorentz force is given by

$$F = qE, \tag{A12}$$

and the energy equation is

$$d\epsilon/dt = qvE. \tag{A13}$$

The field between two charges in one space dimension is a constant. (This corresponds to the field of an infinite sheets of charge in three-dimensional space.)

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[1]A. K. Dewdney, *Two-Dimensional Science and Technology*, London, Ontario (1980). A copy of this privately published book may be obtained by writing to Department of Computer Sciences, University of Western Ontario, London, Ontario N6A 5B9, Canada. Also see the discussion by M. Gardner, Sci. Am. **243**, 18 (1980).

[2]The author has conducted a seminar on the two-dimensional universe and found that the solution of problems in one- and two-dimensional space is indeed an effective pedagogical device for clarifying a number of concepts in physics, chemistry, and other fields.

[3]I. R. Lapidus and J. L. Pietenpol, Am. J. Phys. **28**, 17 (1960).