Principles of

CLASSICAL.ELECTRODYNAMICS.pdf

A "laptop text"

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2001/2002

PREFACE

Preface to the handwritten edition of 1980/81. My *Mathematical Introduction to Electrodynamics* (1972/73) can be fairly described as the work of a "young formalist who should have known better;" it begins with a detailed account of special relativity, continues with detailed surveys of the elements of tensor analysis and the exterior calculus, and treats also the relativistic mechanics of particles before it arrives—after nearly 300 pages of preparation—at the dynamics of the electromagnetic field. Discussion even of the latter topic is marked by frequent "research digressions" of an invariably formalistic nature, digressions which I now consider to be (as even then I did) technically interesting but pedagogically extravagant. The frankly eccentric design of that earlier course can be partially understood if one takes into account the following circumstances:

- I was—by staff decision—under a formal obligation to teach both electrodynamics *and* the "methods of mathematical physics;"
- it was my explicit expectation that my students would be reading one or several of the standard texts collaterally;
- during the decade prior to 1972 Reed College physics students had shown a marked interest in formal/structural matters; the abrupt shift to a more "practical" set of interests and values first became conspicuous about 1973, and caught me by surprise (it anticipated a parallel shift in my own interest and values);
- I had never previously taught electrodynamics, and had "many rocks to turn over" in the service of my own technical education.

When repeated (1973/74 and 1974/75) the course was in fact less eccentric than those old notes suggest, for I omitted much of the formal material, and in its place treated radiation theory, as developed in my *Quantum Perturbations & Radiative Processes* (1973/74). But the formal emphasis and relative absence of phenomenological detail were still (in my present view) excessive.

The course design here recorded arose in part by reaction to that former teaching experience. My goal—as formerly—was not to "rewrite Jackson" but to expose as clearly as I could the structural heart of electrodynamics (and thus to *prepare my students to read* Jackson and similar texts). I was determined "to get Maxwell's equations on the board" as soon as possible (I had recently reviewed a manuscript by Julian Schwinger which indicated how that goal might be accomplished), to treat "formal" problems only when the further elaboration of mainline electrodynamics made such activity unavoidable, and to abandon any explicit attempt to treat comprehensively the miscellaneous "methods of mathematical physics." I wanted also to give relatively more attention to phenomenological matters, and to construct a written record of some of the things I had learned since 1975.

Here (as historically) relativity emerges *in natural consequence of a study of the transformation properties of Maxwell's equations*, and attention is given to the little-known fact that a slight modification of that analytical program leads not to the Lorentz group but to the conformal group, the main properties of which are described in detail (details omitted when the course was repeated). Tensors are treated only in the detail specifically required, and the exterior calculus is (as is all reference to its electrodynamical applications) omitted altogether. The patient reader will still find too-frequent evidence of Wheeler's compulsion to "turn over rocks," and most readers will share my own judgment that the formalism is still too dense, and the reference to phenomenologyn still too slight. The latter defect was in practice somewhat blunted by the fact that students were encouraged to make heavy collateral use of David Griffiths' *Introduction to Electrodynamics* (1981).

Nicholas Wheeler
11 May 1983

Preface to the present electronic edition. This material came into being primarily because I had grown tired of late-night trips to Kinko's to print copies of my old hand-written notes for distribution in class, and of the attendant financial complications. I had become increasingly sensitive also to the circumstance that the material was in fact growing "old" (was already older than my students), and increasingly alert to the advantages of electronic publication, which had been impressed upon me by good experiences in several other courses. So in August 2001 I decided to produce a "revised electronic edition" of my 1980/81 class notes. I imagined the job would keep me busy until about November. In fact it absorbed my almost total attention over an eight-month period.

I found that I was, by and large, still fairly pleased with the basic design and execution of original text, but as I progressed the revisions became progressively more frequent, progressively more radical. Some of the original material has been boiled down or omitted altogether, analytical arguments have often been replaced with *Mathematica*-assisted "mathematical experiments," whether undertaken and reported by me or—at my request—by the students themselves. A fair amount of material (for example: everything having to do with conformal transformations and the covariance of Maxwell's equations), though retained, was omitted from the lectures.

On the other hand, some new material has been introduced. Most conspicuously (and eccentrically), I have allowed myself to draw upon elements of Proca's "theory of massive photons" in order to underscore certain critical respects in which classical electrodynamics is "atypical—poised on the razor's edge." And I have incorporated a theory of "optical beams as electromagnetic objects" that happened to occur to me as I wrote. During the interval 1981–2001 I had fairly frequent occasion to take up electromagnetic topics. *None* of that material was has been folded into these revised notes, though the substantial portion of it that existed already in electronic form was made available to students who cared to do some collateral reading.[1]

It has been my lifelong experience that I learn most effectively not by close reading of what A has to say about the subject, or what B has to say, but by *comparing* A's and B's (and also C's) approaches to the same subject. It has been therefore not willful self-indulgence but something approaching a sense of duty that has led me to organize and approach the subject matter of electrodynamics in ways that many colleagues would consider eccentric. My presumption has been that my students will be comparing what I have to say with what Griffiths, Marion, Jackson, ... have to say—this in their efforts to arrive at their own individual understandings of a complicated subject matter. My intent has been not to sing Griffiths' tune, but—because we are so fortunate as to have David Griffiths among us—to sing in a kind of *obbligato* harmony.

[1] I allude to "Electrodynamic application of the exterior calculus," (1996); "Algebraic theory of spherical harmonics," (1996); "'Electrodynamics' in 2-dimensional spacetime," (1997); "Simplified production of Dirac δ-function identities," (1997); "Theories of Maxwellian design," (1998).

This project began as an effort to solve a distribution problem, and to facilitate future editorial revision. But electronic publishing provides options not available in hard copy, so I soon confronted the question: "Am I generating material intended to be printed (in black and white) or to be read on-screen?" So great did I consider the advantages of using color to eliminate the distracting clutter of primes, superscripts and subscripts that—somewhat tentatively—I selected the latter option. Some information will therefore be lost when the text is laser-printed, but are led to believe that the cost/speed of ink-jet color printing will soon decrease/increase to realistic levels. Some students came to class with black & white hardcopy versions of the text, fewer with colored copy . . . and only one or two with their laptops. It is my hope and expectation that the latter practice will soon become the norm, for it belatedly occurred to me that what I have unwittingly produced is a "laptop text." Once the general run of students become properly equipped (I yesterday made arrangements for the design of the prospective new physics lecture hall to be modified in anticipation of such a development) it will become possible to build animations, links to other documents—in short: the full range of electronic resources—into the design of a future edition of this and other texts.

The text was created with Textures® running TeX on a PowerMac G-3 platform. The TeX code was translated into PostScript by "printing to file," and the final PDF (Portable Document Format) file was created by using Acrobat Distiller® to open the PostScript file (which was then discarded). Some of the figures were drawn by *Mathematica* and exported (to the Textures folder containing the TeX code) as EPS files, others were drawn by hand using FreeHand8,® and some were created by using FreeHand8 to add details to *Mathematica* figures.

The PDF files, as distributed on the Courses Server, are all smaller—often *much* smaller—than the files from which they were created. They are intended to be opened and read with Acrobat Reader,® which is freeware distributed by Adobe. The Acrobat Reader is a powerful tool—capable of much more than simply opening PDF files—and readers are encouraged to familiarize themselves with its search, mark-up and other resources: the Visual QuickStart Guide *PDF with Acrobat* by Ted Alspach (1999) is very useful in this connection.

I am indebted to my students for their patience with a project which for the most part they seem to have taken entirely for granted (one suggested on a class evaluation form that the course might work much better if I adopted a better text), and especially to Eric Lawrence, who brought many typos and misspellings to my attention.

Nicholas Wheeler
25 April 2002

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PHYSICAL & MATHEMATICAL FUNDAMENTALS

Introduction. Since "the world is unitary"—with each part connected (however tenuously) with each other part—it is of some philosophical interest that physics admits of semi-conventional division into semi-autonomous "branches." Most of those branches are concerned with the analysis of fairly *general classes* of physical systems (think, for example, of classical mechanics, or of quantum mechanics, fluid dynamics, thermodynamics), but a few (celestial mechanics, general relativity, . . .) are concerned with relatively particularized systems. It is useful to note that electrodynamics is, for all of its incredible richness and variety, a subject of the latter sort: *all that follows will be motivated by a desire to clarify the structure and dynamical properties of a single physical object—the electromagnetic field*.

Our objective, therefore, is to review progress in a field which achieved a kind of maturity (Maxwell, 1873) just about a century ago, a field to which some of the greatest minds (Einstein, Feynman . . . and many others) have contributed, a field in which "the last word" has certainly *not* been written. Much of great value can be learned from close study of the (ongoing) *history of electrodynamics* . . . but for that I must refer my readers to the relevant literature. A standard source is E. T. Whittaker's *A History of the Theories of Aether & Electricity* (1910). Since this branch of the history of science is currently quite active, it would be well to consult recent issues of (say) *History of Science*. For a good modern account of the "ancient history" of some of the basic notions see Duane & D. H. D. Roller, "The development of the concept of electric charge: electricity from the Greeks to Coulomb" in J. B. Conant (editor), *Harvard Case Histories in Experimental Science* (Volume II, 1966). You should, of course, take this occasion to become acquainted with the Victorian founding fathers (Faraday, Maxwell) of our subject. I urge you therefore to look into *Michael Faraday: A Biography* by L. P. Williams (1965), *James Clerk Maxwell: Physicist and Natural Philosopher* by C. W. F. Everitt (1975) and/or *Contributions of Faraday & Maxwell to Electrical Science* (1966) ... all of which are informative, yet fun to read. Finally, every student of electrodynamics should peruse the pages of Maxwell's own *A Treatise on Electricity & Magnetism* the (posthumous) 3rd edition (1891) of which was reissued by Dover in 1954. While the history of science is its own reward, the history of electrodynamics (as of classical mechanics, quantum mechanics ...) is also of directly utilitarian value, for it illuminates the processes/circumstances/developments which contribute to the maturation of physics—to the discovery/invention of *new* physics.

That electromagnetic phenomenology (and theoretical understanding of that phenomenology) lies at the base of an elaborate technology—think of electrical power grids, the electric light, motorized devices, electronic communication/computation/mealsurement & control ... none of which were known to the founders of the field—is of course not news. Less well known to the general public are the theoretical contributions of classical electrodynamics, which (directly or indirectly) has stimulated the invention/development of

- special relativity
- quantum mechanics
- the modern theory of gravitation (general relativity)
- elementary particle physics
- many of the methods characteristic of modern applied mathematics

... and much else. One could perfectly well base a course such as this on the *technological applications* of our subject: such an approach would be considered standard in schools of engineering, and is reflected in the design of many existing texts. I prefer, however, to let (my personal view of) the theoretical applications/ramifications of electrodynamics govern the selection, arrangement and presentation of the subject matter. Classical electrodynamics provides a unique "classical window" through which can be glimpsed many of the principles which are now recognized to dominate the structure of the micro-world (also the very-large-scale macro-world ... and much that lies in between). But to gain access to that window we must pay close and critical attention to *structural issues* ... and to that end we must from time to time draw upon mathematical methods which, though of growing importance, have heretofore not been considered standard to the undergraduate education of physicists. The latter material will be developed in appropriate detail as needed.

The "historical approach" (recapitulated psuedo-history) which for a long time dominated instruction in classical and—particularly—quantum mechanics has never been popular in the electrodynamical classroom ... and it is certainly not my intention to attempt such an experiment. Nor shall I honor the established practice, which is to proceed "crabwise" into the subject, for a pedagogical strategy which places the (allegedly) "easy parts" (electrostatics, potential theory, ...) first necessarily *displaces the fundamentals* ... with the result that Maxwell's equations tend to get lost in the clutter, and relativity to enter (as historically it did) only as an afterthought.

The design of this introductory chapter proceeds therefore from my desire "to put first things first." My goal, more specifically, is to proceed in all feasible haste to a working understanding—however tentative—of *what kind of a thing electrodynamics is, of the physical and computational issues fundamental to the subject*. This will entail review of material to which you have already had some exposure—the

- conceptual innovations and
- physical phenomenology

which historically led James Clerk Maxwell to the *equations of motion* of the electromagnetic field. But we will also begin what will, as we proceed, become a major activity—"looking under rocks:" conceptual rocks, computational rocks, formal rocks. Our intent at this stage is more *to formulate sharp questions* that to formulate sharp answers (the latter can wait). It is interesting to observe that we will be led, even in this introductory survey, to aspects (both deep and numerous) of electrodynamics of which Maxwell died (5 November 1879, at age 48) unaware.

1. Coulomb's law. The phenomenology here goes back to antiquity, and involves the curious behavior of macroscopic samples of certain biogenic substances (amber, fur, silk, paper, pithballs) which are—except for our story—insignificant constituents of the universe. This speculative tradition (to which an allusion survives in the word "electron," from $\eta\lambda\epsilon\kappa\tau\rho o\nu =$ amber) had by ∼1750—owing largely to the work of Benjamin Franklin (1706–1790)—led to the formulation of a recognizable precorsor of the modern concept of *electric charge*. It is today recognized that *electric charge is—like mass—an attribute not merely of bulk matter (pithballs) but of the elementary constituents of such matter*.

Particles announce their charge by exerting forces (forces of a specific yet-to-be-described structural type: "electromagnetic forces") on each other; *i.e.*, by *interacting—electromagnetically* ... and it is from study of how particles respond to such (postulated) forces that we have learned all that we know concerning the existence and properties of the electromagnetic field. The question—the *experimental* question—therefore arises: How are we to make structurally and quantitatively precise the *force law* latent in the preceding remarks?

Prior to ∼1760 (when this question first moved to centerstage) the only "universal force law" known to physics was Newton's

$$F = G\tfrac{Mm}{r^2}$$

which describes the *instantaneous gravitational interaction-at-a-distance* of mass points M and m. It was widely anticipated that the electrostatic interaction of charged mass points would turn out to be governed by a law of similar form. Experimental evidence in support of this conjecture was published by Daniel Bernoulli in 1760 and by Joseph Priestly in 1767, but the issue was instrumentally delicate, and was definatively resolved only in 1785 by Charles Coulomb (1736–1806), who used sensitive torsion balances and torsion pendula of his own invention (similar to those used years later by Henry Cavendish to measure G). Turning now to the concrete particulars ...

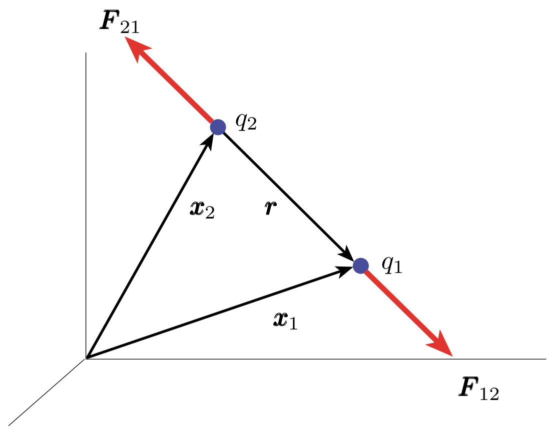

FIGURE 1: *Notation used to describe the relation of one charge to another, and the Coulombic forces which each exerts upon the other.*

Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ describe (relative to an inertial Cartesian frame) the positions of a pair of structureless but electrically charged mass points, (m_1, q_1) and (m_2, q_2). For conceptual convenience (*i.e.*, to circumvent the troublesome action-at-a-distance problem) we *assume the point charges to be at rest* ... both now and in the "relevant past." Experimentally

$$\boldsymbol{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}} = -\boldsymbol{F}_{21} \tag{1}$$

where (see Figure 1) $\boldsymbol{F}_{12}$ is the force exerted <u>on charge #1 by charge #2</u>, and where

$$\boldsymbol{r} \equiv \boldsymbol{r}_{12} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2 = -(\boldsymbol{x}_2 - \boldsymbol{x}_1) \equiv -\boldsymbol{r}_{21} \quad : \quad \text{separation vector } 1 \leftarrow 2$$
$$r^2 \equiv \boldsymbol{r}\cdot\boldsymbol{r} \quad : \quad \text{squared separation}$$
$$\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r \quad : \quad \text{unit separation vector } 1 \leftarrow 2$$

The *gravitational analog of Coulomb's law* (1) reads

$$\boldsymbol{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}} = -\boldsymbol{F}_{21} \tag{2}$$

These familiar results invite several (interconnected) lines of commentary:

1. In the context provided by the 2-body problem we lack the "principle of choice" required to assign $\boldsymbol{F}_{12}$ any direction other than that provided by the "line of centers" (separation vector). The argument would, however, lose its force if

- the point particles were endowed with directionality (spin);
- the forces of interation among n bodies ($n \geqslant 3$) were assembled more complexly than by mere *vectorial superposition*

$$\boldsymbol{F} = \boldsymbol{F}_{12} + \boldsymbol{F}_{13} + \cdots + \boldsymbol{F}_{1n}$$

Happily, complexities of the later type—but not the former!—are *absent from the electrodynamics of* <u>*point*</u> *charges* (though they can be expected to arise when one considers the interaction of—say—charged conductors: why?).

2. Comparison of (2) with (1) suggests that we sould construe m_1 and m_2 to be measures not of intertial mass but of "gravitational charge." It is, however, a deep-reaching and surprising fact that to do so would be to make a "distinction without a difference." For measurements of high precision (performed by Baron L. von Eötvös (1848–1919) and refined by R. H. Dicke) have established (to within about one part in 10^{12}) that

$$\frac{\text{gravitational charge}}{\text{inertial mass}} = \text{universal constant}$$

where "universal" here means "the same for all samples and states of matter." This fact was foreshadowed already in Galileo's alleged "Leaning Tower of Pisa experiment" and in the m-independence of the frequency $\omega = \sqrt{g/\ell}$ of a pendulum (why?). By appropriate adjustment of certain conventions (units) we can arrange that

$$= 1 \pm 10^{-12}$$

Such invariable quantitative identity becomes intelligible only if it proceeds from conceptual identity: "inertial mass" and "gravitational charge" must be *different names for the same thing*. This fundamental insight is built into the structure of (2), and entails that (relative to any prescribed system of mechanical units: cgs, MKS, ...) *G becomes an empirical constant of forced/fixed dimensionality*. Actually

$$G = 6.6732 \times 10^{-8}\ \text{dyn} \cdot \text{cm}^2/\text{g}^2$$

For further discussion of the Eötvös–Dicke experiments and their significance see C. W. Misner, K. S. Thorn & J. A. Wheeler, *Gravitation* (1973), pages 1050–1055.

3. Returning in this light to (1) the question arises: Is the value of electric charge/mass invariable, the same for all charged bodies? The answer is an emphatic "no:"

- for macroscopic bodies q/m is highly variable;
- among elementary particles of a given species q/m is constant, but
- the numerical value of q/m varies from species to species.

So in the real (multi-species) world there is *no electrical analog of the so-called "equivalence principle."* This circumstance entails that we must confront the question: What shall be the physical dimensionality $[Q]$ of electric charge? The answer is semi-*conventional* (there is dimensional trade-off between k and q^2), and presents some useful options:

- We might construe k (whatever its *numerical* value) to be *dimensionless.* Such a convention forces

  $$[Q^2] = \text{force} \cdot \text{length}^2 = \text{energy} \cdot \text{length} = \text{action} \cdot \text{velocity}$$

  whence
  $$[Q] = M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1} \quad : \quad \text{a "derived quantity"}$$

  If we set $k = 1$ and adopt cgs mechanical units we are led thus to the "electrostatic unit" (esu or "statcoulomb") of charge

  $$\text{esu} = \text{g}^{\frac{1}{2}}\text{cm}^{\frac{3}{2}}\text{sec}^{-1}$$

  Evidently charges of one esu, if separated by one centimeter, exert upon each other a force on one dyne. The "rationalized" (or Heaviside–Lorentz) esu arises if—to avoid factors of 4π in the field equations—we set $k = 1/4\pi$.

- Since charges/currents/potentials are most conveniently measured by operations/procedures which are alien to mechanics, we might construe charge to be *dimensionally antonomous*: $(M, L, T) \rightarrow (M, L, T, Q)$. Such a convention forces

  $$[k] = \text{force} \cdot \text{length}^2/\text{charge}^2 = ML^3T^{-2}Q^{-2}$$

  and causes the numerical value of k to assume (like G) the status of an emperical constant. If—following in the footsteps of Faraday—we adopt an electrochemical procedure to define the

  $$\text{ampere} \equiv \text{coulomb}/\text{second}$$

  then we find by measurement that

  $$k = 8.988 \times 10^9 \text{ newton} \cdot \text{meter}^2/\text{coulomb}^2$$

  Circumstances will emerge which make it natural to write

  $$= \frac{1}{4\pi\epsilon_0}$$

  and to call ϵ_0 $(= 8.854 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)$ the "permittivity of free space."

Theoretical physicists tend generally to prefer (rationalized) electrostatic units, and engineers to prefer (rationalized MKS) or "practical" units. Interconversion formulæ follow from

$$\text{coulomb} = 2.997930 \times 10^9 \text{ esu}$$

From the fact that the

$$\begin{aligned}\text{electronic charge } e &= 4.803250 \times 10^{-10} \text{ esu}\\ &= 1.602189 \times 10^{-19} \text{ coulomb}\end{aligned}$$

we see that the coulomb (also for that matter the esu) is, for most fundamental purposes, an impractably large unit. Often it is preferable to measure charge in multiples of e (as is standardly done in elementary particle physics, nuclear physics, chemistry). For further informatrion concerning the notorious (and—in theoretical contexts—usually irrelevant) "problem of units" see J. D. Jackson, *Classical Electrodynamics* (1962), pages 611–621.[1]

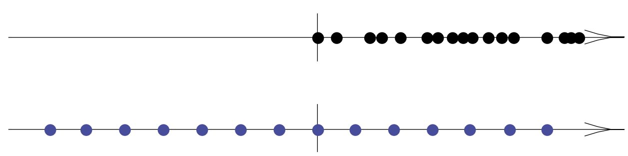

FIGURE 2: *The masses encountered in Nature are shown above, the electric charges below: the former are invariably positive, but are otherwise unconstrained; charges, on the other hand, can occur with either sign, and are always multiples of a fundamental unit.*

4. Gravitational forces are invariably attractive, while charged particles repell or attract each other according as their charges are of the same or opposite sign. These familiar facts trace, *via* the structure of (1) and (2), to the observation that *gravitational charge is invariably positive* while *electric charge can be of either sign.* The situation becomes somewhat more interesting when phrased in the language of elementary particle physics, for in that context the inergial mass concept is somewhat enlarged . . . and an interesting "graininess" reveals itself. One has

$$m \geqslant 0 \quad \text{but} \quad q \gtreqless 0$$

as illustrated in Figure 2. Note that $m \geqslant 0$ applies (according to recent experiments) even to *anti*particles. And while "massless particles" exist (photon, graviton, neutrino?), *there are no charged massless particles*: "charge endows mass" (though not *all* mass arises—as was once supposed—by this complex mechanism).

[1] PROBLEMS 1, 2 & 3

5. In Coulomb's law $F = kQq/r^2$ the "2" is, of course, an *experimental* number. *How accurately* can it be said that electrostatic forces (or, for that matter, gravitational forces) "fall off as the square" of distance? If we write

$$F = kQq\frac{1}{r^{2+\epsilon}}$$

then Coulomb himself knew that $0 < \epsilon < 10^{-1}$. Cavendish (in some unpublished work) showed that $\epsilon < 3 \times 10^{-2}$ and Maxwell, by a refinement of Cavendish's technique, showed (1873) that $\epsilon < 5 \times 10^{-5}$. The most recent work known to me (E. R. Williams, 1971) establishes that $\epsilon < 6 \times 10^{-16}$. Interestingly, the quantum mechanical version of our subject (QED) shows that we can expect to have $\epsilon = 0$ *if the photon mass* μ *is precisely zero* . . . and enables one to convert the sharpest of the results quoted above into the statement that

$$\mu \leqslant 2 \times 10^{-40} \text{ g} \approx (\text{electron mass}) \cdot 10^{-20}$$

For a beautiful discussion of this absorbing topic see A. S. Goldhaber & M. M. Nieto, "Terrestrial and extraterrestrial limits on the photon mass," Rev. Mod. Phys. **43**, 277 (1971).[2] Note finally that the (massless) photon, though it "mediates the electromagnetic interaction of electrically charged particles," is itself *un*charged . . . and moves always "with the speed of light" only because it is massless. I am, however, ahead of my story.

To describe the force $\boldsymbol{F}(\boldsymbol{x})$ experienced by a charge q if situated at a point $\boldsymbol{x}$ in a region of space occupied (see Figure 3) by a static *population* of charges $\{Q_1, Q_2, \ldots\}$ we invoke—but only because it is sanctioned by experience—the *principle of superposition* to write

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{x}) = \sum_i \boldsymbol{F}_i(\boldsymbol{x}) &= \sum_i kq\,Q_i\frac{1}{r_i^2}\,\hat{\boldsymbol{r}}_i \quad \text{with} \quad \boldsymbol{r}_i \equiv \boldsymbol{x} - \boldsymbol{x}_i \\ &= kq\underbrace{\sum_i Q_i\frac{1}{r_i^3}\,\boldsymbol{r}_i}\end{aligned}$$

defines the *electrostatic field* $\boldsymbol{E}(\boldsymbol{x})$ which is established at $\boldsymbol{x}$ by the charges Q_i

The $\boldsymbol{E}$-field is a *force field*, which in electrostatic units (k dimensionless) has the dimensionality

$$[\boldsymbol{E}] = \text{force/charge}$$

[2] While writing this paragraph I chanced (one midnight at the watercooler) to discuss its substance with Richard Crandall, with consequences that can be read about in R. E. Crandall, "Photon mass experiment," AJP **51**, 698 (1983) and R. E. Crandall & N. A. Wheeler, "Klein-Gordon radio and the problem of photon mass," Nuovo Cimento **84B**, 231 (1984): also the splendid thesis of Richard Leavitt, "A photon mass experiment: an experimental verification of Gauss' law" (1983)—on the basis of which Leavitt became Reed's first Apker Award finalist.

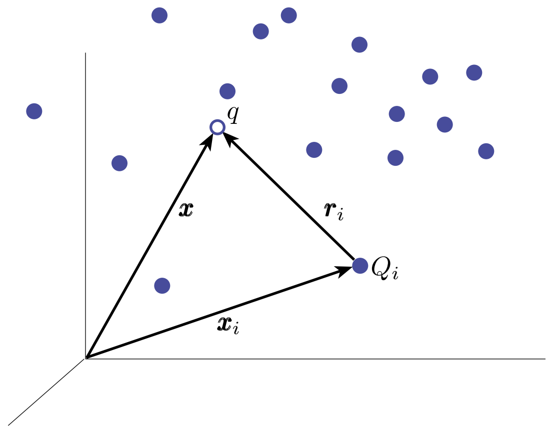

FIGURE 3: *A discrete population of charges acts electrostatically on a test charge*

and which is defined operationally by the dynamical response of the "test charge" (m, q).[3] Mathematically, $\boldsymbol{E}(\boldsymbol{x})$ is a *vector-valued function of position* (which is to say: a "vector field"), given explicitly by

$$\boldsymbol{E}(\boldsymbol{x}) = \sum_i Q_i \frac{\boldsymbol{x} - \boldsymbol{x}_i}{|\boldsymbol{x} - \boldsymbol{x}_i|^3} \tag{3.1}$$

Passing to the *continuous limit* we obtain

$$= \iiint \rho(\boldsymbol{\xi}) \frac{\boldsymbol{x} - \boldsymbol{\xi}}{|\boldsymbol{x} - \boldsymbol{\xi}|^3} d^3\xi \tag{3.2}$$

$\rho(\boldsymbol{\xi})$ is the *charge density* at $\boldsymbol{\xi}$

which gives back (3.1) in the special case

$$\rho(\boldsymbol{\xi}) = \sum_i Q_i \, \delta(\boldsymbol{\xi} - \boldsymbol{x}_i) \tag{4}$$

Though the rich *physics of electrostatic fields* is—in its entirety—latent in (3.2), that equation is susceptible to the criticism that

i) it is, in most contexts, not *computationally* useful

ii) it tells us nothing about the *general structural properties* of $\boldsymbol{E}$-fields.

Thus are we motivated to ask: *What are the <u>differential equations</u> which, in general, constrain/govern/describe the structure of (static)* $\boldsymbol{E}$*-fields*? That question motivates the following

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[3] If the "field sources" Q_i were constrained merely to reside on some prescribed conductors then the presence of the test charge would cause them to rearrange themselves. This effect is minimized by assuming q to become arbitrarily small, though we are in fact constrained by Nature to have $q \geqslant e$ (or at least $q \geqslant \frac{1}{3}e$).

MATHEMATICAL DIGRESSION

For transformation-theoretic reasons which we shall be at pains later to clarify, the differential operators available to us are all latent in the vector-valued "del" operator

$$\boldsymbol{\nabla} \equiv \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \equiv \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

Readers will (I presume) recall that

1) $\boldsymbol{\nabla}$ applied to a scalar field $\varphi(\boldsymbol{x})$ yields the vector-valued *gradient* of φ: $\boldsymbol{\nabla}\varphi \equiv \text{grad}\,\varphi$,
2) $\boldsymbol{\nabla}$ dotted into a vector field $\boldsymbol{A}(\boldsymbol{x})$ yields the scalar-valued *divergence* of $\boldsymbol{A}$: $\boldsymbol{\nabla}\cdot\boldsymbol{A} \equiv \text{div}\,\boldsymbol{A}$, and
3) $\boldsymbol{\nabla}$ crossed into a vector field $\boldsymbol{A}(\boldsymbol{x})$ yields the vector-valued *curl* of $\boldsymbol{A}$: $\boldsymbol{\nabla}\times\boldsymbol{A} \equiv \text{curl}\,\boldsymbol{A}$.

And they should (now's the time to practice!) be able to produce—"on demand" as it were—identities such as the following:

$$\text{grad}(\varphi\psi) = \varphi\ \text{grad}\,\psi + \psi\ \text{grad}\,\varphi \tag{5.1}$$

$$\text{div}(\varphi\boldsymbol{A}) = \varphi\ \text{div}\boldsymbol{A}\ + \boldsymbol{A}\cdot\text{grad}\,\varphi \tag{5.2}$$

$$\text{curl}(\varphi\boldsymbol{A}) = \varphi\ \text{curl}\boldsymbol{A} - \boldsymbol{A}\times\text{grad}\,\varphi \tag{5.3}$$

$$\text{div}(\boldsymbol{A}\times\boldsymbol{B}) = -\boldsymbol{A}\cdot\text{curl}\boldsymbol{B} + \boldsymbol{B}\cdot\text{curl}\boldsymbol{A} \tag{5.4}$$

$$\text{curl}(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}\ \text{div}\boldsymbol{B} \quad - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} - \boldsymbol{B}\ \text{div}\boldsymbol{A} \quad + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} \tag{5.5}$$

$$\text{grad}(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{A}\times\text{curl}\boldsymbol{B} + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{B}\times\text{curl}\boldsymbol{A} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} \tag{5.6}$$

...all of which (though the last three become "easy" only in consequence of some fairly sophisticated technique) are consequences basically of the "product rule:" $\partial(FG) = F\partial G + G\partial F$. Differential expressions of second (and higher) order are obtained from the above by composition. In particular, one has

$$\text{div grad}\,\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\varphi \equiv \nabla^2\varphi = \underbrace{\Big\{\big(\tfrac{\partial}{\partial x}\big)^2 + \big(\tfrac{\partial}{\partial y}\big)^2 + \big(\tfrac{\partial}{\partial z}\big)^2\Big\}}_{\text{Laplacian operator}}\varphi$$

And by explicit calculation[4] one establishes that

$$\text{curl grad}\,\varphi = \boldsymbol{0} \quad \text{for all scalar fields } \varphi(\boldsymbol{x}) \tag{6.1}$$

$$\text{div curl}\boldsymbol{A} = 0 \quad \text{for all vector fields } \boldsymbol{A}(\boldsymbol{x}) \tag{6.2}$$

[4] PROBLEM 4

Turning now from broad generalities to some of their more particular consequences, of which we will soon have specific need . . . let

$$\begin{aligned}\varphi(\boldsymbol{x}) &= f(r)\\ r \equiv r(\boldsymbol{x}) &= \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} = \sqrt{x^2+y^2+z^2}\end{aligned}$$

symbolize the $\boxed{\text{assumption}}$ that <u>x, y and z enter φ only *via* r</u>; *i.e.*, that φ is *constant-valued on spheres* (*radius* r: $0 \leqslant r < \infty$) *which are concentric about the origin in* $\boldsymbol{x}$*-space* (we assume more briefly that φ is <u>rotationally-symmetric</u>). Immediately (by the chain rule)

$$\boldsymbol{\nabla}\varphi = f'(r)\boldsymbol{\nabla}r$$

where the prime signifies differentiation of $f(\bullet)$ with respect to its sole argument. It is elementary[5] that

$$\boldsymbol{\nabla}r = \hat{\boldsymbol{x}} \equiv \frac{\boldsymbol{x}}{r} \tag{7.1}$$

and also that

$$\operatorname{div}\boldsymbol{x} = 3 \tag{7.2}$$

$$\operatorname{curl}\boldsymbol{x} = \mathbf{0} \tag{7.3}$$

so

$$= \frac{1}{r}f'(r)\,\boldsymbol{x} \tag{8}$$

From (5) and (7) it now follows that

$$\begin{aligned}\nabla^2\varphi &= \frac{1}{r}f'\boldsymbol{\nabla}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{\nabla}\Big(\frac{1}{r}f'\Big)\\ &= \frac{3}{r}f' + \underbrace{(\boldsymbol{x}\cdot\boldsymbol{x})\frac{1}{r}\Big(\frac{1}{r}f'\Big)'}\\ &\qquad\qquad = r^2\Big\{\frac{1}{r^2}f'' - \frac{1}{r^3}f'\Big\}\end{aligned}$$

giving

$$= f'' + 2\frac{1}{r}f' \tag{9}$$

It is a notable consequence of this fact that[6]

$$\text{If } \nabla^2\varphi = 0 \text{ then } f(r) = \frac{a}{r} + b \quad : \quad a \text{ and } b \text{ are constants}$$

and if, moreover, it is required that $f(\infty) = 0$ then necessarily $b = 0$.

---

[5] PROBLEM 5

[6] PROBLEM 6

We are in position now to state and prove the results we will need to carry forward the discussion of (3). They are (some trivial generalizations of) the following:

$$\boldsymbol{\nabla}\frac{1}{r} = -\boldsymbol{x}/r^3 \tag{10.1}$$

$$\nabla^2\frac{1}{r} = -4\pi\delta(\boldsymbol{x}) \tag{10.2}$$

The former follows as an immediate corollary of (8). The presence of the *Dirac delta function* $\delta(\boldsymbol{x})$ on the right side of the latter announces that <u>(10.2) has only a *formal* meaning</u>—will be literally meaningful only when encountered in the protective shade of an $\int$-sign—and promises that the proof will be somewhat tricky. To avoid the fact that $1/r$ becomes *singular* at the origin we study the ϵ-parameterized functions

$$g(r\,;\epsilon) \equiv \frac{1}{r+\epsilon}$$

...our plan being to "turn ϵ off" at some appropriate moment(s). Immediately

$$\left.\begin{aligned} g' &= -\frac{1}{(r+\epsilon)^2} \\ g'' &= +\frac{2}{(r+\epsilon)^3} \end{aligned}\right\} \tag{11}$$

so by (9)

$$\begin{aligned} \nabla^2 g &= 2\Big[\frac{1}{(r+\epsilon)^3} - \frac{1}{r(r+\epsilon)^2}\Big] \\ &= -\infty \text{ at the origin (all } \epsilon\text{), but elsewhere vanishes as } \epsilon \downarrow 0 \end{aligned} \tag{12}$$

Next we notice that the result $\iiint \nabla^2 g\, d^3x$ of integrating $\nabla^2 g$ over all space can (by spherical symmetry) be described

$$\begin{aligned} \int_0^\infty \nabla^2 g\, 4\pi r^2\, dr &= \lim_{R\uparrow\infty} 8\pi\int_0^R \left[\frac{r^2}{(r+\epsilon)^3} - \frac{r}{(r+\epsilon)^2}\right] \\ &= \lim_{R\uparrow\infty} 8\pi\left[\frac{\epsilon}{(r+\epsilon)} - \frac{\epsilon^2}{2(r+\epsilon)^2}\right]_0^R \text{ according to } \textit{Mathematica} \\ &= \lim_{R\uparrow\infty} 4\pi\left[1 - \Big(\frac{r}{r+\epsilon}\Big)^2\right]_0^R \\ &= \lim_{R\uparrow\infty} \underbrace{-4\pi\Big(\frac{R}{R+\epsilon}\Big)^2}_{\text{Remarkably, this becomes } \underline{R\text{-independent}} \text{ as } \epsilon\downarrow 0} \quad : \quad \epsilon > 0 \\ &= -4\pi \end{aligned} \tag{13}$$

The function $\nabla^2 g$—see (12)—has, in other words, these seemingly contradictory properties:

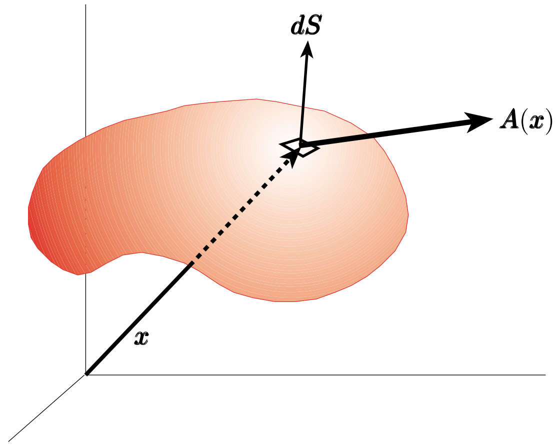

FIGURE 4: *Geometrical context to which* (14) *refers.* $\boldsymbol{x}$ *identifies a point on the boundary* $\partial\mathcal{R}$ *of the "bubble"* $\mathcal{R}$, $\boldsymbol{dS}$ *describes the area and orientation of a surface element, and* $\boldsymbol{A}(\boldsymbol{x})$ *is an arbitrary vector field.*

- it is, for all ϵ (though the fact is masked when $\epsilon = 0$) singular at the origin, but elsewhere
- it vanishes as $\epsilon \downarrow 0$, yet does so in such a way that
- its integral over x-space remains constantly equal to -4π. Finally
- g itself approaches $g(r\,;0) = 1/r$ as $\epsilon \downarrow 0$.

This is precisely the information which the formal equation (10.2) is intended to convey. $\mathcal{QED}$

I should mention that *the preceding line of argument is non-standard*, that the texts argue invariably from the celebrated integral identity

$$\iiint_{\mathcal{R}} \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\,dV = \iint_{\partial\mathcal{R}} \boldsymbol{A}\!\cdot\!\boldsymbol{dS} \tag{14}$$

where (see Figure 4) $\mathcal{R}$ is a "bubble-like" region in 3-dimensional Euclidean space, dV (otherwise denoted d^3x) is an element of volume, $\partial\mathcal{R}$ refers to the (orientable) surface of $\mathcal{R}$, and $\boldsymbol{dS}$ is an outward-directed surface element. That strategy is unavailable to me, since I wish to postpone proof and discussion of *Gauß' theorem* (14) and its relatives. If, however, the reader is content (for the moment) merely to accept (14) then we can

i) take $\mathcal{R}$ to be the sphere of radius R centered at the origin and

ii) take $\boldsymbol{A} = \boldsymbol{\nabla} g$

to obtain

$$\begin{aligned}\iiint_{\mathcal{R}} \nabla^2 g\, d^3x &= \iint_{\partial\mathcal{R}} \boldsymbol{\nabla} g\boldsymbol{\cdot} \boldsymbol{dS}\\ &= -\iint_{\partial\mathcal{R}} \frac{1}{(r+\epsilon)^2}\hat{\boldsymbol{x}}\boldsymbol{\cdot}\boldsymbol{dS} \quad \text{by (8) and (9)}\end{aligned}$$

But $\hat{\boldsymbol{x}}\boldsymbol{\cdot}\boldsymbol{dS} = dS$ since $\hat{\boldsymbol{x}}$ and $\boldsymbol{dS}$ are (for this $\mathcal{R}$) parallel and $\hat{\boldsymbol{x}}$ is a unit vector, so

$$= -\frac{4\pi R^2}{(R+\epsilon)^2}$$

—consistently with a result we obtained *en route* to (13). The surprising fact that this result is (in the limit $\epsilon \downarrow 0$) R-independent is understood as follows: $\nabla^2 g$ is—see again (12)—*singular at the origin* but (in the limit) *vanishes elsewhere*, so $\iiint \nabla^2 g\, d^3x$ *acquires its entire value at/from the singularity* . . . which (again) is the upshot of (10.2). Note finally that by "displacement of the origin" we have

$$\frac{\boldsymbol{x}-\boldsymbol{a}}{|\boldsymbol{x}-\boldsymbol{a}|^3} = -\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}-\boldsymbol{a}|} \tag{15.1}$$

and

$$\nabla^2\frac{1}{|\boldsymbol{x}-\boldsymbol{a}|} = -4\pi\,\delta(\boldsymbol{x}-\boldsymbol{a}) \tag{15.2}$$

as trivial generalizations of (10). Equations (15) are fundamental—the results I have been at such pains to derive. END OF DIGRESSION

Returning now with (15.1) to (3.2) we have

$$\boldsymbol{E}(\boldsymbol{x}) = -\iiint \rho(\boldsymbol{\xi})\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}d^3\xi \tag{16}$$

Here the operator $\boldsymbol{\nabla}$ looks only to the $\boldsymbol{x}$-dependence of its operand, and since $\boldsymbol{x}$ is not the variable of integration we can take the $\boldsymbol{\nabla}$ outside the $\iiint$, writing

$$= -\boldsymbol{\nabla}\varphi(\boldsymbol{x}) \tag{17}$$

$$\varphi(\boldsymbol{x}) \equiv \iiint \rho(\boldsymbol{\xi})\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}d^3\xi \tag{18}$$

$$\equiv \underline{\text{electrostatic potential}}\text{, a } \textit{scalar} \text{ field}$$

Electrostatic $\boldsymbol{E}$-fields are, according to (17), *conservative* (in the sense that they admit of derivation from a scalar "potential," namely the $\varphi(\boldsymbol{x})$ of (18)). The equation

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0} \tag{19}$$

—which follows from (17) by (6.1)—provides a compact formulation of the same fundamental fact (and would motivate a hydrodynamicist to remark that such $\boldsymbol{E}$-fields are "irrotational"). Note, however, that (19)—which contains no reference at all to $\rho(\boldsymbol{x})$—imposes only a weak constraint upon the structure of $\boldsymbol{E}(\boldsymbol{x})$; *i.e.*, that it does not, of itself, enable one to *compute* $\boldsymbol{E}(\boldsymbol{x})$.

Next we take the divergence of (16) to obtain

$$
\begin{aligned}
\boldsymbol{\nabla\cdot E}(\boldsymbol{x}) &= -\iiint \rho(\boldsymbol{\xi})\,\nabla^2\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\,d^3\xi \\
&= 4\pi\iiint \rho(\boldsymbol{\xi})\delta(\boldsymbol{x}-\boldsymbol{\xi})\,d^3\xi \quad \text{by (15.2)} \\
&= 4\pi\rho(\boldsymbol{x}) \qquad (20)
\end{aligned}
$$

And from (20) it follows by (17) that

$$\nabla^2\varphi(\boldsymbol{x}) = -4\pi\rho(\boldsymbol{x}) \qquad (21)$$

Some comments are now in order:

1. Equations (19) and (20)—more specifically

$$
\boxed{
\begin{aligned}
\boldsymbol{\nabla\cdot E} &= 4\pi\rho \quad , \quad \tfrac{\partial}{\partial t}\rho = 0 \\
\boldsymbol{\nabla\times E} &= \boldsymbol{0} \quad , \quad \tfrac{\partial}{\partial t}\boldsymbol{E} = \boldsymbol{0}
\end{aligned}
}
\qquad (22)
$$

where $\partial\rho/\partial t = 0$ and $\partial\boldsymbol{E}/\partial t = \boldsymbol{0}$ formalize the fact that we are here talking about time-independent physics—provide a *complete* <u>*local*</u> *characterization of the laws of electrostatics* . . . where "complete" means that *the solution of* (22) *is, for any prescribed boundary conditions, unique*. From (22) one can, in particular, recover the (non-local) statement (3.2) which provided our point of departure.

2. As will be shown later in greater detail, $\boldsymbol{\nabla\times E} = \boldsymbol{0}$ tells us in effect that "there *exists* a (non-unique) φ such that $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$, while it is the upshot of (21) that <u>in charge-free regions of space</u> φ satisfies *Laplace's equation*

$$\nabla^2\varphi = 0 \qquad (23)$$

In the discrete approximation

$$
\begin{aligned}
&= \frac{\frac{\varphi(x+\epsilon,y,z)-\varphi(x,y,z)}{\epsilon}-\frac{\varphi(x,y,z)-\varphi(x-\epsilon,y,z)}{\epsilon}}{\epsilon} + y\text{-analog} + z\text{-analog} \\
&= \tfrac{6}{\epsilon^2}\Big\{\frac{\varphi(\text{evaluated at 6 “near neighbors” of } \boldsymbol{x})}{6} - \varphi(\boldsymbol{x})\Big\}
\end{aligned}
$$

so (23) tells us that *in the absence of charge* φ *"relaxes" until the value assumed by* φ *at* $\boldsymbol{x}$ *is the* <u>*average of the values assumed by* φ *at the "neighbors" of* $\boldsymbol{x}$</u>. This can be understood to be the "meaning" of Laplace's equation *whatever* the physical/mathematical context in which it is encountered. According to *Poisson's equation*

$$\nabla^2\varphi = -4\pi\rho \qquad (21)$$

the "role" of charge is "to keep φ from relaxing:" φ (locally) exceeds or falls short of the average of neighboring values according as (locally) $\rho \gtrless 0$. Note that if I were to *give* you $\varphi(\boldsymbol{x})$ then you could use (21) to *compute the implied structure of the charge distribution (or "source term")* $\rho(\boldsymbol{x})$.

3. Comparison of (21) with (15.2) shows that we can interpret

$$G(\boldsymbol{x};\boldsymbol{\xi}) \equiv \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}$$

The notation recalls the name of George Green, who ($\sim$1824) was the first to appreciate the power of the general ideas here at issue.

as a description of the *electrostatic potential generated by a unit charge situated at the point* $\boldsymbol{\xi}$ in $\boldsymbol{x}$-space. Now it is fundamental that (see again page 4)

> *Electrodynamics is—like quantum mechanics (but unlike classical mechanics, fluid dynamics, gravitational field physics)—dominated by the principle of superposition.*

This is because the underlying (partial differential) equations are (see (22)) linear: solutions—when

- multiplied by constants and/or
- added to other solutions

—yield solutions. This "build-up principle" pertains, in particular, to (21). Reading the identity

$$\rho(\boldsymbol{x}) = \iiint \rho(\boldsymbol{\xi})\,\delta(\boldsymbol{x}-\boldsymbol{\xi})\,d^3\xi$$

as a formalization of the remark that *arbitrary (even continuous) charge distributions can be synthesized by weighted superposition of point charges*, we infer (by linearity) that $\rho(\boldsymbol{x})$ generates the potential

$$\varphi(\boldsymbol{x}) = \iiint \rho(\boldsymbol{\xi})\,G(\boldsymbol{x};\boldsymbol{\xi})\,d^3\xi \tag{25}$$

Equation (25) is but a notationally disguised re-write of (18), upon which it sheds important new light. If we apply ∇^2 to (25) and insist that the result be—for *all* $\rho(\boldsymbol{x})$—in conformity with (21) then we are forced to the conclusion that

$$\nabla^2 G(\boldsymbol{x};\boldsymbol{\xi}) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{\xi}) \tag{26}$$

which (by (24)) is but a restatement of (15.2), but can be read as the equation that *defines* the *Green's function appropriate to the Poisson equation* (21). Evidently

$$-\boldsymbol{\nabla} G(\boldsymbol{x};\boldsymbol{\xi}) = \text{Coulombic } \boldsymbol{E}\text{-field generated by a unit point charge at } \boldsymbol{\xi}$$

5. To place the preceding remarks in a larger context, and to lend substance to the claim that *the Green's function technique pertains* <u>*generally*</u> *to "linear physics"* ... consider the (inhomogeneous linear partial differential) equation

$$(\nabla^2 - \lambda^2)\,\varphi(\boldsymbol{x}) = -4\pi\rho(\boldsymbol{x}) \tag{27}$$

λ is a constant with (evidently) the physical dimensions of $\frac{1}{\text{length}}$

which gives back (21) in the limit $\lambda^2 \downarrow 0$. Equation (25) serves *still* to describe the general solution φ of (27), but the Green's function $G(\boldsymbol{x};\boldsymbol{\xi})$ is constrained now to satisfy not (26) but

$$(\nabla^2 - \lambda^2)\,G(\boldsymbol{x};\boldsymbol{\xi}) = -4\pi\delta(\boldsymbol{x}-\boldsymbol{\xi})$$

which is readily shown[7] to entail

$$G(\boldsymbol{x};\boldsymbol{\xi}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\,e^{-\lambda|\boldsymbol{x}-\boldsymbol{\xi}|}$$

To reduce the notational clutter and better expose the essence of the matter, write $r \equiv |\boldsymbol{x}-\boldsymbol{\xi}|$ to obtain

$$(\nabla^2 - \lambda^2)G(r) = -4\pi\,\delta(r) \tag{28}$$

$$\Downarrow$$

$$G(r) = \frac{1}{r}\cdot e^{-\lambda r} \tag{29}$$

Equation (29) describes what is sometimes called the "screened Coulomb potential," for reasons that arise from the observation that (28) can be written

$$\left.\begin{aligned} \nabla^2 G(r) &= -4\pi\,\rho(r)\\ \rho(r) &\equiv \delta(r) - \frac{\lambda^2}{4\pi r}\,e^{-\lambda r} \end{aligned}\right\} \tag{30}$$

By quick computation

$$\int_0^\infty \rho(r) 4\pi r^2\,dr = 1 - 1 = 0$$

so (30) can be used to model the *electrostatic environment of a neutral* <u>*atom*</u> (positively charged point-like nucleus that is "screened" by an exponentially attenuated "electron cloud"—the whole being electrically neutral). A visiting test charge feels an $\boldsymbol{E}$-field given by

$$\begin{aligned} \boldsymbol{E} = -\boldsymbol{\nabla} G &= -\Big(\frac{1}{r}e^{-\lambda r}\Big)'\hat{\boldsymbol{x}}\\ &= \frac{1}{r^2}\underbrace{(1+\lambda r)e^{-\lambda r}}_{\text{attenuation factor}}\hat{\boldsymbol{x}} \end{aligned} \tag{31}$$

---

[7] PROBLEM 7

—the strength of which *falls off faster than* $1/r^2$ (which is to say: "faster than geometrically"), with a "characteristic range" given by λ.

> HISTORICAL NOTE: By 1934 it was known that the so-called "strong force" (the force which overcomes electrostatic repulsion to bind nuclei) is of *short range*. Hideki Yukawa—then 27 years old—saw the opportunity to give the λ-term an important physical job. He recognized that *classical physics*—(e,c)-*physics*—*contains no "natural length"*. Neither does its quantized analog ($(e,c,\hbar)$-physics) ...but theories of the latter type would acquire a "natural length"—given on dimensional grounds[8] by
>
> $$\text{natural length} \equiv \lambda^{-1} = \frac{\hbar}{\mu c} \tag{32}$$
>
> —if the analog of the photon (Yukawa's hypothetical—but by now very well established—"meson": the particle which mediates the strong interaction) were assigned a non-zero mass μ. Yukawa was led thus to postulate the existence of an elementary particle (it turned out to be a small *population* of particles—the "π-mesons") with mass
>
> $$\mu = \frac{\hbar}{c}\cdot\frac{1}{\text{range of the strong force}} \sim 265 \text{ electron masses}$$
>
> and to suggest that something like the "Yukawa force law" (31) should (in leading approximation) describe the interaction of nucleons. π-mesons were first observed (in nuclear emulsions by Powell & Occhialini) in 1947, and in 1949 Yukawa received the Nobel Prize.

Note finally that

- the "natural length" of (32) *becomes infinite* as $\mu \downarrow 0$;
- the preceding theory becomes "Coulombic" in that limit ...and could, in particular, be used to construct an alternative to our "non-Gaußian proof" of (10.2);
- we might expect (21) to go over into (27) should it turn out that photons do in fact have a (tiny) mass.

I look finally to the <u>*energetics* of electrostatic fields</u>; *i.e.*, of static chrage configurations. Readers will recall from prior study of elementary mechanics that

1) if $\boldsymbol{F}(\boldsymbol{x})$ describes the forcy environment of a mass point m then the *work that <u>you must perform</u>* to transport m along a prescribed path is given by

$$W[\text{path}] = -\int_{\text{path}} \boldsymbol{F}(\boldsymbol{x})\cdot d\boldsymbol{x}$$

---

[8] PROBLEMS 8 & 9.

If the path is described parametrically $\boldsymbol{x} = \boldsymbol{x}(\lambda)$: $0 \leqslant \lambda \leqslant 1$ then we can (more specifically) write

$$= -\int_0^1 \boldsymbol{F}\big(\boldsymbol{x}(\lambda)\big)\cdot\frac{\boldsymbol{dx}(\lambda)}{d\lambda}d\lambda$$

2) if the force is *"conservative" in the sense that it admits of description as the* *<u>gradient of a scalar potential</u>*

$$\boldsymbol{F}(\boldsymbol{x}) = -\boldsymbol{\nabla}U(\boldsymbol{x})$$

then (by $\boldsymbol{\nabla}U\cdot\boldsymbol{dx} = dU$)

$$\begin{aligned} W[\text{path}] &= \int_0^1 \frac{dU(\boldsymbol{x}(\lambda))}{d\lambda}\,d\lambda \\ &= U(\boldsymbol{x}_1) - U(\boldsymbol{x}_0) \end{aligned}$$

Remarkably, the path-dependence of W has dropped away: W has become (not a "path functional" but) a *<u>function</u> of the <u>endpoints</u>* of the path. A simple argument shows, conversely, that path-independence *implies* the existence of U.

We now ask: *What is the work which <u>you</u> must perform to <u>assemble</u> the constellation of charges Q_i first contemplated on page 8?* . . . the assumption (mainly of convenience) being that *the Q_i reside initially—far from each other and from us—"at infinity"* (*i.e.*, at the only generally available "standard place").

Evidently we can move the 1st charge Q_1 into position "for free." The 2nd charge Q_2 feels (when at $\boldsymbol{x}$) the Coulombic force

$$\boldsymbol{F}_{12}(\boldsymbol{x}) = kQ_1Q_2\frac{1}{r_1^3}\boldsymbol{r}_1 = -kQ_2\boldsymbol{\nabla}\varphi_1(\boldsymbol{x})$$

$$\varphi_1(\boldsymbol{x}) \equiv Q_1\frac{1}{r_1}$$

exerted by Q_1, and from (33) we infer that to bring Q_2 into position we must do work given by

$$W_2 = kQ_2\big\{\varphi_1(\boldsymbol{x}_2) - \underbrace{\varphi_1(\boldsymbol{\infty})}_{0}\big\} = kQ_2Q_1\frac{1}{r_{21}} \equiv W_{21}$$

$$\boldsymbol{r}_{21} \equiv \boldsymbol{x}_2 - \boldsymbol{x}_1$$

Since electrostatic forces conform to the principle of superposition, the force experienced by Q_3 can be described

$$\boldsymbol{F}_3(\boldsymbol{x}) = \boldsymbol{F}_{31}(\boldsymbol{x}) + \boldsymbol{F}_{32}(\boldsymbol{x}) = -kQ_3\boldsymbol{\nabla}\big\{\varphi_1(\boldsymbol{x}) + \varphi_2(\boldsymbol{x})\big\}$$

... and, since $\boldsymbol{F}$ enters *linearly* into the equation $W = \int \boldsymbol{F}\cdot d\boldsymbol{x}$, we infer that to bring Q_3 into position we must do work given by

$$W_3 = kQ_3\big\{\varphi_1(\boldsymbol{x}_3) + \varphi_2(\boldsymbol{x}_3)\big\} = kQ_3Q_1\frac{1}{r_{31}} + kQ_3Q_2\frac{1}{r_{32}} \equiv W_{31} + W_{32}$$

By extension of the same line of argument we obtain

$$W_i = \sum_{j=1}^{i-1} W_{ij}$$

where

$$\left.\begin{aligned} W_{ij} &\equiv kQ_i\varphi_j(\boldsymbol{x}_i) \\ &= kQ_iQ_j\frac{1}{r_{ij}} \quad \text{with} \quad r_{ij} \equiv |\boldsymbol{x}_i - \boldsymbol{x}_j| \\ &= \left\{\begin{array}{l} \text{work done by moving } Q_i \text{ against} \\ \text{the Coulombic force exerted by } Q_j \end{array}\right. \end{aligned}\right\} \tag{34}$$

The <u>energy which we must invest to assemble the entire population</u> is given therefore by

$$\begin{aligned} W = \;& \begin{array}{lr} \bullet & \bullet \\ W_{21} \;\bullet & \bullet \\ +W_{31} + W_{32} \;\bullet & \bullet \\ \vdots & \\ +W_{n1} + W_{n2} + \cdots + W_{n,n-1} & \bullet \end{array} \\ =\;& \sum_{i>j} W_{ij} \end{aligned} \tag{35.1}$$

but would have been given by

$$= \sum_{i>j} W_{ij} \quad : \quad \text{note the reversed inequality} \tag{35.2}$$

had we reversed the order of assembly. Equations (35) are, of course, equivalent (by $W_{ij} = W_{ji}$: see (34)) ... and can be combined to give

$$= \tfrac{1}{2}\sum_i \sum_j{}' W_{ij} \tag{36}$$

└─the prime means that the *self-energy terms* (terms with $i = j$) are to be <u>omitted</u>

which possesses a pleasing formal symmetry. One might be strontly tempted to write

┌─note that the prime is now absent

$$W = \tfrac{1}{2}\sum_i \sum_j W_{ij} - \sum_i W_{ii} \tag{37}$$

were it not for the important fact that *each of the "self-energy terms"* W_{ii} *is numerically infinite*. Equation (37) possesses, in other words, the latently deceptive structure

$$= (W + \infty) - \infty$$

Computationally/conceptually more useful results can be obtained now by appeal to (34): thus

$$W = \tfrac{1}{2}k\sum_i Q_i \underbrace{\Big\{\sum_j{}' \varphi_j(\boldsymbol{x}_i)\Big\}}_{\substack{=\sum_j{}' Q_j \frac{1}{r_{ij}} \\ = \text{potential at } \boldsymbol{x}_i \text{ due to all other charges } Q_j}} \tag{38.1}$$

$$= \tfrac{1}{2}k\sum_i\sum_j{}' Q_i Q_j \frac{1}{r_{ij}} \tag{38.2}$$

In the *continuous limit* equations (38) go over semi-plausibly into

$$W = \tfrac{1}{2}k\int \rho(\boldsymbol{x})\varphi(\boldsymbol{x})\,d^3x \tag{39.1}$$

└─means $\iiint$, and extends over any region $\mathfrak{R}$ big enough to contain all points where the charge distribution $\rho \neq 0$

$$= \tfrac{1}{2}k\iint \rho(\boldsymbol{x})\rho(\boldsymbol{\xi})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\,d^3x\,d^3\xi \tag{39.2}$$

which are (by (18)) equivalent. I say "*semi*-plausibly" becuase equations (39) contain no analogs of the primes which decorate (38). Indeed, if we set

$$\rho(\boldsymbol{\xi}) = \sum_i Q_i\,\delta(\boldsymbol{\xi}-\boldsymbol{x}_i) \tag{4}$$

we can perform the $\iint$ and obtain

$$= \tfrac{1}{2}\sum_i\sum_j W_{ij} = W + \text{self-energy terms}$$

We confront therefore this fundamental question (which I must, for the moment, leave dangling): For *continuous* charge distributions $\rho(\boldsymbol{x})$ do "self-energy terms" (ever? sometimes? always?) automatically *vanish*?

We are in position now to review some ideas which are as fundamental as they are pretty. Introducing $\rho = -\frac{1}{4\pi}\nabla^2\varphi$ into (39.1) we obtain

$$W = -\tfrac{1}{8\pi}k\iiint \varphi\nabla^2\varphi\,d^3x \tag{40}$$

which will strike some readers as reminiscent of the formula

$$\langle E\rangle = \iiint \psi^*\Big\{-\tfrac{\hbar^2}{2m}\nabla^2 + V\Big\}\psi\, d^3x$$

by means of which one computes the *expected value of the average of many energy measurements* if the quantum mechanical system with Hamiltonian $\mathsf{H} = \frac{1}{2m}\mathsf{p}^2 + V(\mathsf{x})$ is known to be in state ψ. Be that as it may . . . it follows from (40) (more directly: introduce $\rho = \frac{1}{4\pi}\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E}$ into (39.1)) that

$$W = \tfrac{1}{8\pi}k\iiint \varphi\,\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E}\, d^3x$$

By (5.2)

$$\varphi\,\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} = -\boldsymbol{E}\!\cdot\!\boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\!\cdot(\varphi\boldsymbol{E})$$

while by (17)

$$\boldsymbol{\nabla}\varphi = -\boldsymbol{E}$$

So—by what is in effect the 3-dimensional analog of an “integration by parts”—we have

$$= \tfrac{1}{8\pi}k\Big\{\iiint E^2\, d^3x + \underbrace{\iiint \boldsymbol{\nabla}\!\cdot(\varphi\boldsymbol{E})\, d^3x}_{=\iint \varphi\boldsymbol{E}\cdot\boldsymbol{dS}\quad\text{by Gauß' theorem (14)}}\Big\}$$

We expect $\varphi\boldsymbol{E}$ to fall off asymptotically as $1/r^3$. This is fast enough to cause the later surface integral to *vanish if the surface of integration is “removed to infinity”* . . . giving

$$= \tfrac{1}{8\pi}k\iiint E^2\, d^3x \tag{41}$$

where $E^2 \equiv \boldsymbol{E}\cdot\boldsymbol{E}$ and where *the* $\iiint$ *ranges over all space* (or at least over all points where $\boldsymbol{E}(\boldsymbol{x}) \neq \mathbf{0}$). Several lines of commentary are now in order:

1. All that has been said concerning W pertains as well to the energetics of *gravitational* (or at least to weak gravitostatic) fields as it does to electrostatic fields. The space-curvature effects associated with very strong fields (whether gravitational or electrostatic) can, of course, be expected to cause our (tacit) Euclidean assumptions to break down . . .

2. More familiar to chemists than to physicists—and so general/powerful that it is difficult to formulate except in words—is

> LE CHATELIER’S PRINCIPLE: When an external force is applied to a system in equilibrium the system adjusts so as to minimize the effect of the applied force.

Somewhat similar—in substance and spirit, in its abstract generality, and in its ever-surprising power—is this

> NAMELESSS PRINCIPLE: If the energy E of a system depends upon an adjustable parameter α (of whatever nature) then an "abstract force" $\mathcal{F} = -\partial E/\partial\alpha$ will be associated with variation of that parameter. If α refers to *spatial position* then $\mathcal{F}$ will have literally the nature of a *mechanical* force.

Our electrostatic W is by nature a function of $\boldsymbol{x}_1,\, Q_1,\, \boldsymbol{x}_2,\, Q_2,\, \ldots, \boldsymbol{x}_n,\, Q_n$ (in the discrete case, and a functional of $\rho(\boldsymbol{x})$ in the continuous case). *What is the (literal) force associated with variation of* $\boldsymbol{x}_i$? Bringing (34) to (36) we have[9]

$$\begin{aligned}
-\boldsymbol{\nabla}_i W &= -\boldsymbol{\nabla}_i \tfrac{1}{2}k \sum_{a,b}{}' Q_a Q_b \frac{1}{r_{ab}} && (42.1)\\
&= -\tfrac{1}{2}kQ_i\boldsymbol{\nabla}_i \Big\{ \underbrace{\sum_a{}' Q_a \frac{1}{r_{ai}} + \sum_b{}' Q_b \frac{1}{r_{ib}}}_{\text{sums identical by } r_{ab} = r_{ba}} \Big\}\\
&= -kQ_i\boldsymbol{\nabla}_i \sum_j{}' Q_j \frac{1}{r_{ij}}\\
&= kQ_i \cdot (\boldsymbol{E}\text{-field at } \boldsymbol{x}_i \text{ due to all other charges})\\
&= \text{force exerted on } Q_i \text{ by the other charges} && (42.2)
\end{aligned}$$

Note that there is a formal sense in which the prime can be dropped from (42.1): Q_i's self-energy W_{ii}—though infinite—does not change when Q_i is moved ... so $\boldsymbol{\nabla} w_{ii} = 0$:

> A charge Q "carries its self-energy with it," so *does not exert an electro<u>static</u> force upon itself.*

Our "nameless principle" can be used to explain why dielectric fluids are lifted into the space between charged capacitor plates, why magnets attract paper clips, where the thermodynamic concepts of "pressure" and "chemical potential" come from ... and much, much else.

3. *Where does W reside*? The structure of (39)—in which the $\int$'s need extend only over that portion of space which contains charge—tempts one to respond "In the charge(s)" ... or perhaps "In the 'Coulombic springs' by which the charges are interconnected." But those "springs" are spooky things, which inhabit empty space. And one is, on the other hand, encouraged by the structure of (41)—where the $\int$ ranges over that portion of space which contains (not charge but) $\boldsymbol{E}$-field—to suppose that W resides "In the $\boldsymbol{E}$-field; *i.e.*, <u>in</u> <u>the empty space which envelops the charge</u>." The question therefore arises: Which viewpoint is correct (= more useful)? The clear answer is "The latter" ... but only on grounds which emerge when one enlarges the conceptual context to contain dynamical (*i.e.*, t-dependent) elements:

It is most useful to *consider W to reside "in the $\boldsymbol{E}$-field."*

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[9] I use a and b as summation indices because i is now otherwise engaged.

We are led thus to speak of *field energy* . . . and to begin to *think of* $\boldsymbol{E}$*-fields (since they possess energy) as "mechanical objects in their own right."* Such "objects" differ from (say) particles mainly in the fact that they (*i.e.*, *their collective properties*) *are not localized but* distributed. Equation (41) can in this light be written

$$\left.\begin{aligned} W &= \iiint \mathcal{E}(\boldsymbol{x})\, d^3x \\ \mathcal{E}(\boldsymbol{x}) &\equiv \tfrac{1}{8\pi}k|\boldsymbol{E}(\boldsymbol{x})|^2 = \text{electrostatic } \textit{energy density} \end{aligned}\right\} \tag{44}$$

4. It is obvious from (44) that *electrostatic energy density* $\mathcal{E}(\boldsymbol{x})$ *is invariably* non-negative:

$$W = W_{\text{interaction}} + W_{\text{self}} \geqslant 0$$

The W described by equations (38) is, on the other hand, clearly of *indefinite* sign. This slight paradox is resolved by the realization that (38)—which applies only to discrete charges—pertains only to the *interaction energy*

$$W_{\text{interaction}} \gtrless 0$$

while

$$W_{\text{self}} > 0$$

Were we to use ($41 \equiv 44$) in problems involving point charges we would (automatically) be taking into explicit account the *energy expended in the* assembly *of those point charges* . . . which since

- we are in fact physically unable to "assemble" electrons
- a result of the form $W = \infty$ is not very useful

would be poor policy. In discrete problems it is essential that one use (38), not (41/44). One begins to see why, for ∼80 years, physicists have spoken balefully of the "self-energy *problem*" . . . which quantum theory transforms, but does not eliminate.[10]

5. According to ($41 \equiv 44$), W—irrespective of how self-energy terms are handled—is a non-linear *number-valued functional of* $\boldsymbol{E}(\boldsymbol{x})$: if $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ give rise to W_1 and W_2, then $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ gives rise not to $W_1 + W_2$ but to

$$W = W_1 + W_2 + 2\tfrac{1}{8\pi}k \iiint \boldsymbol{E}_1(\boldsymbol{x})\cdot\boldsymbol{E}_2(\boldsymbol{x})\, d^3x \tag{45}$$

I have on page 6 drawn attention to the conventional status of k, and wish now (actually for practical reasons) to illustrate how that circumstance might be exploited. If we *think of the field equations (22) as fundamental, and of Coulomb's law as arising from a* particular (spherically symmetric) solution of those equations, then it becomes natural to suppose that all factors of 4π

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[10] PROBLEMS 10 & 11

should attach not to the field equations but to Coulomb's law and its immediate corollaries. Accordingly, we

$$\text{set } k = \tfrac{1}{4\pi} \quad : \quad \text{(dimensionless)} \tag{46.0}$$

so Coulomb's law (1) reads

$$\boldsymbol{F} = \frac{1}{4\pi}\frac{q_1 q_2}{r^2}\,\hat{\boldsymbol{r}} \tag{46.1}$$

. . . which serves, in effect to *define our ("rationalized electrostatic") unit of charge*. We can further—and quite independently—simplify life by absorbing a k into the definition of $\boldsymbol{E}$, writing

$$\boldsymbol{F}(\boldsymbol{x}) = q\boldsymbol{E}(\boldsymbol{x}) = \text{force on the test charge } q \tag{46.2}$$

Equations (3) become

$$\boldsymbol{E}(\boldsymbol{x}) = \tfrac{1}{4\pi}\iiint \rho(\boldsymbol{\xi})\,\frac{\boldsymbol{x}-\boldsymbol{\xi}}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\,d^3\xi \tag{46.3}$$

and its discrete analog (which there is no need to write out). If we insist—conventionally—upon retaining the simplicity of

$$= -\boldsymbol{\nabla}\varphi(\boldsymbol{x}) \tag{17}$$

then (arguing as before from (15.1)) we obtain (compare (18))

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}\iiint \rho(\boldsymbol{\xi})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\,d^3\xi \tag{46.4}$$

It is evident that in place of (20) and (21) we now have

$$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} = -\nabla^2\varphi = \rho \tag{46.5}$$

The 4π-factors, which formerly resided in the field equations, are not attached only to expressions which are clearly and directly "Coulombic in character." In place of (39.1) we now have

$$W = \tfrac{1}{2}\int \rho(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,d^3x$$

giving

$$\begin{aligned} &= -\tfrac{1}{2}\int \varphi\,\nabla^2\varphi\,d^3x \\ &= \tfrac{1}{2}\int \varphi\,\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E}\,d^3x = \tfrac{1}{2}\int E^2\,d^3x \end{aligned} \tag{46.6}$$

. . . all of which are conspicuously cleaner that their counterparts on pages 21/22, and which suggest that cleanliness invested in the field equations tends to

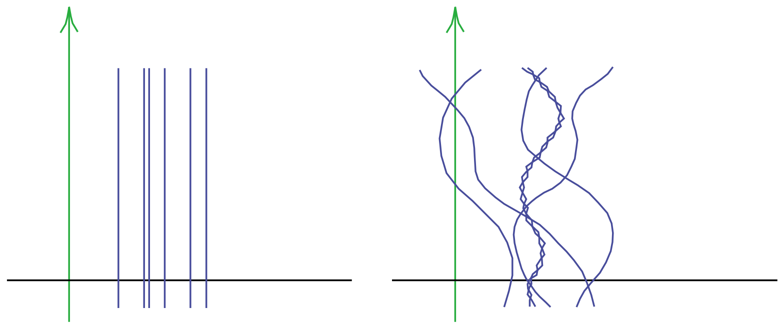

Figure 5: *Spacetime diagrams of (on the left) the worldlines traced by a static population of charges, and (on the right) by a dynamical population. The time axis is indicated* ↑.

persist: the best way to clean up a theory is to scour its most fundamental statements. The exercise has involved a good deal of slip and slide: we have

1) assigned to k a conventional value;
2) conventionally adjusted the relationship between $\boldsymbol{F}$ and $\boldsymbol{E}$;
3) conventionally retained the relationship between $\boldsymbol{E}$ and φ.

The whole business admits obviously of many variations . . . and is never much fun.

Our work thus far contains no reference to *time*, no reference to *magnetism*. It turns out—surprisingly, and independently of whether one proceeds in the laboratory or on paper—that to make provision for either is (semi-automatically) to make provison for the other: that

$$\text{electrostatics} \longrightarrow \text{electro}\textit{dynamics}$$

is a program conceptually equivalent to

$$\text{electrostatics} \longrightarrow \text{electro}\textit{magnetism}$$

Now, electrostatics has been seen to proceed from essentially three assumptions:

1) the field sources Q_i don't move; *i.e.*, that they trace worldlines of the form illustrated in the first of the following figures, and that within that specialized context
2) they interact *via* Coulomb's law; moreover
3) electrostatic forces compose by superposition.

Our forward progress requires that we relax the immobility assumption . . . allowing the Q_i to trace worldlines like those shown on the right side of Figure 5.

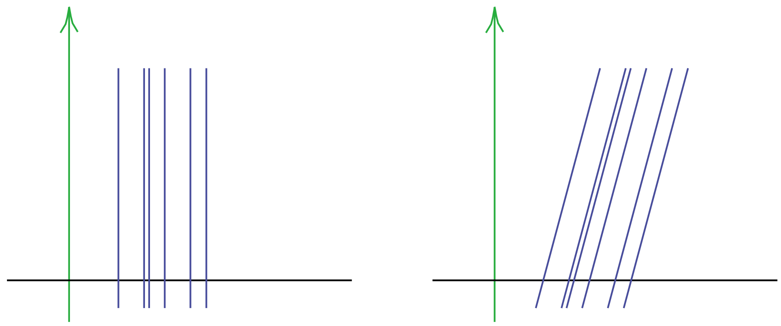

FIGURE 6: *Spacetime diagrams of (on the left) a static population as it appears to O and (on the right) the same population as it appears to us, who see O to be moving by with uniform velocity $\boldsymbol{v}$.*

How to proceed? *How do charged particles interact when they are in relative motion*?

2. Bootstrapping our way to Maxwell's equations. Since my ultimate intent is simply to illuminate the formal/physical ramifications of the *structural properties of the electromagnetic field* (and—as opportunities arise—to illustrate some of the heuristic devices characteristic of modern theoretical physics) . . . I need not apologize for the fact that the discussion which follows is grossly ahistorical. The essential pattern of the argument is due to Julian Schwinger (unpublished notes: ∼1976)[11] . . . but several closely related lines of argument have been around for decades, have been reinvented many times by many people, and have been promoted in the classrooms of Reed College by Dennis Hoffman.

What follows is by nature a "theoretical *bootstrap*" operation, which draws heavily (if interestingly) upon "plausibility arguments" and which leads to results which would remain *merely* plausible in the absence of supporting observational data . . . of which, as it turns out, there is a great deal. The success of the program can itself be read as evidence either of

- the power of hindsight or
- the *extraordinary simplicity of electrodynamics*.

Turning now from anticipatorty generalities to the curious details of our argument . . . let O be an *inertial observer*

1) whom we see to be gliding by with constant velocity $\boldsymbol{v}$;

[11] An edition of those notes was prepared posthumously by several of Schwinger's former associates, and was recently published: see J. Schwinger, L. L. DeRaad, K. A. Milton & W. Tsai, *Classical Electrodynamics* (1998)—especially Chapter 1—and also the review by Jagdish Mehra: AJP **68**, 296 (2000).

2) who possesses the usual "good clock and Cartesian frame" with the aid of which he assigns coordinates $(t, \boldsymbol{x})$ to points in spacetime; *i.e.*, to "events" in his inertial neighborhood;[12]

3) who possesses "complete knowledge of electro*statics*," as developed in §1 and summarized on page 25.

We, on the other hand, possess

1) our own "good clock and Cartesian frame";

2) enough knowledge of physics to know that if O is inertial then so are we ... and (more specifically) enough knowledge of Galilean relativity to "know" that *the coordinates* $(t, \boldsymbol{x})$ *which we assign to an event are related to the coordinates* $(t, \boldsymbol{x})$ *which* O *assigns to that same event* by the equations

$$\left.\begin{aligned} t &= t(t, \boldsymbol{x}) = t \\ \boldsymbol{x} &= \boldsymbol{x}(t, \boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{v}t \end{aligned}\right\} \tag{47}$$

3) no prior knowledge of electrostatics.

Our simple goal—at least at the outset—is to translate O's electrostatic equations in to our variables. The circumstance which makes the enterprise interesting is (see Figure 6) that *while* O*'s charges are at rest with respect both to* O <u>*and to each other*</u> ... *they are* <u>*in (uniform) motion with respect to us*</u>.[13] We confront therefore a situation intermediate between those depicted in Figure 5. Though the figures refer (as a matter of graphic convenience) to point charges, we shall find it analytically most convenient to work with continuous charge distributions ρ—a convention which entails no essential loss of generality.

In 2-dimensional spacetime (to which I retreat for merely notational convenience) it would follow from (47) that

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial t}{\partial t}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} &= \frac{\partial t}{\partial x}\frac{\partial}{\partial t} + \frac{\partial x}{\partial x}\frac{\partial}{\partial x} = \qquad\quad \frac{\partial}{\partial x} \end{aligned}$$

while from (47) itself it follows (similarly) that[14]

$$\left.\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} \\ \boldsymbol{\nabla} &= \qquad\quad \boldsymbol{\nabla} \end{aligned}\right\} \tag{48}$$

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[12] The extent of such a neighborhood is set by curvature effects; *i.e.*, by the structure of the gravitational field. We shall eliminate such (typically quite small) effects by *supposing gravitation to have been "switched off"* ($G \downarrow 0$), so all "neighborhoods" become infinite and coextensive: spacetime becomes (not just locally but) *globally flat*.

[13] From O's point of view we are in effect asking: "How does an electrostatic field look to a moving observer (namely: us)?

[14] Here $\boldsymbol{\nabla}$ denotes "del with respect to $\boldsymbol{x}$," while $\boldsymbol{\nabla}$ denotes "del with respect to $\boldsymbol{x}$."

If we $\boxed{\text{assume}}$ —plausibly?—that O's charge density can by us be described

$$\rho(\boldsymbol{x},t) = \rho(\boldsymbol{x}-\boldsymbol{v}t) = \rho(\boldsymbol{x}) \tag{49}$$

—*i.e.*, that (relative to the $\boldsymbol{v}$-parameterized Galilean transformations (47)) ρ transforms as a scalar field—then O's equation

$$\tfrac{\partial}{\partial t}\rho = 0 \tag{22}$$

can by us (according to (48)) be written

$$\tfrac{\partial}{\partial t}\rho + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho = 0$$

The $\boldsymbol{x}$-independence of $\boldsymbol{v}$ entails $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ so (by (5.2))

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\rho = \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})$$

and if we define

$$\boldsymbol{j} \equiv \rho\boldsymbol{v} \equiv \text{(electric) } \textit{current density} \tag{50}$$

we have

$$\boxed{\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0} \tag{51}$$

Postponing (here and below) all physical/formal commentary, I can report that (51) provides a local formulation of the principle of *charge conservation*.

If we $\boxed{\text{assume}}$—plausibly in view of (49) and what we know from mechanics about the Galilean transform properties of force—that O's $\boldsymbol{E}$-field can by us be described

$$\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{E}(\boldsymbol{x}-\boldsymbol{v}t) = \boldsymbol{E}(\boldsymbol{x})$$

—*i.e.*, that the *individual components* of $\boldsymbol{E}$ respond to (47) like *scalar* fields—then O's equation

$$\frac{\partial \boldsymbol{E}}{\partial t} = 0 \tag{22}$$

can by us be written

$$\frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{E} = 0$$

Expressions of the form $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{E}$ are—since $\boldsymbol{\nabla}$ is neither dotted nor crossed into $\boldsymbol{E}$—"funny," but they are in fact familiar already from (5). It follows in fact from (5.5) that

$$\begin{aligned}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{E} = \boldsymbol{v}(\boldsymbol{\nabla}\cdot\boldsymbol{E}) &- \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{E})\\ &- \underbrace{\boldsymbol{E}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + (\boldsymbol{E}\cdot\boldsymbol{\nabla})\boldsymbol{v}}_{0 \text{ by } \boldsymbol{x}\text{-independence of } \boldsymbol{v}}\end{aligned}$$

O's equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$$

can by us (and without the assistance of any additional assumptions) be written

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho} \tag{53}$$

It follows therefore by (50) that

$$= \boldsymbol{j} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{E})$$

So we have

$$\boxed{\boldsymbol{\nabla}\times\boldsymbol{B} = \tfrac{1}{c}\boldsymbol{j} + \tfrac{1}{c}\tfrac{\partial\boldsymbol{E}}{\partial t}} \tag{54}$$

where now

$$\boldsymbol{B} \equiv \tfrac{1}{c}(\boldsymbol{v}\times\boldsymbol{E}) \tag{55}$$

Here c—a *constant with the dimensions of velocity* (it turns out in a moment to be associated with the speed of light)—has been introduced for no more fundamental purpose than to insure that $\boldsymbol{E}$ and $\boldsymbol{B}$ are dimensionally identical: $[\boldsymbol{E}] = [\boldsymbol{B}]$. $\boldsymbol{B}(\boldsymbol{x}, t)$ itself is a vector field which turns out to be associated with the phenomenology of *magnetism*.[15]

In view of the structure of (54) it becomes natural to inquire after the value of $\boldsymbol{\nabla}\cdot\boldsymbol{B}$. Drawing upon (5.4) we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{B} &= \tfrac{1}{c}\boldsymbol{\nabla}\cdot(\boldsymbol{v}\times\boldsymbol{E})\\ &= -\tfrac{1}{c}\boldsymbol{v}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) + \underbrace{\tfrac{1}{c}\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v})}_{0 \text{ by } \boldsymbol{x}\text{-independence of } \boldsymbol{v}}\end{aligned}$$

But O's equation

$$\boldsymbol{\nabla}\times\boldsymbol{E} = 0 \tag{22}$$

can by us (and again without the assistance of any additional assumptions) be written

$$\boldsymbol{\nabla}\times\boldsymbol{E} = 0 \tag{56}$$

So we have

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0} \tag{57}$$

It is a striking fact that the preceding boxed equations contain no reference to $\frac{\partial\boldsymbol{B}}{\partial t}$. But it follows from (55) by differentiation that

$$\tfrac{\partial\boldsymbol{B}}{\partial t} = \tfrac{1}{c}\big(\boldsymbol{v}\times\tfrac{\partial\boldsymbol{E}}{\partial t}\big)$$

which by (54) becomes

$$= \boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) - \underbrace{\tfrac{1}{c}(\boldsymbol{v}\times\boldsymbol{j})}_{0 \text{ because } \boldsymbol{v} \text{ and } \boldsymbol{j} \text{ are, by (50) parallel}}$$

Reading from (5.6) we have

$$\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{B}) - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} - \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v}$$

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[15] PROBLEM 12.

The leading term on the right presents $(\boldsymbol{v}\cdot\boldsymbol{B}) = \frac{1}{c}\boldsymbol{v}\cdot(\boldsymbol{v}\times\boldsymbol{B})$, which vanishes because $\boldsymbol{v}\perp(\boldsymbol{v}\times\boldsymbol{B})$. And the two final terms vanish because $\boldsymbol{v}$ is $\boldsymbol{x}$-independent. The surviving term can be developed

$$-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} = -\boldsymbol{v}\boldsymbol{\nabla}\cdot\boldsymbol{B} + \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) + \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{v} - (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v}$$

The leading term on the right vanishes by (57): $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. And again: the two final terms vanish because $\boldsymbol{v}$ is $\boldsymbol{x}$-independent. So we have

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) \tag{58}$$

of which, as we saw *en route*,

$$\frac{\partial\boldsymbol{B}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} = \boldsymbol{0} \tag{59}$$

provides an alternative formulation.[16,17]

Reviewing our progress ... we note that equations structurally similar to ours would be written by each of our inertial colleauges (our relation to O being entirely "typical"). Since we are related to O by a Galilean transformation (more specifically: by a *Galilean boost*, as rotation-free transformations of the form (47) are standardly called), and since the compose of two (or more) Galilean transformations is itself Galilean,[18] what we have in effect constructed is a *Galilean-covariant formulation of electrostatics*— a theory which gives back the standard theory (O's theory, as summarized on page 25) when $\boldsymbol{v}$ (whence also $\boldsymbol{j}$ and $\boldsymbol{B}$, by (50) and (55)) vanishes.

But the theory we seek is more ambitious. We seek a theory capable of describing the *electromagnetic interaction of charged mass points Q_i which are free to experience <u>arbitrarily complex relative motions</u>*. Looking in this light to the theory in hand, we note that

1) the $\boldsymbol{v}$ which enters into the definition (50) of $\boldsymbol{j}$ may be interpreted as referring to *our perception of the velocity of an existential thing* (a charged mass point), but that
2) every other reference to $\boldsymbol{v}$ is a source of acute embarrassment, for it is a reference to our perception of the velocity of a non-entity: an observer who sees all charges to be at rest. The inertial observers who perceive any *particular* Q_i to be *momentarily* at rest are easily discovered. But an observer O who sees *all* Q_i to be *constantly* at rest *does (in the general case) <u>not exist</u>*!

How to get along without the assistance of our "preferred observer"? How—with minimal formal damage—to eliminate the embarrassing $\boldsymbol{v}$-terms from our theory?

[16] If we apply $\boldsymbol{\nabla}\cdot$ to (58) we obtain $\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ which, while it does not imply, is certainly consistent with (57): $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$.

[17] All that has been said thus far pertains as accurately to our perception of O's *gravitostatics* as to our perception of his electrostatics.

[18] PROBLEM 13.

It is here that we have recourse to the "bootstrap" mentioned on page 27. And here, by the way, that we take leave of Newtonian gravitostatics.

We seem to be forced—both formally (at (55)) and phenomenologically (of which more later)—to retain something like $\boldsymbol{B}$ in our theory. But we eliminate one embarrassing $\boldsymbol{v}$ if

1) we *deny the invariable/general validity of* $\boldsymbol{B} \equiv \frac{1}{c}(\boldsymbol{v}\times\boldsymbol{E})$. This has the effect of *promoting* $\boldsymbol{B}(\boldsymbol{x},t)$ *to the status of an autonomous (if—at the moment—operatonally undefined) field* . . . entitled to all the privileges and respect that we are in the habit of according to $\boldsymbol{E}(\boldsymbol{x},t)$.

The proof of (57) now breaks down, but (note that (57) contains no $\boldsymbol{v}$-term, and must retain at least its electrostatic validity) the situation is saved if

2) we *promote* $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ *to the status of a law.*

3) We have every reason to *retain* $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$ *as it stands.* Noting that (53) and (54) conjointly imply charge conservation (which we wish to retain) and that (54) contains as it stands no $\boldsymbol{v}$-term, it seems to make conservative good sense if (tentatively)

4) we *promote* $\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$ *to the status of a law.*

Our problem assumes at last its full force in this question:

What to do with (58): $\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})$?

It was at just such a juncture that we were motivated at (55) to define $\boldsymbol{B}$. One is therefore tempted to write

$$\begin{aligned} \boldsymbol{C} &\equiv \tfrac{1}{c}(\boldsymbol{v}\times\boldsymbol{B}) \\ &= \tfrac{1}{c^2}(\boldsymbol{v}\times(\boldsymbol{v}\times\boldsymbol{E})) = \tfrac{1}{c^2}\big[(\boldsymbol{v}\cdot\boldsymbol{E})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{v})\boldsymbol{E}\big] \end{aligned} \tag{60}$$

and then to declare $\boldsymbol{C}(\boldsymbol{x},t)$ "autonomous." But such a program (which would amount to sweeping the dirt under the carpet) must—because of its allusion to $\boldsymbol{v}$—now be dismissed as conceptually unattractive. Besides, it would oblige us to search (by the methods of Galilean electrostatics?) for the field equations satisfied by $\boldsymbol{C}$. Such activity would certainly lead us to the field $\boldsymbol{v}\times\boldsymbol{C}$, and thus oblige us to *keep on* introducing such fields . . . a process which would terminate if and only if it were to turn out that at some stage the resulting "$\boldsymbol{Z}$-field" were a ($\boldsymbol{v}$-independent) linear combination of fields previously introduced, which is unlikely/impossible. How, therefore, to proceed?

When in a theoretical jam, it is never unfair to ask Nature for assistance. In this spirit (following Schwinger) we observe that

> There is abundant observational evidence—none of which was known to Maxwell!—that *light is an electromagnetic phenomenon*, that *charge-motion can give rise to radiation*, that *in charge-free regions of spacetime the electromagnetic field equations must possess wave-like solutions*.

where the *wave equation*, is, we recall, a 2nd-order partial differential equation of the form

$$\underbrace{\Big\{\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}-\nabla^2\Big\}}_{\substack{\text{the ``wave operator,'' sometimes called}\\ \text{the } d'Alembertian \text{ and denoted } \Box^2.}} f(\boldsymbol{x},t)=0$$

In charge-free regions of spacetime the equations in hand (set $\rho=0$ and $\boldsymbol{j}=\boldsymbol{0}$) read

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=0 \tag{61.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \tag{61.2}$$

$$\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E}-\boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{0} \tag{61.3}$$

$$\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B}-\boldsymbol{\nabla}\times\underbrace{\big(\tfrac{\boldsymbol{v}}{c}\times\boldsymbol{B}\big)}_{\text{problematic term}}=\boldsymbol{0} \tag{61.4}$$

Application of $\frac{1}{c}\frac{\partial}{\partial t}$ to (61.3) gives

$$\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\boldsymbol{E}-\boldsymbol{\nabla}\times\tfrac{1}{c}\tfrac{\partial\boldsymbol{B}}{\partial t}=\boldsymbol{0}$$

which by (61.4) becomes

$$\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\boldsymbol{E}-\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\big(\tfrac{\boldsymbol{v}}{c}\times\boldsymbol{B}\big)\big)=\boldsymbol{0}$$

Drawing now upon the general identity[19]

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\underbrace{\nabla^2\boldsymbol{A}}_{\substack{\text{means that } \nabla^2 \text{ acts separately}\\ \text{on each of the components of } \boldsymbol{A}}} \tag{62}$$

we obtain

$$\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}\boldsymbol{E}-\boldsymbol{\nabla}\Big\{\boldsymbol{\nabla}\cdot\big(\tfrac{\boldsymbol{v}}{c}\times\boldsymbol{B}\big)\Big\}+\nabla^2\big(\tfrac{\boldsymbol{v}}{c}\times\boldsymbol{B}\big)=\boldsymbol{0}$$

This would (by (61.1)) go over into the *vectorial wave equation*

$$\Big\{\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2}-\nabla^2\Big\}\boldsymbol{E}=\boldsymbol{0} \tag{63}$$

provided we set

$$\boldsymbol{E}=-\tfrac{1}{c}(\boldsymbol{v}\times\boldsymbol{B}) \tag{64}$$

Equation (64) is, however, *unacceptable*: it contains—as did the rejected equation (55)—an objectionable allusion to $\boldsymbol{v}$ (and would, moreover, imply $\boldsymbol{E}\to\boldsymbol{0}$ as $\boldsymbol{v}\to 0$: we would be *out of business*!). But our objective—(63)—would in fact be realized if we assumed (64) to hold in the specific context afforded by (61.4). Thus are we led—tentatively—

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[19] PROBLEM 14.

5) to *write* $\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}$ *in place of* (61.4)≡ (58).

By this strategy we have, in effect, *short-circuited at first opportunity* the "infinite regress problem" which (in connection with the $\boldsymbol{C}$-field) was discussed earlier. But in so doing we have (as will emerge) also done much else.[20]

The field equations that emerge from the heuristic arguments just outlined are precisely **Maxwell's equations**

$$\boxed{\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \rho \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{\nabla}\times\boldsymbol{B} &= \tfrac{1}{c}\big(\boldsymbol{j} + \tfrac{\partial}{\partial t}\boldsymbol{E}\big) \\ \boldsymbol{\nabla}\times\boldsymbol{E} &= \tfrac{1}{c}\big(\boldsymbol{0} - \tfrac{\partial}{\partial t}\boldsymbol{B}\big) \end{aligned}} \qquad \begin{aligned} &(65.1)\\ &(65.2)\\ &(65.3)\\ &(65.4)\end{aligned}$$

Here as always, heuristically generated statements could be dismissed out of hand if it could be shown that they were internally inconsistent. Once that test is passed, we acquire the obligation to *to show that our statements conform to physical experience*. It is those two complementary lines of activity—especially the latter—that will absorb our energy in all the pages which follow.

It is—on methodological grounds, and in view of the preceding allusion to "physical experience"—interesting to notice that the heuristic arguments which led us from

$$\text{electrostatics} \longrightarrow \text{Maxwellian electrodynamics}$$

give rise to *observationally incorrect physics* when applied to Newtonian gravitostatics. Electrodynamics and gravitodynamics "share Coulomb's law in the static approximation" but otherwise differ profoundly. Owing to the equivalence principle, the gravitational field acts as its own source—*i.e.*, is "self-coupled"—and so must be governed by *non-linear* partial differential equations . . . but *Maxwell's equations are* <u>linear</u> *partial differential equations*. The circumstance force to this obvious—but sometimes overlooked—point: *heuristic arguments cannot in general be promoted convincingly to the status of "proof;" by bootstrapping one* <u>*may infer but cannot expect to demonstrate*</u> *the "unique and necessary structure" of the enveloping theory*.

Maxwell's equations (+ boundary & initial data) enaable us in principle to compute the (generally *dynamical*) *electromagnetic* ($\boldsymbol{E},\boldsymbol{B}$)*-fields which are generated by prescribed source activity* (the latter described by ρ and $\boldsymbol{j}$), but tell us nothing about the converse problem: *How do charged mass points move in response to prescribed ambient electromagnetic fields*? More sharply: *What—given* $\boldsymbol{E}(\boldsymbol{x},t)$ *and* $\boldsymbol{B}(\boldsymbol{x},t)$ *is the force* $\boldsymbol{F}$ *experienced by a test charge* q *situated at* $(\boldsymbol{x},t)$? If we see q to be <u>at rest</u> then—by definition!—

$$\boldsymbol{F} = q\boldsymbol{E} \quad : \quad \text{note the } \boldsymbol{B}\text{-independence} \tag{66}$$

[20] In particular we have denied the universal validity of (56): $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$. Note, however, that we recover (56) when $\boldsymbol{B}$ is t-independent: $\partial\boldsymbol{B}/\partial t = \boldsymbol{0}$.

. . . but *What if we see q to be in motion*? An inertial observer O' who sees q to be momentarily at rest would—since (66) is "shared physics"—write $\boldsymbol{F}' = q'\boldsymbol{E}'$. The question therefore arises:

> How do the quantities $(\boldsymbol{E}', \boldsymbol{B}', \ldots)$ seen by O' relate to the quantities $(\boldsymbol{E}, \boldsymbol{B}, \ldots)$ seen by another inertial observer O?

The surprising answer to this question shows that we should in the general case write

$$\boldsymbol{F} = q\big(\boldsymbol{E} + \tfrac{1}{c}\underset{\uparrow}{\boldsymbol{v}}\times\boldsymbol{B}\big) \tag{67}$$

$\llcorner$refers to our perception of q's instantaneous velocity

The *Lorentz force law* (67) is of importance partly because it removes a problem which has been a source of embarrassment ever since we declared the $\boldsymbol{B}$-field to be "autonomous:" it makes possible an *operational definition* of $\boldsymbol{B}$.

The resolution of the transformation-theoretic question

$$(\boldsymbol{E}', \boldsymbol{B}', \ldots) \xrightarrow{\quad\textbf{?}\quad} (\boldsymbol{E}, \boldsymbol{B}, \ldots)$$

posed above turns out to be "surprising" in this profoundly consequential sense: The heuristic arguments which *led* us to Maxwell's equations (65) drew strongly upon the Galilean transformation (47). But the Maxwell equations themselves are (as will be shown, and the statements (49) and (52) notwithstanding) *not Galilean covariant; i.e.*, they do not preserve their form under tha action of (47/48). It is attention to this critical point which will lead us—as historically it led Lorentz and Einstein—to the formulation of **special relativity**. One need only glance at the history of 20^{th} Century physics ($t \geqslant 1905$)—indeed: of 20^{th} Century civilization—to get a sense of how incredibly consequential a formal subtlety can be![21]

My objective in the next few sections will be to describe, if only in the most preliminary terms, some of the most *characteristic phenomenological consequences of Maxwell's equations.* Note in this connection that if in (65) we set $\boldsymbol{j} = \boldsymbol{B} = \mathbf{0}$ we obtain (as remarked already on page 31) the equations

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{E} &= \rho \\ \frac{\partial\boldsymbol{E}}{\partial t} = \nabla\times\boldsymbol{E} &= \mathbf{0} \\ \therefore\ \frac{\partial\rho}{\partial t} &= 0 \end{aligned}\right\} \tag{69}$$

which were seen at (22) to be fundamental to *electrostatics*, and of which the phenomenological consequences were discussed in some (by no means exhaustive) detail already in §1.

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[21] PROBLEMS 15 & 16

3. Current, and the principle of charge conservation. We begin by discussing a simple *corollary* of Maxwell's equations. If (recall PROBLEM 12) we construct $\frac{\partial}{\partial t}(65.1) + c\boldsymbol{\nabla\cdot}$**(65.3)** we obtain

$$\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla\cdot j} = 0 \tag{70 ≡ 51}$$

Equations of this particular structure are (for reasons which will emerge) called "continuity equations:" it is because we attach specific interpretations to ρ and $\boldsymbol{j}$ that (70) becomes the "charge conservation equation."

Important insight into the meaning of (70)—and of continuity equations generally—can be obtained as follows: Let $\mathcal{R}$ be a t-independent "bubble" in $\boldsymbol{x}$-space, and let $\rho(\boldsymbol{x},t)$ and $\boldsymbol{j}(\boldsymbol{x},t)$—fields which we shall assume to be in conformity with (70)—be given. The *total charge* $Q(t)$ *contained within* $\mathcal{R}$ can be described

$$Q(t) = \iiint_{\mathcal{R}} \rho(\boldsymbol{x},t)\,d^3x$$

Looking now to the *rate of temporal variation of* Q we have (see Figure 7)

$$\dot{Q} = \iiint_{\mathcal{R}} \frac{\partial\rho(\boldsymbol{x},t)}{\partial t}\,d^3x$$

NOTE: An additional term—describing the "rate at which $\mathcal{R}$ gobbles up charge"—would be required had we allowed $\mathcal{R}$ to be t-dependent.

$$= -\iiint_{\mathcal{R}} \boldsymbol{\nabla\cdot j}\,d^3x \quad \text{by (70)}$$

$$= -\iint_{\partial\mathcal{R}} \underbrace{\boldsymbol{j\cdot dS}} \quad \text{by Gauß' theorem: (14)}$$

charge flux through the surface element $\boldsymbol{dS}$. Since $\boldsymbol{dS}$ is "outward directed," we have

$$= -\big\{\text{total flux outward through the surface } \partial\mathcal{R} \text{ of } \mathcal{R}\big\}$$

The implication is that *no "birth" or "death" processes contribute to* $\dot{Q}$ . . . which is what we mean when we say that "charge is conserved." The generality of the argument follows from the observation that it works whenever

- ρ is a *density* and
- $\boldsymbol{j}$ is the corresponding *flux density*.

We see that

$$\frac{d}{dt}\iiint_{\mathcal{R}} \rho(\boldsymbol{x},t)\,d^3x + \iint_{\partial\mathcal{R}} \boldsymbol{j}(\boldsymbol{x},t)\boldsymbol{\cdot dS} = 0 \tag{71}$$

expresses globally the information which (70) expresses locally. From the requirement that (71) hold for *all* t-independent bubbles $\mathcal{R}$ one can in fact *recover* (70).

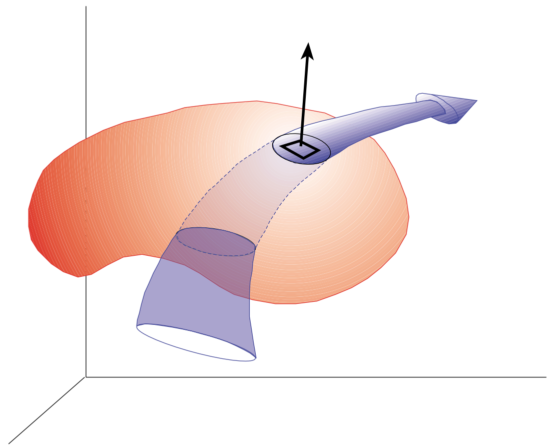

FIGURE 7: *In a spatial region occupied by a drifting charge cloud* $\boldsymbol{j}(\boldsymbol{x},t)$*—represented here by the fat blue arrow—the argument on the preceding page asks us to designate a "bubble"* $\mathcal{R}$*, and to identify the rate of change of enclosed charge with the rate at which charge is transported into* $\mathcal{R}$ *through its surface* $\partial\mathcal{R}$*.*

What, specifically, is the meaning of the statement that $\boldsymbol{j}$ is by nature a measure of (electric) "flux density"? It follows from (70)—whence ultimately from (65)—that

$$\begin{aligned}[\,\boldsymbol{j}\,] = [\rho\,]\cdot\text{velocity} &= \text{charge}\cdot\text{velocity}/\text{volume}\\ &= \text{charge}/\text{area}\cdot\text{time}\end{aligned}$$

We infer that

$$\boldsymbol{j}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{dS} = \begin{cases}\text{instantaneous rate (at time } t\text{) at which}\\ \text{charge is being transported through a}\\ \text{little "window" } \boldsymbol{dS} \text{ situated at position } \boldsymbol{x}\end{cases}$$

Recalling the definition of "$\boldsymbol{\cdot}$" we have

$$\begin{aligned}&= j\cdot dS\cdot\cos\theta\\ &\qquad\qquad \theta \equiv \text{angle between } \hat{\boldsymbol{j}} \text{ and } \boldsymbol{S};\ i.e., \text{ the}\\ &\qquad\qquad\quad \text{window's "presentation angle"}\end{aligned}$$

It is important to appreciate that the $\boldsymbol{j}$ here under consideration is a more general conception than the $\boldsymbol{j}\equiv\rho\,\boldsymbol{v}$ contemplated at (50) . The latter is literally appropriate only if the charge which flows through the window $\boldsymbol{dS}$ does so coherently—as a unitary entity endowed with a single, well-defined velocity—while the $\boldsymbol{j}$ contemplated in (70) refers only to the *effective mean drift* of the charge at $(\boldsymbol{x},t)$. The distinction is illustrated in Figure 8.

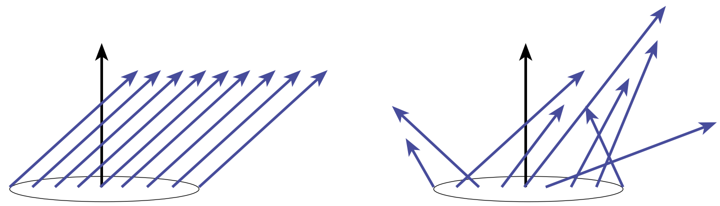

FIGURE 8: *Charges q stream through an inspection window with identical velocities (or "coherently") at left, and with statistically distributed velocities ("incoherently") at right.*

The fields ρ and $\boldsymbol{j}$ pertain most naturally to "continuum physics," and have to strain a bit to accommodate the microscopic physical fact that charge always rides around on localized bits of matter. Let $\boldsymbol{x}(t)$ and $\boldsymbol{v}(t) \equiv \dot{\boldsymbol{x}}(t)$ describe the motion of a charged mass point (m, q). To describe the associated ρ and $\boldsymbol{j}$ we might write

$$\begin{aligned} \rho(\boldsymbol{x}, t) &= q\,\delta(\boldsymbol{x} - \boldsymbol{x}(t)) \\ \boldsymbol{j}(\boldsymbol{x}, t) &= q\,\delta(\boldsymbol{x} - \boldsymbol{x}(t))\,\boldsymbol{v}(\boldsymbol{x}, t) \end{aligned} \tag{72}$$

These singular fields—can you show that they satisfy (70)?—acquire the correct physical dimensionality from the circumstance that

$$[\delta(\boldsymbol{x})] = (\text{volume})^{-1}$$

The restrictive equation

$$\tfrac{\partial}{\partial t}\rho = 0 \tag{73}$$

is familiar from §1, where it was interpreted as referring to "charges that don't move." If, however, we reflect upon the meaning of its mate

$$\tfrac{\partial}{\partial t}\boldsymbol{j} = \boldsymbol{0} \tag{74}$$

we see that (73) admits of a more relaxed interpretation. For (74) *requires that the charges move, but in such a stereotyped manner that they keep replacing each other*. Phrased another way: $\frac{\partial}{\partial t}\rho = 0 \Rightarrow \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$ (by (70)), but does not, of itself, require $\boldsymbol{j} = \boldsymbol{0}$. Equations (73/74) may be satisfied momentarily, "accidentally," at isolated spacetime points, but tend to be of practical importance only when they hold *globally*. Source fields $\rho(\boldsymbol{x})$ and $\boldsymbol{j}(\boldsymbol{x})$ are—for the reason just mentioned—best described not as "static" but as t-independent or *steady*. In view of the fact that it is so easy to build steady ρ-fields with an isolated point charge, it becomes interesting to note that (except in the trivial sense $\boldsymbol{j} = \boldsymbol{0}$) one *cannot build a steady $\boldsymbol{j}$-field with a single charge*. Interesting to note also that the reason appears to be not logical, not electrodynamical . . . but (see Figure 9) *mechanical*: one runner can't (in the continuous limit) "keep running by with velocity $\boldsymbol{v}$"—even if the racetrack is infinitely short. The

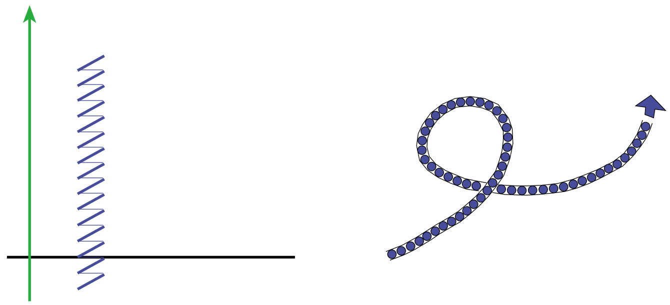

FIGURE 9: *At left, a single charge attempts in vain to "keep running past" an inspection point. At right, entrained charges achieve the intended effect (production of a steady* $\boldsymbol{j}$*) by serially replacing one another. But even with the latter arrangement we cannot produce a steady* $\boldsymbol{j}$ *which vanishes everywhere except* <u>at a point</u>.

problem would disappear if Nature provided not only point charges but true *line charges* (charged strings). Absent those, we are forced to build our steady $\boldsymbol{j}$-fields with the aid of *entrained point charges*: we "glue charges on a string, pull the string ... and pretend not to notice the microscopic granularity." The operation (see again the preceding figure) is most commonly called "sending a current through a wire."

In many practical contexts—particularly those which arise from engineering—it is more common to speak of the *current* I than of the current density $\boldsymbol{j}$. These concepts are related as follows: Let $\mathcal{D}$ be (topologically equivalent—see Figure 10—to) a "disk," and let $\partial\mathcal{D}$ denote its boundary (a closed curve). Given $\boldsymbol{j}(\boldsymbol{x},t)$, we form

$$I(t;\mathcal{D}) \equiv \iint_D \boldsymbol{j}\cdot\boldsymbol{dS} \tag{75}$$

to obtain a measure of the *instantaneous rate at which charge is (at time t) being transported through* $\mathcal{D}$, *i.e.*, of the <u>total charge flux</u> through $\mathcal{D}$. Evidently

$$[\,I\,] = [\,\boldsymbol{j}\,]\cdot\text{area} = \text{charge/time}$$

Engineers perefer to measure currents I in *Amperes* $\equiv$ *Coulombs/second*. Note that on a disk $\boldsymbol{dS}$ is sign-ambiguous ("outside" being undefined). A disk $\mathcal{D}$ endowed with a sign convention is said to be "oriented." Evidently we are, for the purposes of (75), obliged to require that $\mathcal{D}$ be orient*able*: no Möbius strips allowed! For a given $\boldsymbol{j}$-field one expects to have

$$I(t;\mathcal{D}_1) \neq I(t;\mathcal{D}_2) \quad \text{even when} \quad \partial\mathcal{D}_1 \equiv \partial\mathcal{D}_2$$

It is therefore of some interest that one can show without difficulty[22] that if ρ is steady and if, moreover, $\mathcal{D}_1$ and $\mathcal{D}_2$ share the same boundary ($\partial\mathcal{D}_1 \equiv \partial\mathcal{D}_2$),

[22] PROBLEM 17.

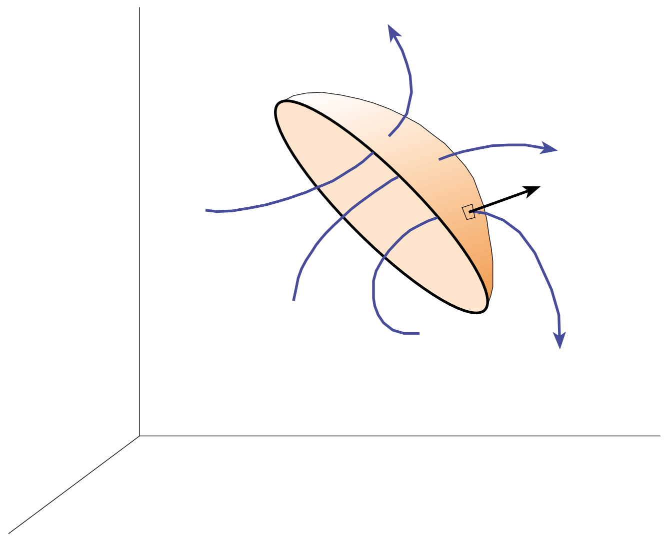

FIGURE 10: *The net current through a hypothetical cap* $\mathcal{D}$ *is, according to* (75), *found by integrating* $\boldsymbol{j}_\perp$*—the normal component of current density, the vector field represented here by (blue) directed curves.*

then $I(t;\mathcal{D}_1) = I(t;\mathcal{D}_2)$. In such (frequently encountered) contexts there is no reason to retain any explicit allusion either to t or to $\mathcal{D}$: it becomes more natural to write $I(\partial\mathcal{D})$, and then to drop the $\partial\mathcal{D}$ as "obviouis from the context." Thus does one acquire the privilege of referring simply to "the current I."

To describe the *current I in a wire*[23] we have only to suppose (see Figure 11) that $\partial\mathcal{D}$ circumscribes the wire. Phenomenologically, the current I in a wire of cross-sectiuonal area A can be described

$$I = nevA \qquad \begin{cases} n \text{ is the number of charge carriers per unit volume} \\ e \text{ is the charge per carrier (= elecronic charge)} \\ v \text{ is the mean drift velocity of the charge carriers} \end{cases}$$

People are often surprised to discover that (because n is typically quite large) v is typically quite small. If in (76) we assign I, n, e and A the values appropriate to a *1 amp current in a 14-gauge copper wire* (radius $R = 0.0814$ cm) we find that the drift velocity $v = 3.55\times10^{-3}$ cm/sec: evidently *the physics of electrical signal propagation has very little to do with the physics of charge carrier drift*.

It will be appreciated that the currents encountered in Nature, and of fundamental interst to physicists, are for the most part *not confined to wires*

[23] We agree here to overlook the "charge accumulation effects" which may arise at high frequencies: we agree, in other words, to "think DC."

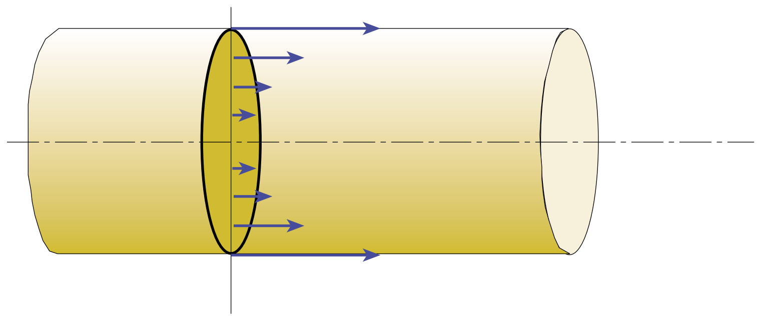

FIGURE 11: *Variant of the preceding figure, adapted to the problem of evaluating the current* <u>*in a wire*</u>*. The "cap" has in this instance become simply a cross-section of the wire. The presumption in the figure is that* $\boldsymbol{j}$ *is axially symmetric but non-uniform, being strongest near the "skin" of the wire.*

. . . wires—and nerves—being "rare objects" in the universe. "Unconfined currents" are found in (for example) lightning bolts and throughout the natural world, and in some engineering applications (arcs welders, vacuum tubes, electrochemical process vats, particle accelerators).

Returning now to more theoretical matters . . . the *interests of symmetry* would clearly be served if in place of (65) one had

$$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} = \rho \quad , \quad \boldsymbol{\nabla}\!\times\!\boldsymbol{B} = +\tfrac{1}{c}\big(\boldsymbol{j} \quad + \tfrac{\partial \boldsymbol{E}}{\partial t}\big) \tag{77.1}$$

$$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{B} = \rho_m\,, \quad \boldsymbol{\nabla}\!\times\!\boldsymbol{E} = -\tfrac{1}{c}\big(\boldsymbol{j}_m + \tfrac{\partial \boldsymbol{B}}{\partial t}\big) \tag{77.2}$$

where the subscript $_m$ means "magnetic." Then the argument which when applied to (77.1) gave

$$\tfrac{\partial}{\partial t}\rho \quad + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{j} \quad = 0 \tag{70}$$

would when applied to (77.2) give

$$\tfrac{\partial}{\partial t}\rho_m + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{j}_m = 0$$

We would, in charge-free regions, still have $\Box^2\boldsymbol{E} = \Box^2\boldsymbol{B} = \boldsymbol{0}$, etc. and all would be well. From this point of view *the actual structure of Maxwell's equations (65) is seen to contain an informative surprise*: (65.2)—$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{B} = 0$—states in effect that "point magnetic charges" or (as they are called)

$$\text{"magnetic monopoles"} \ \ldots \underline{\text{do not exist}} \tag{78}$$

It is, therefore, not at all surprising that the $\boldsymbol{j}_m$ term is absent from (65.4); *i.e.*, that "neither do magnetic currents exist." Several questions—particularly since Maxwell's equations seem in this respect to be so "permissive"—now arise:

1. *Is* (78) *factually true*? The answer must come from the laboratory. Many sophisticated searches have thus far failed to detect a single magnetic monopole. Such activity continues (if as a fairly low-priority item), and reports of the results are always received with interest by the international community of physicists.[24]

2. Is it possible that magnetic *monopoles exist but cannot be observed*? A weak instance of such a situation would arise if monopoles were bound to oppositely charged monopoles by forces so strong that they *cannot be feasibly dislodged and studied in isolation*. The "strong" interpretation—that "monopoles exist but cannot be observed *in principle*—would appear to *strike at the philosophical foundations of science*, to be latently "unscientific" . . . unless it were argued that monopoles (like quarks?) *announce themselves not in isolation but indirectly—by their effects*.

3. Physicists have come to adhere generally—if informally—to the view that

"all which is not forbidden is mandatory"

This heuristic principle suggests that monopoles—if *not* forbidden (by some yet-undiscovered conservation law?)—will eventually (by their direct or indirect effects) be detected, and in the contrary case gives rise to this sharp question: *What (presently unknown) principle effectively "forbids" the existence of magnetic monopoles*?

There is (as will emerge in a subsequent chapter) an interesting—if but little-known—sense in which (78) misrepresents the physical situation: (78) expresses not a fact (?) but a "fact wrapped in a convention." If the (elementary) particles found in Nature carried magnet charge p as well as electric charge q, then to describe the compound charge structure (q,p) of a particle population one might present something like the topmost of the following figures. It is, however, a surprising fact of Nature that (central figure) the observed points lie *on a line; i.e.*, that

$$p/q \equiv \tan\theta$$

has a <u>value shared by all known elementary particles</u>. *This* is the elemental fact which awaits explanation. It is by (seldom remarked) operational *convention*

[24] . . . Particularly since, in the two decades since this paragraph was written, it has become the clear tendency of theoretical developments (in elementary particle physics, cosmology) to *demand* the physical existence of magnetic monopoles!

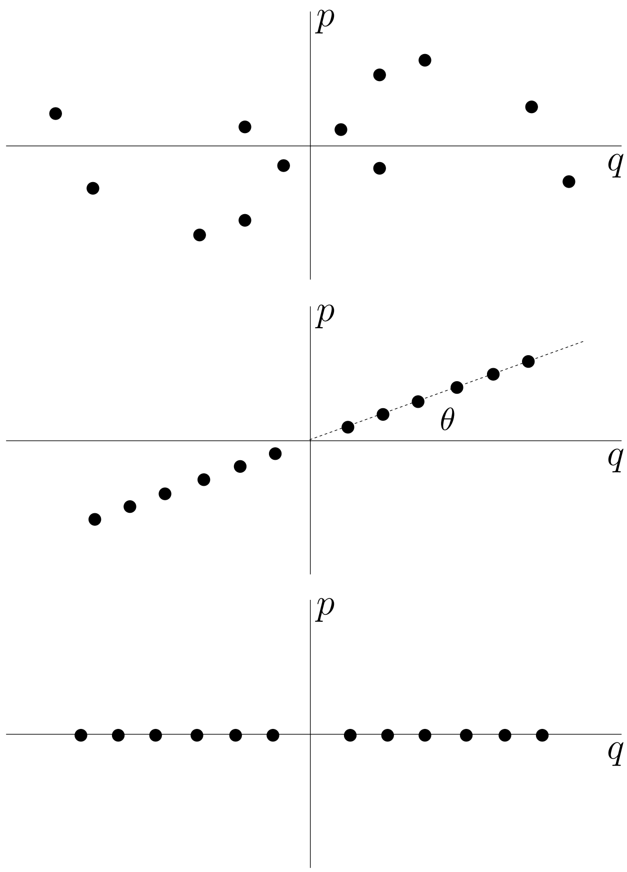

FIGURE 12: *At top: the kind of (q, p)-distribution that one naively might expect to encounter in Nature. In the center: the distribution one in fact encounters. At bottom: θ-rotational invariance has been used to eliminate the magnetic components from all (q, p)-pairs. The surprising fact is that the same rotation works in all cases.*

that we have—essentially by (67): $\boldsymbol{F} = q\big(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\big)$—set $\theta = 0$ (bottom figure). It is, in other words, *by convention that we have associated the observed "(q, p)-line" with the "electric axis" in "charge space."* Later I will have occasion to discuss the deep formal symmetry (θ-rotational invariance) of Maxwell's equations which permits one to *exercise* such an option.

I turn finally to some historical points. It is roughly—but *only* roughly—correct to state that at some point in the later developmental stages of his electrodynamical work Maxwell realized that

- GAUSS' LAW (of which Coulomb's law is a corollary, and which is not to be confused with Gauß' theorem) can be rendered

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \tfrac{1}{\epsilon_0}\rho \tag{79.1}$$

- The NON-EXISTENCE OF MAGNETIC MONOPOLES can be rendered

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{79.2}$$

- AMPERE'S LAW[25] can be rendered

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{j} \tag{79.3}$$

- FARADAY'S LAW[26] can be rendered

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\tfrac{\partial\boldsymbol{B}}{\partial t} \tag{79.4}$$

Here ϵ_0 and μ_0 are empirical constants made necessary by the fact that Maxwell —working close to laboratory experience—used practical units to quantify electromagnetic variables . . . while the subscript $_0$ reflects his tendency (born of the same circumstance) to view the vacuum as "degenerate matter." Maxwell realized more particularly that equations (79), while they account for most of the phenomenology known to him, cannot be correct . . . for it follows from (79.3) that $\boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$, which conforms to (70), *i.e.*, to the principle of charge conservation, *only in the steady case*.[27] Thus was Maxwell led at length to propose—on no direct observational evidence!—that in place of (79.3) one should write[28]

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\big(\boldsymbol{j} + \epsilon_0\tfrac{\partial\boldsymbol{E}}{\partial t}\big) \tag{80}$$

> REMARK: The new term $\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$—because it enters as a kind of companion to $\boldsymbol{j}$—was called by Maxwell the "displacement current." It's introduction represents a somewhat spooky modification of (79.3), for *it involves no charge motion*. Oddly, Maxwell felt no obligation to attach a similar name/interpretation to the $\frac{\partial\boldsymbol{B}}{\partial t}$-term in Faraday's law (79.4).

The first writing of (80) was, in my view, one of the most seminal events in 19^{th} Century physics: indeed, in the entire history of physics. For it gave rise —automatically—to a fully detailed *electromagnetic theory of light* . . . and thus by implication to relativity, quantum mechanics and all that follows therefore. How did this come about?

[25] Describes the $\boldsymbol{B}$-fields generated by steady currents. . . of which more later.

[26] Describes the $\boldsymbol{E}$-fields generated by changing $\boldsymbol{B}$-fields. . . of which again: more later.

[27] This is hardly surprising in view of the fact that *Ampere's law was abstracted from steady-case observations*.

[28] PROBLEM 20.

In charge-free regions of space Maxwell's own equations (79)—as modified —read[29]

$$\nabla\cdot\boldsymbol{E} = 0 \tag{81.1}$$
$$\nabla\cdot\boldsymbol{B} = 0 \tag{81.2}$$
$$\nabla\times\boldsymbol{B} = \mu_0\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t} \tag{81.3}$$
$$\nabla\times\boldsymbol{E} = \quad -\frac{\partial\boldsymbol{B}}{\partial t} \tag{81.4}$$

REMARK: From (81.4) we see that in Maxwell's units

$$[\boldsymbol{E}\,] = \text{velocity}\cdot[\boldsymbol{B}\,]$$

It follows therefore from (81.3) that

$$[\,\mu_0\epsilon_0\,] = (\text{velocity})^{-2}$$

Equations (81) can be "separated by differentiation,"[30] giving

$$\big(\mu_0\epsilon_0\tfrac{\partial^2}{\partial t^2} - \nabla^2\big)\boldsymbol{E} = \big(\mu_0\epsilon_0\tfrac{\partial^2}{\partial t^2} - \nabla^2\big)\boldsymbol{B} = \boldsymbol{0}$$

It was the observation that the measured values of μ_0 and ϵ_0 entail

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} \approx 3\times 10^{10}\ \text{cm/sec}$$

which led Maxwell (1863) to write that "we can scarcely avoid the inference that *light consists of undulations in the medium which is the cause of electric and magnetic phenomena"* [his italics]. This was an idea which had fallen out of the blue into Maxwell's lap, but in which he obviously had great confidence[31] . . . though it remained *merely* an idea at his death, and for eight years thereafter: electromagnetic radiation was first generated/detected by H. Hertz in 1887.

HISTORICAL REMARK: The excitement of discovery experienced by Maxwell would today be impossible . . . because the upshot of his discovery has—by recent international convention—been made a cornerstone of physical metrology:

$$\mu_0 = 4\pi\times 10^{-7} = 12.566370614\ldots\ \text{N A}^{-1}$$

and

$$c = 299792458\ \text{m s}^{-1}$$

are both now held to be *exact*, and

$$\epsilon_0 = (\mu_0 c^2)^{-1}\ \underline{\text{by modern definition}}!$$

---

[29] Note that the right side of (81.3) was introduced by Maxwell to salvage an equation—(70)—which is *not even relevant* in charge-free space!

[30] The procedure was encountered already in PROBLEM 15.

[31] See Maxwell's curiously understated discussion in Chapter XX of his *Treatise on Electricity & Magnetism*. C. W. F. Everitt, in his *James Clerk Maxwell: Physicist & Natural Philosopher* (1975), provides a good brief account of the history of Maxwell's discovery, and on page 101 reports that in 1865 Maxwell ". . . in a rare moment of unveiled exuberance to wrote to [a friend that] 'I have also a paper afloat, containing an electromagnetic theory of light, which, till I am convinced to the contrary, I hold to be great guns.' "

I draw attention finally to one formal point which was only recently brought to light,[32] but which I find to be of deep interest. If in (81.3) we set $\mu_0\epsilon_0 = 1/c^2$ and then proceed to the limit $c\uparrow\infty$ we find that one and only one thing happens: the $\frac{\partial \boldsymbol{E}}{\partial t}$-term, which Maxwell was at such pains to introduce, is *extinguished*! We recover precisely the charge-free version of (79). What Jean-Marc Lévy-Leblond was evidently the first to notice (1967) is that (as the reader may verify) the equations

$$\begin{aligned} \boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} &= 0\\ \boldsymbol{\nabla}\!\cdot\!\boldsymbol{B} &= 0\\ \boldsymbol{\nabla}\!\times\!\boldsymbol{B} &= \boldsymbol{0}\\ \boldsymbol{\nabla}\!\times\!\boldsymbol{E} &= -\tfrac{\partial \boldsymbol{B}}{\partial t}\end{aligned}$$

are covariant with respect to[33] the following extension

$$\left.\begin{aligned} t &\longmapsto t' = t\\ \boldsymbol{x} &\longmapsto \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t\\ \tfrac{\partial}{\partial t} &\longmapsto \tfrac{\partial}{\partial t'} = \tfrac{\partial}{\partial t} + \boldsymbol{v}\!\cdot\!\boldsymbol{\nabla}\\ \boldsymbol{\nabla} &\longmapsto \boldsymbol{\nabla}' = \boldsymbol{\nabla}\\ \boldsymbol{E} &\longmapsto \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}\!\times\!\boldsymbol{B}\\ \boldsymbol{B} &\longmapsto \boldsymbol{B}' = \boldsymbol{B}\end{aligned}\right\} \tag{82}$$

of what at (47/48) we meant by a "Galilean boost." We conclude that—though the point was not appreciated by Maxwell himself—"Maxwell's trick" enforced the *abandonment of Galilean relativity, & the adoption of Einsteinian relativity*.

I hope readers will by now understand why it seems to me not entirely frivolous to suggest that "20$^{\text{th}}$ Century physics is a *grandchild of the principle of charge conservation*" ... or, more precisely, of the *symmetry principle* of which charge conservation is the physical manifestation.

4. Generation of B-fields: Ampere's law. Having reviewed already the "physical upshot" of $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} = \rho$, $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{B} = 0$ and $\partial\rho/\partial t + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{j} = 0$ we turn now to a similarly preliminary discussion of the physical significance of the statement

$$\boldsymbol{\nabla}\!\times\!\boldsymbol{B} = \tfrac{1}{c}\big(\boldsymbol{j} + \underbrace{\tfrac{\partial \boldsymbol{E}}{\partial t}}_{\text{Maxwell's stroke of genius}}\big) \tag{65.3 ≡ 80 ≡ 83}$$

... which is, unlike the statements studied previously, *vector*-valued.

[32] See Section V, §C.1 (page 267) of J-M. Lévy-Leblond's "Group theory and Galilean invariance" in E. M. Loebl (ed.), *Group Theory and Its Applications II* (1971). Also §2 in P. G. Bergmann, "The special theory of relativity" in Volume IV of *Handbüch der Physik* (1962).

[33] Compare (48).

Hans Christian Oersted had expressed his intuitive conviction that "electricity & magnetism must be interrelated" already in 1812 (in his *View of Chemical Laws*), but it was during (!) a lecture in the spring of 1820 that he discovered "electromagnetism;" *i.e.*, that *electric currents give rise to magnetic fields*. Oersted's discovery immediately engaged the excited attention of the leading scientists of the day (J. B. Biot, F. Savant, H. Davy, the young M. Faraday, . . .), and when (11 September 1820) Oersted repeated his lecture/ demonstration before members of the Académie des Sciences it came to the attention of André Marie Ampere (1775–1836). I mention these facts partly in order to suggest that it is somewhat inappropriate that we associate with (83) the name of Ampere, rather than that of Oersted. Since Ampere's own work was concerned mainly with the *magnetic interaction of currents* (*i.e.*, of current-carrying wires), it might more appropriately be attached to what we now call the "Biot-Savart law" (see below).

The experimental work to which I have just referred involved <u>steady currents</u>—made possible by Volta's then-recent invention (1800) of the *voltaic cell*.[34] When the sources (whence also their associated fields) are steady the $\frac{\partial \boldsymbol{E}}{\partial t}$-term drops away from (83) and we obtain

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \tfrac{1}{c}\boldsymbol{j} \tag{84}$$

It is with the phenomenological implications (not of (83) but) of (84) that will mainly concern us in the paragraphs which follow. And it is the analytical problem posed by equations of the form (84) that motivates the following

MATHEMATICAL DIGRESSION

A population of elementary theorems of exceptional beauty and power (which could, until rather recently, have been described as "well known to every student of analytical geometry") follows from the idea developed in

Problem 21. Show that the area A of a triangle (012), which is oriented and coordinatized as indicated in the following Figure 13, can be described

$$A(012) = \tfrac{1}{2}\begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} \tag{85}$$

This can be accomplished in many ways: you might, for example, try assembling the triangle from simpler triangles, then drawing upon elementary propeties of determinants. Note that (85) refers

[34] Alessandro Volta' work in this area was stimulated by Luigi Galvani's famous chance observation (1791) that electrical discharges caused the legs of dead frogs, laid out for dissection, to twitch. We may therefore add a frog to the items which already repose in (page 3) our little "museum of biogenic relics."

implicitly to an *orientation convention*, which supplies

$$\begin{aligned} A(012) &= A(120) = A(201) \\ &= -A(021) = -A(210) = -A(102) \end{aligned}$$

and tells us that $A \gtrless 0$ according as the triangle is right or left handed (↺ or ↻).

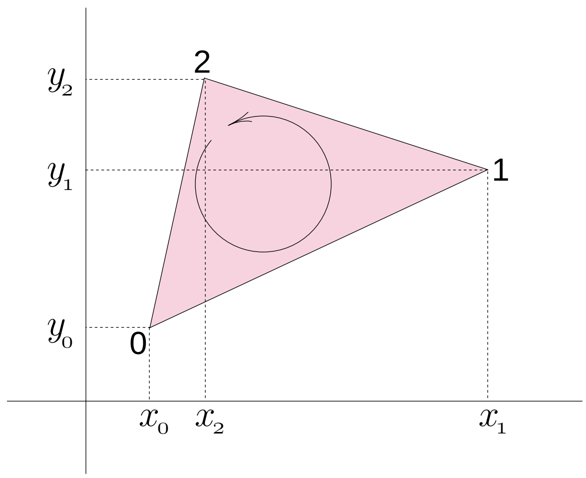

FIGURE 13: *Labeled geometrical construction used in* PROBLEM 21 *to establish the the determinantal description of area.*

Drawing now upon (85) we infer that the *area* $A(\mathcal{D})$ *of an arbitrary plane region* $\mathcal{D}$ (see Figure 14) can be described either

$$A(\mathcal{D}) = \iint_{\mathcal{D}} dxdy \tag{86.1}$$

or

$$= \text{limit of sum of areas } \underbrace{\tfrac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & x & y \\ 1 & x+dx & y+dy \end{vmatrix}} \text{ of triangular slivers}$$

But

$$= \tfrac{1}{2}(x\,dy - y\,dx)$$

so

$$= \tfrac{1}{2}\oint_{\partial\mathcal{D}} (x\,dy - y\,dx) \tag{86.2}$$

According to (86.2) one can compute area by operations that are <u>restricted to the boundary</u> $\partial\mathcal{D}$ of the region $\mathcal{D}$ in question. This surprising fact provides the

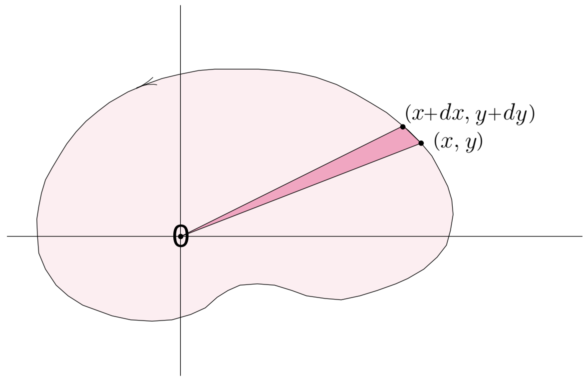

FIGURE 14: *Computing areas by adding triangular slivers—the geometrical basis of* (86.2).

operating principle of the *polar planimeter*—a wonderful device used mainly by architects and engineers.[35]

Let the preceding construction be considered now to be inscribed on the xy-plane in Euclidean 3-space, and let us agree that x, y and z refer henceforth to a *right*-handed frame. Readers will find it very easy to verify that (86)—thus situated—can be formulated

$$\iint_{\mathcal{D}} (\boldsymbol{\nabla}\times\boldsymbol{A})\boldsymbol{\cdot}\boldsymbol{dS} = \oint_{\partial\mathcal{D}} \boldsymbol{A}\boldsymbol{\cdot}\boldsymbol{d\ell} \tag{87}$$

provided we set

$$\boldsymbol{d\ell} = \begin{pmatrix} dx \\ dy \\ 0 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} -y \\ +x \\ 0 \end{pmatrix}$$

$$\boldsymbol{dS} = \begin{pmatrix} 0 \\ 0 \\ dxdy \end{pmatrix}$$

This result provides an instance—and its derivation provides some insight into the proof—of *Stokes' theorem*, according to which (87) holds *generally ... for <u>all</u> (even non-flat) disks* $\mathcal{D}$ *in 3-space, and for <u>all</u> vector fields* $A(x, y, z)$.

[35] For a masterful discussion of this pretty topic see Felix Klein's *Elementary Mathematics from an Advanced Standpoint: Geometry* (1925), pages 11–15. The subject is treated also on pages 57–65 of Chapter I in my QUANTUM TOPICS (2000).

It is evident that (see again pages 13 & 19) the following statements

$$\begin{aligned}
\text{Gauss} &: \quad \iiint_{\mathcal{R}} \text{div}\boldsymbol{A}\, d^3x = \iint_{\partial\mathcal{R}} \boldsymbol{A}\cdot\boldsymbol{dS} \\
\text{Stokes} &: \quad \iint_{\mathcal{D}} \text{curl}\boldsymbol{A}\cdot\boldsymbol{dS} = \oint_{\partial\mathcal{D}} \boldsymbol{A}\cdot\boldsymbol{d\ell} \\
\text{Newton} &: \quad \int_{\mathcal{C}} \text{grad}\,\varphi\cdot\boldsymbol{d\ell} = \varphi\Big|_{\text{endpoints of }\mathcal{C}}
\end{aligned}$$

are "of a type." They originate in the work of many 19th Century physicist-mathematicians (Gauß, Green, Kelvin, Tait, Maxwell, Cauchy, Stokes, . . .), and have come to bear collectively the name of George Gabriel Stokes (1819–1903) for curious reasons that are explained on page *viii* of M. Spivak's *Calculus on Manifolds* (1965: see particularly the cover illustration!). Such identities were first studied *in unified generality* by H. Poincaré (1887), whose work was deepened and given its modern formulation—of which more later—mainly by Élie Cartan (∼1920).[36] "Stokes' theorems" are available even on n-dimensional non-Euclidean manifolds (where there are n such things), and all share the design

$$\int_{\text{region}} \text{differentiated object} = \int_{\text{boundary of region}} \text{undifferentiated object}$$

foreshadowed already in the

$$\textsc{fundamental theorem of the calculus} \quad : \quad \int_a^b f'x\,dx = f(b) - f(a)$$

END OF DIGRESSION

Just as

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho \tag{65.1}$$

and

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{65.2}$$

give rise by Gauß' theorem (14) to

$$\iint_{\partial\mathcal{R}} \boldsymbol{E}\cdot\boldsymbol{dS} = \iiint_{\mathcal{R}} \rho\, d^3x \tag{90.1}$$

$$\Updownarrow$$

total "electric flux" through $\partial\mathcal{R}$ = total charge interior to $\mathcal{R}$

and

$$\iint_{\partial\mathcal{R}} \boldsymbol{B}\cdot\boldsymbol{dS} = 0 \tag{90.2}$$

[36] PROBLEMS 22, 23 & 24. Equations (88) and (89)—absent from the text—appear in the first two of those exercises.

so do

$$\nabla\times\boldsymbol{B} = \tfrac{1}{c}\big(\boldsymbol{j} + \tfrac{\partial \boldsymbol{E}}{\partial t}\big) \tag{65.3}$$

and

$$\nabla\times\boldsymbol{E} = \tfrac{1}{c}\big(\boldsymbol{0} - \tfrac{\partial \boldsymbol{B}}{\partial t}\big) \tag{65.4}$$

give rise by Stokes' theorem (87) to

$$\begin{aligned}\int_{\partial\mathcal{D}} \boldsymbol{B}\cdot\boldsymbol{d\ell} &= \tfrac{1}{c}\iint_{\mathcal{D}} \big(\boldsymbol{j} + \tfrac{\partial \boldsymbol{E}}{\partial t}\big)\cdot\boldsymbol{dS}\\ &= \tfrac{1}{c}\Big\{I + I_d\Big\}\end{aligned} \tag{90.3}$$

and

$$\int_{\partial\mathcal{D}} \boldsymbol{E}\cdot\boldsymbol{d\ell} = -\tfrac{1}{c}\iint_{\mathcal{D}} \tfrac{\partial \boldsymbol{B}}{\partial t}\cdot\boldsymbol{dS} \tag{90.4}$$

where

$$I \equiv \iint_{\mathcal{D}} \boldsymbol{j}\cdot\boldsymbol{dS}$$

is the *conduction current* through $\mathcal{D}$, and

$$\begin{aligned}I_d &\equiv \iint_{\mathcal{D}} \tfrac{\partial \boldsymbol{E}}{\partial t}\cdot\boldsymbol{dS}\\ &= \tfrac{\partial}{\partial t}\underbrace{\iint_{\mathcal{D}} \boldsymbol{E}\cdot\boldsymbol{dS}}_{\text{“electric flux”}} \quad \text{if } \mathcal{D} \text{ is not itself moving}\end{aligned}$$

is the *displacement current*.[37]

Equations (90) comprise the so-called "integral formulation of Maxwell's equations," and in some conceptual/computational contexts—particularly those which possess a high degree of symmetry—prove more directly informative than their differential counterparts (65).

Example. *What is the $\boldsymbol{E}$-field generated by a static point charge q?* Let the "Gaussian pillbox" $\mathcal{R}$ be spherical, or radius r and centered on q. A familiar symmetry argument implies $\boldsymbol{E} = E(r)\,\hat{\boldsymbol{r}}$, so the $\iint$

[37] "Displacement current" is standardly given that name but not a symbol; I have borrowed my $_d$ convention from E. M. Purcell, *Electricity & Magnetism: Berkeley Physics Course, Volume 2* (1963), page 261. Its magnetic analog

$$\iint_{\mathcal{D}} \tfrac{\partial \boldsymbol{B}}{\partial t}\cdot\boldsymbol{dS} = \tfrac{\partial}{\partial t}\underbrace{\iint_{\mathcal{D}} \boldsymbol{B}\cdot\boldsymbol{dS}}_{\text{“magnetic flux”}} \quad \text{if } \mathcal{D} \text{ is not itself moving}$$

is standardly given *neither* a name nor a symbol.

on the left side of (90.1) acquires immediately the value $E(r)4\pi r^2$. From (90.1) we are led thus back again to Coulomb's force law

$$\boldsymbol{E} = E(r)\,\hat{\boldsymbol{r}} \quad \text{with} \quad E(r) = q/4\pi r^2$$

The $\boldsymbol{E}$-field generated by an *arbitrtary* charge distribution ρ could now be assembled by superposition.

Example. *What is the $\boldsymbol{B}$-field generated by a steady current I in an infinitely long straight wire*? Resolve $\boldsymbol{B}$ into parallel, radial and tangential components

$$\boldsymbol{B} = \boldsymbol{B}_{\parallel} + \boldsymbol{B}_r + \boldsymbol{B}_t$$

as indicated in the figure. By symmetry, the magnitude of each can

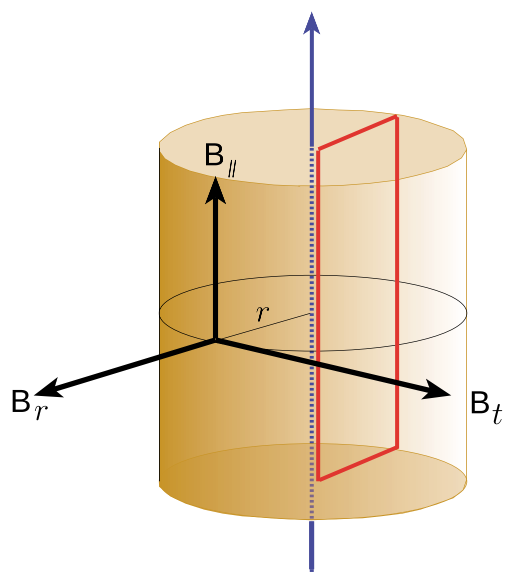

FIGURE 15: *Cylindrical pillbox concentric about a straight wire carrying a steady current. The box has radius r and height h.*

depend only upon r. Equation (90.2) supplies

$$\iint_{\mathcal{R}} \boldsymbol{B}\cdot d\boldsymbol{S} = B_r(r)2\pi r h = 0 \quad \Rightarrow \quad \boldsymbol{B}_r = \boldsymbol{0}$$

while by (90.3) we have

$$\oint_{\text{red rectangle}} \boldsymbol{B}\cdot d\boldsymbol{\ell} = h\big[B_{\parallel}(r_2) - B_{\parallel}(r_1)\big] = 0 \quad \Rightarrow \quad \boldsymbol{B}_{\parallel} = \text{constant}$$

and since we expect to have $\boldsymbol{B}(\infty) = \boldsymbol{0}$ this entails $\boldsymbol{B}_{\parallel} = \boldsymbol{0}$. Finally

$$\oint_{\text{circular cap}} \boldsymbol{B}\cdot d\boldsymbol{\ell} = B_t(r)2\pi r = \tfrac{1}{c}I \tag{91}$$

The implication that the magnetic field "wraps around" the wire, and has a strength that falls off as $1/r$ (*i.e.*, "geometrically," since the system is effectively 2-dimensional). Whether the $\boldsymbol{B}$-field generated by an *arbitrary* steady $\boldsymbol{j}$ could now "be assembled by superposition" (of current-carrying straight wires) remains an interesting open question.[38]

We are in position now to confront the generality of this fundamental question: *What is the $\boldsymbol{B}$-field generated by an arbitrary steady current*? It proves most efficient to proceed not from the integral formulation (90) but from the differential formulation (65) of Maxwell's equations. Just as

the equations

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$$

and $$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$$

give rise to electrostatics, so do

the equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

and $$\boldsymbol{\nabla}\times\boldsymbol{B} = \tfrac{1}{c}\boldsymbol{j}$$

give rise to *magnetostatics*,

the conditions $\partial\rho/\partial t = \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$ being *shared* by the two subjects in question.

The equation $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$ can, by (6.1), be read as stating that there exists a *scalar potential* φ such that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$$

We note that φ is determined only to within a *gauge transformation*

$$\varphi \to \varphi' = \varphi + \text{constant}$$

and that one can thus arrange that φ vanishes at some given "reference point." Similarly

the equation $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ can, by (6.2), be read as stating that there exists a *vector potential* $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \tag{92}$$

We note that $\boldsymbol{A}$ is determined only to within a *gauge transformation*

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \text{grad}\,\chi$$

and that one can thus arrange that $\boldsymbol{A}$ shall in particular satisfy

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0 \tag{93}$$

---

[38] See page 193 in Purcell's celebrated textbook.[37]

The field equation $\boldsymbol{\nabla}\times\boldsymbol{E}=\boldsymbol{0}$ follows automatically from $\boldsymbol{E}=-\boldsymbol{\nabla}\varphi$, while $\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho$ becomes

$$\nabla^2\varphi=-\rho$$

This is an inhomogeneous linear equation, the solution of which can, as we have seen (page 16) be described

$$\varphi(\boldsymbol{x})=\iiint G(\boldsymbol{x}-\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,d^3\xi$$

where

$$\nabla^2 G(\boldsymbol{x}-\boldsymbol{\xi})=-\delta(\boldsymbol{x}-\boldsymbol{\xi})$$

entails

$$G(\boldsymbol{x}-\boldsymbol{\xi})=\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{\xi}|}$$

The $\boldsymbol{E}$-field itself is given therefore by

$$\boldsymbol{E}(\boldsymbol{x})=\iiint -\boldsymbol{\nabla}G(\boldsymbol{x}-\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,d^3\xi$$

where $-\boldsymbol{\nabla}G(\boldsymbol{x}-\boldsymbol{\xi})=\dfrac{\boldsymbol{x}-\boldsymbol{\xi}}{4\pi|\boldsymbol{x}-\boldsymbol{\xi}|^3}$

Similarly

The field equation $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ follows automatically from $\boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A}$, while $\boldsymbol{\nabla}\times\boldsymbol{B}=\frac{1}{c}\boldsymbol{j}$ becomes

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\tfrac{1}{c}\boldsymbol{j}$$

But $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}$ so if we install the gauge condition $\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$ we obtain

$$\nabla^2\boldsymbol{A}=\tfrac{1}{c}\boldsymbol{j}\tag{94}$$

This is a trio of inhomogeneous linear equations, the solutions of which can evidently be described

$$\boldsymbol{A}(\boldsymbol{x})=\tfrac{1}{c}\iiint G(\boldsymbol{x}-\boldsymbol{\xi})\,\boldsymbol{j}(\boldsymbol{\xi})\,d^3\xi\tag{95}$$

where

$$\nabla^2 G(\boldsymbol{x}-\boldsymbol{\xi})=-\delta(\boldsymbol{x}-\boldsymbol{\xi})$$

entails

$$G(\boldsymbol{x}-\boldsymbol{\xi})=\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{\xi}|}$$

The $\boldsymbol{B}$-field itself is given therefore by

$$\boldsymbol{B}(\boldsymbol{x})=\tfrac{1}{c}\iiint\begin{pmatrix}0&-\partial_3&\partial_2\\ \partial_3&0&-\partial_1\\ -\partial_2&\partial_1&0\end{pmatrix}G(\boldsymbol{x}-\boldsymbol{\xi})\,\boldsymbol{j}(\boldsymbol{\xi})\,d^3\xi$$

By straightforward calculation[39] we are led thus to

$$=\tfrac{1}{4\pi c}\iiint\frac{\boldsymbol{j}(\boldsymbol{\xi})\times\hat{\boldsymbol{r}}}{r^2}\,d^3\xi\tag{96}$$

with $\boldsymbol{r}\equiv\boldsymbol{r}(\boldsymbol{x},\boldsymbol{\xi})\equiv\boldsymbol{x}-\boldsymbol{\xi}$.

[39] PROBLEM 25.

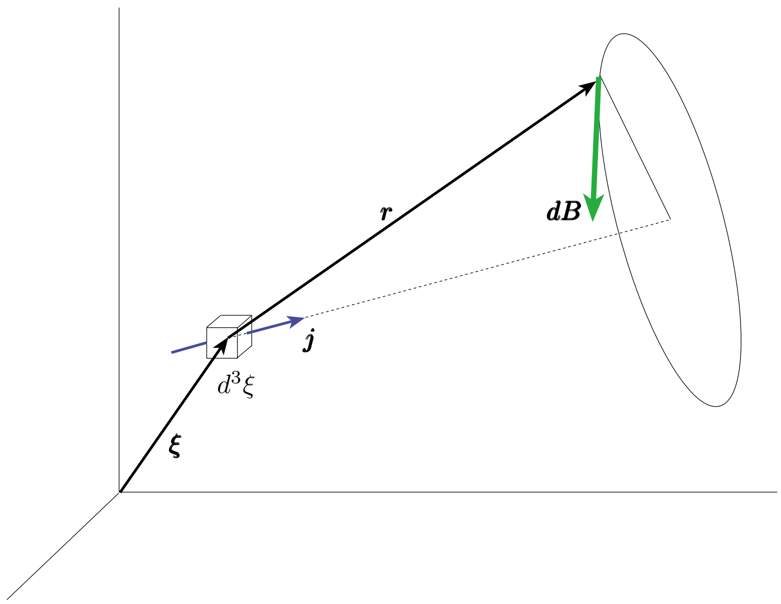

FIGURE 16: *Geometrical meaning of the notations used at* (97) *to describe the differential contribution* $\boldsymbol{dB}$ *to the magnetic field* $\boldsymbol{B}(\boldsymbol{x})$ *at a typical field-point* $\boldsymbol{x}$ *arising from the current differential* $\boldsymbol{j}(\boldsymbol{\xi})\,d^3\xi$ *at a typical source-point* $\boldsymbol{\xi}$.

Equation (96)—though analytically a corollary of Ampere's law (84)—is known standardly (and with more historical justice) as the *Biot–Savart law*. It describes <u>the $\boldsymbol{B}$-field generated by an arbitrary steady current distribution $\boldsymbol{j}$</u>, and invites "interpretation-by-superposition" along lines which emerge if (see the figure) we write

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{x}) &= \int \boldsymbol{dB}(\boldsymbol{x},\boldsymbol{\xi}\,) \\ \boldsymbol{dB}(\boldsymbol{x},\boldsymbol{\xi}\,) &= \tfrac{1}{4\pi c}\frac{[\boldsymbol{j}(\boldsymbol{\xi})\,d^3\xi\,]\times\hat{\boldsymbol{r}}}{r^2} \end{aligned} \tag{97}$$

The interpretation of $\boldsymbol{j}(\boldsymbol{\xi})\,d^3\xi$ is, however, a little bit odd. The object in question is perfectly meaningful *in context* (*i.e.*, under the $\int$), but—for the reasons remarked already on page 38—*could not be realized in isolation*.

Later we shall have occasion to study illustrative applications of (96), but for the moment must rest content with a single

> **Example.** *What—according to (96)—is the $\boldsymbol{B}$-field generated by a steady current I in an infinitely long straight wire?* Taking our notation from the following figure, it is immediate that

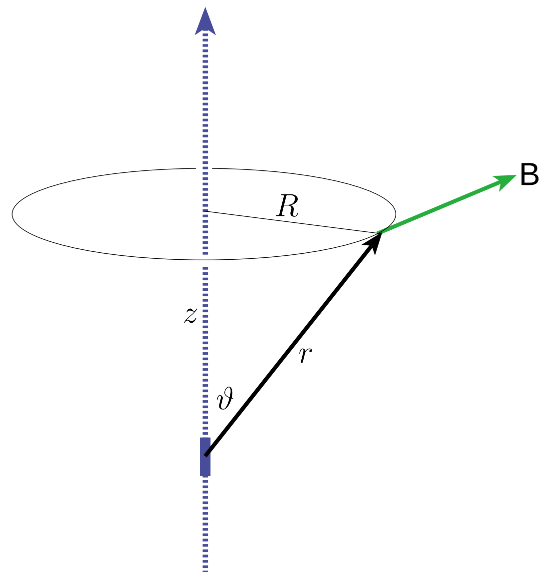

FIGURE 17: *Notation employed in computing the magnetic field* $\boldsymbol{B}$ *generated by current in an infinitely long straight wire.*

$$\begin{aligned}
B(R) &= \tfrac{1}{4\pi c}I\int_{-\infty}^{+\infty}\tfrac{1}{r^2}\sin\vartheta\,dz\\
&= \tfrac{1}{4\pi c}I\underbrace{\int_{-\infty}^{+\infty}\frac{R}{(R^2+z^2)^{\frac{3}{2}}}\,dz}\\
&\qquad\qquad = \frac{z}{R\sqrt{R^2+z^2}}\bigg|_{-\infty}^{+\infty} = \frac{2}{R}\\
&= \frac{I}{2\pi cR}
\end{aligned}$$

—which agrees precisely with the result (91) obtained previously by other means.

It should be noticed that if the Biot-Savart law were *postulated* (*i.e.*), abstracted from laboratory experience then the equations $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ and $\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c}\boldsymbol{j}$—our starting point—could have been recovered as *corollaries* of (96).

I turn now to discussion of the question which was central to Ampere's own work in this area: *What is the force which (steady) currents exert upon one another by virtue of the magnetic fields which they generate*? Suppose, by way of preparation, that

1) impressed fields $\boldsymbol{E}$ and $\boldsymbol{B}$
2) source functions ρ and $\boldsymbol{j}$

are defined on the neighborhood d^3x of a representative point $\boldsymbol{x}$. From the Lorentz force law (67)

$$\boldsymbol{F} = q(\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{B})$$

we infer that the total force experienced by the charges which momentarily occupy d^3x can be described

$$\begin{aligned}\boldsymbol{dF} &= \boldsymbol{\mathfrak{F}}(\boldsymbol{x})d^3x\\ \boldsymbol{\mathfrak{F}} &= \rho\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{j}\times\boldsymbol{B} \equiv \textit{force density}\end{aligned}$$

If we look more specifically to the situation typical of <u>wires</u> (where "charge carriers" drift through a population of oppositely charges) we have $\rho = 0$ whence

$$= \tfrac{1}{c}\boldsymbol{j}\times\boldsymbol{B} \tag{98}$$

> Wires—even wires carrying current—are standardly uncharged, and therefore don't feel ambient $\boldsymbol{E}$-fields.

If (see the first of the following figures) we integrate (98) over a snippet $\boldsymbol{d\ell}$ of wire we obtain

$$\boldsymbol{df} = \int_{\text{snippet}} \boldsymbol{dF} = \tfrac{1}{c}I\boldsymbol{d\ell}\times\boldsymbol{B} \tag{99}$$

Suppose now that $I \equiv I_1$ and $\boldsymbol{d\ell} \equiv \boldsymbol{d\ell}_1$ refer (see the second of the following figures) to a closed <u>loop</u> $\mathcal{L}_1$ of wire, and that $\boldsymbol{B}$ arises from a (steady) current I_2 in a second loop $\mathcal{L}_2$. From (97) and (99) we conclude that the *force* $\boldsymbol{f}_{12}$ *exerted* <u>*on* $\mathcal{L}_1$ *by* $\mathcal{L}_2$</u> can be described

$$\boldsymbol{f}_{12} = \tfrac{1}{4\pi c^2}I_1 I_2 \oint_{\mathcal{L}_1}\oint_{\mathcal{L}_2} \frac{\boldsymbol{d\ell}_1\times(\boldsymbol{d\ell}_2\times\hat{\boldsymbol{r}}_{12})}{r_{12}^2} \tag{100}$$

It is to this implausible, non-local (*i.e.*, distributed, whence geometry-dependent) result that the name of Ampere is most properly attached. Looking now to some of the implications of (100) ...from $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ we have

$$= \tfrac{1}{4\pi c^2}I_1 I_2\left\{\oint_{\mathcal{L}_1}\left[\oint_{\mathcal{L}_2}\frac{\boldsymbol{r}_{12}\cdot\boldsymbol{d\ell}_1}{r_{12}^3}\right]\boldsymbol{d\ell}_2 - \oint_{\mathcal{L}_1}\oint_{\mathcal{L}_2}\frac{\boldsymbol{r}_{12}}{r_{12}^3}(\boldsymbol{d\ell}_1\cdot\boldsymbol{d\ell}_2)\right\}$$

But (recall (15.1)) $\boldsymbol{r}_{12}/r_{12}^3 = -\boldsymbol{\nabla}_1(1/r_{12})$ so we have

$$\Big[\text{etc.}\Big] = -\frac{1}{r_{12}}\Big|_{\text{starting point}}^{\text{endpoint}} = 0 \quad \text{for a loop}$$

giving

$$\begin{aligned}\boldsymbol{f}_{12} &= -\tfrac{1}{4\pi c^2}I_1 I_2\oint_{\mathcal{L}_1}\oint_{\mathcal{L}_2}\frac{\boldsymbol{r}_{12}}{r_{12}^3}(\boldsymbol{d\ell}_1\cdot\boldsymbol{d\ell}_2) \qquad (101)\\ &= \begin{cases}\text{net force on circuit } \mathcal{L}_1 \text{ due to interaction with}\\ \text{magnetic field generated by current in circuit } \mathcal{L}_2\end{cases}\end{aligned}$$

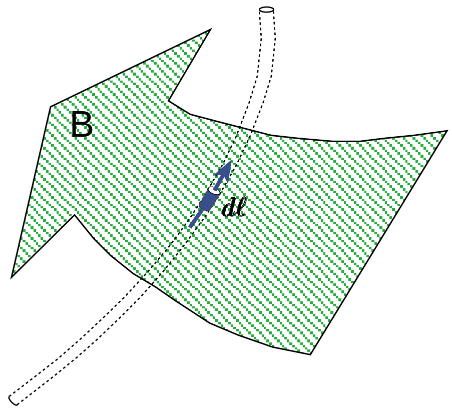

FIGURE 18: *Snippet $d\boldsymbol{\ell}$ of wire carrying a current I in the presence of an ambient magnetic field $\boldsymbol{B}$.*

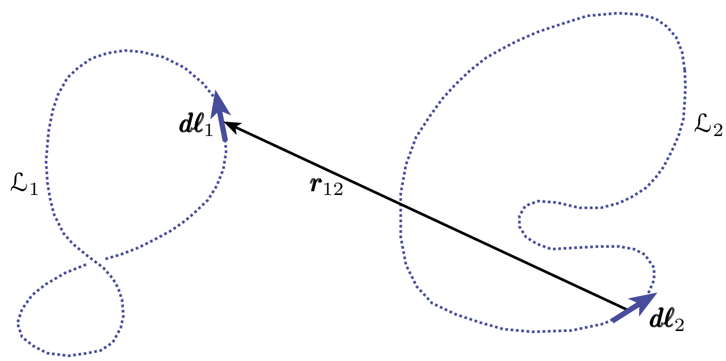

FIGURE 19: *Interaction of current I_1 in snippet $d\boldsymbol{\ell}_1$ with magnetic generated by current I_2 in snippet $d\boldsymbol{\ell}_2$.*

From $\boldsymbol{r}_{12} = -\boldsymbol{r}_{21}$ we conclude that *the forces of interaction between steady current loops conform to Newton's* 3^{rd} *law*:

$$\boldsymbol{f}_{12} = -\boldsymbol{f}_{21} \tag{102}$$

The structure of (100) encourages one to suppose that the equation in question arises by superposition from a statement of the form

$$\text{force on } I_1 d\boldsymbol{\ell}_1 \text{ by } I_2 d\boldsymbol{\ell}_2 = \tfrac{1}{4\pi c^2}\frac{I_1 d\boldsymbol{\ell}_1 \times (I_2 d\boldsymbol{\ell}_2 \times \hat{\boldsymbol{r}}_{12})}{r_{12}^3} \tag{103.1}$$

Observing that the vector on the right lies in the plane spanned by $I_2 d\boldsymbol{\ell}_2$ and $\boldsymbol{r}_{12}$, we conclude that

$$\neq -\text{force on } I_2 d\boldsymbol{\ell}_2 \text{ by } I_1 d\boldsymbol{\ell}_1 \tag{103.2}$$

i.e., that *the element-element interaction which purportedly lies at the root of (101) does* <u>*not*</u> *conform to Newton's* 3^{rd} *law*.

> People frequently proceed from this fact to the (in my view) profoundly misguided conclusion that Newton's 3^{rd} law is "soft ...that it holds except when it doesn't." The correct conclusion, it seems to me, is that <u>the isolated current element $I\boldsymbol{d\ell}$ is a hazardous abstraction</u>.

This surprising result is illustrated in the following figure. The figure suggests also that an unknotted current-carrying loop will tend (by magnetic self-interaction) to deform until circular ... which for a closed loop means "as nearly

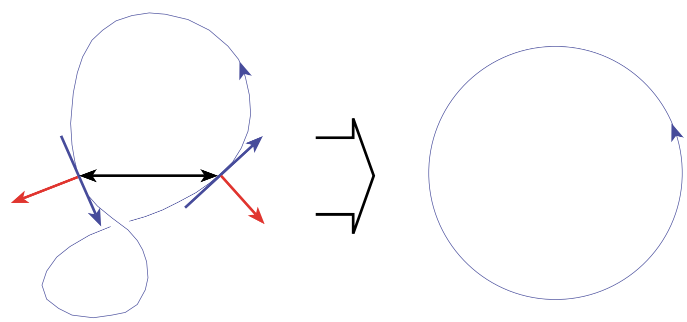

FIGURE 20: *Red arrows in the figure at left refer to the interaction* (103) *of two current elements in a filamentary circuit. If the filament is flexible we expect it to assume the circular form shown at right.*

straight as possible." From (103) we see that current elements in a *straight* wire *do not interact at all*—whence again the inference: "current-carrying wires like, for magnetic reasons, to be as straight as possible." We come away with the impression that electrical devices in which the

1) geometry and/or
2) operative I-values

favor the production of substantial $\boldsymbol{B}$-fields ... must be strongly constructed, for they will be subjected generally to a tendency to explode![40]

Since wires and electrical devices are "unnatural/artificial" in the sense that they more often the work of engineers than of Nature, it is attractive to suppose that (103) arises as a corollary from

$$\text{force on } \boldsymbol{j}(\boldsymbol{x}_1)d^3x_1 \text{ by } \boldsymbol{j}(\boldsymbol{x}_2)d^3x_2 = \tfrac{1}{4\pi c^2}\frac{\boldsymbol{j}(\boldsymbol{x}_1)\times\big(\boldsymbol{j}(\boldsymbol{x}_2)\times\hat{\boldsymbol{r}}_{12}\big)}{r_{12}^3}d^3x_1d^3x_2 \tag{104}$$

and to view (104) as the *magnetic analog of Coulomb's law*.

[40] PROBLEM 26.

The pattern provided by our prior discussion of electrostatics (see especially pages 19–24) makes it natural to inquire finally into the <u>*energetics* of *magneto*static fields</u>. But we encounter at once some unexpected conceptual difficulties: it is unnatural (taking the argument of pages 19–20 as our model) to attempt to position the current elements $\boldsymbol{j}(\boldsymbol{x})d^3x$ "one at a time" because

1) "isolated point currents" *do not exist*;
2) we would stand in *violation of charge conservation* (*i.e.*, of $\boldsymbol{\nabla\cdot j} = 0$) until the assembly is complete;
3) the assembly process entails that we work against forces which *violate Newton's* 3rd *law*.

It is better practice to *build the* $\boldsymbol{j}$*-field by slowly <u>turning it on</u>* . . . but this, by (65.4), involves "Faraday emf effects" which we are not presently in position to calculate. My plan, therefore, will be simply to *present* the formula in question (several lines of supporting argument will be reviewed later) and to develop its formal relationship to its electrostatic counterpart:

In electrostatics we obtained

$$W = \int \mathcal{E}(\boldsymbol{x})\, d^3x$$

where $\mathcal{E} \equiv \frac{1}{2}\boldsymbol{E\cdot E}$ defines the electrostatic energy density. Thus

$$W = \tfrac{1}{2}\int \boldsymbol{E\cdot E}\, d^3x$$

which arose (at page 22) from

$$W = -\tfrac{1}{2}\int \underbrace{\boldsymbol{E}\cdot\boldsymbol{\nabla}\varphi}\, d^3x$$

This in turn came—use

$$\equiv -\varphi\boldsymbol{\nabla\cdot E} + \boldsymbol{\nabla}(\varphi\boldsymbol{E})$$

and discard the surface term—from

$$W = \tfrac{1}{2}\int \varphi\,\boldsymbol{\nabla\cdot E}\, d^3x$$

which we got (by $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$) from

$$\begin{aligned} &= -\tfrac{1}{2}\int \varphi\,\nabla^2\varphi\, d^3x \\ &= \tfrac{1}{2}\int \rho\,\varphi\, d^3x \\ &= \tfrac{1}{8\pi}\iint \rho(\boldsymbol{x})\rho(\boldsymbol{\xi})\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\, d^3x d^3\xi \end{aligned}$$

Proceeding similarly (but in reverse), → in magnetostatics we write

$$W = \int \mathcal{B}(\boldsymbol{x})\, d^3x$$

where $\mathcal{B} \equiv \frac{1}{2}\boldsymbol{B\cdot B}$ defines the *magnetostatic energy density*. Thus

$$W = \tfrac{1}{2}\int \boldsymbol{B\cdot B}\, d^3x \tag{105}$$

giving

$$W = \tfrac{1}{2}\int \underbrace{\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})}\, d^3x$$

But

$$\equiv \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) + \boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B})$$

so—discarding the surface term—

$$W = \tfrac{1}{2}\int \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})\, d^3x$$

From $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and the gauge condition $\boldsymbol{\nabla\cdot A} = 0$ it now follows that

$$\begin{aligned} &= -\tfrac{1}{2}\int \boldsymbol{A}\cdot\nabla^2\boldsymbol{A}\, d^3x \\ &= -\tfrac{1}{2c}\int \boldsymbol{j\cdot A}\, d^3x \\ &= \tfrac{1}{8\pi c^2}\iint \boldsymbol{j}(\boldsymbol{x})\cdot\boldsymbol{j}(\boldsymbol{\xi})\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\, d^3x d^3\xi \end{aligned} \tag{106}$$

We note that the formal parallel is perfect. Also

1) that the $\int$ in (105) ranges over the field, while the $\int$ in (106) ranges only over its source;
2) that the W of (105) is a *non-negative & non-linear* number-valued functional of $\boldsymbol{B}$;
3) that true line currents give rise to a variant of the familiar *self-energy problem*, and that so (for other reasons) do currents which have their termini at ∞.

The previously-remarked tendency of current-carrying wires to move around can be considered now to follow—by the "nameless principle" of page 23—from the fact that in so doing they may reduce the energy stored in the associated $\boldsymbol{B}$-field.[41]

To conclude: the discussion in recent pages derives mainly from Ampere's law

$$\nabla\times\boldsymbol{B} = \tfrac{1}{c}\boldsymbol{j} \tag{84}$$

—the phenomenological consequences of which have been seen to conform to the diverse physical facts, and to come to this: currents generate and respond to magnetic fields. Maxwell's modification

$$\nabla\times\boldsymbol{B} = \tfrac{1}{c}\big(\boldsymbol{j} + \tfrac{\partial\boldsymbol{E}}{\partial t}\big)$$

implies that $\boldsymbol{B}$-fields are generated also by *temporal variation of* $\boldsymbol{E}$ ... of which more later.

5. Faraday's law. Coulomb had argued (from evidence) that "electrical and magnetic phenomena are unrelated." It is because Coulomb's view had come to be widely shared that news of Oersted's discovery (of "electromagnetism:" 1820) produced such perplexed excitement among French physicists ... and generated the developments reported in the preceding seciton of these notes. Across the Channel, Sir Humphry Davy (1778–1829) was receiving regular reports—jumbled and contradictory as they at the time seemed—of the work of his French colleagues. His assistant (Michael Faraday) repeated the basic experiments and entered into correspondence with Ampere, whose work had earned him a position of acknowledged leadership among the French. Faraday confessed openly that he could not understand the mathematical aspects of Ampere's work ... but took intuitive exception to some of Ampere's interpretive comments/ideas, particularly those concerning the microscopic meaning of "current." Absent the observational data required to settle the issue, Ampere and Faraday "agreed to disagree," and Faraday turnjed to other matters.

By 1830, Faraday (under the influence of Charles Wheatstone: 1802–1875) had developed an interest in physical acoustics ... and particularly in the Chladni patterns which are set up on one membrane when another (distant)

[41] PROBLEM 27.

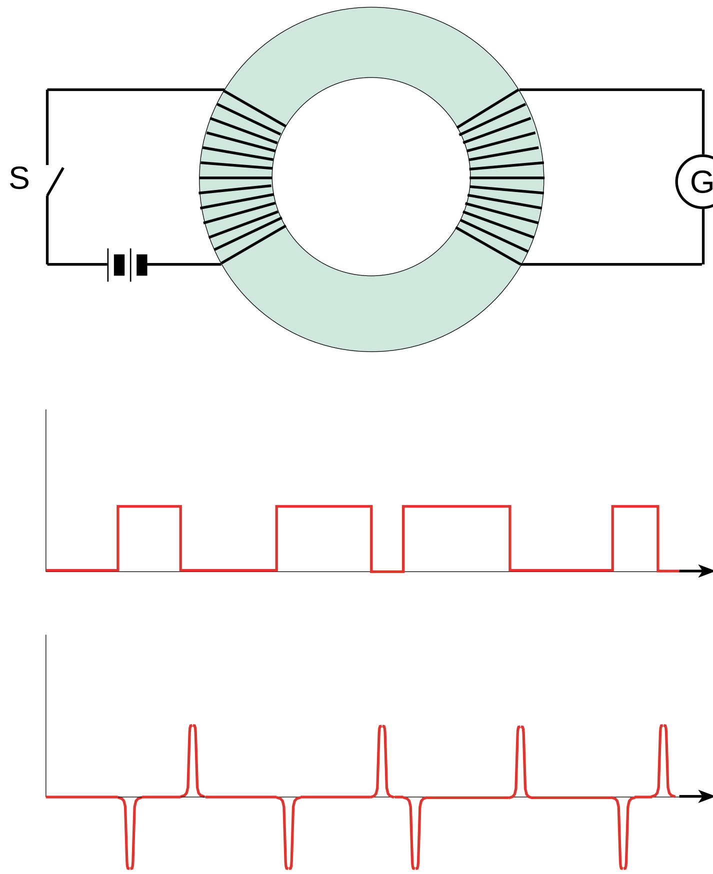

Figure 21: *At top, the experimental set-up used by Faraday. The idea of using a soft iron ring to link the primary and secondary coils was borrowed from Joseph Henry; without it the induced current would have been undetectably small. The middle figure shows the current in the primary that results from closing/opening the switch S. The graph of the induced current (lower figure) does not mimic the primary current, but shows spikes synchronized with the switch activity; i.e., with the moments when $\frac{d}{dt}I_{\text{primary}} \neq 0$. Jacque d'Arsonval's dates, by the way, are 1851–1950; the question therefore arises*: What kind of ballistic galvinometer was available to Faraday in 1831?

membrane is stimulated. Faraday's interest in this topic was reenforced by his reading of an essay by John Herschel ("A preliminary discourse on the study of natural philosophy;" 1830) in which it was argued that the physics of light and the physics of sound must be similar . . . in the sense that both must have root in the vibratory motion of an elastic medium. Faraday speculated that such an analogy might pertain also to electrical and magnetic phenomena. Faraday was aware that such a view—though out of fashion among the French—had been advanced already in 1806 by Oersted (who had himself cited Chladni patterns as "analogs of electrical action"), and reasoned that electrical/magnetic effects (if such a view were correct)

1) could be understood only in terms of the dynamics of the "elastic medium;" *i.e.*, in the language of a *field theory*
2) entailed *delayed* action-at-a-distance.

Thus did time t become for Faraday a relevant dynamical variable. Faraday's problem was to discover observational evidence which would support or contradict the weight of his intuition.

It was at about this point (1831) that Faraday learned of the *strong electromagnets* which Joseph Henry (of Albany, New York, and later first director of the Smithsonian Institution: 1797–1878) had achieved by *replacing the traditional air core with a soft iron core*. Faraday knew that

1) currents give rise to (and feel) $\boldsymbol{B}$-fields, and anticipated (with an intuition rooted partly in his religious convictions) that
2) $\boldsymbol{B}$-fields should give rise (after some brief delay?) to currents.

More or less thus[42] was Faraday led (September & October, 1831) to the experimental arrangement and discovery outlined in Figure 21. Previous efforts to detect "the currents generated by $\boldsymbol{B}$-fields" had always yielded a negative result. What Faraday had in effect discovered was that currents arise not from $\boldsymbol{B}$ but from $\partial\boldsymbol{B}/\partial t$. The qualitative/quantitative upshot of Faraday's experiments—which were many and diverse, and were in some respects anticipated (1830) by Henry (who, however, was slow to publish his findings)—can be summarized

$$\nabla\times\boldsymbol{E} = -\tfrac{1}{c}\tfrac{\partial\boldsymbol{B}}{\partial t} \tag{107}$$

which was encountered already at (65.4) and (81.4), and is an expression of *Faraday's law of electromagnetic induction*. Several comments are now in order:

1. Comparison of (107) with

$$\nabla\times\boldsymbol{B} = \begin{cases} \tfrac{1}{c}\,\boldsymbol{j} & : \quad \text{steady case} \\ \tfrac{1}{c}\big(\boldsymbol{j} + \tfrac{\partial\boldsymbol{E}}{\partial t}\big) & : \quad \text{general case} \end{cases} \tag{83}$$

shows that the geometry of the $\boldsymbol{E}$-field generated by $\partial\boldsymbol{B}/\partial t$ resembles the geometry of the $\boldsymbol{B}$-field generated (in the steady case) by $\boldsymbol{j}$. We see also that the structure of Faraday's law (107) provides formal precedent for Maxwell's $\partial\boldsymbol{E}/\partial t$-term.

2. Faraday's law (107) presents—*Lenz' law*—a "stabilizing minus sign" which is absent from the Ampere-Maxwell law (83) . . . of which more later.

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[42] The ture history of his thought is not known, and I have provided only the grossest outline of what *is* known: see Chapter 4 in L. P. Williams, *Michael Faraday* (1964) or Chapter 3 in R. A. R. Tricker, *The Contributions of Faraday & Maxwell to Electrical Science* (1966) for further details.

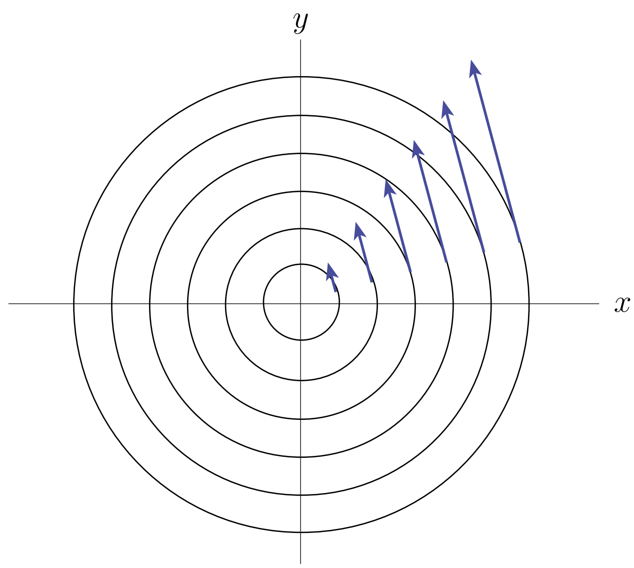

FIGURE 22: *The* $\boldsymbol{E}$*-field encountered in the following example. The field is divergenceless, but has obvious circulation (or "curl"). Such fields cannot be produced electrostatically, but are typical of the fields produced by time-dependent magnetic fields.*

3. Equation (107)—surprisingly, in view of Faraday's intent (and the nature of his observationss)—contains *no direct reference to current*. It says that $\partial\boldsymbol{B}/\partial t$ generates an $\boldsymbol{E}$-field, which *in the presence of charge* may give rise to charge flow. Suppose, for example, that $\boldsymbol{B}(\boldsymbol{x},t)$ has the (physically implausible) form

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ -c\beta t \end{pmatrix} \quad : \quad \begin{cases} \text{uniformly ramped,} \\ \boldsymbol{x}\text{-independent,} \\ \text{everywhere} \parallel \text{to the } z\text{-axis} \end{cases}$$

It then follows from (107) that

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x},t) &= \begin{pmatrix} -\frac{1}{2}\beta y \\ +\frac{1}{2}\beta x \\ 0 \end{pmatrix} + \text{grad}\,\varphi \\ &= \boldsymbol{E}_{\text{faraday}} + \boldsymbol{E}_{\text{electrostatic}} \end{aligned}$$

where—by (6)—$\boldsymbol{E}_{\text{faraday}}$ is divergenceless, but $\boldsymbol{E}_{\text{electrostatic}}$ is curlless (and is fixed not by (107) but by $\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$ and the physically appropriate boundary conditions). The structure of the induced field $\boldsymbol{E}_{\text{faraday}}$ (which, it is important to notice, is not conservative: $\boldsymbol{\nabla}\times\boldsymbol{E}_{\text{faraday}} \neq \boldsymbol{0}$) is indicated in Figure 22.

If a charge—let us, for simplicity, say a *solitary* charge—q were released it would move off initially in response to the $\boldsymbol{E}$-field, but after it had gained some velocity it would—by

$$\boldsymbol{F} = q\big(\boldsymbol{E} + \tfrac{1}{c}\,\boldsymbol{v}\times\boldsymbol{B}\big) \tag{67}$$

—also feel the $\boldsymbol{B}$-field. We would, therefore, expect the trajectory of q to be in general quite complicated ... and even more so if (instead of being solitary) q has companions. Faraday was himself protected from the full force of the complications just mentioned by two experimental circumstances:

- his currents were *confined to wires*
- his wires were mechanically *constrained to resist Amperean forces*.

But how—in detail—do the circumstances just noted serve to "simplify" the implications of the Maxwellian field equations (65) and of the Lorentzian force law (67)? To resolve this important question we must digress to acquire some familiarity with

1) a phenomenological law (Ohm's law) and
2) a poorly named but useful concept ("electromotive force").

THEORETICAL STATUS OF OHM'S LAW It is to Georg Simon Ohm (1789–1854) that we owe the experimental discovery (1826) that for samples of the most commonly encountered conductive materials

$$\frac{\text{impressed voltage}}{\text{resulting current}} = \text{constant}$$

i.e.,

$$V/I = R \equiv \text{sample } \textit{resistance}$$

—resistance <u>between those particular contact points</u> (see upper Figure 23). To formulate this result in geometry-independent terms specific to the material in question let the sample be of "standard shape" (*i.e.*, cylindrical, of length L and cross-sectional area A). Using $V = EL$ and $I = JA$ we have

$$\begin{aligned} E &= \rho J \\ &\rho \equiv RA/L \equiv \textit{resistivity} \text{ of the material} \end{aligned}$$

which is more usefully[43] notated

$$\begin{aligned} \boldsymbol{j} &= \sigma \boldsymbol{E} \\ &\sigma \equiv \textit{conductivity} = \frac{1}{\text{resistivity}} \end{aligned} \tag{108}$$

OHM'S LAW (108) provides our first instance of what is called a "constitutive relation." Such relations are denied "fundamental" status not because they are approximate (even Maxwell's field equations[44] are, strictly speaking, only approximate) but because they are <u>subject in (in)appropriately chosen materials</u>

[43] And at less risk of confusing ρ with "charge density"!

[44] What we call "Maxwell's equations" were abstracted from Maxwells' work by Heaviside, Lorentz and others over a period of nearly twenty years. I was surprised to discover that the equations proposed by Maxwell himself *included Ohm's law as a full-fledged partner*; see "Theories of Maxwellian design" (1998).

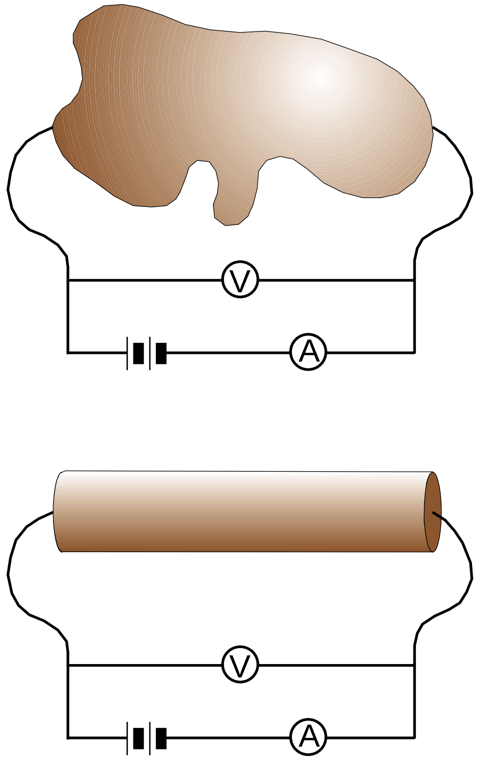

FIGURE 23: *Above: the arrangement used to measure the resistance R between two specified points on the surface of an arbitrary material blob. Below: the standardized sample of homogeneous material used to measure "resistivity" (or "conductivity"), which is an intrinsic property of that material.*

to gross violation. Constitutive relations have always—sooner or later—to be derived from first principles: the task is seldom easy, and entails that *such relations have always the character of macroscopic averages over microscopic complexities*. For *anisotropic* materials (108) assumes the form

$$\begin{aligned} \boldsymbol{j} &= \sigma\boldsymbol{E} \\ \sigma &\equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \equiv \text{conductivity matrix} \end{aligned} \tag{109}$$

Equations (109) look like a truncated version of the still more general relation

$$j_n = \sum_a \sigma_{na} E_a + \underbrace{\tfrac{1}{2}\sum_{a,b} \sigma_{nab} E_a E_b + \tfrac{1}{3!}\sum_{a,b,c} \sigma_{nabc} E_a E_b E_c + \cdots}_{\substack{\text{important in specialized materials, or} \\ \text{when the } \boldsymbol{E}\text{-field is sufficiently strong}}}$$

Evidently Ohm's law refers merely[45] to the "tip of a phenomenological iceberg." The surprising fact, from this point of view, is that (108) contains <u>no reference at all to $\boldsymbol{B}$</u> . . . for reasons which have evidently to do with the fact that the drift velocity v is typically so small that the $(\boldsymbol{v}\times\boldsymbol{B})$-forces experienced by individual charge carriers are negligible . . . though the facts are, as will emerge, somewhat more subtle. Bringing (76) to (108) we obtain $nev = \sigma E$ or

$$e\boldsymbol{E} = \frac{ne^2}{\sigma}\,\boldsymbol{v} \tag{110}$$

. . . according to which the impressed force $e\boldsymbol{E}$ is proportional not to the acceleration but to the (mean) *velocity* of the charge carriers. The situation is (roughly) this: the charge carriers keep *trying* to accelerate, but keep *running into things and getting stopped* (releasing their recently acquired kinetic energy to the obstacle-matrix, which gets hot). The situation can be modeled (Drude, 1900) by writing

$$e\boldsymbol{E} - \underbrace{b\,\boldsymbol{v}}_{\text{damping force: models the effect of collisions}} = m\boldsymbol{a} \tag{111}$$

and supposing that the (mean) acceleration $\boldsymbol{a} = \mathbf{0}$. The drift velocity acquires thus the status of a kind of "terminal velocity," and it follows in fact from (110/111) that

$$\sigma = \frac{ne^2}{b}$$

The implication is that *charge carriers keep moving because the ambient $\boldsymbol{E}$-field <u>keeps doing work</u>* on them. How about the $\boldsymbol{B}$-field? It does work at the temporal rate given by

$$P_{\text{magnetic}} = \boldsymbol{v}\cdot\tfrac{e}{c}(\boldsymbol{v}\times\boldsymbol{B}) = 0 \tag{112}$$

Magnetic fields <u>do no work</u> on moving charges and so cannot assist in the transport of charge carriers through a wire . . . except perhaps indirectly: one can imagine far-fetched circumstances in which $\boldsymbol{B}$-fields (by deforming the carrier trajectories) might cause charge carriers to hit/miss appropriately deployed obstacles. This would lend $\boldsymbol{B}$-dependence to σ, but would *not* cause an additive $\sigma_{\text{mag}}\boldsymbol{B}$-term to appear on the right side of Ohm's law (108).

ELECTROMOTIVE FORCE Given, therefore, that charge carriers flow through material wires because constantly worked on by $\boldsymbol{E}$-fields, the question arises: *How much* work (per unit charge) do the $\boldsymbol{E}$-fields do? Let the circuit in question be modeled by a closed curve (of loop) $\mathcal{C}$. Immediately

$$q\oint_{\mathcal{C}} \boldsymbol{E}\cdot d\boldsymbol{\ell} = \text{work done in transporting } q \text{ virtually around } \mathcal{C}$$

[45] . . . but importantly: the conductivities of common materials range over at least 23 orders of magnitude. Few indeed are the "laws of Nature" that can claim such dynamic range.

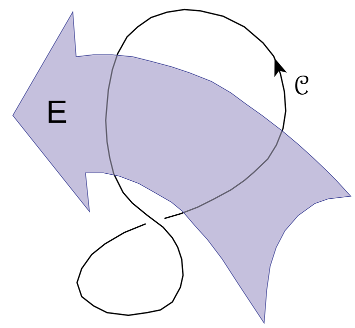

FIGURE 24: *The physical* $\boldsymbol{E}$*-field and "mental loop"* $\mathcal{C}$ *that enter at (113) into the definition of* "emf." *In practical applications it is often natural to identify the "mental loop" with a* metal *loop (wire).*

where the "virtually" means that the pransport takes place "mentally," not physically (*i.e.*, not in real time, with the attendant accelerations, etc.). The $\oint$ defines what is called the "electromotive force" associated with the given circuit and field. It is standardly denoted $\mathcal{E} \equiv \mathcal{E}(\mathcal{C}, \boldsymbol{E})$, and *has actually not the dimensions of "force" but of "work/charge."* I prefer therefore to call

$$\mathcal{E} \equiv \oint_{\mathcal{C}} \boldsymbol{E}\cdot\boldsymbol{d\ell} \tag{113}$$

the "emf" of the circuit/field in question (and to put out of mind the fact that "emf" came into the world as an acronym). What *is* the $\boldsymbol{E}$-field contemplated at (113)? It is the "field experienced by the virtually transported test charge"—a field which (since the interior of matter is a complicated place) is actually *unknown*. Happily, *the complication just noted is—to the (substantial) extent that it is microelectrostatic in origin—irrelevant* . . . for this simple reason: electrostatic fields are curlless

$$\boldsymbol{\nabla}\times\boldsymbol{E}_{\text{electrostatic}} = \boldsymbol{0}$$

. . . from which it follows by Stokes' theorem (87) that (for all circuits $\mathcal{C}$)

$$\mathcal{E}_{\text{electrostatic}} = 0 \tag{114}$$

This means that *the (generally unknown) electrostatic component of the "fields experienced by the transported charge q" can be dropped from all* emf-*calculations*. To make the same point another way: purely electrostatic $\boldsymbol{E}$-fields cannot be used to drive currents in circuits.[46] But while $\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}$ pertains universally to electrostatic fields, it does *not* pertain

- to the $\boldsymbol{E}$-fields generated by chemical action in batteries;

[46] What, in this light, do you make of the physics of *lightening bolts*?

- to the $\boldsymbol{E}$-fields produced by thermal/optical/mechanical action in diverse solid-state devices;

...and in particular it does not pertain

- to the Faraday $\boldsymbol{E}$-fields which, according to (107), are induced by time-dependent $\boldsymbol{B}$-fields.

So non-zero values of $\mathcal{E}$ are certainly attainable.[47] Drawing finally upon Ohm's law, we have

$$\begin{aligned}\mathcal{E} \equiv \oint_{\mathcal{C}} \boldsymbol{E}\boldsymbol{\cdot} \boldsymbol{d\ell} &= \oint_{\mathcal{C}} \rho\, \boldsymbol{j}\boldsymbol{\cdot} \boldsymbol{d\ell} \\ &= \rho \frac{I}{A} L \quad \text{for wires of uniform cross section} \\ &\downarrow\end{aligned}$$

giving
$$\mathcal{E} = IR \tag{115}$$

It should, in view of (115), not be necessary to belabor the claim that $\mathcal{E}$ is—at least for the purposes of practical/applied physics—a "useful[48] concept."

The question posed near the top of page 65 now "answers itself." The integral formulation

$$\int_{\partial\mathcal{D}} \boldsymbol{E}\boldsymbol{\cdot} \boldsymbol{d\ell} = -\tfrac{1}{c} \iint_{\mathcal{D}} \tfrac{\partial \boldsymbol{B}}{\partial t} \boldsymbol{\cdot} \boldsymbol{dS} \tag{90.4}$$

of Faraday's law (107) can now be formulated

$$\mathcal{E}_{\text{faraday}} = -\tfrac{1}{c}\dot{\Phi} \tag{116}$$
$$\Phi \equiv \iint_{\mathcal{D}} \boldsymbol{B}\boldsymbol{\cdot} \boldsymbol{dS} \equiv \textit{magnetic flux} \text{ through } \mathcal{D}$$

> REMARK: Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be distinct caps that share the same boundary $\mathcal{C} = \partial\mathcal{D}_1 = \partial\mathcal{D}_2$. It is (recall the formal upshot of PROBLEM 17) a consequence of $\boldsymbol{\nabla\cdot B} = 0$ that
>
> $$\text{magnetic flux through } \mathcal{D}_1 = \text{magnetic flux through } \mathcal{D}_2$$
>
> and better, therefore, to speak of the "magnetic flux entrapped by $\mathcal{C}$."

If the loop $\mathcal{C}$ is realized physically by a <u>wire of resistanc R</u> then the so-called "flux theorem" (116) states that $\Phi(t)$ and the induced current $I(t)$ stand in the following relationship:

$$IR = -\tfrac{1}{c}\dot{\Phi}(t) \tag{117}$$

This is the physics to which Faraday's induction experiments directly speak. One does not deny the utility of (117)—but does gain a more vivid sense of

[47] They arise from *non-conservative* $\boldsymbol{E}$-fields.

[48] For "useful" read "indispensable"!

Maxwell's physics/formal genius—when one stresses that $\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{c}\dot{\boldsymbol{B}}$ is a much <u>deeper</u> statement than (117): it is independent of the dirty physics which underlies Ohm's law, and it tells us "what is going on" even in the absence of mobile charge. Equation (117) is susceptible also to the criticism that it *does not quite represent the facts* ... for reasons which emerge from the following

EXAMPLE: Let an impressed $\boldsymbol{B}$-field have the spatially uniform and temporally ramped structure

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ -c\beta t \end{pmatrix}$$

encountered already on page 64, and let $\mathcal{C}$ refer to a circular wire ring of radius r and resistance R, oriented as shown in

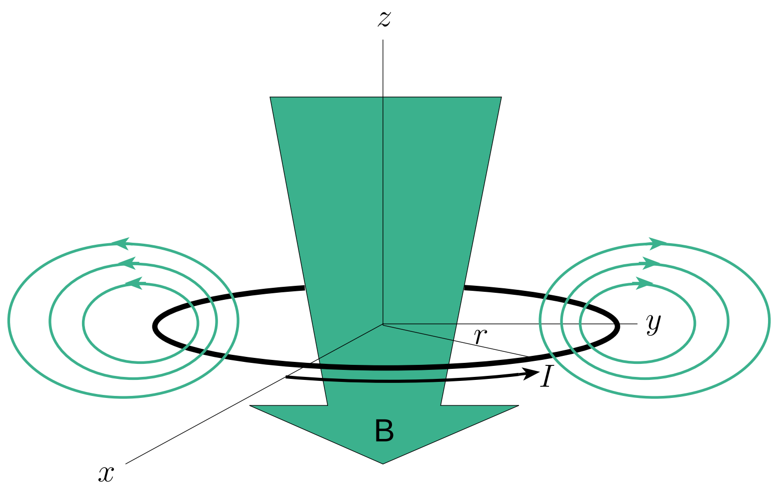

FIGURE 25: *A surging magnetic field stimulates current in the conductive ring, which generates an oppositely oriented time-dependent toroidal field, which ...*

the figure. Time-dependent flux $\Phi(t) = \pi r^2 B(t) = -\pi r^2 c\beta t$ is encircled by the wire, which by (117) induces a current $I = -(cR)^{-1}\dot{\Phi} = \pi r^2\beta/R$. That current itself generates a toroidal magnetic field $\boldsymbol{B}'$ and an associated Φ'. In general, we must take into account the so-called

$$\text{"back emf"}\ \mathcal{E}' = -\tfrac{1}{c}\dot{\Phi}'$$

when computing $I(t)$. We are here released from that infinite regress only because in the present (highly artificial) context $\dot{\Phi}' = 0$. Generally, however, we confront this question: *How to describe the quantitative physics of the self-interactive effect just noted?*

It is to get a handle on that issue that we digress now to acquire familiarity with the concept of

INDUCTANCE Let it be arranged/assumed that steady (!) currents $I_1, I_2, \dots, I_N$ circulate in material loops $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_N$:

FIGURE 26: *Current-carrying loops interact magnetically. We are not concerned at the moment with the mechanism (batteries?) that in reality would be required to* maintain *the steady currents* I_n*.*

the figure. The $\boldsymbol{B}$-field at any point $\boldsymbol{x}$ (if we dismiss as irrelevant any $\boldsymbol{B}$-field of extrinsic origin) can be described

$$\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2 + \cdots + \boldsymbol{B}_n + \cdots + \boldsymbol{B}_N$$
$$\boldsymbol{B}_n \equiv \boldsymbol{B}\text{-field generated by current } I_n$$

From the Biot-Savart law (96) it follows in parlticular that

$$= I_n \cdot \underbrace{\tfrac{1}{4\pi c}\oint_{\mathcal{C}_n} \frac{\boldsymbol{d\ell}\times\hat{\boldsymbol{r}}}{r^2}}_{\text{vector-valued factor which relates } \boldsymbol{x} \text{ to the geometry of } \mathcal{C}_n} \tag{118}$$

Let

$$\begin{aligned}\Phi_{mn} &\equiv \text{magnetic flux through } \mathcal{C}_m \text{ due to field generated by } I_n\\ &= \iint_{\mathcal{D}_m} \boldsymbol{B}_n\boldsymbol{\cdot} \boldsymbol{dS}_m \quad : \quad \mathcal{D}_m \text{ is any cap with } \partial\mathcal{D}_m = \mathcal{C}\end{aligned} \tag{119}$$

Introducing (118) into (119) we conclude that Φ_{mn} is proportional to I_n through <u>a factor which depends mutually and exclusively upon the *geometries*</u> of the loops $\mathcal{C}_m$ and $\mathcal{C}_n$:

$$\Phi_{mn} = M_{mn} I_n \tag{120}$$

The analytical <u>evaluation of M_{mn}</u> is—even in simple cases—typically quite difficult[49] ... but some formal progress is possible. Appealing to (92) we have

$$\boldsymbol{B}_n = \boldsymbol{\nabla}\times\boldsymbol{A}_n$$

[49] See Richard Verbeck, "S & M Induction Formulæ" (Reed College 1981).

so

$$\varPhi_{mn} = \iint_{\mathcal{D}_m} (\boldsymbol{\nabla}\times\boldsymbol{A}_n)\boldsymbol{\cdot}\boldsymbol{dS}_m = \oint_{\mathcal{C}_m} \boldsymbol{A}_n\boldsymbol{\cdot}\boldsymbol{d\ell}_m$$

But it follows from (95)[50] that

$$\boldsymbol{A}_n = \tfrac{1}{4\pi c}\oint_{\mathcal{C}_n} \tfrac{1}{r}\boldsymbol{d\ell}_n \cdot I_n \tag{121}$$

Thus do we obtain

$$M_{mn} = \tfrac{1}{4\pi c}\oint_{\mathcal{C}_m}\oint_{\mathcal{C}_n} \tfrac{1}{r}\,\boldsymbol{d\ell}_m\boldsymbol{\cdot}\boldsymbol{d\ell}_n \tag{122}$$

$$r \equiv \text{distance between } \boldsymbol{d\ell}_m \text{ and } \boldsymbol{d\ell}_n$$

This pretty result (subject, however, to an alternative interpretation) was first achieved (∼1845) by Franz Neumann[51] (1798–1895). It is known as "Neumann's formula," and carries with it the important implication that

$$M_{mn} = M_{nm} \tag{123}$$

The real numbers M_{mn}—which, though electrodynamically important, refer exclusively to the geometry and relative placement of the loops $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N$ —are called *coefficients of mutual inductance* when $m \neq n$, and *coefficients of self-inductance* when $m = n$. In the latter case it is standard to adjust the notation:

$$M_{mm} \longmapsto L_m \equiv \text{self-inductance of the } m^{\text{th}} \text{ loop}$$

From the fact (see again page 61) that $\boldsymbol{B} \to \boldsymbol{0}$ near a "filamentary current" (current in a wire of zero radius) we conclude—the associated "self-fluxes" being anavoidably infinite—that

The self-inductance of a *filamentary* loop is —irrespective of the loop's geometry—infinite. (124)

. . . which I take to be Nature's way of reminding us that "filamentary currents" are a (physical unrealizable and) latently *dangerous abstraction*.

[50] The $\iiint$ ranges only over the volume of the wire, since the integrand vanishes elsewhere. Integration over cross-sections converts current density to current. The surviving integral is a $\oint$ along the length of the wire.

[51] Neumann was the *inventor of the vector potential* $\boldsymbol{A}$ (and of much else), but how he obtained (122) at such an early date—and without knowledge of Stokes' theorem—is beyond my understanding! Notice that in (122) all reference to $\boldsymbol{A}$ has dropped away.

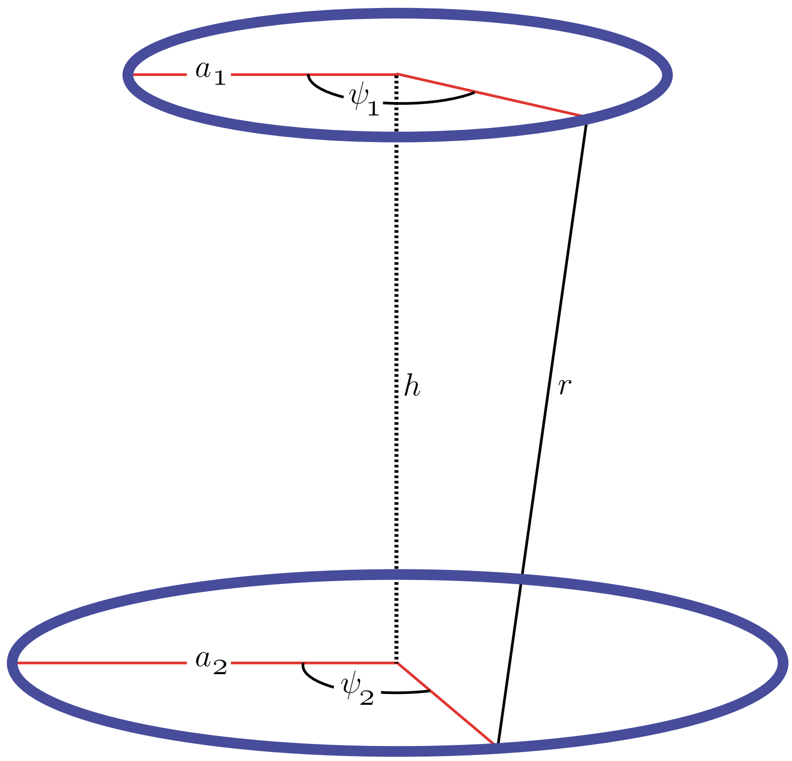

FIGURE 27: *Coaxial filamentary rings. In the text we compute the mutual inductance M—an arduous task made (barely) feasible by the high symmetry of the system.*

I turn now to review of a line of argument which leads to a description of the mutual inductance $M \equiv M_{12} = M_{21}$ of a pair of coaxial filamentary rings.[52] First we establish by geometrical argument that the distance between point ψ_1 on $\mathcal{C}_1$ and point ψ_2 on $\mathcal{C}_2$ can be described

$$r = \sqrt{h^2 + a_1^2 + a_2^2 - 2a_1a_2\cos(\psi_1 - \psi_2)}$$

so (122) supplies

$$\begin{aligned}
M &= \tfrac{1}{4\pi c}\int_0^{2\pi}\!\!\int_0^{2\pi} \frac{a_1a_2\cos(\psi_2-\psi_1)}{\sqrt{h^2+a_1^2+a_2^2-2a_1a_2\cos(\psi_2-\psi_1)}}\,d\psi_1 d\psi_2\\
&= \tfrac{1}{2c}\int_0^{2\pi} \frac{a_1a_2\cos\theta}{\sqrt{h^2+a_1^2+a_2^2-2a_1a_2\cos\theta}}\,d\theta\\
&= -\tfrac{k}{c}\sqrt{a_1a_2}\int_0^{\frac{1}{2}\pi} \frac{\cos 2\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi
\end{aligned}$$

[52] See PROBLEM 28, where you are asked to work out the details of the individual steps.

where $\phi \equiv \frac{1}{2}(\theta - \pi)$ and

$$k^2 \equiv \frac{4a_1a_2}{h^2 + (a_1 + a_2)^2}$$

The integral is tabulated, and supplies

$$M = \tfrac{1}{c}\sqrt{a_1a_2}\Big\{\big(\tfrac{2}{k} - k\big)\mathbf{K}(k) - \tfrac{2}{k}\mathbf{E}(k)\Big\} \tag{125.1}$$

where

$$\mathbf{K}(k) \equiv \int_0^{\frac{1}{2}\pi} \frac{1}{\sqrt{1 - k^2\sin^2\phi}}\,d\phi$$

$$\mathbf{E}(k) \equiv \int_0^{\frac{1}{2}\pi} \sqrt{1 - k^2\sin^2\phi}\,d\phi$$

define the "complete elliptic integrals of 1^{st} and 2^{nd} kinds.[53] Writing

$$k^2 = \frac{4a_1a_2}{h^2}\cdot\left[1 + \frac{(a_1+a_2)^2}{h^2}\right]^{-1} = \frac{4a_1a_2}{h^2}\left\{1 - \frac{(a_1+a_2)^2}{h^2} + \cdots\right\}$$

we see that $k^2 \sim 0$ corresponds physically to the case in which the rings are *widely separated* ($h \gg \sqrt{4a_1a_2}$). But for k^2 small the handbooks supply[54]

$$\left.\begin{aligned}\mathbf{K} &= \tfrac{2}{\pi}\Big[1 + 2\tfrac{k^2}{8} + 9\big(\tfrac{k^2}{8}\big)^2 + \cdots\Big]\\ \mathbf{E} &= \tfrac{2}{\pi}\Big[1 - 2\tfrac{k^2}{8} - 3\big(\tfrac{k^2}{8}\big)^2 + \cdots\Big]\end{aligned}\right\} \quad : \quad k^2 \text{ just greater than } 0$$

From these facts it follows that <u>for loosely coupled coaxial rings</u>

$$M \approx \tfrac{\pi}{16c}\sqrt{a_1a_2}\,k^3 \quad \text{with} \quad k = \sqrt{4a_1a_2}/h \tag{125.2}$$

On the other hand . . . we observe that

$$1 - k^2 = \frac{h^2 + (a_1 - a_2)^2}{h^2 + (a_1 + a_2)^2}$$

which shows that $k^2 \sim 1$ corresponds physically to the case in which the rings are *very close together* ($h \sim 0$ and $a_1 \sim a_2$). The handbooks now supply[54]

$$\left.\begin{aligned}\mathbf{K} &= \Lambda + \tfrac{1}{4}(\Lambda - 1)\kappa^2 + \tfrac{9}{64}(\Lambda - \tfrac{7}{6})\kappa^4 + \cdots\\ \mathbf{E} &= 1 + \tfrac{1}{2}(\Lambda - \tfrac{1}{2})\kappa^2 + \tfrac{3}{16}(\Lambda - \tfrac{13}{12})\kappa^4 + \cdots\end{aligned}\right\} \quad : \quad k^2 \text{ just less than } 1$$

with $\kappa \equiv \sqrt{1 - k^2}$ and $\Lambda \equiv \log(4/\kappa)$. From these (more intricate) facts it follows

[53] For discussion of the properties of these famous functions see, for example, E. Jahnke & F. Emde, *Tables of Functions* (1945), pages 73–85 or J. Spanier & K. B. Oldham, *An Atlas of Functions* (1987), Chapter 61.

[54] See Jahnke & Emde, page 73.

that <u>for tightly coupled coaxial rings</u>

$$M \approx \tfrac{1}{c}\sqrt{a_1 a_2}\big(\Lambda - 2\big)$$
$$\Lambda = \log \frac{4}{\sqrt{1-k^2}}$$

and that this (by $0 \sim h \ll a_1 \sim a_2 \sim a$) can be formulated

$$\approx \tfrac{1}{c}a\Big(\log \tfrac{8a}{b} - 2\Big) \tag{125.3}$$

where $b = r_{\min} = \sqrt{h^2 + (a_1 - a_2)^2}$ is the shortest distance between the two rings.

Inductance calculations can sometimes (*i.e.*, in a few favorable cases) be accomplished by more elementary means. Suppose, for example, that in the "coaxial 2-ring problem" <u>one ring is very much smaller than the other</u>: $a_1 \ll a_2$. The small ring lies then in the "axial region," where the magnetic field $\boldsymbol{B}_2$ generated by current I_2 in the large ring is easy to calculate:[55] one finds that $\boldsymbol{B}_2$ runs parallel to the axis (in the sense given by the ↺-rule) and is of magnitude

$$B_2(h) = \frac{I_2}{2c}\frac{a_2^2}{(h^2 + a_2^2)^{3/2}}I_2$$
$$= \begin{cases} \frac{I_2}{2ca_2}\Big[1 - \frac{3}{2}\big(\frac{h}{a_2}\big)^2 + \frac{15}{8}\big(\frac{h}{a_2}\big)^4 + \cdots\Big] & : \quad h \ll a_2 \qquad (126.1) \\ \frac{I_2}{2ch}\big(\frac{a_2}{h}\big)^2\Big[1 - \frac{3}{2}\big(\frac{a_2}{h}\big)^2 + \frac{15}{8}\big(\frac{a_2}{h}\big)^4 + \cdots\Big] & : \quad h \gg a_2 \qquad (126.2) \end{cases}$$

If we conceptualize the present 2-ring problem as indicated in Figure 28 then it follows immediately from (126.2) that in leading approximation

$$\Phi_{12} = \pi a_1^2 \cdot \frac{I_2 a_2^2}{2ch^3}$$

whence

$$M_{12} = \pi a_1^2 a_2^2 / 2ch^3$$

which agrees precisely with (125.2). It is instructive to note that the problem would *not* have seemed easy had we on the other hand tried to evaluate M_{21}, for while

- the field $\boldsymbol{B}_2$ intercepted by the small ring $\mathcal{C}_1$ is nearly uniform
- the same cannot be said of the $\boldsymbol{B}_1$ intercepted by the large ring $\mathcal{C}_2$.

Nevertheless—and from this point of view somewhat surprisingly—we know on general grounds that $M_{12} = M_{21}$. I turn now from the calculation of mutual inductances to the *calculation of <u>self</u>-inductances—a problem which* (*because bedeviled by* ∞*'s*) *tends to be rather more difficult.*

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[55] PROBLEM 29.

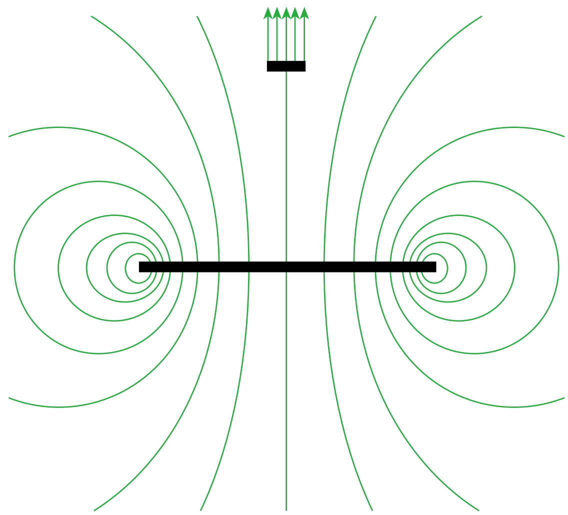

FIGURE 28: *Coaxial ring system, made exceptionally tractable by the circumstance that the small ring intercepts an essentially* uniform *sample of the* $\boldsymbol{B}$*-field generated by the large ring.*

INTERLUDE: It is useful—pedagogically, and because it will gain me a result of which I will have need—to rise for a moment above the details of battle to ask: *Why* are we studying inductance? The answer" *So that we can deal effectively with the magnetic interactions of currents confined to wires*—with one another, and with themselves.[56] And it is the *self*-interaction problem which has just moved to center stage. How are we doing? We are in position now to state that a t-dependent current I_n in $\mathcal{C}_n$ produces in $\mathcal{C}_m$ an emf given (return with (120) to (116)) by

$$\mathcal{E}_{mn} = -\tfrac{1}{c}M_{mn}\tfrac{d}{dt}I_n \tag{127}$$

Setting $m = n$, we expect the "back emf" to be given by an equation of the form

$$\text{back emf} = -\tfrac{1}{c}L\tfrac{d}{dt}I \tag{128}$$

and it is L which we desire now to compute. We proceed, as before, in terms of particular examples.

[56] We take no embarrassment from the fact that "currents confined to wires" are of more importance to people than to God and other physicists. It is, after all, by the activities of men (Faraday) that we *learn* the ways of God: our real intent at the moment is *to understand Faraday's laboratory experience*.

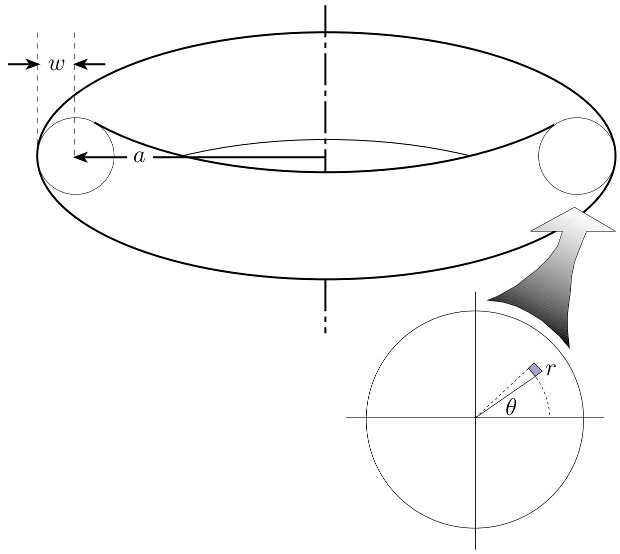

FIGURE 29: *Variables used to describe the gross form and cross-sectional elements of a conductive ring. Our objective is to compute the self-inductance of such a ring.*

Let a ring of radius a be formed from wire of radius w ($w \ll a$). We will assume $w > 0$ (*i.e.*, we exclude the filamentary idealization $w = 0$) in order

1) to avoid the ∞ mentioned at (124), and
2) the better to model engineering reality

but have purchased thus a conceptual problem: *How* to model such a wire? This we do as follows: we agree (tentatively) to . . .

> Think of the ring as a "cable" made up of *filamentary sub-rings*, each of cross-sectional area $dA = r dr d\theta$. The current carried by the filament with coordinates (r, θ) can be described

$$dI = j(r, \theta) r dr d\theta \tag{129}$$

> and we will assume that the *ratio* of the currents carried by *any pair* of filaments is time-independent; *i.e.*, that they *fluctuate in concert*. This entails

$$j(r, t) = J(r) \cdot K(t) \tag{130}$$

> REMARK: Though it makes physical sense, it is really only for analytical convenience that I have assume the current density j to be θ-independent. In that same spirit one could—though I for the moment won't—assume further that $J(r)$ is r-independent.

Now it follows from (130) that

$$I(t) = K(t)\cdot\underbrace{\int_0^{2\pi}\!\!\int_0^w J(r)r\,drd\theta}$$

—It is convenient to scale $J(r)$ so this factor equals *unity*. This is in effect to write

$$j(r,t) = I(t)\cdot J(r) \tag{131}$$

and to interpret $J(r)rdrd\theta$ as the <u>fraction</u> of the total current I which circulates in the filament with coordinates (r,θ).

The magnetic flux $\Phi(r,\theta)$ through the (r,θ)-filament—produced by the currents circulating in all the other filaments—can be described

$$\Phi(r,\theta) = \int \underbrace{M(r,\theta;r',\theta')}\,dI'$$

—This function has (see below) the structure implied by (125.2).

NOTE: In the discrete approximation we would have to write $\sum{}'$, signaling our intention to omit the infinite self-fields that arise when "filaments of zero cross-section" are imagined to carry finite currents. That problem *does not arise in the present context because our filaments carry currents <u>proportional</u> to their cross-sections.*

$$\begin{aligned} &= \Big\{\int_0^{2\pi}\!\!\int_0^w M(r,\theta;r',\theta')J(r')\,r'\,dr'd\theta'\Big\}\cdot I(t)\\ &\equiv W(r,\theta)I(t) \end{aligned} \tag{132}$$

Next—looking to (127/128) for guidance—we note that temporal variation of the current I produces in the (r,θ)-filament an emf

$$\begin{aligned} \mathcal{E}(r,\theta) &= -\tfrac{1}{c}\dot{\Phi}(r,\theta)\\ &= -\tfrac{1}{c}W(r,\theta)\dot{I}(t) \quad \text{by (132)} \end{aligned} \tag{133}$$

which would stimulate a current

$$dI(r,\theta) = \frac{1}{R(r,\theta)}\mathcal{E}(r,\theta)$$

where

$$\begin{aligned} R(r,\theta) &\equiv \text{resistance of the } (r,\theta)\text{-filament} \\ &= \rho\frac{1}{rdrd\theta}2\pi(a+\underbrace{r\cos\theta}_{\text{neglect because } r \leqslant w \ll a}) \end{aligned}$$

can be used to give

$$dI(r,\theta) = \underbrace{\frac{1}{2\pi\rho a}}_{=\frac{1}{R\cdot\pi w^2} \quad \text{where } R \equiv \text{total ring resistance}}\cdot\mathcal{E}(r,\theta)\,rdrd\theta$$

So we <u>do have</u>

$$\begin{aligned} I = \int dI &= \frac{1}{R\cdot\pi w^2}\iint \mathcal{E}(r,\theta)\,rdrd\theta \\ &= -\frac{1}{R}\Big\{\frac{1}{\pi w^2 c}\iint W(r,\theta)\,rdrd\theta\Big\}\dot{I} \quad \text{by (133)} \end{aligned}$$

but by the effective *definition* (128) of self-inductance <u>expect to have</u>

$$= -\frac{1}{Rc}L\dot{I}$$

Comparison gives

$$\begin{aligned} L &= \tfrac{1}{\pi w^2}\iint W(r,\theta;)\,rdrd\theta \\ &= \tfrac{1}{\pi w^2}\iiiint M(r,\theta;r',\theta')J(r')\,r'rdr'd\theta'drd\theta \end{aligned} \tag{134.1}$$

where according to (125.3)

$$M(r,\theta;r',\theta') = \tfrac{a}{c}\big(\log\tfrac{8a}{s} - 2\big) \tag{134.2}$$

$$\begin{aligned} s &\equiv \text{distance between } (r,\theta) \text{ and } (r',\theta') \\ &= \sqrt{r^2 + r'^2 - 2rr'\cos(\theta'-\theta)} \end{aligned} \tag{134.3}$$

Equation (134.1) is susceptible to some simplificaiton. From $\iint rdrd\theta = \pi w^2$ and $\iint J(r')\,r'dr'd\theta' = 1$ it follows almost immediately that

$$L = \tfrac{a}{c}\Big\{\log 8a - 2 - \tfrac{1}{w^2}\iiint J(r')\,r'r\log\big[r^2 + r'^2 - 2rr'\cos\theta\big]dr'drd\theta\Big\}$$

but to obtain a more concrete result let us now assume the current to be *uniformly* distributed: $J(r') = 1/\pi w^2$. Then

$$L = \tfrac{a}{c}\Big\{ \log 8a - 2 - Q(w) \Big\}$$

$$Q(w) \equiv \tfrac{1}{\pi w^4}\int_0^w\!\!\int_0^w\!\!\int_0^{2\pi} r' r \log\big[r'^2 + r^2 - 2rr'\cos\theta\big]\,dr'drd\theta$$

and by interesting calculation[57] we obtain

$$= \log w - \tfrac{1}{4}$$

We are brought thus to the conclusion that the *self-inductance of a homogeneous ring* can—in the approximation

$$\text{wire radius} \ll \text{ring radius}$$

—be described

$$L = \tfrac{a}{c}\big(\log\tfrac{8a}{w} - \tfrac{7}{4}\big) \tag{135}$$

I apologize for the analytical tedium which has attended the derivation of (135), but in real physics one *frequently* encounters points of principle which can be clarified in no other way than by computational labor.[58] What have we learned?

1. Just as the electrostatic self-energy problem disappears when charged particles are modeled not as "points" but as "pithballs of finite radius," so does the magnetic self-energy problem disappear when currents are imagined to be carried not by idealized "filaments" but by more realistic "wires *of non-zero cross-section*."
2. Such realistic models serve to display self-inductance as integrated mutual inductance.
3. Our progress hinged on our willingness to make certain approximations, of which the physically most interesting was that the r-dependence of $j(r)$ could be neglected. This (in the language of Figure 11) amounts to an assumption that

   $$\text{skin depth} \gg \text{wire radius} \tag{136}$$

   In point of physical fact, skin depth decreases as frequency increases; we should therefore look upon (135) as the low-frequency approximation to a function $L(\omega)$ ...except that at *very* high frequencies—frequencies so high that

   $$\text{period} < \text{optical transit time across the circuit}$$

   we expect the very *concept* of mutual/self-inductance to lose its utility.

---

[57] PROBLEM 30.

[58] The formal simplicity of (135) suggests the possibility of a "simple derivation"... which—if it exists—is unknown to me.

4. Equation (135) provides a sharpened version of (124). It states that L diverges only *logarithmically* as the wire becomes filamentary ($w \to 0$). This can be understood as reflecting the fact that *logarithmic potentials give* $1/r$ *force laws, which correspond to "geometrical fall-off in a two-dimensional world"—the "world" defined by a linear source in 3-space*.We may expect the logarithmic divergence of L to pertain (not just to rings but) generally. to loops of *every* figure.

5. (Self)-inductance calculations are essentially *geometrical* in nature. They stand prior to electrodynamical calculation just as (say) moment of inertia calculations stand prior to the dynamics of rigid bodies. The question arises: Does the self-inductance of a loop stand in any invariable relationship to any other physically important "shape-sensitive" parameters (for example: the least area and/or fundamental frequency of a spanning membrane, the moments of inertia, etc.)? Can one anticipate on general grounds what happens to $L(\mathcal{C})$ when $\mathcal{C}$ is deformed? Or—see again Figure 20—what $\mathcal{C}$ will minimize $L(\mathcal{C})$?[59]

A surprisingly limited population of *analytical induction formulæ* can be found scattered (sparcely) throughout the literature—particularly the older electrical engineering literature.[49] Experimentally inclined readers may ask: If physically reliable analytical inductance formulæ are so difficult to obtain ... *why bother*? *Why not must* <u>*measure*</u> *the inductance*? I would remind such readers of our primary goal, which is *review the classical basis of the claim that* <u>*Maxwell's equations do in fact provided a representation of electromagnetic reality*</u>... and for that we must be in position to <u>*compare*</u> *theory with experiment*. Returning now to the physical question which precipated this digression ...

Figure 30 presents a schematic diagram of Faraday's experimental set-up (see again Figure 21). Working from the diagram, we have

$$\left.\begin{aligned} V(t) - \tfrac{1}{c}L_1\dot{I}_1 - \tfrac{1}{c}M\,\dot{I}_2 &= R_1I_1 \\ - \tfrac{1}{c}M\,\dot{I}_1 - \tfrac{1}{c}L_2\dot{I}_2 &= R_2I_2 \end{aligned}\right\} \tag{137}$$

and have interest in the currents $I_1(t)$ and $I_2(t)$ that result when the battery is switched on at time $t = 0$:

$$I_1(0) = I_2(0) = 0 \quad \text{and} \quad V(t) = \begin{cases} 0 & : \quad t < 0 \\ V & : \quad t \geqslant 0 \end{cases}$$

We confront at this point the (purely mathematical) problem of solving a coupled system of 1$^{\text{st}}$-order ordinary differential equations, which can be notated

$$\mathbb{L}\,\tfrac{d}{dt}\boldsymbol{I} + \mathbb{R}\,\boldsymbol{I} = \boldsymbol{V}(t)$$

with

$$\boldsymbol{V}(t) \equiv \begin{pmatrix} V(t) \\ 0 \end{pmatrix}, \ \boldsymbol{I}(t) \equiv \begin{pmatrix} I_1(t) \\ I_2(t) \end{pmatrix}, \ \mathbb{L} \equiv \tfrac{1}{c}\begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix}, \ \mathbb{R} \equiv \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$$

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[59] Questions of precisely this nature are explored in G. Polya & G. Szegö, *Isoperimetric Inequalities in Mathematical Physics* (1951)—an extraordinary monograph which I recommend very highly to your attention.

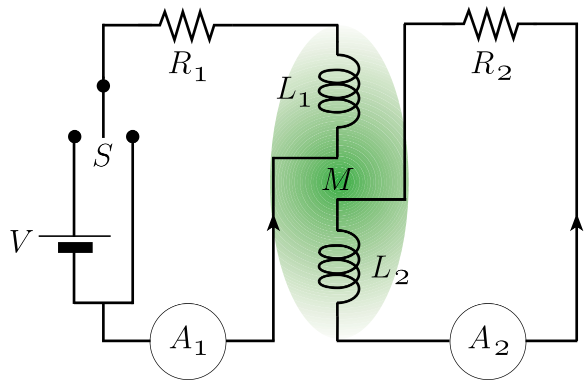


FIGURE 30: *Schematic representaion of Faraday's experimental set-up. The switch S premits one to insert/remove the battery from the primary circuit. The principal effect of the soft iron core was to increase the value of the mutual inductance M, which serves to couple the two circuits. The green shading represents magnetic field. A (usually tacit) presumption—here as always in circuit analysis—arises from the circumstance that "electromagnetic news is propagated with finite speed," that a charge will not be instantly aware of non-local field variations: we assume that* all characteristic circuit times are much greater than the time required for light to transit the circuit. *This, by the way, is the reason ultrahigh frequency devices must be physically small.*

Multiplication by $\mathbb{L}^{-1}$ is possible provided

$$\det \mathbb{L} = \tfrac{1}{C^2}\big\{L_1L_2 - M^2\big\} \neq 0$$

which will later be shown on very general grounds to be invariably the case. We are led thus to

$$\begin{aligned} (\mathbb{D}+\mathbb{W})\boldsymbol{I}(t) = \boldsymbol{F}(t) \quad &: \quad \text{inhomogeneous differential equation} \qquad (138)\\ \Downarrow \quad & \\ \boldsymbol{I}(t) = (\mathbb{D}+\mathbb{W})^{-1}\boldsymbol{F}(t) \quad & \end{aligned}$$

$$(\mathbb{D}+\mathbb{W})\boldsymbol{I}(t) = \boldsymbol{0} \quad : \quad \text{homogeneous companion of (138)}$$

Here $\boldsymbol{F}(t) \equiv \mathbb{L}^{-1}\boldsymbol{V}(t)$ and

$$\mathbb{D} \equiv \begin{pmatrix} \partial_t & 0 \\ 0 & \partial_t \end{pmatrix}, \quad \mathbb{W} \equiv \mathbb{L}^{-1}\mathbb{R} = \underbrace{c(L_1L_2 - M^2)^{-1} \cdot \begin{pmatrix} L_2R_1 & MR_2 \\ MR_1 & L_1R_2 \end{pmatrix}}_{\text{elements have dimensionality of "frequency"}}$$

What meaning are we to assign to $(\mathbb{D}+\mathbb{W})^{-1}$? From the identity

$$\tfrac{d}{dt}e^{Wt}f = e^{Wt}(\tfrac{d}{dt}+W)f \quad : \quad \text{all } f(t)$$

one obtains the "shift rule"

$$\big(\tfrac{d}{dt}+W\big)\bullet = e^{-Wt}\tfrac{d}{dt}e^{Wt}\,\bullet \quad : \quad \text{to be read as an } \textit{operator identity}$$

of which

$$\big(\tfrac{d}{dt}+W\big)^n\bullet = e^{-Wt}\big(\tfrac{d}{dt}\big)^n e^{Wt}\,\bullet$$

is a corollary and (in the case $n=-1$)

$$(\mathbb{D}+\mathbb{W})^{-1}\bullet = e^{-\mathbb{W}t}\int^t e^{\mathbb{W}s}\bullet\,ds \tag{139}$$

the matrix analog.[60] The theory of linear differential equations supplies this general proposition:

general solution of inhomogeneous equation
= any particular solution of inhomogeneous equation
+ general solution of associated homogeneous equation

Bringing these remarks together, we conclude that the general solution of (138) can be described

$$\boldsymbol{I}(t) = e^{-\mathbb{W}t}\int_0^t e^{\mathbb{W}s}\boldsymbol{F}(s)ds + e^{-\mathbb{W}t}\boldsymbol{I}(0)$$

In the present instance $\boldsymbol{I}(0)=\mathbf{0}$ and $\boldsymbol{F}(t)$ is (for $t>0$) a *constant* vector, so we can *perform* the integration, and obtain[61]

$$= \frac{\mathbb{I}-e^{-\mathbb{W}t}}{\mathbb{W}}\boldsymbol{F} = \big(\mathbb{I}-e^{-\mathbb{W}t}\big)\mathbb{R}^{-1}\boldsymbol{V} \tag{140}$$

Our analytical task reduces therefore to the evaluation of $(\mathbb{I}-e^{-\mathbb{W}t})/\mathbb{W}$. This can be accomplished in a great variety of ways, two of which are described on pages 124–129 of the 1980/81 edition of these notes. The details are amusing, and of some methodological interest . . . but distract from the *physical* points at issue: here I will be content to

- assign representative values to the circuit parameters (you are encouraged to try other values) and
- entrust the computational labor to *Mathematica*.

[60] In preceding equations the $\bullet$'s are placeholders for the "operands"—*i.e.*, for unspecified functions of t

[61] PROBLEM 31.

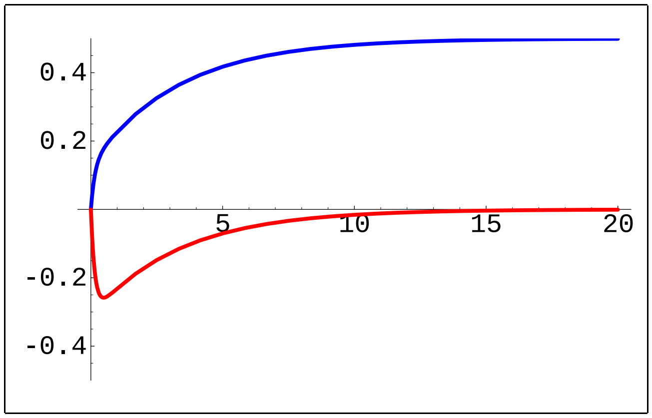


FIGURE 31: *At time $t = 0$ the battery is switched* ON *and the current I_1 in the primary circuit rises (as the sum to two exponentials, one "fast" and the other "slow") to the steady value V/R_1. The current I_2 induced in the secondary circuit is dipping transcient, present only while $\frac{d}{dt}I_1 \neq 0$.*

Specifically, I (semi-randomly) set

$$\boldsymbol{V} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \text{unit} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbb{L} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \ \mathbb{R} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

which entail

$$\mathbb{W} = \begin{pmatrix} 2 & -2 \\ -4 & 5 \end{pmatrix} \quad : \quad \det \mathbb{W} \neq 0$$

The command `(unit - MatrixExp[-t W]).Inverse[R].V` instantaneously supplies

$$\left.\begin{aligned} I_1(t) &= \tfrac{1}{2}\Big\{1 - \tfrac{\sqrt{41}-3}{2\sqrt{41}}e^{-\omega_1 t} - \tfrac{\sqrt{41}+3}{2\sqrt{41}}e^{-\omega_2 t}\Big\} \\ I_2(t) &= \tfrac{1}{2}\Big\{\qquad\quad \tfrac{4}{\sqrt{41}}e^{-\omega_1 t} - \quad \tfrac{4}{\sqrt{41}}e^{-\omega_2 t}\Big\} \end{aligned}\right\} \tag{141.1}$$

where

$$\omega_1 = \tfrac{7+\sqrt{41}}{2} = 6.70156$$
$$\omega_2 = \tfrac{7-\sqrt{41}}{2} = 0.29844$$

are observed to be precisely the *eigenvalues* of $\mathbb{W}$ (of which more later). The "primary" and "secondary" currents $I_1(t)$ and $I_2(t)$ are plotted in Figure 31. Asymtotically the system approaches a steady state, with (as is obvious already from (137))

$$I_1(\infty) = V/R_1 \quad \text{and} \quad I_2(\infty) = 0$$

That state having been achieved, let us stitch the battery *off* (and at the same time restart the clock). We then have $\boldsymbol{I}(t) = e^{-\mathbb{W}t}\boldsymbol{I}_{\text{steady}}$, which in our

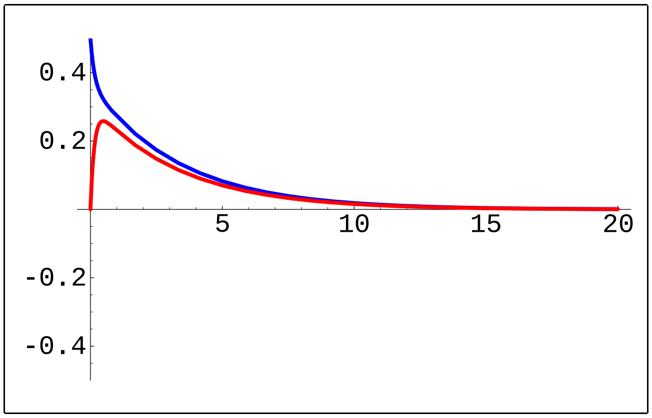


FIGURE 32: *At time $t = 0$ the battery is switched* OFF *and the previously steady current in the primary circuit drops exponentially back to zero. The current I_2 induced in the secondary circuit is now a rising transcient—again present only while $\frac{d}{dt}I_1 \neq 0$.*

numerical example supplies

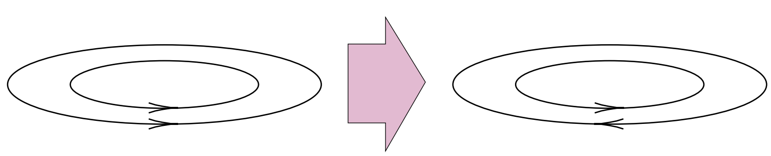
$$\left.\begin{aligned} I_1(t) &= \tfrac{1}{2}\Big\{ \tfrac{\sqrt{41}-3}{2\sqrt{41}}e^{-\omega_1 t} + \tfrac{\sqrt{41}+3}{2\sqrt{41}}e^{-\omega_2 t}\Big\} \\ I_2(t) &= \tfrac{1}{2}\Big\{ -\tfrac{4}{\sqrt{41}}e^{-\omega_1 t} + \tfrac{4}{\sqrt{41}}e^{-\omega_2 t}\Big\} \end{aligned}\right\} \tag{141.2}$$

These functions are displayed in Figure 32. It is the *contrary transcience* of the induced current (see again Figure 21) that lies at the heart of Faraday's surprising experimental discovery, and a wonder that Maxwell was able in

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\tfrac{1}{c}\tfrac{\partial \boldsymbol{B}}{\partial t} \tag{107}$$

to capture its formal essence. Several remarks are now in order:

1. The diagonal elements W_{11} and W_{22} of $\mathbb{W}$ arise from *self*-inductance, and are therefore invariably positive. But the off-diagonal elements $W_{12} = W_{21}$ refer to *mutual*-inductance, and reverse sign when we reverse either of the sign conventions attached to $\mathcal{C}_1$ and $\mathcal{C}_2$:



Mutual inductance is, in other words, *sign-indefinite and convention-dependent.*

2. The temporal aspects of Faraday induction are evidently *under the control of the eigenvalues* of $\mathbb{W}$, which in the general case[62] read

$$\begin{aligned}\omega_\pm &= \frac{R_1L_2 + R_2L_1 \pm \sqrt{(R_1L_2 + R_2L_1)^2 - 4R_1R_2(L_1L_2 - M^2)}}{L_1L_2 - M^2}\\ &= \frac{R_1L_2 + R_2L_1 \pm \sqrt{(R_1L_2 - R_2L_1)^2 + 4R_1R_2M^2}}{L_1L_2 - M^2}\end{aligned}$$

These are clearly real in all cases, but will both be positive—as is required if neither of the factors $e^{-\omega_\pm t}$ is to blow up catastrophically—if and only if

$$L_1L_2 - M^2 > 0 \tag{142}$$

Soon we will be in position to show that the inequality (142)—though not at all an obvious implication of (122)!—follows with elegant simplicity from first principles. Note that "catastrophic blow-up" would result also if the minus sign were dropped from (107): it was with that point in mind that I referred on page 63 to the "stabilizing minus sign." It is a common practice—but, as I will argue, misleading—to point to that minus sign and say "That is Lenz' law." For the minus sign is *always* correct, while Lenz' law (which might better be called "Lenz' rule of thumb") is sometimes violated.

3. Recall the statement of

> LENZ' LAW: The directionality of I_{induced} tends to be such that the resulting magnetic flux $\varPhi_{\text{induced}}$ counteracts the $\dot{\varPhi}_{\text{impressed}}$ from which the induced current itself derives.

The word "tends"—though as sharp as it gets in some fields—tells a physicist next to nothing. Where does it come from, and what does it mean? Replace the battery with a signal generator designed to produce some/any *prescribed* $I_1(t)$. Reading from (137) we see that the induced current $I_2(t)$ satisfies $\frac{1}{c}L_2\dot{I}_2 + R_2I_2 = -\frac{1}{c}M\dot{I}_1$ which we may write

$$\big(\tfrac{d}{dt} + \Omega\big)I_2(t) = -f(t), \text{ some } \underline{\text{prescribed}} \text{ function}$$

with $\Omega \equiv cR_2/L_2$ and $f(t) \equiv (M/L_2)\dot{I}_1(t)$. Arguing as on page 83 we have

$$\begin{aligned}I_2(t) &= -\int_0^t e^{-\Omega(t-s)}f(s)\,ds + e^{-\Omega t}I_2(0)\\ &= -\begin{pmatrix}\text{weighted summary of the}\\ \text{recent history of } f(t)\end{pmatrix} + \begin{pmatrix}\text{start-up transcient}\\ \text{that soon dies}\end{pmatrix}\end{aligned}$$

Evidently Lenz' law speaks to the minus sign, and is made fuzzy by the allusion to "recent history," since the "weighted summary" can be of

[62] Use the *Mathematica* command `Eigenvalues[W]`.

either sign, depending upon *details* of that history. The induced current might, in particular, be found to be flowing momentarily in the "wrong" (anti-Lenzian) direction as a kind of "inertial" effect. Arguing now in somewhat finer detail, we have

$$I_2(t) = -(M/L_2)\int_0^t e^{-\Omega(t-s)}\dot{I}_1(s)\,ds + e^{-\Omega t}I_2(0)$$

which upon integration-by-parts becomes

$$\begin{aligned}&= -(M/L_2)\Big\{I_1(t) - \Omega\int_0^t e^{-\Omega(t-s)}I_1(s)\,ds\Big\}\\ &\downarrow\\ &= -(M/L_2)I_1(t) \quad \text{as the "recall time" } \Omega^{-1}\to 0\end{aligned}$$

In that limit we have $L_2I_2 = -MI_1$ or (to say the same thing another way) $\Phi_{\text{induced}} = -\Phi_{\text{impressed}}$. This we might call "Lenz' exaggeration," because "short recall time" means large Ω means small L_2, and by (142) there is a limit to how small L_2 can be: $L_2 > M^2/L_1$.

The preceding discussion—which began on page 61, and has involved digressive looks at several important subtopics

- the physics of Ohmic materials
- emf
- mutual and self-inductance
- techniques for solving coupled circuit equations
- Lenz' rule of thumb

—shows that Maxwell's equations do indeed account for Faraday's experimental results. We have proceeded deductively, but the historic route was (no pun intended) *inductive* (experiment $\longrightarrow \boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{c}\partial\boldsymbol{B}/\partial t$) . . . and clearly required genious of an exceptionally high order. Faraday's work—as experimentalist and as intuitive father of the field concept[63]—was clearly critical to the development of Maxwellian electrodynamics. Remarkably, it opened also some doors which I have not yet mentioned.

6. Some technological & theoretical ramifications of Faraday's law. Faraday was perfectly well aware from the outset that he had discovered a point of fundamental physical principle. He was obliged, however, to leave the theoretical elaboration of his discovery to others (namely to Maxwell, who was his junior by nearly 40 years: Maxwell was, in fact, only three months old when Faraday performed his famous experiements) . . . for while Ampere was celebrated for his mathematical virtuosity, Faraday was, by his own admission, a mathematical ignoramus. The *technological* ramifications of his discovery—that one might

[63] For a good discussion, see Mary B. Hesse, *Forces & Fields: A Study of Action at a Distance in the History of Physics* (1965).

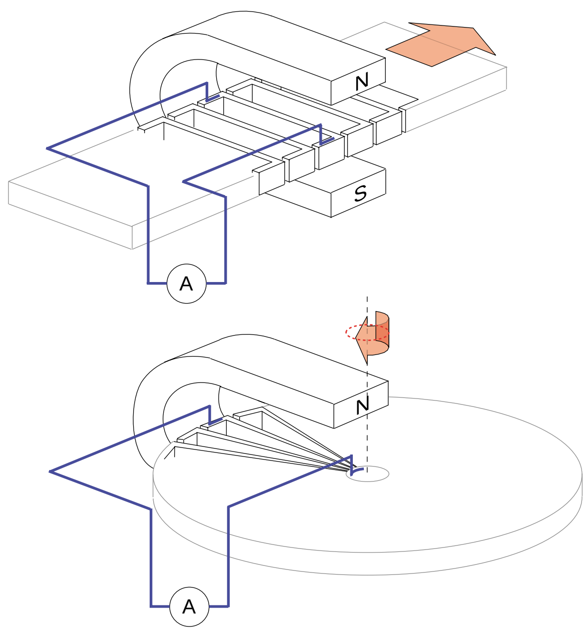


Figure 33: *Simple dynamos. At top: an infinite train of "staples" is dragged through the field of a permanent magnet. The magnetic flux enveloped by the circuit (shown in* blue*) is time-dependent, so an* emf *is developed, which produces a current. The rotational variant of the same device (below) presents no such absurdity as an "infinite train," and could actually be constructed. The placement of the magnet is, in both figures, schematic: in practice one would want to slide the magnet back until the maximal field is positioned to have the maximal effect. Notice that both devices involve sliding contacts—realized in practice by "brushes," which are a source of wear and of electrical noise. For description of a wonderfully ingenious escape from that limitation, see the Reed College thesis "A dynamo without slip rings" by Evan Wedell* (1982).

expect to be able to use not Voltaic cells but changing magnetic fields to generate practical currents—was, on the other hand, instantly apparent to Faraday (who, however, did not immediately foresee that his idea was to have profound ramifications outside the laboratory: he did not imagine rivers strangled by hydroelectric stations, forests dissected by power lines). The invention of the dynamo was essentially *simultaneous* with the discovery of electromagnetic induction, and was followed very swiftly by the first steps toward the "electrification" of the world.

In the figure I show an imagined early chapter in the history of the invention of the dynamo. The second (rotational) design is conceptually so simple that it

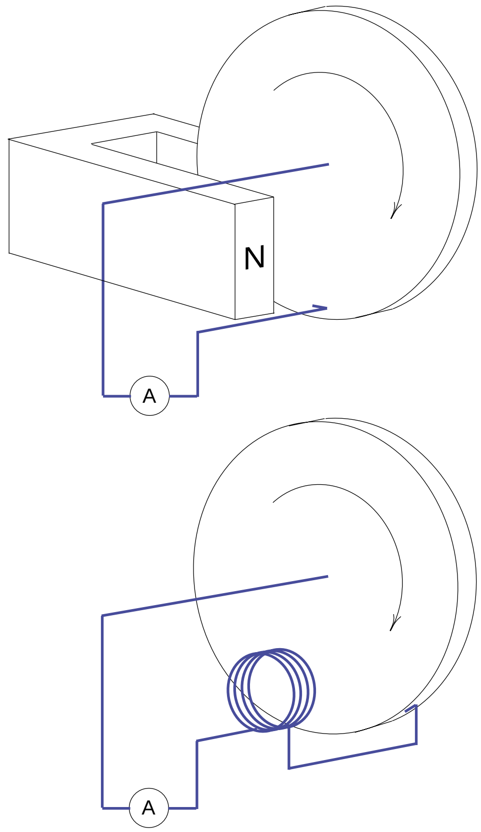


FIGURE 34: *"Homopolar disk dynamos." The design at the top can be looked upon as the "continuous limit" of the design shown at the bottom in Figure 33. Here the lower figure illustrates the operating principle of the* "self-excited *homopolar dynamo:" the permanent magnet has been replaced by an electromagnet that draws its current* from the dynamo itself.

almost "invents itself." But I think most physicists would, on intuitive grounds, be somewhat doubtful that the "homopolar disk dynamos" shown in Figure 34 —evolved from the previous design by proceeding "to the limit of infinitely many fins"—would even work, for they involve *currents which are not confined to moving wires*.[64] They depend, to be more precise, upon the (evidently quite complex) *physics of <u>eddy currents</u>* (interior to the rotating conductive disk). Actually, Faraday was led almost at once to the homopolar design (which works!), and for interesting reasons. In 1824 Dominique F. J. Arago had discovered that *a rotating copper disk exerts torque upon a suspended magnet*, and (see Figure 35) conversely. It seemed clear that some kind of "induced

[64] That circumstance makes it awkward to argue—at least on the geometrical face of the matter—that $\dot{\Phi} \neq 0$.

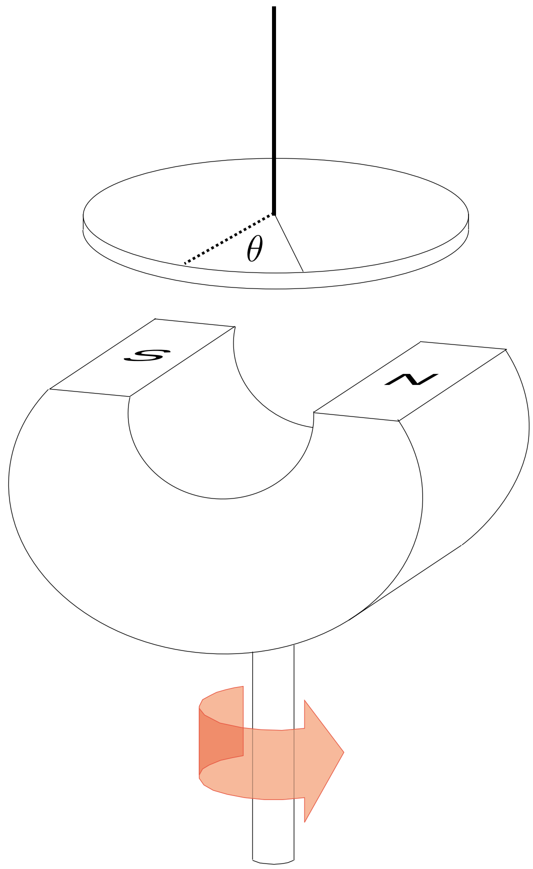


Figure 35: *Arago's apparatus. The copper disk is suspended by a torsion fiber. The spinning magnet is found to exert torque on the disk, but the effect is reduced/extinguished when radial slots perforate the disk. The first satisfactory account of the phenomenon was given by Faraday.*

magnetism" was involved, but this mode of explanation could not account for the observation that *when radial slots are cut in the disk the Arago effect is extinguished*. "Arago's extraordinary experiment" was much on the minds of physicists in the late 1820's, and was very well known to Faraday (to whom we owe the correct explanation: the "induced magnetism" arises from induced eddy currents, which Faraday called "whirl currents," and which the slots served to inhibit by "opening the circuits"). So Faraday had "disks on the brain." His homopolar disk dynamo can be understood as a variant of Arago's experimental configuration.

The self-excited homopolar disk-dynamo is a device of such elemental simplicity that it becomes natural to inquire whether it occurs spontaneously in Nature. I do not know enough about "biomotors" (such as twirl flagella) to know whether they provide examples, but a magnetohydrodynamic analog is

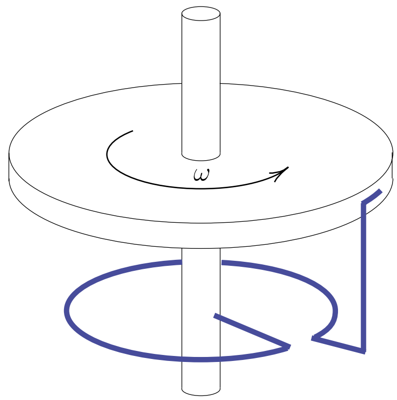


FIGURE 36: *Schematic representation of a self-exciting homopolar disk dynamo, abstracted from the device shown at the bottom of Figure 34.*

believed to be responsible for the geomagnetic field and for the magnetic fields of certain classes of stars.[65] Similar principles may operate at a galactic level. I would like therefore to consider briefly how the physics of such a device might be formalized. Proceeding in reference to Figure 36 . . . let an external mechanical agency maintain the constant angular velocity ω of a conductive disk. Evidently

$$\tfrac{1}{c}L\dot{I} + RI = \mathcal{E} \tag{143}$$

where L and R refer to the self-inducatance and resistance of the electrical circuit, and where

$$\mathcal{E} \equiv \text{dynamo emf}$$

We expect[66] $\mathcal{E}$ to be proportional to the rate (set by ω) at which "filamentary constituents of the eddy current cut field lines" of the $\boldsymbol{B}$-field generated by the solenoid. Since $\boldsymbol{B}$ is proportional to I, we expect to have $\mathcal{E} \sim \omega I$. This we will express

$$= \tfrac{1}{c}M\omega I \tag{144}$$

where M is dimensionally some kind of "mutual inductance." We are in no position to *compute* M since

- we don't know *how current is distributed* in the disk (*i.e.*, we lack a "theory of eddy currents") and
- we don't know anything about the $\boldsymbol{B}$-field *interior* to the disk.

[65] See W. M. Elsasser, "Hydromagnetism," AJP **23**, 590 (1955) & **24**, 85 (1956) and "Hydromagnetic dynamo theory," Rev. Mod. Phys. **28**, 135 (1956). For a good and very detailed review of more recent work see H. K. Moffatt, *Magnetic Fields Generation in Electrically Conducting Fluids* (1978).

[66] This is clearest if one argues from the second figure on page 88.

If, however, we assume (144) to be qualitatively correct, then (143) becomes

$$\tfrac{1}{c}L\dot{I} + RI = \tfrac{1}{c}M\omega I$$

—the solution of which

$$I(t) = I_0\, e^{\frac{M\omega - cR}{L}t} \quad : \quad \begin{cases} \text{grows exponentially} & \text{if } \omega > cR/M \\ \text{dies} & \text{if } \omega < cR/M \end{cases}$$

(Physically, exponential growth would proceed only until the rate I^2R of Joule dissipation becomes equal to the power of the external agency which drives the disk.) We conclude that the homopolar dynamo *becomes self-excited only if spun fast enough ... and not at all if spun in the wrong direction*.

There is general agreement among geophysicists that some kind of "dynamo action" (slow convection within the earth's electrically conductive core) must be responsible for the principal component of the geomagnetic field, though details of the mechanism remain inaccessible. Suggestive insight into a characteristic feature of the phenomenon—aperiodic polarity reversal—was obtained by T. Rikitake,[67] who studied the system of *coupled* disk dynamos shown in Figure 37. The two circuits are assumed to have identical resistances R and self-inductances L. The "external agency" is asked not to maintain constant angular velocity but to apply constant and identical torques N to the two disks, which are assumed to have (relative to their spin axes) identical moments of inertia A: $\omega_1(t)$ and $\omega_2(t)$ have joined $I_1(t)$ and $I_1(t)$ as functions to be determined. With these simplifying assumptions one has

$$\left.\begin{aligned} \tfrac{1}{c}L\dot{I}_1 + RI_1 &= \tfrac{1}{c}M\omega_1 I_2 \\ \tfrac{1}{c}L\dot{I}_2 + RI_2 &= \tfrac{1}{c}M\omega_2 I_1 \end{aligned}\right\} \tag{145.1}$$

which describe the *electrical* properties of the system, and

$$\left.\begin{aligned} A\dot{\omega}_1 &= N - \tfrac{1}{c}MI_1I_2 \\ A\dot{\omega}_2 &= N - \tfrac{1}{c}MI_2I_1 \end{aligned}\right\} \tag{145.2}$$

which describe its *mechanical* properties: here $-\frac{1}{c}MI_2I_1$ describes the torque which arises from the Lorentz forces experienced by the eddy current in one disk due to the magnetic field generated by the other ... and *vice versa*. The constant M quantifies the strength of that effect, and acquires its name from the circumstance that dimensionally $[M]$ = "inductance." With Rikitaki, we

[67] "Oscillations of a system of disk dynamos," Proc. Camb. Phil. Soc. **54**, 89 (1958). See also A. E. Cook & P. H. Roberts, "The Rikitake two-disk synamo system," Proc. Camb. Phil. Soc. **68**, 547 (1970) and the final pages of Moffatt's monograph.[65]

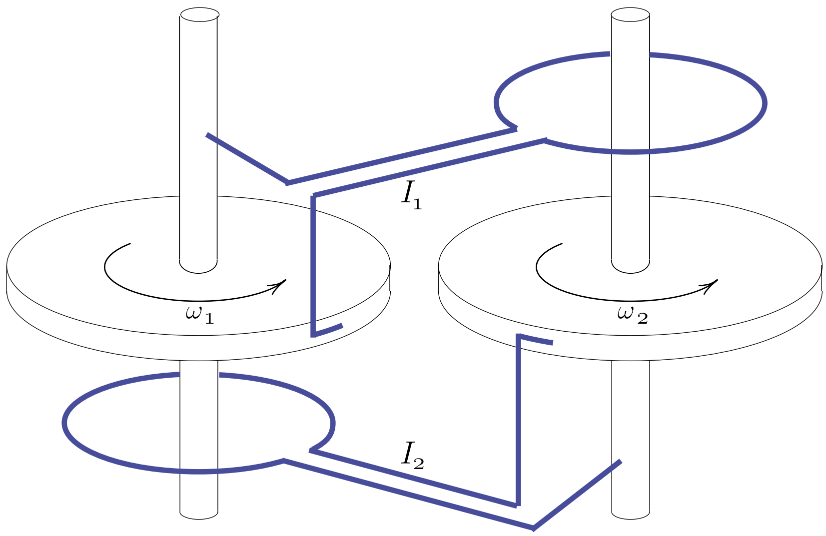


FIGURE 37: *Rikitake's system of cross-coupled disk dynamos, in which the magnetic field experienced by each results from current generated by the other. The external agency, instead of controlling the angular velocities* ω_1 *and* ω_2 *of the disks, now applies to each the same constant torque* N*. Simple though the system is, its behavior is shown in the text to be sometimes* chaotic.

introduce dimensionless variables

$$\tau \equiv \sqrt{NM/AL}\cdot t \qquad : \quad \text{dimensionless } \textit{time}$$

$$\left.\begin{aligned} U &\equiv \sqrt{AM/NL}\cdot \omega_1 \\ V &\equiv \sqrt{AM/NL}\cdot \omega_2 \end{aligned}\right\} \quad : \quad \text{dimensionless } \textit{angular velocities}$$

$$\left.\begin{aligned} X &\equiv \sqrt{M/cN}\cdot I_1 \\ Y &\equiv \sqrt{M/cN}\cdot I_2 \end{aligned}\right\} \quad : \quad \text{dimensionless } \textit{currents}$$

and find that equations (145) can be written

$$\left.\begin{aligned} \dot{X} &= -\mu X + UY \\ \dot{Y} &= -\mu Y + VX \\ \dot{U} &= \quad 1 - XY \\ \dot{V} &= \quad 1 - XY \end{aligned}\right\} \tag{146}$$

where $\mu \equiv cR\sqrt{A/LMN}$ is a solitary adjustable parameter, and where the dot now signifies differentiation with respect to τ. Trivially $U - V = \alpha$, where α is

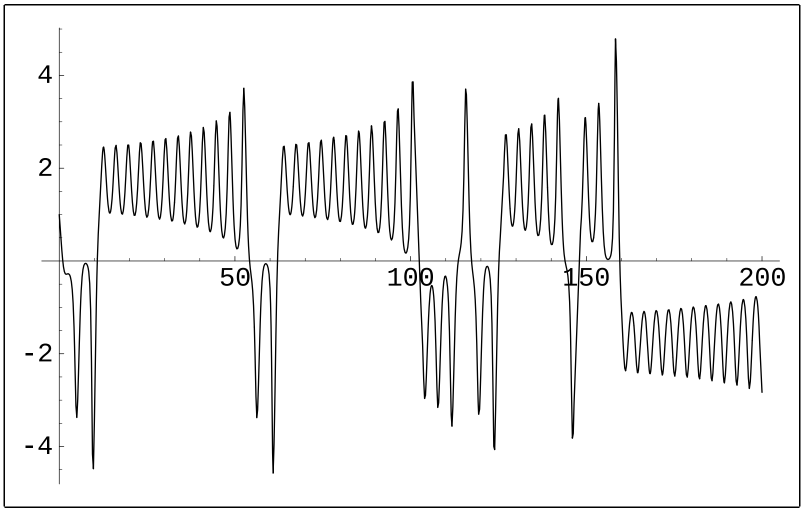


FIGURE 38: *Graph of $X(\tau)$, derived from (147) in the case $\mu = 1.0$, $\mu = 2.7$ with initial conditions $X(0) = 1.0$, $Y(0) = 0$, $U(0) = 0.5$.*

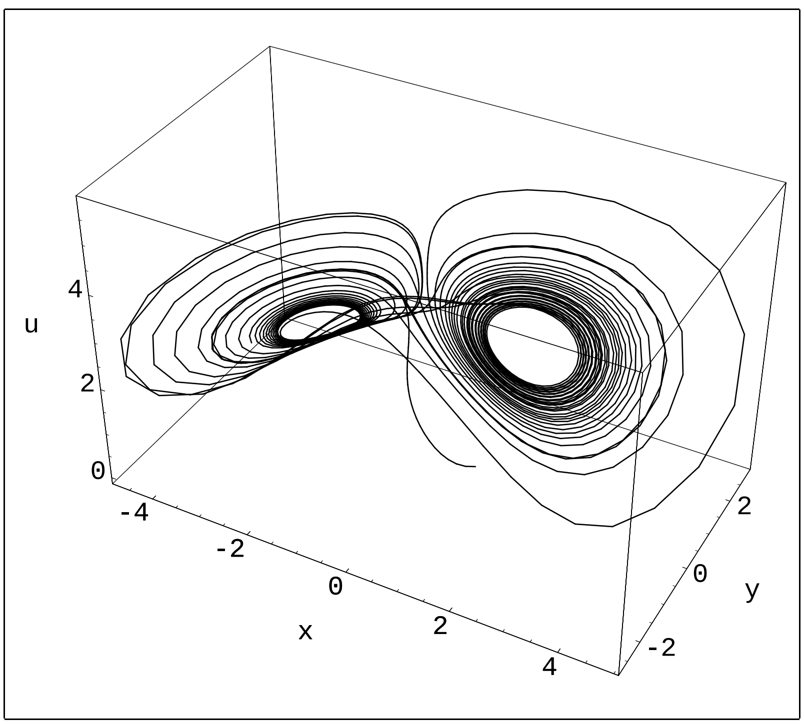


FIGURE 39: *3-dimensional parametric plot of $\{X(\tau), Y(\tau), U(\tau)\}$ under those same assumptions.*

a constant which we may without loss of generality assume to be non-negative. Returning with this information to (146) we obtain

$$\left.\begin{aligned} \dot{X} &= \qquad\quad -\mu X + UY \\ \dot{Y} &= -\alpha X - \mu Y + UX \\ \dot{U} &= \quad 1 \qquad\quad - XY \end{aligned}\right\} \tag{147}$$

which is a triplet of coupled *non-linear* 1^{st}-order differential equations. They defy analytical solution, must be solved numerically ... which in 1958 was a highly non-trivial undertaking, but today lies within the capability of every sophomore. In "Physicist's Introduction to *Mathematica*" (2000) I describe[68] how the resources of *Mathematica* can be brought to bear on the problem, and produce Figures 38 & 39. The point to which Rikitaki drew the attention of his geophysical colleagues was the surprising *aperiodicity of the sign reversals* evident in Figure 38.[69]

I mention finally H. Gruenberg's accidental discovery[70] of a "motor" of astounding simple design. The device can be thought of as a disk dynamo run "backwards—in motor mode."[71]

Returning our "curious devices" to the shelves from which they came, I return now to the theoretical mainline of our subject ... looking specifically to the description of the energy which resides in a magnetostatic field. We saw (pages 18–22) that the analogous electrostatic problem could be formulated as a study of the energetic details of the "source assembly process." But we have seen also (page 60) *the source* $\boldsymbol{j}(\boldsymbol{x})$ *of an magnetostatic field cannot be "assembled": it must be turned on*. This is a process the energetic details of which we are only now—thanks to Faraday—in position to examine. By way of preparation ...

[68] See Laboratory 6, Part A

[69] Rikitaki's work did not engage the attention of the broader population of applied mathematicians. But several years later the meterological work of E. N. Lorenz led him (in "Deterministic nonperiodic flow," J. Atmos. Sci. **20**, 130 (1963)) to a triplet of equations

$$\begin{aligned} &\dot{x} - \sigma x + \sigma y \\ &\dot{y} = rx - y - rxz \quad : \quad \sigma > 0,\ r \text{ and } b > 0 \text{ are parameters} \\ &\dot{z} = xy - bz \end{aligned}$$

which is structurally quite similar to (147), and which yield qualitatively similar solution curves. Lorenz' discovery contributed importantly to the development of the modern *theory of chaotic systems*, but Rikitaki's remains—even today—largely unknown.

[70] "The ball bearing as a motor," AJP **46**, 1213 (1978).

[71] See the Reed College thesis of Peter Miller: "The ball bearing motor: strange torques in spinning conductors" (1981).

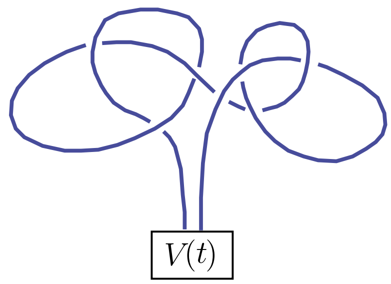


FIGURE 40: *A power supply—drawing is power from some external source (the wall socket, not shown) is used to create and maintain current in a wire loop of resistance R and self-inductance L. $V(t)$ is under the control of the experimenter.*

We have already remarked (page 67) that magnetic fields do no work on moving charges. Time-dependent $\boldsymbol{B}$-fields give rise, however, (by Faraday's law) to $\boldsymbol{E}$-fields, and $\boldsymbol{E}$-fields (of *whatever* origin) do work at a temporal rate given locally by

$$\mathcal{P} = \boldsymbol{E}\cdot\boldsymbol{j} \tag{148}$$

Suppose that $\boldsymbol{j}(\boldsymbol{x})$ refers to the steady current I which circulates in a loop of wire. We then have

$$\begin{aligned} P &\equiv \iiint_{\text{volume of wire}} \mathcal{P}\, d^3x \\ &= \oint_{\text{loop}} \left\{ \iint_{\text{cross section}} \boldsymbol{E}\cdot\boldsymbol{j}\, dA \right\} d\ell \\ &= I \oint \boldsymbol{E}\cdot\boldsymbol{d\ell} \\ &= I\mathcal{E} \end{aligned} \tag{149}$$

as a description of the *temporal rate at which* $\boldsymbol{E}$ *does work* on the charge carriers that comprise the current I. In the *steady* case the power invested by the $\boldsymbol{E}$-field is dissipated (Joule heating) at the familiar rate

$$= I^2 R$$

but in the *non-steady* case some fraction of P may be invested in the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields which are associated with the capacitive and inductive features of the circuit (while another fraction may be dispatched as electromagnetic radiation). Which brings us back to the problem at hand:

Consider (Figure 40) a loop of wire (resistance R, self-inductance L) into which we have introduced an adjustable DC power supply, and let $V(t)$ denote

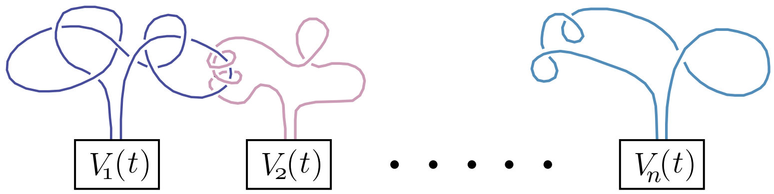


FIGURE 41: *A system of magnetically interactive circuits, each with its own power supply.*

the power supply's *output voltage at time t* (which is under our control). The current $I(t)$ can be computed from

$$V(t) = RI + \tfrac{1}{c}L\dot{I}$$

and the specified value of $I(0)$. The power supply is doing work at a rate given instantaneously by

$$\begin{aligned}\tfrac{d}{dt}W &= VI\\ &= RI^2 + \tfrac{d}{dt}\big(\tfrac{1}{2c}LI^2\big)\end{aligned}$$

so the total energy delivered by the power supply since $t = 0$ becomes

$$\begin{aligned}W \equiv \int_0^t \dot{W}\,dt &= \int_0^t RI^2\,dt + \tfrac{1}{2c}LI^2(t)\\ &= (\text{heat dissipated in } R) + (\text{energy stored in the magnetic field})\end{aligned}$$

Note that the latter term (but not the former) represents an investment which the power supply *recovers when V is turned down/off*. Note also that if the wire were replaced by an idealized "filament" then (since for a filament $L = \infty$) the latter term would become infinite (which is to say: the powerless supply would find itself "powerless to drive a current"): this again is the *magnetic version of the self-energy problem*.

Consider now a *system* of wire loops, each with its own adjustable power supply (Figure 41). The currents at time t can be computed from

$$\begin{aligned}V_1 &= R_1 I_1 + \tfrac{1}{c}\big\{\ \ L_1\dot{I}_1 + M_{12}\dot{I}_2 + \cdots + M_{1n}\dot{I}_n\big\}\\ V_2 &= R_2 I_2 + \tfrac{1}{c}\big\{M_{21}\dot{I}_1 + \ \ L_2\dot{I}_2 + \cdots + M_{1n}\dot{I}_n\big\}\\ &\ \ \vdots\\ V_n &= R_n I_n + \tfrac{1}{c}\big\{M_{n1}\dot{I}_1 + M_{n2}\dot{I}_2 + \cdots + \ \ L_n\dot{I}_n\big\}\end{aligned}$$

of which

$$\boldsymbol{V} = \mathbb{R}\,\boldsymbol{I} + \tfrac{1}{c}\mathbb{M}\,\dot{\boldsymbol{I}}$$

is a handy abbreviation. The power supplies are (collectively) doing work at the rate

$$\begin{aligned}\dot W &= \dot W_1 + \dot W_2 + \cdots + \dot W_n\\ &= I_1V_1 + I_2V_2 + \cdots + I_nV_n \equiv \boldsymbol{I}^{\mathsf T}\boldsymbol{V}\\ &\qquad\qquad = \boldsymbol{I}^{\mathsf T}\mathbb{R}\,\boldsymbol{I} + \tfrac{1}{c}\boldsymbol{I}^{\mathsf T}\mathbb{M}\,\dot{\boldsymbol{I}}\\ &\qquad\qquad = \boldsymbol{I}^{\mathsf T}\mathbb{R}\,\boldsymbol{I} + \tfrac{d}{dt}\big(\tfrac{1}{2c}\boldsymbol{I}^{\mathsf T}\mathbb{M}\,\boldsymbol{I}\big) \quad\text{by}\quad \mathbb{M}^{\mathsf T} = \mathbb{M}\end{aligned}$$

so (if we assume that $\boldsymbol{I}(0) = \boldsymbol{0}$)

$$\begin{aligned}W = \int_0^t \dot W\,dt &= \int_0^t \boldsymbol{I}^{\mathsf T}\mathbb{R}\,\boldsymbol{I}\,dt + \tfrac{1}{2c}\boldsymbol{I}^{\mathsf T}\mathbb{M}\,\boldsymbol{I}\\ &= (\text{heat}) + (\text{recoverable }\underline{\text{magnetic field energy}})\end{aligned}$$

The Joule dissipation term is (though physically important) for our present purposes uninteresting. Accordingly . . .

We restrict our attention henceforth to the term

$$W_{\text{magnetic}} = \tfrac{1}{2c}\boldsymbol{I}^{\mathsf T}\mathbb{M}\,\boldsymbol{I} \tag{150}$$

This is work which our power supplies would have to perform *even if the wires were non-resistive*. Physically, it records our *effort to overcome the* emf *which results from Faraday induction*. It is (to reemphasize a point already on page 96) the effects not of $\boldsymbol{B}$ (which does no work) but of $\dot{\boldsymbol{B}}$ which lie at the foundation of (150). The question arises:

What—if any—is the relationship between (150) and the formulæ developed (on merely analogical grounds) on page 60? Returning with Neuman's formula (122) to (150) we obtain

$$W_{\text{magnetic}} = \tfrac{1}{8\pi c^2}\sum_m\sum_n \oint\!\oint \frac{I_m\boldsymbol{d\ell}_m\boldsymbol{\cdot}I_n\boldsymbol{d\ell}_n}{r_{mn}}$$

Evidently

$$= \tfrac{1}{8\pi c^2}\iint \boldsymbol{j}(\boldsymbol{x})\boldsymbol{\cdot}\boldsymbol{j}(\boldsymbol{\xi})\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\,d^3x\,d^3\xi \tag{151}$$

when the $\boldsymbol{j}$-field is not confined to the interior of wires. But this is precisely (106) and, by the arguments of page 60 (traced in reverse), is known to entail

$$\begin{aligned}&= \int \mathcal{B}(\boldsymbol{x})\,d^3x\\ &\quad \mathcal{B} \equiv \tfrac12\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{B} \quad:\quad \text{magnetic energy density}\end{aligned}$$

It is on this formal basis that we allow ourselves to state (as we did on the preceding page) that W_{magnetic} describes "energy stored *in the magnetic field*." Several comments are now in order:

1. It was emphasized on page 71 and again on page 81 that the components M_{mn} of $\mathbb{M}$ are "geometrical in nature." They summarize all that is "magnetically relevant" about the current configuration. First encountered in the description

$$\varPhi_{mn} = M_{mn}I_n \tag{120}$$

of the magnetic flux which interlinks a population of current loops, their occurrence in

$$W_{\text{magnetic}} = \tfrac{1}{2c}\sum_{m,n} I_m M_{mn} I_n \tag{150}$$

is equally fundamental . . . and provides in fact an efficient framework within which to address questions such as those posed on page 81.

2. M_{mn} refers more particularly to the magnetostatics of steady current <u>loops</u>. Since *not every* $\boldsymbol{j}$*-field admits of conceptualization as a "bundle of filamentary loops* ($\boldsymbol{\nabla}\!\cdot\!\boldsymbol{j} = 0$ states that "$\boldsymbol{j}$-lines do not have ends," but that does not of itself entail loop-structure), *the concept of inductance would appear to have only limited relevance to the magnetostatics of <u>distributed</u> currents* (such as eddy currents).[72]

3. From results already in hand we have

$$W_{\text{magnetic}} = \tfrac{1}{2c}\boldsymbol{I}^{\mathsf{T}}\mathbb{M}\,\boldsymbol{I} = \tfrac{1}{2}\iiint \boldsymbol{B}\!\cdot\!\boldsymbol{B}\,d^3x \geqslant 0$$

from which we conclude that

$$\boldsymbol{I}^{\mathsf{T}}\mathbb{M}\,\boldsymbol{I} \text{ is a } \underline{\text{positive definite}} \text{ quadratic form}$$

i.e., that the inductance matrix $\mathbb{M}$ is positive definite. This amounts to a statement that the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation

$$\det(\mathbb{M} - \lambda\mathbb{I}) = 0$$

(which are the "eigenvalues" of $\mathbb{M}$, and which are—by the reality and symmetry of $\mathbb{M}$—necessarily real) are necessarily all positive: $\lambda_i > 0$. An equivalent (and—since they do not require that we *solve* the characteristic equation—more useful) set of "positivity conditions" are[73] the following:

$$L_1 > 0\,,\quad \begin{vmatrix} L_1 & M_{12} \\ M_{12} & L_2 \end{vmatrix} > 0\,,\quad \begin{vmatrix} L_1 & M_{12} & M_{13} \\ M_{13} & L_2 & M_{23} \\ M_{13} & M_{23} & L_3 \end{vmatrix} > 0\,,\ldots,\ \det\mathbb{M} > 0$$

Equivalent inequalities can be obtained by permuting the indices. Thus does

$$L_1L_2 - M^2 > 0 \tag{142}$$

---

[72] PROBLEM 32.

[73] See G. E. Shilov, *Linear Algebra* (1977), page 209. For a particularly clear discussion of this classic result see (of all people!) Paul Samuelson, *Foundations of Economic Analysis* (1967), pages 365–375.

—seen now to be one of a large *set* of inequalities—follow "with elegant simplicity from first principles," as was asserted on page 86.[74] From this general proposition

$$\text{arithmetic mean} \geqslant \text{geometric mean}$$

it now follows, by the way, that

$$\tfrac{1}{2}(L_1 + L_2) \geqslant \sqrt{L_1 L_2} \geqslant |M|$$

4. Equation (150) can sometimes be used "backwards" to compute the elements of $\mathbb{M}$ without confronting the $\oint\oint$-complexities of (122): one uses $\frac{1}{2}\iiint B^2\,d^3x$ to compute W_{magnetic} and then infers the values of the desired coefficients M_{mn}.[75]

Let currents I_1 and I_2 flow in a pair of loops. The energy which resides in the associated $\boldsymbol{B}$-field can, by (150), be described

$$W = \tfrac{1}{2c}\Big\{L_1 I_1^2 + 2MI_1 I_2 + L_2 I_2^2\Big\} \tag{152}$$

Suppose we work to displace of one loop with respect to the other, bringing about (let us for convenience say) of a rotation-free *translation* of $\mathcal{C}_1$, as illustrated in Figure 42. Our effort has two effects:

- it *modifies the value of M (but not of L_1 and L_2)* and
- it produces Faraday emf's which, if uncompensated, would serve to *modify the values of I_1 and I_2.*

We accept as a condition of the problem that $\underline{I_1 \text{ and } I_2 \text{ are to be held constant}}$, and it is to realize that constraint (also to compensate for I^2R-losses) that we have inserted smart power supplies into the circuits. From the conditions just stipulated and (152) it follows that

$$dW = \tfrac{1}{c}I_1 I_2 dM \tag{153}$$

Working from Neumann's formula (122) we have

$$dM = \tfrac{1}{4\pi c}\bigg\{\oint\!\!\oint \tfrac{1}{|\boldsymbol{r}_{12}+\boldsymbol{dx}|}\boldsymbol{d\ell}_1\!\cdot\boldsymbol{d\ell}_2 - \oint\!\!\oint \tfrac{1}{r_{12}}\boldsymbol{d\ell}_1\!\cdot\boldsymbol{d\ell}_2\bigg\}$$

and it was established already at (10.1) that by Taylor's theorem

$$\tfrac{1}{|\boldsymbol{r}_{12}+\boldsymbol{dx}|} = \tfrac{1}{r_{12}} - \tfrac{\boldsymbol{r}_{12}}{r_{12}^3}\cdot\boldsymbol{dx} + \cdots$$

so

$$= \bigg\{-\tfrac{1}{4\pi c}\oint\!\!\oint \tfrac{\boldsymbol{r}_{12}}{r_{12}^3}\,\boldsymbol{d\ell}_1\!\cdot\boldsymbol{d\ell}_2\bigg\}\cdot\boldsymbol{dx} \tag{154}$$

[74] PROBLEM 33.
[75] PROBLEM 34.

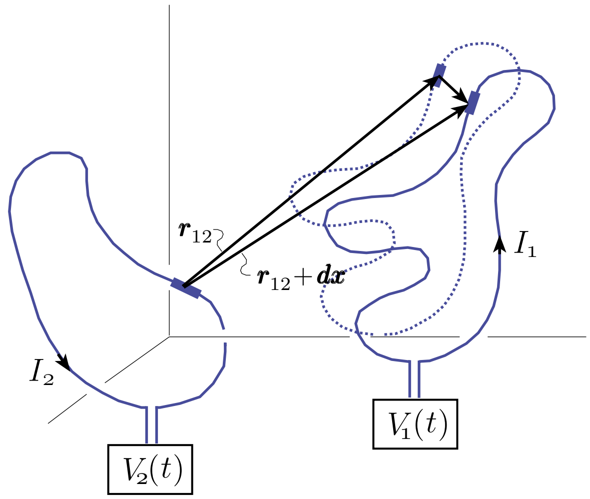


FIGURE 42: *Currents I_1 and I_2 circulate in a pair of loops. We ask*: How is the magnetic field energy altered when one of the loops is displaced with respect to the other? *Resolution of the question leads back—by a tricky argument—to Ampere's description of the* mechanical force *which one loop exerts upon the other.*

But we have encountered $\{$etc.$\}$ before—in Ampere's description (101) of the force

$$\boldsymbol{f}_{12} = I_1 I_2 \Big\{ -\tfrac{1}{4\pi c^2} \oint\!\!\oint \frac{\boldsymbol{r}_{12}}{r_{12}^3}\, \boldsymbol{d\ell}_1 \cdot \boldsymbol{d\ell}_2 \Big\}$$

that circuit $\mathcal{C}_2$ exerts on circuit $\mathcal{C}_1$. Returning with this information to (154), we find that (153) can be expressed

$$dW = \boldsymbol{f}_{12} \cdot \boldsymbol{dx} \tag{155}$$

and <u>appear to have encounted a sign problem</u>[76] . . . for the work which *we* do, struggling *against* that Amperean force—the energy which we *inject into the magnetic field*—is given by

$$dW_{\text{performed by us}} = -\boldsymbol{f}_{12} \cdot \boldsymbol{dx} \tag{156}$$

Ah! But we are not the sole workers in this story! The <u>power supplies</u> (over and above their obligations to pay the costs of I^2R-losses) have been *working to maintain the constancy of the currents*; *i.e.*, to compensate for the Faraday

[76] To be a physicist is to spend much of a lifetime chasing signs and errant factors of $\frac{1}{2}$, and we have encountered here a classic instance.

inductive effects that arise from the circumstance that the displacement of $\mathcal{C}_1$ takes place (not "virtually" but) in real time. Specifically

$$\begin{aligned}
V_1(t) &= I_1R_1 + I_1\dot{\Phi}_{12} = I_1R_1 + \tfrac{1}{c}\dot{M}_{12}I_2\\
V_2(t) &= I_2R_2 + I_2\dot{\Phi}_{21} = I_2R_2 + \tfrac{1}{c}\dot{M}_{21}I_1
\end{aligned}$$

The power supplies are delivering energy at instantaneous rates given by

$$\begin{aligned}
P_1(t) &= I_1V_1(t) = I_1^2R_1 + \tfrac{1}{c}I_1\dot{M}_{12}I_2\\
P_2(t) &= I_2V_2(t) = I_2^2R_2 + \tfrac{1}{c}I_2\dot{M}_{21}I_1
\end{aligned}$$

The I^2R terms will be dismissed as irrelevant to the present discussion: they describe energy dissipated as heat, the unrecoverable "cost of doing business." We are left with

$$\left.\begin{array}{r}\text{rate at which the power supplies are collectively}\\ \text{investing energy in redesign of the magnetic field}\end{array}\right\} = 2\cdot I_1I_2\tfrac{1}{c}\dot{M}$$

where use has been made of $M = M_{12} = M_{21}$. Clearly, the argument that gave (154) gives

$$\dot{M} = \left\{-\frac{1}{4\pi c}\oint\!\!\oint\frac{\boldsymbol{r}_{12}}{r_{12}^3}\,\boldsymbol{d\ell}_1\!\cdot\!\boldsymbol{d\ell}_2\right\}\!\cdot\!\frac{\boldsymbol{dx}}{dt}$$

The energy that the power supplies collectively/recoverably invest in time dt is given therefore by

$$\begin{aligned}
dW_{\text{performed by power supplies}} &= 2\cdot I_1I_2\tfrac{1}{c}dM\\
&= 2\cdot\boldsymbol{f}_{12}\!\cdot\!\boldsymbol{dx}\\
&= 2\,dW \qquad\qquad \text{by (155)}
\end{aligned}$$

In short: the power supplies collectively invest <u>twice</u> the energy dW that shows up in the redesigned magnetic field. But

$$\begin{aligned}
dW &= dW_{\text{performed by us}} + dW_{\text{performed by power supplies}}\\
&= dW_{\text{performed by us}} + 2\,dW
\end{aligned}$$

from which we immediately recover the desired statement (156).

The preceding argument exposes the sense in Ampere's formula (101) and Neumann's formula (122) make *equivalent statements*. We have used the latter to recover the former. Proceeding similarly, we could study the response dW of the field energy to differential *rotation* of $\mathcal{C}_1$ to obtain a description of the *torque* $\boldsymbol{\tau}_{12}$ which $\mathcal{C}_2$ exerts upon $\mathcal{C}_1$.

The argument shows that we can expect to recover

$$W_{\text{magnetostatic}} = \frac{1}{2c}\sum_{i,j}{}' I_iM_{ij}I_j = \tfrac{1}{2}\int \boldsymbol{B}\!\cdot\!\boldsymbol{B}\,d^3x$$

(not by "turning on" the currents in the already-positioned loops, but) by assembly of the loop system if we take sufficiently careful account of the *work done against Amperean inter-loop forces* . . . just as (on pages 19 *et seq*) we achieved

$$W_{\text{electrostatic}} = \tfrac{1}{8\pi}\sum_{i,j}{}' Q_i \tfrac{1}{r_{ij}} Q_j = \tfrac{1}{2}\int \boldsymbol{E}\!\cdot\!\boldsymbol{E}\, d^3x$$

by taking account of the work done against *Coulombic* forces. The idea is

1) to fabricate the loops "at infinity" and
2) there to invest the self-energy $W_{\text{self}} = \frac{1}{2c}\sum_i L_i I_i^2$ (infinite, if the loops are "filamentary") required to "switch on" the currents
3) then—quasistatically—to bring the pre-assembled current-carrying loops into their desired local configuration.

It is by "pre-assembly" that we escape the absurdities (both physical and formal: see again pages 58 & 59) that would attend "snippet by snippet" assembly. If the assembly process were "brisk" rather than quasistatic then *radiative effects* would complicate the energetic analysis: a similar restriction pertains to the electrostatic assembly process, since accelerated charges radiate.

7. Recapitulation . . . and a glance ahead. We have—by Schwingerean bootstrap —"derived" Maxwell's equations, and have shown that those equations do account correctly for the experimental discoveries of Coulomb, Oersted, Ampere, Faraday . . . and for some related phenomenology. The foundations of our subject are now in our possession, and many/most of the major formal/ phenomenological ramifications have been hinted at, if *only* hinted at. We have now to examine the details . . . which is quite an assignment, for in terms of

- the subtlety and variety of the relevant points of principle
- its power to inspire mathematical invention
- the diversity and importance of its physical applications

classical electrodynamics stands apart from virtually every other branch of physics.

Here follows—for purposes of orientation—a list of some of the specialized topics into which one might want to inquire. Looking first to formal matters . . .

1. We will want to understand the sense and ramifications of the statement that *electrodynamics is a* <u>*relativistic*</u> *classical field theory*. Exploration of this topic leads to certain *reformulations* of the standard theory, which in specialized contexts sometimes prove useful.It leads also to sharpened perception of some fundamental points of principle. And it motivates study of some aspects of *tensor analysis*.

So far as concerns mathematical technique

2. We will want to sharpen our ability actually to *solve* Maxwell's equations. In t-independent contexts (electrostatics, magnetostatics) this objective motivates study of *potential theory* (and of associated mathematics: partial differential equations, higher functions, . . .). In dynamical contexts the theory of potentials gives rise to the theory of *Green's functions*—a topic of practical but also of deep theoretical importance. The solution of Maxwell's equations has in recent decades acquired also an obvious *numerical* aspect.

The physical ramifications of Maxwellean electrodynamics are so diverse as to require discussion under several headings:

3. We have preferred thus far to work in vacuum, and have alluded to "stuff" only in begrudging recognition of the circumstance that the currents encountered in laboratories tend generally to be confined to wires. That bulk matter is held together by electromagnetic forces (wearing quantum mechanical hats)—and so is inevitably "electromagnetically active"—is, however, a fact of Nature which we cannot forever ignore. We are obliged, therefore, to develop an *electromagnetics of media*. This is a highly model-dependent topic, which fragments into a great variety of subtopics: the solid state physics of dielectrics, of dia/para/ferromagnetic materials, magnetohydrodynamics, . . . the list goes on and on, and each subtopic can be approach in various levels of depth.
4. On the other hand . . . the electromagnetic field is a highly structured and very "busy" object even in source-free regions of space. We would like to acquire detailed knowledge of the *electrodynamics of light* (physical optics, geometrical optics) . . . and of the important "mathematical technology" to which this subject has given rise. We note in particular that it was upon some thermodynamic aspects of this subject that Planck based the theory of *blackbody radiation* which gave rise to quantum mechanics. Also that "optics" must be understood in a sense broad enough to include *radio*. In this connection . . .
5. We would like also to study details of the *radiation production/detection process* and of related topics (scattering theory, antenna theory). At issue here is the physics of *fields and sources in dynamic interaction* . . . which is electrodynamics in its purest/deepest form. It is at this level that the conceptual limitations of classical electrodynamics come most sharply into focus. The subject exhibits a marked "proto-quantum mechanical" tendency, and inspires some of the imagery fundamental to the physics of elementary particles.
6. If we consider (not the sources but) the *field to be prescribed* then we confront the question: "What is the motion of a charged particle in an impressed field (electron optics, accelerator design)?" And what, more generally, can be said concerning the motion of bulk samples (solid/liquid/gas) of "electromagnetically active" matter?

What I have been describing are some of the principal limbs of a large tree, that dominates its central place in a dense forest. We are not surprised that the limbs, on closer scrutiny, resolve into branches, the branches into twigs . . . that intricately intertwine, forming shifting patterns . . . which, however, will remain impossible even to begin to describe until we acquire a command of some of the details.

2

From Electrodynamics to

SPECIAL RELATIVITY

Introduction. We have already had occasion to note that "Maxwell's trick" implied—tacitly but inevitably—the *abandonment of Galilean relativity*. We have seen how this development came about (it was born of Maxwell's desire to preserve charge conservation), and can readily appreciate its revolutionary significance, for

> To the extent that Maxwellean electrodynamics is physically correct, Newtonian dynamics—which *is* Galilean covariant—must be physically in error.

. . . but have now to examine the more detailed ramifications of this formal development. The issue leads, of course, to *special relativity*.

That special relativity is—though born of electrodynamics—"bigger" than electrodynamics (*i.e.*, that it has non-electrodynamic implications, applications —and roots) is a point clearly appreciated by Einstein himself (1905). Readers should understand, therefore, that my intent here is a limited one: my goal is not to produce a "complete account of the special theory of relativity" but only *to develop those aspects of special relativity which are specifically relevant to our electrodynamical needs* . . . and, conversely, to underscore those aspects of electrodynamics which are of a peculiarly "relativistic" nature.

In relativistic physics c —born of electrodynamics and called (not quite appropriately) the "velocity of light"—is recognized for what it is: a *constant* *of Nature* which would retain its relevance and more fundamental meaning "even if electrodynamics—light—did not exist." From

$$[\,c\,] = \text{velocity} = LT^{-1}$$

we see that in "c-physics" we can, if we wish, *measure temporal intervals in the units of spatial length*. It is in this spirit—and because it proves formally to be very convenient—that we agree henceforth to write

$$x^0 \equiv ct \quad \text{and} \quad \begin{cases} x^1 \equiv x \\ x^2 \equiv y \\ x^3 \equiv z \end{cases}$$

To indicate that he has used his "good clock and Cartesian frame" to assign coordinates to an "event" (*i.e.*, to a point in space at a moment in time: briefly, to a point in spacetime) an inertial observer O may write x^μ with $\mu \in \{0, 1, 2, 3\}$. Or he may (responding to the convenience of the moment) write one of the following:

$$x \equiv \begin{pmatrix} x^0 \\ \boldsymbol{x} \end{pmatrix} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

We agree also to write

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,, \quad \text{and also} \quad \partial \equiv \begin{pmatrix} \partial_0 \\ \boldsymbol{\nabla} \end{pmatrix} \equiv \begin{pmatrix} \partial_0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$$

Note particularly that $\partial_0 = \frac{1}{c}\partial_t$. We superscript x's but subscript ∂'s in anticipation of a fundamental transformation-theoretic distinction that will be discussed in §2.

It is upon this notational base—simple though it is—that we will build.

1. Notational reexpression of Maxwell's equations. Even simple thoughts can be rendered unintelligible if awkwardly expressed ... and Maxwell's was hardly a "simple thought." It took physicists the better part of 40 years to gain a clear sense of the essentials of the theory that Maxwell had brought into being (and which he himself imagined to be descriptive of the mechanical properties of an imagined but elusive "æther"). Running parallel to the ever-deepening physical insight were certain notational adjustments/simplifications inspired by developments in the world of pure mathematics.[77]

During the last decade of that formative era increasing urgency attached to a question

[77] See "Theories of Maxwellian design" (1998).

> *What are the (evidently non-Galilean) transformations which preserve the form of Maxwell's equations?*

was first posed (1899) and resolved (1904) by H. A. Lorentz (1853–1928), who was motivated by a desire to avoid the *ad hoc* character of previous attempts to account for the results of the Michelson–Morley, Trouton–Noble and related experiments. Lorentz' original discussion[78] strikes the modern eye as excessively complex. The discussion which follows owes much to the mathematical insight of H. Minkowski (1864–1909),[79] whose work in this field was inspired by the accomplishments of one of his former students (A. Einstein), but which has roots also in Minkowski's youthful association with H. Hertz (1857–1894), and is distinguished by its notational modernism.

Here we look to the notational aspects of Minkowski's contribution, drawing tacitly (where Minkowski drew explicitly) upon the notational conventions and conceptual resources of tensor analysis. In a reversal of the historical order, I will in §2 let the pattern of our results serve to *motivate* a review of tensor algebra and calculus. We will be placed then in position to observe (in §3) the sense in which special relativity almost "invents itself." Now to work:

Let Maxwell's equations (65) be notated

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \rho \\
\boldsymbol{\nabla}\times\boldsymbol{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} &= \tfrac{1}{c}\boldsymbol{j} \\[6pt]
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\
\boldsymbol{\nabla}\times\boldsymbol{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} &= \boldsymbol{0}
\end{aligned}
$$

where, after placing all fields on the left and sources on the right, we have grouped together the "sourcy" equations (Coulomb, Ampere), and formed a second quartet from their sourceless counterparts. Drawing now upon the notational conventions introduced on the preceding page we have

$$
\left.
\begin{aligned}
\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 &= \tfrac{1}{c}j^0 \equiv \rho \\
-\partial_0 E_1 \qquad\quad + \partial_2 B_3 - \partial_3 B_2 &= \tfrac{1}{c}j^1 \\
-\partial_0 E_2 - \partial_1 B_3 \qquad\quad + \partial_3 B_1 &= \tfrac{1}{c}j^2 \\
-\partial_0 E_3 + \partial_1 B_2 - \partial_2 B_1 \qquad\quad &= \tfrac{1}{c}j^3
\end{aligned}
\right\} \tag{157.1}
$$

[78] Reprinted in English translation under the title "Electromagnetic phenomena in a system moving with any velocity less than that of light" in *The Principle of Relativity* (1923), a valuable collection reprinted classic papers which is still available in paperback (published by Dover).

[79] See §7 of "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpen" (1907) in Minkowski's *Collected Works*.

$$\left.\begin{aligned}
-\partial_1 B_1 - \partial_2 B_2 - \partial_3 B_3 &= 0\\
+\partial_0 B_1 \qquad\qquad + \partial_2 E_3 - \partial_3 E_2 &= 0\\
+\partial_0 B_2 - \partial_1 E_3 \qquad\qquad + \partial_3 E_1 &= 0\\
+\partial_0 B_3 + \partial_1 E_2 - \partial_2 E_1 \qquad\qquad &= 0
\end{aligned}\right\} \tag{157.2}$$

where we have found it formally convenient to write

$$j \equiv \begin{pmatrix} j^0 \\ \boldsymbol{j} \end{pmatrix} = \begin{pmatrix} j^0 \\ j^1 \\ j^2 \\ j^3 \end{pmatrix} \quad \text{with} \quad j^0 \equiv c\rho \tag{158}$$

It is evident that (157.1) could be written in the following remarkably compact and simple form

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c} j^\nu$$

NOTE : Here as always, *summation* $\sum_0^3$ *on the repeated index is understood.*

provided the $F^{\mu\nu}$ are defined by the following scheme:

$$\mathbb{F} \equiv \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{pmatrix} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\equiv \mathbb{A}(\boldsymbol{E}, \boldsymbol{B}) \tag{159}$$

Here the $\mathbb{A}$-notation is intended to emphasize that the 4×4 matrix in question is <u>antisymmetric</u>; as such, it has $\begin{smallmatrix}\cdot&\cdot&\cdot\\&\cdot&\cdot\\&&\cdot\end{smallmatrix}$ or 6 independently-specifiable components, which at (159) we have been motivated to identify in a specific way with the six components of a pair of 3-vectors. The statement

$$F^{\nu\mu} = -F^{\mu\nu} \quad : \quad \text{more compactly} \quad \mathbb{F}^{\mathsf{T}} = -\mathbb{F} \tag{160}$$

evidently holds *at every spacetime point*, and will play a central role in our work henceforth.

It follows by inspection from results now in hand that the sourceless field equations (157.2) can be formulated

$$\partial_\mu G^{\mu\nu} = 0$$

with

$$\mathbb{G} \equiv \|G^{\mu\nu}\| = \mathbb{A}(-\boldsymbol{B}, \boldsymbol{E}) = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{pmatrix} \tag{161}$$

. . . but with this step we have acquired an obligation to *develop the sense in which* $\mathbb{G}$ *is a "natural companion" of* $\mathbb{F}$. To that end:

Let the *Levi-Civita symbol* $\epsilon_{\mu\nu\rho\sigma}$ be defined

$$\epsilon_{\mu\nu\rho\sigma} \equiv \begin{cases} +1 & \text{if } (\mu\nu\rho\sigma) \text{ is an \underline{even} permutation of } (0123) \\ -1 & \text{if } (\mu\nu\rho\sigma) \text{ is an \underline{odd} permutation of } (0123) \\ 0 & \text{otherwise} \end{cases}$$

and let quantities $F^{\star}_{\mu\nu}$ be constructed with its aid:

$$F^{\star}_{\mu\nu} \equiv \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \quad \text{where } \sum_{\rho,\sigma} \text{ is understood} \tag{162}$$

By computation we readily establish that

$$\begin{aligned}\mathbb{F}^{\star} \equiv \|F^{\star}_{\mu\nu}\| &= \begin{pmatrix} 0 & F^{23} & -F^{13} & F^{12} \\ & 0 & F^{03} & -F^{02} \\ & (-) & 0 & F^{01} \\ & & & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix} = \mathbb{A}(\boldsymbol{B}, \boldsymbol{E})\end{aligned}$$

which would become $\mathbb{G}$ if we could change the sign of the B-entries, and this is readily accomplished: multiply $\mathbb{A}(\boldsymbol{B}, \boldsymbol{E})$ by

$$g \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{163}$$

on the right (this leaves the 0^{th} column unchanged, but changes the sign of the 1^{st}, 2^{nd} and 3^{rd} columns), and again by another factor of g on the left (this leaves the 0^{th} row unchanged, but changes the sign of the 1^{st}, 2^{nd} and 3^{rd} rows, the 1^{st}, 2^{nd} and 3^{rd} elements of which have now been restored to their *original* signs). We are led thus to $g\mathbb{A}(\boldsymbol{B}, \boldsymbol{E})g = \mathbb{A}(-\boldsymbol{B}, \boldsymbol{E})$ which—because

$$g = \begin{cases} g^{\mathsf{T}} & : \quad g \text{ is its own transpose } (\textit{i.e.}, \text{ is } \textit{symmetric}) \\ g^{-1} & : \quad g \text{ is its own inverse} \end{cases} \tag{164}$$

—can also be expressed $\mathbb{A}(\boldsymbol{B}, \boldsymbol{E}) = g\mathbb{A}(-\boldsymbol{B}, \boldsymbol{E})g$. In short,[80]

$$\mathbb{F}^{\star} = g\,\mathbb{G}g^{\mathsf{T}} \quad \text{equivalently} \quad \mathbb{G} = g^{-1}\mathbb{F}^{\star}(g^{-1})^{\mathsf{T}} \tag{165}$$

---

[80] PROBLEM 35.

Let the elements of $\mathcal{g}$ be called $g_{\mu\nu}$, and the elements of $\mathcal{g}^{-1}$ (though they happen to be numerically identical to the elements of $\mathcal{g}$) be called $g^{\mu\nu}$:

$$\mathcal{g} \equiv \|g_{\mu\nu}\| \quad \text{and} \quad \mathcal{g}^{-1} \equiv \|g^{\mu\nu}\| \quad \Rightarrow \quad g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu{}_\nu \equiv \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases}$$

We then have

$$F^\star_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}G^{\alpha\beta} \quad \text{or equivalently} \quad G^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F^\star_{\alpha\beta}$$

To summarize: we have

$$F^{\mu\nu} \xrightarrow[\star]{\hspace{6em}} F^\star_{\mu\nu} \xrightarrow{\text{lift indices with } \mathcal{g}^{-1}} F^{\star\mu\nu} = G^{\mu\nu}$$

which in $(\boldsymbol{E}, \boldsymbol{B})$-notation reads

$$\mathbb{F} = \mathbb{A}(\boldsymbol{E}, \boldsymbol{B}) \longrightarrow \mathbb{A}(\boldsymbol{B}, \boldsymbol{E}) \longrightarrow \mathbb{A}(-\boldsymbol{B}, \boldsymbol{E}) = \mathbb{G}$$

Repetition of the process gives

$$\mathbb{G} = \mathbb{A}(-\boldsymbol{B}, \boldsymbol{E}) \longrightarrow \mathbb{A}(\boldsymbol{E}, -\boldsymbol{B}) \longrightarrow \mathbb{A}(-\boldsymbol{E}, -\boldsymbol{B}) = -\mathbb{F}$$

$G^{\mu\nu}$ is said to be the "dual" of $F^{\mu\nu}$, and the process $F^{\mu\nu} \longrightarrow G^{\mu\nu}$ is called "dualization;" it amounts to a kind of "*rotation* in $(\boldsymbol{E}, \boldsymbol{B})$-space," in the sense illustrated below:

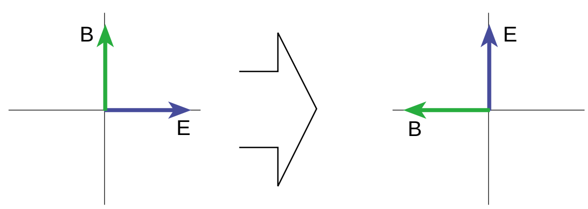


FIGURE 45: *The "rotational" effect of "dualization" on* $\boldsymbol{E}$ *and* $\boldsymbol{B}$.

Preceding remarks lend precise support and meaning to the claim that $F^{\mu\nu}$ and $G^{\mu\nu}$ are "natural companions," and very closely related.

We shall—as above, but more generally (and for the good tensor-theoretic reasons that will soon emerge) *use* $g^{\mu\nu}$ *and* $g_{\mu\nu}$ *to raise and lower—in short, to "manipulate"—indices*, writing (for example)[81]

$$\partial^\mu = g^{\mu\alpha}\partial_\alpha\,, \quad \partial_\mu = g_{\mu\alpha}\partial^\alpha$$
$$j^\mu = g^{\mu\alpha}j_\alpha\,, \quad j_\mu = g_{\mu\alpha}j^\alpha$$
$$F_{\mu\nu} = g_{\mu\alpha}F^\alpha{}_\nu = g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta}$$

[81] PROBLEM 36.

We are placed thus in position to notice that the sourceless Maxwell equations (157.2) can be formulated[82]

$$\left.\begin{aligned}\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= 0\\ \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} &= 0\\ \partial_0 F_{13} + \partial_1 F_{30} + \partial_3 F_{01} &= 0\\ \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} &= 0\end{aligned}\right\} \tag{166.1}$$

where the *sums over cyclic permutations* are sometimes called "windmill sums." More compactly, we have[83]

$$\epsilon_{\mu\alpha\rho\sigma}\partial^\alpha F^{\rho\sigma} = 0 \tag{166.2}$$

There is no new physics in the material presented thus far: our work has been merely reformulational, notational—old wine in new bottles. Proceeding in response mainly to the *linearity* of Maxwell's equations, we have allowed ourselves to play linear-algebraic and notational games intended to maximize the formal symmetry/simplicity of Maxwell's equations...so that the transformation-theoretic problem which is our real concern can be posed in the simplest possible terms. Maxwell himself[84] construed the electromagnetic field to involve a *pair of 3-vector fields:* $\boldsymbol{E}$ *and* $\boldsymbol{B}$. We have seen, however, that

- one can construe the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ to be the accidentally distinguished names given to the six independently-specifiable non-zero components of an *antisymmetric tensor*[85]*field* $F^{\mu\nu}$. The field equations then read

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c}j^\nu \quad\text{and}\quad \epsilon_{\mu\alpha\rho\sigma}\partial^\alpha F^{\rho\sigma} = 0 \tag{167}$$

  provided the $g^{\alpha\beta}$ that enter into the definition $\partial^\alpha \equiv g^{\alpha\beta}\partial_\beta$ are given by (163). Alternatively ...

- one can adopt the view that the electromagnetic field to involves a *pair* of antisymmetric tensor fields $F^{\mu\nu}$ and $G^{\mu\nu}$ which are constrained to satisfy not only the field equations

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c}j^\nu \quad\text{and}\quad \partial_\mu G^{\mu\nu} = 0 \tag{168.1}$$

  but also the algebraic condition

$$G^{\mu\nu} = \tfrac{1}{2}g^{\mu\alpha}g^{\nu\beta}\epsilon_{\alpha\beta\rho\sigma}F^{\rho\sigma} \tag{168.2}$$

  Here again, the "index manipulators" $g_{\mu\nu}$ and $g^{\mu\nu}$ must be assigned the specific meanings implicit in (163).

---

[82] PROBLEM 37.

[83] PROBLEM 38.

[84] Here I take some liberty with the complicated historical facts of the matter: see again the fragmentary essay[77] cited earlier.

[85] For the moment "tensor" simply means "doubly indexed."

It will emerge that Lorentz' question (page 107), if phrased in the terms natural to either of those descriptions of Maxwellian electrodynamics, virtually "answers itself." But to see how this comes about one must possess a command of the basic elements of tensor analysis—a subject with which Minkowski (mathematician that he was) enjoyed a familiarity not shared by any of his electrodynamical predecessors or contemporaries.[86]

2. Introduction to the algebra and calculus of tensors. Let P be a *point in an N-dimensional manifold* $\mathcal{M}$.[87] Let $(x^1, x^2, \ldots, x^N)$ be coordinates assigned to P by a coordinate system $\mathcal{X}$ inscribed on a neighborhood[88] containing P, and

---

[86] Though (167) and (168) serve optimally my immediate purposes, the reader should be aware that there exist also many alternative formulations of the Maxwellian theory, and that these may afford advantages in specialized contexts. We will have much to say about the formalism that proceeds from writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and considering the fundamental object of electrodynamic analysis to be a *single 4-vector field*. Alternatively, one might construct and study the "6-vector"

$$f = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \\ f^5 \\ f^6 \end{pmatrix} \equiv \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

(see §26 in Arnold Sommerfeld's *Electrodynamics* (English translation 1964) or my *Classical Field Theory* (1999), Chapter 2, pages 4–6). Or one might consider electrodynamics to be concerned with the properties of a *single* <u>*complex*</u> *3-vector*

$$\boldsymbol{V} \equiv \boldsymbol{E} + i\boldsymbol{B}$$

(see Appendix B in my "On some recent electrodynamical work by Thomas Wieting" (2001)). And there exist yet many other formalisms. Maxwell himself gave passing attention to a "quaternionic" formulation of his theory.

[87] Think "surface of a sphere," "surface of a torus," *etc.* or of their higher-dimensional counterparts. Or of N-dimensional Euclidean space itself. Or—as soon as you can—4-dimensional spacetime. I intend to proceed quite informally, and to defer questions of the nature "What *is* a manifold?" until such time as we are able to look back and ask "What properties *should* we fold into our definitions? What did we need to make our arguments work?"

[88] I say "neighborhood" because it may happen that every coordinate system inscribed on $\mathcal{M}$ necessarily displays one or more singularities (think of the longitude of the North Pole). It is our announced intention to stay away from such points.

let $(x^1, x^2, \ldots, x^N)$ be the coordinates assigned to that same point by a second coordinate system $\mathfrak{X}$. We seek to develop <u>rules according to which objects defined in the neighborhood of P respond to coordinate transformations:</u> $\mathfrak{X} \rightarrow \mathfrak{X}$.

The statement that "$\phi(x)$ transforms as a *scalar field*" carries this familiar meaning:

$$\phi(x) \longrightarrow \phi(x) \equiv \phi(x(x)) \tag{169}$$

Here and henceforth: $x(x)$ alludes to the functional statements

$$x^m = x^m(x^1, x^2, \ldots, x^N) \quad : \quad m = 1, 2, \ldots N \tag{170}$$

that describe how $\mathfrak{X}$ and $\mathfrak{X}$ are, in the instance at hand, specifically related. How do the partial *derivatives* of ϕ transform? By calculus

$$\frac{\partial \phi}{\partial x^m} = \frac{\partial x^a}{\partial x^m}\frac{\partial \phi}{\partial x^a} \tag{171.1}$$

where (as always) $\sum_a$ is understood. Looking to the 2nd derivatives, we have

$$\frac{\partial^2 \phi}{\partial x^m \partial x^n} = \frac{\partial x^a}{\partial x^m}\frac{\partial x^b}{\partial x^n}\frac{\partial^2 \phi}{\partial x^a \partial x^b} + \frac{\partial^2 x^a}{\partial x^m \partial x^n}\frac{\partial \phi}{\partial x^a} \tag{171.2}$$

Et cetera. Such are the "objects" we encounter in routine work, and the transformation rules which we want to be able to manipulate in a simple manner.

The quantities $\partial x^a/\partial x^m$ arise directly and exclusively from the equations (170) that describe $\mathfrak{X} \leftarrow \mathfrak{X}$. They constitute the elements of the "transformation matrix"

$$\begin{aligned} \mathbb{W} &\equiv \|W^n{}_m\| \\ W^n{}_m &\equiv \partial x^n/\partial x^m \end{aligned} \tag{172.1}$$

—the value of which will in general vary from point to point. Function theory teaches us that the coordinate transformation will be *invertible* (*i.e.*, that we can proceed from $x^n = x^n(x)$ to equations of the form $x^n = x^n(x)$) if and only if $\mathbb{W}$ is non-singular: $\det \mathbb{W} \neq 0$, which we always assume to be the case (in the neighborhood of P). The inverse $\mathfrak{X} \rightarrow \mathfrak{X}$ of $\mathfrak{X} \leftarrow \mathfrak{X}$ gives rise to

$$\begin{aligned} \mathbb{M} &\equiv \|M^m{}_n\| \\ M^m{}_n &\equiv \partial x^m/\partial x^n \end{aligned} \tag{172.2}$$

It is important to notice that

$$\mathbb{W}\mathbb{M} = \left\| \sum_a \frac{\partial x^n}{\partial x^a}\frac{\partial x^a}{\partial x^m} \right\| = \left\| \frac{\partial x^n}{\partial x^m} \right\| = \left\| \delta^n{}_m \right\| = \mathbb{I} \tag{173}$$

i.e., that the matrices $\mathbb{M}$ and $\mathbb{W}$ are inverses of each other.

Objects $X^{m_1\dots m_r}{}_{n_1\dots n_s}$ are said to comprise the "components of a (mixed) tensor of contravariant rank r and covariant rank s if and only if they respond to $\mathfrak{X} \to \mathfrak{X}$ by the following multilinear rule:

$$\begin{aligned} &X^{m_1\dots m_r}{}_{n_1\dots n_s} \\ &\downarrow \\ &X^{m_1\dots m_r}{}_{n_1\dots n_s} = M^{m_1}{}_{a_1}\cdots M^{m_r}{}_{a_r} W^{b_1}{}_{n_1}\cdots W^{b_s}{}_{n_s} X^{a_1\dots a_r}{}_{b_1\dots b_s} \end{aligned} \tag{174}$$

All indices range on $\{1, 2, \dots, N\}$, N is called the "dimension" of the tensor, and summation on repeated indices is (by the "Einstein summation convention") understood. The covariant/contravariant distinction is signaled notationally as a subscript/superscript distinction, and alludes to whether it is $\mathbb{W}$ or $\mathbb{M}$ that transports the components in question "across the street, from the $\mathfrak{X}$-side to the $\mathfrak{X}$-side."

If

$$X^m \longrightarrow X^m = M^m{}_a X^a$$

then the X^m are said to be "components of a contravariant *vector*." Coordinate differentials provide the classic prototype:

$$dx^m \longrightarrow dx^m = \sum_a \frac{\partial x^m}{\partial x^a} dx^a \tag{175}$$

If, on the other hand,

$$X_n \longrightarrow X_n = W^b{}_n X_b$$

then the X_n are said to be "components of a *co*variant vector." Here the first partials $\phi_{,n} \equiv \partial_n \phi$ of a scalar field (components of the **gradient**) provide the classic prototype:

$$\phi_{,n} \longrightarrow \phi_{,n} = \sum_b \phi_{,b} \frac{\partial x^b}{\partial x^n} \tag{176}$$

That was the lesson of (171.1).

Look, however, to the lesson of (171.2), where we found that

$$\phi_{,mn} \longrightarrow \phi_{,mn} = \sum_b \phi_{,ab} \frac{\partial x^a}{\partial x^m}\frac{\partial x^b}{\partial x^n} + \text{extraneous term}$$

The intrusion of the "extraneous term" is typical of the *differential calculus of tensors*, and arises from an elementary circumstance: hitting

$$X^m{}_n = M^m{}_a W^b{}_n X^a{}_b \quad \text{(say)}$$

with $\partial_p = W^q{}_p \partial_q$ gives

$$\begin{aligned} X^m{}_{n,p} &= M^m{}_a W^b{}_n X^a{}_{b,q} W^q{}_p + W^q{}_p \frac{\partial(M^m{}_a W^b{}_n)}{\partial x^q} X^a{}_b \\ &= \text{(term with covariant rank increased by one)} \\ &\qquad\qquad + \text{(extraneous term)} \end{aligned}$$

The "extraneous term" vanishes if the M's and W's are *constant*; *i.e.*, if the functions $x^n(x)$ depend at most *linearly* upon their arguments $x^n = M^n{}_a x^a + \xi^a$. And in a small number of (electrodynamically important!) cases the extraneous terms *cancel* when derivatives are *combined* in certain ways . . . as we will soon have occasion to see. But in general, effective management of the extraneous term must await the introduction of some powerful new ideas—ideas that belong not to the algebra of tensors (my present concern) but to the calculus of tensors. For the moment I must be content to emphasize that, on the basis of evidence now in hand,

Not every multiply-indexed object transforms tensorially!

In particular, the x^n themselves do not transform tensorially except in the linear case $x^n = M^n{}_a x^a$.

A conceptual point of major importance: the $X^{m_1 \dots m_r}{}_{n_1 \dots n_s}$ *refer* to a tensor, but do not themselves *comprise* the tensor: they are the <u>components</u> of the tensor X with respect to the coordinate system $\mathcal{X}$, and collectively serve to *describe* X. Similarly $X^{m_1 \dots m_r}{}_{n_1 \dots n_s}$ with respect to $\mathcal{X}$. The tensor itself is a coordinate-independent object that lives "behind the scene." The situation is illustrated in Figure 46.

To lend substance to a remark made near the top of the page: Let X_m transform as a covariant vector. Look to the transformation properties of $X_{m,n}$ and obtain

$$X_{m,n} = W^a{}_m W^b{}_n X_{a,b} + \underbrace{\frac{\partial^2 x^a}{\partial x^n \partial x^m} X_a}_{\text{extraneous term, therefore non-tensorial}}$$

Now construct $A_{mn} \equiv X_{m,n} - X_{n,m} = -A_{nm}$ and obtain

$$A_{mn} = W^a{}_m W^b{}_n A_{ab} \quad \text{because the } \underline{\text{extraneous terms cancel}}$$

We conclude that the antisymmetric construction A_{mn} (which we might call the **curl** of the covariant vector field $X_m(x)$) *does*—"accidentally"—transform tensorially.

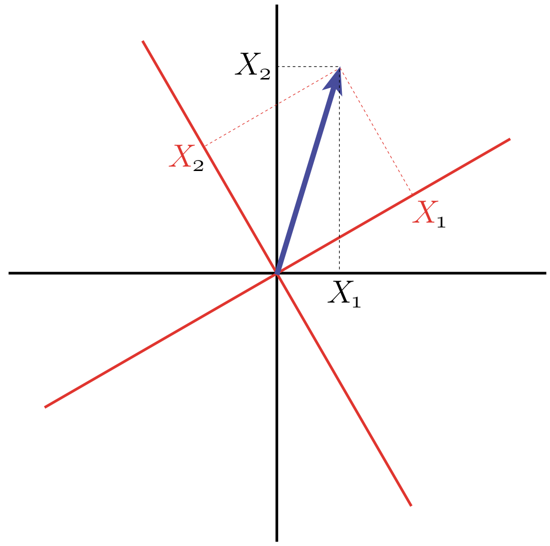


FIGURE 46: *The X_m serve to describe the blue arrow with respect to the black coordinate system $\mathcal{X}$, as the X_m serve to describe the blue arrow with respect to the red coordinate system $\mathcal{X}$. But neither X_m nor X_m will be confused with the blue arrow itself: to do so would be to confuse descriptors with the thing described. So it is with tensors in general. Tensor analysis is concerned with* relationships among alternative descriptors, not with "things in themselves."

The following points are elementary, but fundamental to *applications* of the tensor concept:

1) If the components $X^{\cdots}{}_{\cdots}$ of a tensor (all) vanish one coordinate system, then they vanish in *all* coordinate systems—this by the *homogeneity* of the defining statement (174).

2) Tensors can be added/subtracted if and only if $X^{\cdots}{}_{\cdots}$ and $Y^{\cdots}{}_{\cdots}$ are of the same covariant/contravariant rank and dimension. Constructions of (say) the form $A^m + B_m$ "come unstuck" when transformed; for that same reason, statements of (say) the form $A^m = B_m$—while they may be valid in some given coordinate system—do not entail $A^m = B_m$. But . . .

3) If $X^{\cdots}{}_{\cdots}$ and $Y^{\cdots}{}_{\cdots}$ are of the *same* rank and dimension, then

$$X^{\cdots}{}_{\cdots} = Y^{\cdots}{}_{\cdots} \quad\Longrightarrow X^{\cdots}{}_{\cdots} = Y^{\cdots}{}_{\cdots}$$

It is, in fact, because of the remarkable transformational stability of tensorial equations that we study this subject, and try to formulate our physics in tensorial terms.

4) If $X^{\cdots}{}_{\cdots}$ and $Y^{\cdots}{}_{\cdots}$ are co-dimensional tensors of ranks $\{r', s'\}$ and $\{r'', s''\}$ then their product $X^{\cdots}{}_{\cdots}Y^{\cdots}{}_{\cdots}$ is tensorial with rank $\{r' + r'', s' + s''\}$: tensors of the same dimension can be multiplied irrespective of their ranks.

If $X^{\cdots}{}_{\cdots}$ is tensorial of rank $\{r, s\}$ then a the operation of

CONTRACTION: Set a superscript equal to a subscript, and add

yields components of a tensor of rank $\{r-1, s-1\}$. The mechanism is exposed most simply by example: start from (say)

$$X^{jk}{}_{\ell} = M^j{}_a M^k{}_b W^c{}_{\ell} X^{ab}{}_c$$

Set (say) $k = \ell$ and obtain

$$\begin{aligned} X^{jk}{}_{k} &= M^j{}_a M^k{}_b W^c{}_{k} X^{ab}{}_c \\ &= M^j{}_a \quad \delta^c{}_b \quad X^{ab}{}_c \qquad \text{by} \quad \mathbb{M}\,\mathbb{W} = \mathbb{I} \\ &= M^j{}_a X^{ab}{}_b \end{aligned}$$

according to which $X^j \equiv X^{jk}{}_k$ transforms as a contravariant vector. Similarly, the twice-contracted objects $X^{jk}{}_{jk}$ and $X^{jk}{}_{kj}$ transform as (generally distinct) *invariants*.[89] Mixed tensors of high rank can be singly/multiply contracted in many distinct ways. It is also possible to "contract one tensor into another;" a simple example:

$$A_k B^k \quad : \quad \begin{cases} \text{invariant formed by contracting a covariant} \\ \text{vector into a contravariant vector} \end{cases}$$

The "Kronecker symbol" $\delta^m{}_n$ is a number-valued object[90] with which all readers are familiar. If "transformed tensorially" it gives

$$\begin{aligned} \delta^m{}_n \longrightarrow \delta^m{}_n &= M^m{}_a W^b{}_n \delta^a{}_b \\ &= M^m{}_a W^a{}_n \\ &= \delta^m{}_n \qquad \text{by} \quad \mathbb{M}\,\mathbb{W} = \mathbb{I} \end{aligned}$$

and we are brought to the remarkable conclusion that *the components $\delta^m{}_n$ of the Kronecker tensor have the <u>same numerical values in every coordinate system</u>*. Thus does $\delta^m{}_n$ become what I will call a "universally available object"—to be joined soon by a few others. With this ...

We are placed in position to observe that if the quantities g_{mn} transform as the components of a 2nd rank covariant tensor

$$g_{mn} \longrightarrow g_{mn} = W^a{}_m W^b{}_n g_{ab} \tag{177}$$

[89] The "theory of invariants" was a favorite topic among 19th Century mathematicians, and provided the founding fathers of tensor analysis with a source of motivation (see pages 206–211 in E. T. Bell's *The Development of Mathematics* (1945)).

[90] See again the top of page 110.

then

1) the equation $g^{ma}g_{an}=\delta^m{}_n$, if taken as (compare page 110) a definition of the contravariant tensor g^{mn}, makes good coordinate-independent tensor-theoretic sense, and
2) so do the equations
$$\begin{aligned}X^{\cdots m\cdots}_{\cdots\ \ \cdots} &\equiv g^{ma}X^{\cdots\ \ \cdots}_{\cdots a\cdots}\\ X^{\cdots\ \ \cdots}_{\cdots m\cdots} &\equiv g_{ma}X^{\cdots a\cdots}_{\cdots\ \ \cdots}\end{aligned}$$
by means of which we have proposed already on page 110 to raise and lower indices.[91] To insure that $g^{ma}X^{\cdots\ \ \cdots}_{\cdots a\cdots}$ and $g^{am}X^{\cdots\ \ \cdots}_{\cdots a\cdots}$ are identical we will require that
$$g_{mn}=g_{nm}\quad:\quad\text{implies the symmetry also of } g^{mn}$$

The transformation equation (177) admits—uncharacteristically–of matrix formulation
$$\mathbb{g}\longrightarrow \mathbb{g}=\mathbb{W}^{\mathsf{T}}\mathbb{g}\,\mathbb{W}$$
Taking determinant of both sides, and writing
$$g\equiv\det\mathbb{g}\ ,\quad W\equiv\det\mathbb{W}=1/\det\mathbb{M}=M^{-1}$$
we have
$$g\longrightarrow g=W^2g \tag{178.1}$$
The statement that $\phi(x)$ transforms as a *scalar density of weight w* carries this meaning:
$$\phi(x)\longrightarrow\phi(x)=W^w\cdot\phi(x(x))$$
We recover (169) in the "weightless" case $w=0$ (and for arbitrary values of w when it happens that $W=1$). Evidently
$$g\equiv\det\mathbb{g}\text{ transforms as a scalar density of weight }w=2 \tag{178.2}$$
The more general statement that $X^{m_1\dots m_r}{}_{n_1\dots n_s}$ transforms as a *tensor density of weight w* means that
$$X^{m_1\dots m_r}{}_{n_1\dots n_s}=W^w\cdot M^{m_1}{}_{a_1}\cdots M^{m_r}{}_{a_r}W^{b_1}{}_{n_1}\cdots W^{b_s}{}_{n_s}X^{a_1\dots a_r}{}_{b_1\dots b_s}$$
We can multiply/contract tensors of dissimilar weight, but must be careful not to try to add them or set them equal. The "tensor/tensor density distinction" becomes significant only in contexts where $W\neq 1$.

Familiarity with the tensor density concept places us in position to consider the tensor-theoretic significance of the Levi-Civita symbol

[91] Note, however, that we work now N-dimensionally, and have stripped g_{mn} of its formerly specialized (Lorentzian) construction (163): it has become "generic."

$$\epsilon_{n_1 n_2 \ldots n_N} \equiv \text{sgn}\begin{pmatrix} 1 & 2 & \cdots & N \\ n_1 & n_2 & \cdots & n_N \end{pmatrix}$$

where "sgn" refers to the "signum," which reports (see again page 109) whether $\{n_1, n_2, \ldots, n_N\}$ is an even/odd permutation of $\{1, 2, \ldots, N\}$ or no permutation at all. The tentative assumption that $\epsilon_{n_1 n_2 \ldots n_N}$ transforms as a (totally antisymmetric) tensor density of unspecified weight w

$$\begin{aligned} \epsilon_{n_1 n_2 \ldots n_N} &= W^w \cdot \underbrace{W^{a_1}{}_{n_1} W^{a_2}{}_{n_2} \cdots W^{a_N}{}_{n_N} \epsilon_{a_1 a_2 \ldots a_N}}_{= \epsilon_{n_1 n_2 \ldots n_N} \det \mathbb{W} \text{ by definition of the determinant!}} \\ &= W^{w+1} \cdot \epsilon_{n_1 n_2 \ldots n_N} \end{aligned}$$

brings us to the remarkable conclusion that *the components of the Levi-Civita tensor will have the <u>same numerical values in every coordinate system</u> provided* $\epsilon_{n_1 n_2 \ldots n_N}$ *is assumed to transform as a <u>density of weight $w = -1$</u>*. The Levi-Civita tensor thus joins our short list of "universally available objects."[92]

I have remarked that $\epsilon_{n_1 n_2 \ldots n_N}$ is "totally antisymmetric." It is of importance to notice in this connection that—more generally—statements of the forms

$$X^{\cdots m \cdots n \cdots}{}_{\cdots} = \pm X^{\cdots n \cdots m \cdots}{}_{\cdots}$$

and

$$X^{\cdots}{}_{\cdots m \cdots n \cdots} = \pm X^{\cdots}{}_{\cdots n \cdots m \cdots}$$

have tensorial (or coordinate system independent) significance, while symmetry statements of the hybrid form

$$X^{\cdots m \cdots}{}_{\cdots n \cdots} = \pm X^{\cdots n \cdots}{}_{\cdots m \cdots}$$

—while they might be valid in some particular coordinate system—"become unstuck" when transformed. Note also that

$$X^{mn} = \tfrac{1}{2}(X^{mn} + X^{nm}) + \tfrac{1}{2}(X^{mn} - X^{nm})$$

serves to resolve X^{mn} tensorially into its *symmetric and antisymmetric parts*.[93]

---

[92] The (weightless) "metric tensor" g_{mn} is *not* "universally available," but must be introduced "by hand." In contexts where g_{mn} *is* available (has been introduced to facilitate index manipulation) it becomes natural to construct

$$\sqrt{g}\,\epsilon_{n_1 n_2 \ldots n_N} \quad : \quad \underline{\text{weightless}} \text{ totally antisymmetric tensor}$$

—the values of which range on $\{0, \pm\sqrt{g}\}$ in all coordinate systems.

[93] PROBLEM 39.

We have now in our possession a command of **tensor algebra** which is sufficient to serve our immediate needs, but must sharpen our command of the **differential calculus of tensors**. This is a more intricate subject, but one into which—surprisingly—we need not enter very deeply to acquire the tools needed to achieve our electrodynamical objectives. I will be concerned mainly with the development of a short list of "accidentally tensorial derivative constructions,"[94] and will glance only cursorily at what might be called the "non-accidental aspects" of the tensor calculus.

CATALOG OF ACCIDENTALLY TENSORIAL DERIVATIVE CONSTRUCTIONS

1. We established already at (171.1) that if ϕ transforms as a weightless scalar field then the components of the *gradient* of ϕ

$$\partial_m\phi \text{ transform tensorially} \tag{179.1}$$

2. And we observed on page 115 that if X_m transforms as a weightless covariant vector field then the components of the *curl* of X_m transform tensorially.

$$\partial_n X_m - \partial_m X_n \text{ transform tensorially} \tag{179.2}$$

3. If X_{jk} is a weightless tensor field, how do the $\partial_i X_{jk}$ transform? Immediately

$$\begin{aligned}\partial_i X_{jk} &= W^b{}_j W^c{}_k \cdot W^a{}_i\,\partial_a X_{bc} + X_{bc}\partial_i\big\{W^b{}_j W^c{}_k\big\}\\ &= W^a{}_i W^b{}_j W^c{}_k\,\partial_a X_{bc} + \underbrace{X_{bc}\Big\{\frac{\partial^2 x^b}{\partial x^i\partial x^j}\frac{\partial x^c}{\partial x^k} + \frac{\partial x^b}{\partial x^j}\frac{\partial^2 x^c}{\partial x^k\partial x^i}\Big\}}_{\text{extraneous term}}\end{aligned}$$

so $\partial_i X_{jk}$ transforms tensorially only under such circumstances as cause the "extraneous term" to vanish: this happens when $\mathcal{X}\to\mathcal{X}$ is "affine;" *i.e.*, when the $\mathbb{W}$-matrix is x-independent. Notice, however, that we now have

$$\begin{aligned}\partial_i X_{jk} + \partial_j X_{ki} + \partial_k X_{ij} = W^a{}_i W^b{}_j W^c{}_k(\partial_a X_{bc} + \partial_a X_{bc} + \partial_a X_{bc})&\\ + X_{bc}\Big\{\frac{\partial^2 x^b}{\partial x^i\partial x^j}\frac{\partial x^c}{\partial x^k} + \frac{\partial x^b}{\partial x^j}\frac{\partial^2 x^c}{\partial x^k\partial x^i}&\\ + \frac{\partial^2 x^b}{\partial x^j\partial x^k}\frac{\partial x^c}{\partial x^i} + \frac{\partial x^b}{\partial x^k}\frac{\partial^2 x^c}{\partial x^i\partial x^j}&\\ + \frac{\partial^2 x^b}{\partial x^k\partial x^i}\frac{\partial x^c}{\partial x^j} + \frac{\partial x^b}{\partial x^i}\frac{\partial^2 x^c}{\partial x^j\partial x^k}\Big\}&\end{aligned}$$

in which $\big\{\text{etc.}\big\}$ is bc-symmetric; if X_{bc} were *anti*symmetric the extraneous term would therefore drop away. We conclude that if X_{jk} is an *antisymmetric* weightless covariant tensor field then the components of the windmill sum

$$\partial_i X_{jk} + \partial_j X_{ki} + \partial_k X_{ij} \text{ transform tensorially} \tag{179.3}$$

[94] The possibility and electrodynamical utility of such a list was brought first to my attention when, as a student, I happened upon the discussion which appears on pages 22–24 of E. Schrödinger's *Space-time Structure* (1954). This elegant little volume (which runs to only 119 pages) provides physicists with an elegantly succinct introduction to tensor analysis. I recommend it to your attention.

4. If X^m is a vector *density of unspecified weight* w how does $\partial_m X^m$ transform? Immediately

$$
\begin{aligned}
\partial_m X^m &= W^w \cdot \underbrace{M^m{}_a \partial_m}_{\partial_a} X^a + X^a \partial_m \{W^w \cdot M^m{}_a\} \\
&= W^w \cdot \partial_a X^a + X^a \Big\{ W^w \frac{\partial}{\partial x^m} \frac{\partial x^m}{\partial x^a} + w W^{w-1} \frac{\partial W}{\partial x^a} \Big\}
\end{aligned}
$$

An important LEMMA[95] asserts that

$$
\begin{aligned}
\frac{\partial}{\partial x^m}\frac{\partial x^m}{\partial x^a} &= \frac{\partial}{\partial x^a} \log\det \left\| \frac{\partial x^m}{\partial x^n} \right\| \\
&= \partial_a \log M = -\partial_a \log W \\
&= -W^{-1} \partial_a W
\end{aligned}
$$

so

$$
= W^w \cdot \partial_a X^a + \underbrace{X^a (w-1) W^{w-1} \frac{\partial W}{\partial x^a}}_{\text{extraneous term}}
$$

The extraneous term vanishes (for all w) when $\mathfrak{X} \rightarrow \mathfrak{X}$ has the property that W is x-independent,[96] and it vanishes unrestrictedly if $w = 1$. We conclude that if X^m is a contravariant vector density of unit weight then its *divergence*

$$
\partial_m X^m \text{ transforms tensorially (by invariance)} \tag{179.4}
$$

5. If X^{mn} is a vector density of unspecified weight w how does $\partial_m X^{mn}$ transform? Immediately

$$
\partial_m X^{mn} = \underbrace{W^w \cdot M^m{}_a M^n{}_b (W^c{}_m \partial_c X^{ab})}_{= W^w \cdot M^n{}_b\, \partial_a X^{ab} \quad \text{by } M^m{}_a W^c{}_m = \delta^c{}_a} + \underbrace{X^{ab} \partial_m (W^w \cdot M^m{}_a M^n{}_b)}_{\text{extraneous term}}
$$

The extraneous term can be developed

$$
X^{ab} \Big\{ M^n{}_b w W^{w-1} (M^m{}_a \partial_m) W + W^w \Big[M^n{}_b \underbrace{\partial_m M^m{}_a}_{= -W^{-1}\partial_a W \quad \text{by the LEMMA}} + (M^m{}_a \partial_m) M^n{}_b \Big] \Big\}
$$

so by $M^m{}_a \partial_m = \partial_a$ we have

$$
\text{extraneous term} = X^{ab} \Big\{ M^n{}_b (w-1) W^{w-1} \partial_a W + W^w \frac{\partial^2 x^n}{\partial x^a \partial x^b} \Big\}
$$

---

[95] For the interesting but somewhat intricate proof, see CLASSICAL DYNAMICS (1964/65), Chapter 2, page 49.

[96] This is weaker than the requirement that $\mathbb{W}$ be x-independent.

The second partial is ab-symmetric, and makes no net contribution if we assume X^{ab} to be ab-*anti*symmetric. The surviving fragment of the extraneous term vanishes (all w) if W is constant, and vanishes unrestrictedly if $w = 1$. We are brought thus to the conclusion that if X^{mn} is an antisymmetric density of unit weight then

$$\partial_m X^{mn} \text{ transforms tensorially} \tag{179.5}$$

"Generalized divergences" $\partial_m X^{mn_1\cdots n_p}$ yield to a similar analysis, but will not be needed.

6. Taking (179.5) and (179.4) in combination we find that under those same conditons (*i.e.*, if X^{mn} is an antisymmetric density of unit weight) then

$$\partial_m\partial_n X^{mn} \text{ transforms tensorially}$$

but this is hardly news: the postulated antisymmetry fo X^{mn} combines with the manifest symmetry of $\partial_m\partial_n$ to give

$$\partial_m\partial_n X^{mn} = 0 \text{ automatically}$$

The evidence now in hand suggests—accurately—that *antisymmetry has a marvelous power to dispose of what we have called "extraneous terms."* The calculus of antisymmetric tensors is in fact much easier than the calculus of tensors-in-general, and is known as the **exterior calculus**. That independently developed sub-branch of the tensor calculus supports not only a differential calculus of tensors but also—uniquely—an integral calculus, which radiates from the theory of determinants (which are antisymmetry-infested) and in which the fundamental statement is a vast generalization of Stokes' theorem.[97]

> REMARK: Readers will be placed at no immediate disadvantage if, on a first reading, they skip the following descriptive comments, which have been inserted only in the interest of a kind of "sketchy completeness" and which refer to material which is—remarkably!—inessential to our electrodynamical progress (though indispensable in many other physical contexts).

In more general (antisymmetry-free) contexts one deals with the non-tensoriality of $\partial_m X^{\cdots}{}_{\cdots}$ by <u>modifying the concept of differentiation</u>, writing (for example)

$$\begin{aligned} D_j X_k &\equiv \underbrace{W^b{}_j W^c{}_k\, \partial_b X_c}_{\text{tensorial transform of } \partial_j X_k} \\ &\equiv \text{components of the } \textbf{covariant derivative} \text{ of } X_k \end{aligned}$$

[97] See again the MATHEMATICAL DIGRESSION that culminates on page 50. A fairly complete and detailed account of the exterior calculus can be found in "Electrodynamical applications of the exterior calculus" (1996).

where by computation

$$= \partial_j X_k - X_i \Gamma^i{}_{jk}$$

with

$$\Gamma^i{}_{jk} \equiv \frac{\partial x^i}{\partial x^p}\frac{\partial^2 x^p}{\partial x^j \partial x^k}$$

By extension of the notational convention $X_{k,\,j} \equiv \partial_j X_k$ one writes $X_{k;\,j} \equiv D_j X_k$. It is a clear that $X_{j;\,k}$—since created by "tensorial continuation" from the "seed" $\partial_j X_k$—transforms tensorially, and that it has something to do with familiar differentiation (*is* differentiation, but with built-in compensation for the familiar "extraneous term," and reduces to ordinary differentiation in the root coordinate system $\mathcal{X}$). The quantities $\Gamma^i{}_{jk}$ turn out not to transform tensorially, but by the rule

$$= M^i{}_a W^b{}_j W^c{}_k \Gamma^a{}_{bc} + \frac{\partial x^i}{\partial x^p}\frac{\partial^2 x^p}{\partial x^j \partial x^k}$$

characteristic of "affine connections." Finally, one gives up the assumption that there exists a coordinate system (the $\mathcal{X}$-system of prior discussion) in which D_j and ∂_j have coincident (*i.e.*, in which $\Gamma^i{}_{jk}$ vanishes globally). The affine connection $\Gamma^i{}_{jk}(x)$ becomes an object that we are free to *deposit* on the manifold $\mathcal{M}$, to create an "affinely connected manifold"... just as by deposition of $g_{ij}(x)$ we create a "metrically connected manifold." But when we do *both* things[98] a compatability condition arises, for we expect

- index manipulation followed by covariant differentiation, and
- covariant differentiation followed by index manipulation

to yield the same result. This is readily shown to entail $g_{ij;k} = 0$, which in turn entails

$$\Gamma^i{}_{jk} = \tfrac{1}{2} g^{ia}\Big(\frac{\partial g_{aj}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^a}\Big)$$

The affine connection has become implicit in the metric connection—it has become the "Christoffel connection," which plays a central role in Riemannian geometry and its applications (general relativity): down the road just a short way lies the Riemann-Christoffel curvature tensor

$$R^m{}_{nij} = \frac{\partial \Gamma^m{}_{nj}}{\partial x^i} - \frac{\partial \Gamma^m{}_{ni}}{\partial x^j} + \Gamma^m{}_{ai}\Gamma^a{}_{nj} - \Gamma^m{}_{aj}\Gamma^a{}_{ni}$$

which enters into statements such as the following

$$X_{n;ij} - X_{n;ji} = X_a R^a{}_{nij}$$

which describes the typical *inequality of crossed covariant derivatives*. The "covariant derivative" was invented by Elwin Christoffel (1829–1900) in 1869.

[98] Notice that we *need* both if we want to construct such things as the

$$\text{covariant Laplacian of } \phi \equiv g^{mn}\phi_{;mn}$$

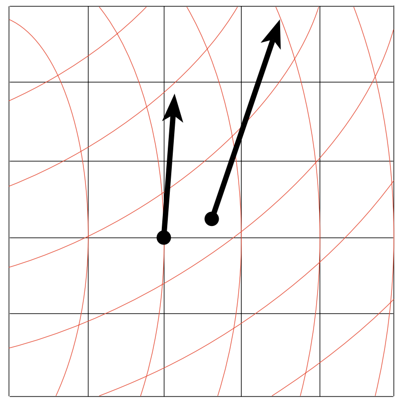


Figure 47: *Any attempt to construct a transformationally coherent theory of differentiation by comparing such neighboring vectors is doomed unless* $\mathcal{X} \rightarrow \mathcal{X}$ *gives rise to a transformation matrix that is* constant *on the neighborhood.*

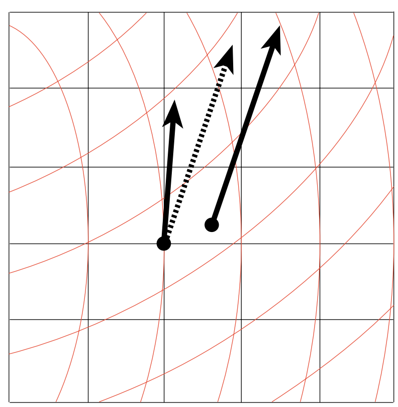


Figure 48: *The problem just noted is resolved if one compares one vector* with the local parallel transport *of the other—a "stand-in" rooted to the same point as the original vector. For then only a single transformation matrix enters into the discussion.*

Sharp insight into the meaning of the covariant derivative was provided in 1917 by Levi-Civita,[99] who pointed out that when one works from Figure 47 one *cannot realistically expect* to obtain a transformationally sensible result, for the

[99] The fundamental importance of Levi-Civita's idea was immediately appreciated and broadcast by Hermann Weyl. See §14 in his classic *Space, Time & Matter* (4th edition 1920, the English translation of which has been reprinted by Dover).

transformation matrices $\mathbb{W}(x)$ and $\mathbb{W}(x+dx)$ that act upon (say) $X_m(x)$ and $X_m(x+dx)$ are, in general, distinct. Levi-Civita observed that a workable procedure does, however, result if one looks not $X_m(x+dx) - X_m(x)$ but to $\mathcal{X}_m(x) - X_m(x)$, where

$\mathcal{X}_m(x)$ results from **parallel transport**
of $X_m(x+dx)$ from $x+dx$ back to x

He endowed the intuitive concept "parallel transport" (Figure 48) with a precise (natural) meaning, and immediately recovered the standard theory of covariant differentiation. But he obtained also much else: he showed, for example, that "geodesics" can be considered to arise not as "shortest" curves—curves produced by minimization of arc length

$$\int ds \quad \text{with} \quad (ds)^2 = g_{mn}dx^m dx^n$$

—but as curves whose tangents can be got one from another by parallel transportation: head off in some direction and "follow your nose" was the idea. Levi-Civita's idea so enriched a subject previously known as the "absolute differential calculus" that its name was changed ... to "tensor analysis."

Our CATALOG (pages 120–122) can be looked upon as an ennumeration of circumstances in which—"by accident"—the Γ-apparatus falls away. Look, for example, to the "covariant curl," where we have

$$\begin{aligned} X_{m;n} - X_{n;m} &= (X_{m,n} - X_a\Gamma^a{}_{nm}) - (X_{n,m} - X_a\Gamma^a{}_{mn}) \\ &= X_{m,n} - X_{n,m} \qquad \text{by } \Gamma^a{}_{mn} = \Gamma^a{}_{nm} \end{aligned}$$

The basic principles of the "absolute differential calculus" were developed between 1884 and 1894 by Gregorio Ricci-Curbastro (1853–1925), who was a mathematician in the tradition of Riemann and Christoffel.[100] In 1896 his student, Tullio Levi-Civita (1873–1941), published "Sulle transformazioni della eqazioni dinamiche" to demonstrate the *physical* utility of the methods which Ricci himself had applied only to differential geometry. In 1900—at the urging of Felix Klein, in Göttingen—Ricci and Levi-Civita co-authored "Méthodes de calcul différentiel absolus et leurs applications," a lengthy review of the subject ... but they were Italians writing in French, and published in a German periodical (Mathematische Annalen), and their work was largely ignored: for nearly twenty years the subject was known to only a few cognoscente (who included Minkowski at Göttingen), and cultivated by fewer. General interest in the subject developed—explosively!—only in the wake of Einstein's *general theory of relativity* (1916). Tensor methods had been brought to the reluctant attention of Einstein by Marcel Grossmann, a geometer who had been a classmate of Einstein's at the ETH in Zürich (Einstein reportedly used to study

[100] Ricci had interest also in physics, and as a young man published (in Nuovo Cimento) the first Italian account of Maxwellian electrodynamics.

Grossmann's class notes instead of attending Minkowski's lectures) and whose father had been instrumental in obtaining for the young and unknown Einstein a position in the Swiss patent office.

Acceptence of the tensor calculus was impeded for a while by those (mainly mathematicians) who perceived it to be in competition with the exterior calculus—an elegant French creation (Poincaré, Goursat, Cartan, . . .) which treats (but more deeply) a narrower set of issues, but (for that very reason) supports also a robust *integral* calculus. The exterior calculus shares the Germanic pre-history of tensor analysis (Gauss, Grassmann, Riemann, . . .) but was developed semi-independently (and somewhat later), and has only fairly recently begun to be included among the work-a-day tools of mathematical physicists. Every physicist can be expected today to have some knowledge of the tensor calculus, but the exterior calculus has yet to find a secure place in the pedagogical literature of physics, and for that (self-defeating) reason physicists who wish to be understood still tend to avoid the subject . . . in their writing and (at greater hazard) in their creative thought.

3. Transformation properties of the electromagnetic field equations. We will be led in the following discussion from Maxwell's equations to—first and most easily—the group of "Lorentz transformations," which by some fairly natural interpretive enlargement detach from their electrodynamic birthplace to provide the foundation of Einstein's *Principle of Relativity*. But it will emerge that

> The covariance group of a theory depends
> in part upon *how the theory is expressed*:

slight adjustments in the formal rendition of Maxwell's equations will lead to transformation groups that differ radically from the Lorentz group (but that contain the Lorentz group as a subgroup) . . . and that also is a lesson that admits of "enlargement"—that pertains to fields far removed from electrodynamics. The point merits explicit acknowledgement because it relates to how casually accepted conventions can exert unwitting control on the *development* of physics.

FIRST POINT OF VIEW Let Maxwell's equations be notated[101]

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c} j^\nu \tag{180.1}$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \tag{180.2}$$

where $F^{\mu\nu}$ is antisymmetric and where

$$F_{\mu\nu} \equiv g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} \quad \text{with} \quad \mathbb{g} \equiv \|g_{\mu\nu}\| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{181}$$

is (automatically) also antisymmetric. From $F_{\mu\nu} = -F_{\nu\mu}$ it follows, by the way,

---

[101] Compare (167).

that (180.2) reduces to the triviality $0 = 0$ unless μ, ν and λ are distinct, so the equation in question is just a condensed version of the sourceless Maxwell equations as they were encountered on page 111.[102] In view of entry (179.5) in our CATALOG it becomes natural to

> **assume** that $F^{\mu\nu}$ and j^μ transform as the components of tensor densities of unit weight:
>
> $$F^{\mu\nu} \longrightarrow F^{\mu\nu} = W \cdot M^\mu{}_\alpha M^\nu{}_\beta F^{\alpha\beta} \qquad \mathbf{A}_1$$
> $$j^\mu \longrightarrow j^\mu = W \cdot M^\mu{}_\alpha j^\alpha \qquad \mathbf{A}_2$$
>
> We note[103] that it makes coordinate-independent good sense to assume of the field tensor that it is antisymmetric:
>
> $$F^{\mu\nu} \text{ antisymmetric} \Longrightarrow F^{\mu\nu} \text{ antisymmetric}$$

The unrestricted covariance (in the sense "form-invariance under coordinate transformation") of (180.1) is then assured

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c} j^\nu \longrightarrow \partial_\mu F^{\mu\nu} = \tfrac{1}{c} j^\nu$$

On grounds that it would be intolerable for the description (181) of g to be "special to the coordinate system $\mathcal{X}$" we

> **assume** $g_{\mu\nu}$ to transform as a symmetric tensor of zero weight
>
> $$g_{\mu\nu} \longrightarrow g_{\mu\nu} = W^\alpha{}_\mu W^\beta{}_\nu g_{\alpha\beta} \qquad \mathbf{B}_1$$
>
> but impose upon $\mathcal{X} \rightarrow \mathcal{X}$ the *constraint* that
>
> $$= g_{\mu\nu} \qquad \mathbf{B}_2$$

This amounts in effect to imposition of the requirement that $\mathcal{X} \rightarrow \mathcal{X}$ be of such a nature that

$$\mathbb{W}^{\mathsf{T}} g \mathbb{W} = g \quad \text{everywhere} \tag{182}$$

---

[102] We might write

$$\|F^{\mu\nu}\| \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\therefore \|F_{\mu\nu}\| = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

to establish explicit contact with orthodox 3-vector notation and terminology (and at the same time to make antisymmetry manifest), but such a step would be extraneous to the present line of argument.

[103] See again page 119.

Looking to the determinant of the preceding equation we obtain

$$W^2 = 1$$

from which (arguing from continuity) we conclude that

$$W \text{ is } \begin{cases} \text{everywhere equal to } +1, \text{ else} \\ \text{everywhere equal to } -1. \end{cases} \tag{183}$$

This result protects us from a certain embarrassment: assumptions $\mathbf{A}_1$ and $\mathbf{B}_1$ jointly imply that $F_{\mu\nu}$ transforms as a tensor of unit weight, while covariance of the windmill sum in (180.2) was seen at (179.3) to require $F_{\mu\nu}$ to transform as a weightless tensor. But (183) reduces all weight distinctions to empty trivialities. Thus does $\mathbf{B}_2$ insure the covariance of (180.2):

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \longrightarrow \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

From (182) we will extract the statement that

$$\mathcal{X} \to \mathcal{X} \text{ is a } \textbf{Lorentz transformation} \tag{184}$$

and come to the conclusion that *Maxwellian electrodynamics—as formulated above—is Lorentz covariant*. Lorentz (1904) and Einstein (1905) were the independent co-discoverers of this fundamental fact, which they established by two alternative (and quite distinct) lines of argument.

SECOND POINT OF VIEW Retain both the field equations (180) and the assumptions **A** but—in order to escape from the above-mentioned "point of embarrassment"—agree in place of $\mathbf{B}_1$ to

> **assume** that $g_{\mu\nu}$ transforms as a symmetric tensor *density* of weight $w = -\frac{1}{2}$

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} = W^{-\frac{1}{2}} \cdot W^\alpha{}_\mu W^\beta{}_\nu g_{\alpha\beta} \tag*{$\mathbf{B}_1^*$}$$

for then $F_{\mu\nu}$ becomes weightless, as (179.3) requires. Retaining

$$= g_{\mu\nu} \tag*{$\mathbf{B}_2$}$$

we obtain

$$W^{-\frac{1}{2}} \cdot \mathbb{W}^{\mathsf{T}} \mathbb{g} \mathbb{W} = \mathbb{g} \quad \text{everywhere} \tag{185.1}$$

If spacetime were N-dimensional the determinantal argument would now give

$$W^{2-\frac{N}{2}} = 1$$

which (uniquely) in the physical case $(N = 4)$ reduces to a triviality: $W^0 = 1$. The constraint (183) therefore drops away, with consequences which I will discuss in a moment.

$\boxed{\text{THIRD POINT OF VIEW}}$ This differs only superficially from the viewpoint just considered. *Retain* $\mathbf{B}_1$ but in place of $\mathbf{B}_2$

assume that

$$g_{\mu\nu} = \Omega g_{\mu\nu} \qquad \mathbf{B}_2^*$$

Then

$$\mathbb{W}^{\mathsf{T}} g\, \mathbb{W} = \Omega g \tag{185.2}$$

and the determinantal argument supplies

$$\begin{aligned} \Omega &= W^{\frac{2}{N}} \\ &\downarrow \\ &= W^{\frac{1}{2}} \quad \text{in the physical case } N = 4 \end{aligned}$$

Equations (185.1) and (185.2) evidently say the same thing: the Lorentzian constraint (183) drops away and in place of (184) we have

$$\mathcal{X} \rightarrow \mathcal{X} \text{ is a } \textbf{conformal transformation} \tag{186}$$

The conformal covariance of Maxwellian electrodynamics was discovered independently by Cunningham[104] and Bateman.[105] It gives rise to ideas which have a curious past[106] and which have assumed a central place in elementary particle physics at high energy. Some of the electrodynamical implications of conformal covariance are so surprising that they have given rise to vigorous controversy.[107] A transformation is said (irrespective of the specific context) to be "conformal" if it preserves angles locally . . . though such transformations do not (in general) preserve non-local angles, nor do they (even locally) preserve length. Engineers make heavy use of the conformal recoordinatizations of the plane that arise from the theory of complex variables *via* the statement

$$z \rightarrow z = f(z) \quad : \quad f(z) \text{ analytic}$$

The bare bones of the argument: write $z = x + iy$, $z = u + iv$ and obtain

$$\left.\begin{aligned} u &= u(x, y) \\ v &= v(x, y) \end{aligned}\right. \quad \text{giving} \quad \begin{aligned} du &= u_x dx + u_y dy \\ dv &= v_x dx + v_y dy \end{aligned}$$

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[104] E. Cunningham, "The principle of relativity in electrodynamics and an extension thereof," Proc. London Math. Soc. **8**, 223 (1910).

[105] H. Bateman, "The transformation of the electrodynamical equations," Proc. London Math. Soc. **8**, 223 (1910).

[106] T. Fulton, F. Rohrlich & L. Witten, "Conformal invariance in physics," Rev. Mod. Phys. **34**, 442 (1962).

[107] See "Radiation in hyperbolic motion" in R. Peierls, *Surprises in Theoretical Physics* (1979), page 160.

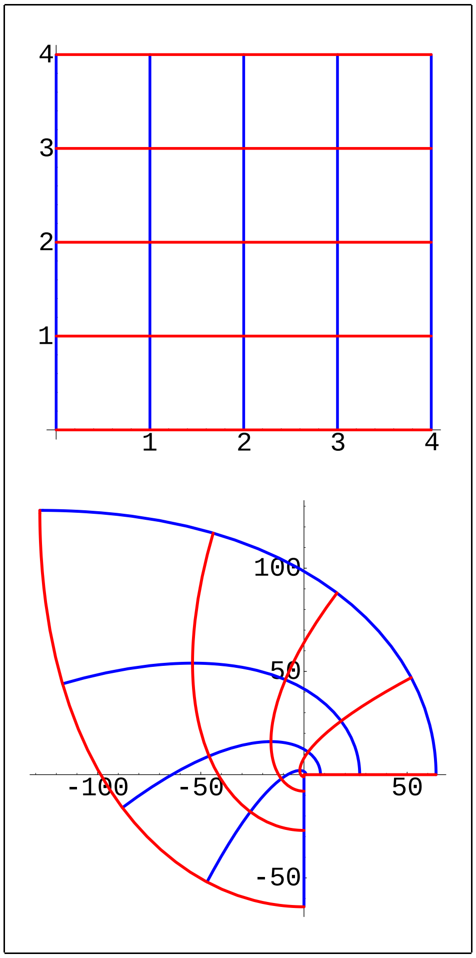

FIGURE 49: *Cartesian grid (above) and its conformal image (below) in the case $f(z) = z^3$, which supplies*

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 \\ v(x,y) &= 3x^2y - y^3 \end{aligned}$$

The command `ParametricPlot` *was used to construct the figure.*

But

$$\text{analyticity of } f(z) \iff \text{CAUCHY-RIEMANN CONDITIONS}: \begin{cases} u_x = +v_y \\ u_y = -v_x \end{cases}$$

so

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$$

which is to say: curves of constant u are everywhere $\perp$ to curves of constant v, just as curves of constant x were everywhere normal to curves of constant y. The situation is illustrated in the preceding figure. The 2-dimensional case—in which one can conformally transform in as infinitely many ways as one can select $f(z)$—is, however, exceptional:[108] in the cases $N > 2$ conformality arises from a less esoteric circumstance, and the possibilities are described by a finite set of parameters. Let A^m and B^m be weightless vectors, let the inner product be defined $(A, B) \equiv g_{mn}A^m B^n$, and suppose g_{mn} to transform as a symmetric tensor *density of weight* w. Then (A, B) and the "squared lengths" (A, A) and (B, B) of all transform (not as invariants but) as *scalar densities*. But the

$$\text{angle between } A^m \text{ and } B^m \equiv \arccos\left\{\frac{(A,B)}{\sqrt{(A,A)(B,B)}}\right\}$$

clearly *does* transform by invariance. Analysis of (185.2) gives rise in the physical case ($N = 4$) to a 15-parameter *conformal group* that contains the 6-parameter *Lorentz group* as a subgroup.

FOURTH POINT OF VIEW Adopt the (unique) affine connection $\Gamma^\lambda{}_{\mu\nu}$ which vanishes here in our inertial $\mathfrak{X}$-coordinate system. For us there is then no distinction between ordinary differentiation and covariant differentiation. So in place of (180) we can, if we wish, write

$$F^{\mu\nu}{}_{;\mu} = \tfrac{1}{c}j^\nu \tag{187.1}$$

$$F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} + F_{\mu\nu;\lambda} = 0 \tag{187.2}$$

Which is to say: we can elect to "tensorially continuate" our Maxwell equations to other coordinate systems or arbitrary (moving curvilinear) design. We retain the description (181) of $g_{\mu\nu}$, and we retain

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} = W^\alpha{}_\mu W^\beta{}_\nu g_{\alpha\beta} \qquad \mathbf{B}_1$$

But we have no longer any reason to retain $\mathbf{B}_2$, no longer any reason to impose any specific constraint upon the design of $g_{\mu\nu}$. We arrive thus at a formalism in which

$$F^{\mu\nu}{}_{;\mu} = \tfrac{1}{c}j^\nu \longrightarrow F^{\mu\nu}{}_{;\mu} = \tfrac{1}{c}j^\nu$$

$$F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} + F_{\mu\nu;\lambda} = 0 \longrightarrow F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} + F_{\mu\nu;\lambda} = 0$$

and in which

$$\mathfrak{X} \rightarrow \mathfrak{X} \text{ is } \textbf{unrestricted} \tag{188}$$

No "natural weights" are assigned within this formalism to $F^{\mu\nu}$, j^μ and $g_{\mu\nu}$, but formal continuity with the conformally-covariant formalism (whence with the Lorentz-covariant formalism) seems to require that we assign weights $w = 1$ to $F^{\mu\nu}$ and j^μ, weight $w = -\frac{1}{2}$ to $g_{\mu\nu}$.

[108] See page 55 of "The transformations which preserve wave equations" (1979) in TRANSFORMTIONAL PHYSICS OF WAVES (1979–1981).

Still other points of view are possible,[109] but I have carried this discussion already far enough to establish the validity of a claim made at the outset: the only proper answer to the question "What transformations $\mathfrak{X} \to \mathfrak{X}$ preserve the structure of Maxwell's equations?" is "It depends—depends on how you have chosen to *write* Maxwell's equations."

We have here touched, in a physical setting, upon an idea—look at "objects," and the groups of transformations which preserve relationships among those objects—which Felix Klein, in the lecture given when (in 1872, at the age of 23) he assumed the mathematical professorship at the University of Erlangen, proposed might be looked upon as *the* organizing principle of *all* pure/applied mathematics—a proposal which has come down to us as the "Erlangen Program." It has been supplanted in the world of pure mathematics, but continues to illuminate the historical and present development of physics.[110]

4. Lorentz transformations, and some of their implications. To state that $\mathfrak{X} \leftarrow \mathfrak{X}$ is a *Lorentz transformation* is, by definition, to state that the associated transformation matrix $\mathbb{M} \equiv \|M^\mu{}_\nu\| \equiv \|\partial x^\mu/\partial x^\nu\|$ has (see again page 127) the property that

$$\mathbb{M}^{\mathsf{T}} g\, \mathbb{M} = g \quad \text{everywhere} \tag{182}$$

where by fundamental assumption $g = g^{\mathsf{T}} = g^{-1}$ possesses at each point in spacetime the specific structure indicated at (181).

I begin with the observation that $\mathbb{M}$ must necessarily be a *constant* matrix. The argument is elementary: hit (182) with ∂_λ and obtain

$$(\partial_\lambda \mathbb{M})^{\mathsf{T}} g \mathbb{M} + \mathbb{M}^{\mathsf{T}} g (\partial_\lambda \mathbb{M}) = \mathbb{O} \quad \text{because } g \text{ is constant}$$

This can be rendered

$$g_{\alpha\beta} M^\alpha{}_{\lambda\mu} M^\beta{}_\nu + g_{\alpha\beta} M^\alpha{}_\mu M^\beta{}_{\nu\lambda} = 0$$

where $M^\alpha{}_{\lambda\mu} \equiv \partial^2 x^\alpha/\partial x^\lambda \partial x^\mu = M^\alpha{}_{\mu\lambda}$. More compactly

$$\Gamma_{\mu\nu\lambda} + \Gamma_{\nu\lambda\mu} = 0$$

where $\Gamma_{\mu\nu\lambda} \equiv g_{\alpha\beta} M^\alpha{}_\mu M^\beta{}_{\nu\lambda}$. Also (subjecting the $\mu\nu\lambda$ to cyclic permutation)

$$\begin{aligned} \Gamma_{\nu\lambda\mu} + \Gamma_{\lambda\mu\nu} &= 0 \\ \Gamma_{\lambda\mu\nu} + \Gamma_{\mu\nu\lambda} &= 0 \end{aligned}$$

so

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Gamma_{\lambda\mu\nu} \\ \Gamma_{\mu\nu\lambda} \\ \Gamma_{\nu\lambda\mu} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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[109] See D. van Dantzig, "The fundamental equations of electromagnetism, independent of metric geometry," Proc. Camb. Phil. Soc. **30**, 421 (1935).

[110] For an excellent discussion see the section "Codification of geometry by invariance" (pages 442–453) in E. T. Bell's *The Development of Mathematics* (1945). The Erlangen Program is discussed in scholarly detail in T. Hawkins, *Emergence of the theory of Lie Groups* (2000): see the index. For a short history of tensor analysis, see Bell's Chapter 9.

The 3×3 matrix is non-singular, so we must have

$$\Gamma_{\lambda\mu\nu} = M^\alpha{}_\lambda g_{\alpha\beta}\partial_\mu M^\beta{}_\nu = 0 \quad : \quad \text{ditto cyclic permutations}$$

which in matrix notation reads

$$\mathbb{M}^\mathsf{T}\mathbb{g}(\partial_\mu\mathbb{M}) = \mathbb{O}$$

The matrices $\mathbb{M}$ and $\mathbb{g}$ are non-singular, so we can multiply by $(\mathbb{M}^\mathsf{T}\mathbb{g})^{-1}$ to obtain

$$\partial_\mu\mathbb{M} = \mathbb{O} \quad : \quad \text{the elements of } \mathbb{M} \text{ must be } \underline{\text{constants}}$$

The functions $x^\mu(x)$ that describe the transformation $\mathcal{X} \leftarrow \mathcal{X}$ must possess therefore the <u>inhomogeneous linear</u> structure[111]

$$x^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad : \quad \text{the } \Lambda^\mu{}_\nu \text{ and } a^\mu \text{ are constants}$$

The transformation matrix $\mathbb{M}$, with elements given now by constants $\Lambda^\mu{}_\nu$, will henceforth be denoted $\mathbb{\Lambda}$ to emphasize that it is no longer generic but has been specialized (and also to suggest "Lorentz"). We shall (when the risk of confusion is slight) write

$$x = \mathbb{\Lambda}x + \underbrace{a}_{\text{describes a }\underline{\text{translation}}\text{ in spacetime}} \tag{189.1}$$

to describe an ("inhomogeneous Lorentz" or) **Poincaré transformation**, and

$$x = \mathbb{\Lambda}x \tag{189.2}$$

to describe a (simple homogeneous) **Lorentz transformation**, the assumption in both cases being that

$$\mathbb{\Lambda}^\mathsf{T}\mathbb{g}\,\mathbb{\Lambda} = \mathbb{g} \tag{190}$$

> IMPORTANT REMARK: Linearity of a transformation—constancy of the transformation matrix—is sufficient in itself to kill all "extraneous terms," without the assistance of weight restrictions.

It was emphasized on page 119 that "not every indexed object transforms tensorially," and that, in particular, the x^μ themselves do not transform tensorially *except in the linear case*. We have now in hand just such a case, and for that reason relativity becomes—not just locally but globally—an exercise in linear algebra. Spacetime has become a 4-dimensional vector space; indeed, it has become an *inner product space*, with

$$\left.\begin{aligned}(x,y) &\equiv g_{\mu\nu}x^\mu y^\nu\\ &= (y,x) \quad \text{by} \quad g_{\mu\nu} = g_{\nu\mu}\\ &= x^\mathsf{T}\mathbb{g}\,y\\ &= x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3 = x^0y^0 - \boldsymbol{x}\cdot\boldsymbol{y}\end{aligned}\right\} \tag{191.1}$$

[111] Einstein (1905)—on the grounds that what he sought was a *minimal modification of the Galilean transformations* (which are themselves linear)—was content simply to *assume* linearity.

The *Lorentz inner product* (interchangeably: the "Minkowski inner product") described above is, however, "pathological" in the sense that it gives rise to an "indefinite norm;" *i.e.*, to a norm

$$\left.\begin{aligned}(x,x) &= g_{\mu\nu}x^\mu x^\nu\\ &= x^{\mathsf{T}}\mathbb{g}\,x\\ &= (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2=(x^0)^2-\boldsymbol{x}\cdot\boldsymbol{x}\end{aligned}\right\}\tag{191.2}$$

which (instead of being positive unless $x=0$) can assume either sign, and can vanish even if $x\neq 0$. From this primitive fact radiates much—arguably all—that is most distinctive about the geometry of spacetime ... which, as Minkowski was the first to appreciate (and as will emerge) lies at the heart of the theory of relativity.

If A^μ, B^μ and $g_{\mu\nu}$ transform as weightless tensors, then basic tensor algebra informs us that $g_{\mu\nu}A^\mu B^\nu$ transforms by invariance:

$$g_{\mu\nu}A^\mu B^\nu \longrightarrow g_{\mu\nu}A^\mu B^\nu = g_{\mu\nu}A^\mu B^\nu \quad \text{unrestrictedly}$$

What distinguishes Lorentz transformations from transformations-in-general is that

$$g_{\mu\nu}=g_{\mu\nu}$$

To phrase the issue as it relates not to things (like A^μ and B^μ) "written on" spacetime but to the structure of spacetime itself, we can state that the linear transformation

$$x\longrightarrow x=\mathbb{\Lambda}x$$

describes a Lorentz transformation if and only if

$$x^{\mathsf{T}}\mathbb{g}\,y = x^{\mathsf{T}}\mathbb{\Lambda}^{\mathsf{T}}\mathbb{g}\,\mathbb{\Lambda}\,y = x^{\mathsf{T}}\mathbb{g}\,y \quad \text{for all } x \text{ and } y \quad : \quad \text{entails} \quad \mathbb{\Lambda}^{\mathsf{T}}\mathbb{g}\,\mathbb{\Lambda}=\mathbb{g}$$

where, to be precise, we require that $\mathbb{g}$ has the specific design

$$\mathbb{g}\equiv\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$$

that at (163) was impressed upon us by our interest in the transformation properties of Maxwell's equations (*i.e.*, by some narrowly prescribed specific physics).

We come away with the realization that *Lorentz transformations have in fact only incidentally to do with electrodynamics*: they are the transformations that <u>preserve Lorentzian inner products</u>, which is to say: that <u>preserve the metric properties of spacetime</u> ...just as "rotations" $\boldsymbol{x}\longrightarrow\boldsymbol{x}=\mathbb{R}\boldsymbol{x}$ are the linear transformations that preserve Euclidean inner products

$$\boldsymbol{x}^{\mathsf{T}}\mathbb{I}\,\boldsymbol{y} = \boldsymbol{x}^{\mathsf{T}}\mathbb{R}^{\mathsf{T}}\mathbb{I}\,\mathbb{R}\,\boldsymbol{y} = \boldsymbol{x}^{\mathsf{T}}\mathbb{I}\,\boldsymbol{y} \quad \text{for all } \boldsymbol{x} \text{ and } \boldsymbol{y} \quad : \quad \text{entails} \quad \mathbb{R}^{\mathsf{T}}\mathbb{R}=\mathbb{I}$$

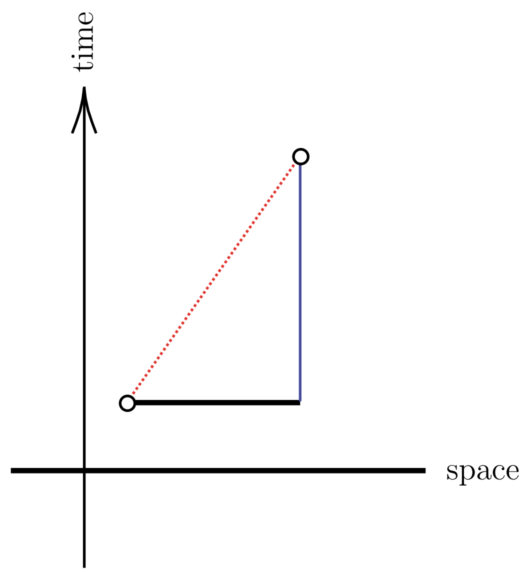

FIGURE 50: *Two "events" identify a triangle in the spacetime. Relativity asks each inertial observer to use metersticks and clocks to assign traditional meanings to the "Euclidean length" of the black side (here thickened to suggest that space is several-dimensional) and to the "duration" of the blue side—meanings which (as will emerge) turn out, however, to yield observer-dependent numbers—but assigns (Lorentz-invariant!) meaning also to the <u>squared length of the hypotenuse</u>.*

and in so doing preserve the lengths/angles/areas/volumes . . . that endow Euclidean 3-space with its distinctive metric properties.

That spacetime can be said to <u>possess</u> metric structure is the great surprise, the great discovery. In pre-relativistic physics one could speak of the duration (quantified by a clock) of the temporal interval $\Delta t = t_a - t_b$ separating a pair of events, and one could speak of the length

$$\Delta\ell = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2}$$

(quantified by a meter stick) of the spatial interval separating a pair of points; one spoke of "space" and "time," but "spacetime" remained an abstraction of the design space $\otimes$ time. Only with the introduction g did it become possible (see Figure 50) to speak of the (squared) length

$$(\Delta s)^2 = c^2(t_a - t_b)^2 - (x_a - x_b)^2 - (y_a - y_b)^2 - (z_a - z_b)^2$$

of the interval separating $(t_a, \boldsymbol{x}_a)$ from $(t_b, \boldsymbol{x}_b)$:

$$\text{“space} \otimes \text{time”} \quad \text{had become} \quad \text{“spacetime”}$$

The first person to recognize the profoundly revolutionary nature of what had been accomplished was (not Einstein but) Minkowski, who began an address to the Assembly of German Natural Scientists & Physicians (21 September 1908) with these words:

> *"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."*

Electrodynamics had led to the first clear perception of the geometrical design of the spacetime manifold upon which all physics is written. The symmetries inherent in that geometry were by this time know to be reflected in the design of Maxwell's equations. Einstein's *Principle of Relativity* holds that they must, in fact, be reflected in the design of *all* physical theories—irrespective of the specific phenomenology to which any individual theory may refer.

Returning now to the technical mainstream of this discussion ... let the Lorentz condition (190) be written

$$/\!\!\Lambda^{-1} = g^{-1}/\!\!\Lambda^{\mathsf{T}} g \tag{192}$$

Generally inversion of a 4×4 matrix is difficult, but (192) shows that *inversion of a Lorentz matrix* $/\!\!\Lambda$ *can be accomplished very easily*.[112].

Equations (190/192) impose a multiplicative condition upon $/\!\!\Lambda$. It was to reduce multiplicative conditions to additive conditions (which are easier) that logarithms were invented. Assume, therefore, that $/\!\!\Lambda$ can be written

$$/\!\!\Lambda = e^{\mathbb{A}} = \mathbb{I} + \mathbb{A} + \tfrac{1}{2!}\mathbb{A}^2 + \cdots$$

It now follows that

$$/\!\!\Lambda^{-1} = e^{-\mathbb{A}} \quad \text{while} \quad g^{-1}/\!\!\Lambda^{\mathsf{T}} g = g^{-1} e^{\mathbb{A}^{\mathsf{T}}} g = e^{g^{-1}\mathbb{A}^{\mathsf{T}} g}$$

Evidently $/\!\!\Lambda$ will be a Lorentz matrix if

$$-\mathbb{A} = g^{-1}\mathbb{A}^{\mathsf{T}} g$$

which (by $g^{\mathsf{T}} = g$) can be expressed

$$(g\mathbb{A})^{\mathsf{T}} = -(g\mathbb{A})$$

This is an *additive* condition (involves negation instead of inversion) and amounts simply to the statement that $g\mathbb{A} \equiv \|A_{\mu\nu}\|$ is *antisymmetric*. Adopt this notation

$$g\mathbb{A} = \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ -A_1 & 0 & -a_3 & a_2 \\ -A_2 & a_3 & 0 & -a_1 \\ -A_3 & -a_2 & a_1 & 0 \end{pmatrix}$$

[112] PROBLEM 40

where $\{A_1, A_2, A_3, a_1, a_2, a_3\}$ comprise a *sextet of adjustable real constants*. Multiplication on the left by g^{-1} gives a matrix of (what I idiosyncratically call) the "g-antisymmetric" design[113]

$$\mathbb{A} \equiv \|A^\mu{}_\nu\| = \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ A_1 & 0 & a_3 & -a_2 \\ A_2 & -a_3 & 0 & a_1 \\ A_3 & a_2 & -a_1 & 0 \end{pmatrix}$$

We come thus to the conclusion that matrices of the form

$$/\!\!\Lambda = \exp \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ A_1 & 0 & a_3 & -a_2 \\ A_2 & -a_3 & 0 & a_1 \\ A_3 & a_2 & -a_1 & 0 \end{pmatrix} \tag{193}$$

are Lorentz matrices; *i.e.*, they satisfy (190/192), and when inserted into (189) they describe Poincaré/Lorentz transformations.

Does every Lorentz matrix $/\!\!\Lambda$ admit of such representation? Not quite. It follows immediately from (190) that $(\det /\!\!\Lambda)^2 = 1$; *i.e.*, that

$$\Lambda \equiv \det /\!\!\Lambda = \pm 1, \text{ according as } /\!\!\Lambda \text{ is } \begin{cases} \text{"proper"} \\ \text{"improper"} \end{cases}$$

while the theory of matrices supplies the lovely identity[114]

$$\det(e^{\mathbb{M}}) = e^{\operatorname{tr}\mathbb{M}} \quad : \quad \mathbb{M} \text{ is } \textit{any} \text{ square matrix} \tag{194}$$

We therefore have $\Lambda = \det(e^{\mathbb{A}}) = 1$ by $\operatorname{tr}\mathbb{A} = 0$:

> Every Lorentz matrix $/\!\!\Lambda$ of the form (193) is necessarily *proper*; moreover (as will emerge), every proper $/\!\!\Lambda$ admits of such an "exponential representation." (195)

It will emerge also that when one has developed the structure of the matrices $/\!\!\Lambda = e^{\mathbb{A}}$ one has "cracked the nut," in the sense that it becomes easy to describe their improper companions.[115]

What it means to "develop the structure of $/\!\!\Lambda = e^{\mathbb{A}}$" is exposed most simply in the (physically artificial) case $N = 2$. Taking

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad : \quad \text{Lorentz metric in 2-dimensional spacetime}$$

---

[113] Notice that g-antisymmetry becomes literal antisymmetry when the metric g is Euclidean. Notice also that while it makes tensor-algebraic good sense to write $\mathbb{A}^2 = \|A^\mu{}_\alpha A^\alpha{}_\nu\|$ it would be hazardous to write $(g\mathbb{A})^2 = \|A_{\mu\alpha}A_{\alpha\nu}\|$.

[114] PROBLEM 41.

[115] PROBLEM 42.

as our point of departure, the argument that gave (193) gives

$$\mathbb{\Lambda} = \exp\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} = e^{A\mathbb{J}} \tag{196.1}$$

where evidently

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

By quick calculation (or, more elegantly, by appeal to the *Cayley-Hamilton theorem*, according to which every matrix satisfies its own characteristic equation) we find $\mathbb{J}^2 = \mathbb{I}$, from which it follows that

$$\mathbb{J}^n = \begin{cases} \mathbb{I} & \text{if } n \text{ is even} \\ \mathbb{J} & \text{if } n \text{ is odd} \end{cases}$$

So

$$\begin{aligned} \mathbb{\Lambda} &= \underbrace{\left\{1 + \tfrac{1}{2!}A^2 + \tfrac{1}{4!}A^4 + \cdots\right\}}_{\cosh A}\mathbb{I} + \underbrace{\left\{A + \tfrac{1}{3!}A^3 + \tfrac{1}{5!}A^5 + \cdots\right\}}_{\sinh A}\mathbb{J} \\ &= \begin{pmatrix} \cosh A & \sinh A \\ \sinh A & \cosh A \end{pmatrix} \qquad (196.2) \\ &\equiv \mathbb{\Lambda}(A) \quad : \quad \text{Lorentzian for all real values of } A \end{aligned}$$

It is evident—whether one argues from (196.2) of (more efficiently) from (196.1) —that

$$\begin{aligned} \mathbb{I} &= \mathbb{\Lambda}(0) && : \text{ existence of identity} && (197.1) \\ \mathbb{\Lambda}(A_2)\mathbb{\Lambda}(A_1) &= \mathbb{\Lambda}(A_1 + A_2) && : \text{ compositional closure} && (197.2) \\ \mathbb{\Lambda}^{-1}(A) &= \mathbb{\Lambda}(-A) && : \text{ existence of inverse} && (197.3) \end{aligned}$$

and that all such $\mathbb{\Lambda}$-matrices *commute*.

We are now—but *only* now—in position to consider the <u>kinematic meaning</u> of A, and of the action of $\mathbb{\Lambda}(A)$. We are, let us pretend, a "point PhD" who—having passed the physical tests required to establish our inertiality—use our "good clock and Cartesian frame" to assign coordinates $x \equiv \{x^0, x^1, x^2, x^3\}$ to events. O—a second observer, similarly endowed, who we see to be gliding by with velocity $\boldsymbol{v}$—assigns coordinates $x \equiv \{x^0, x^1, x^2, x^3\}$ to those same events. O shares our confidence in the validity of Maxwellian electrodynamics: we can therefore write $x = \mathbb{\Lambda}x + a$. In the interests merely of simplicity we will *assume that O's origin and our origin coincide*: the translational terms a^μ then drop away and we have $x = \mathbb{\Lambda}x$ . . . which in the 2-dimensional case reads

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} \cosh A & \sinh A \\ \sinh A & \cosh A \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \tag{198}$$

To describe the successive "ticks of the clock at his origin" O writes

$$\begin{pmatrix} ct \\ 0 \end{pmatrix}$$

while—to describe those same events—*we* write

$$\begin{pmatrix} ct \\ vt \end{pmatrix}$$

Immediately $vt = ct \cdot \sinh A$ and $ct = ct \cdot \cosh A$ which, when we divide the former by the latter, give

$$\tanh A = \beta \tag{199}$$

with

$$\beta \equiv v/c \tag{200}$$

These equations serve to *assign kinematic meaning to* A*, and therefore to* $\mathbb{\Lambda}(A)$. Drawing now upon the elementary identities

$$\cosh A = \frac{1}{\sqrt{1-\tanh^2 A}} \qquad \text{and} \qquad \sinh A = \frac{\tanh A}{\sqrt{1-\tanh^2 A}}$$

we find that (198) can be written

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \tag{201}$$

with

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = 1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \cdots \tag{202}$$

Evidently γ becomes singular (see Figure 51) at $\beta^2 = 1$; *i.e.*, at $v = \pm c$ . . . with diverse consequences which we will soon have occasion to consider. The *non-relativistic limit* arises physically from $\beta^2 \ll 1$; *i.e.*, from $v^2 \ll c^2$, but can be considered formally to arise from $c \uparrow \infty$. One must, however, take careful account of the c that lurks in the definitions of x^0 and x^0: when that is done, one finds that (201) assumes the (less memorably symmetric) form

$$\begin{aligned} \begin{pmatrix} t \\ x \end{pmatrix} &= \gamma \begin{pmatrix} 1 & v/c^2 \\ v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \\ \text{giving} \qquad &\downarrow \\ &= \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \quad \text{as} \quad c \uparrow \infty \end{aligned} \tag{203}$$

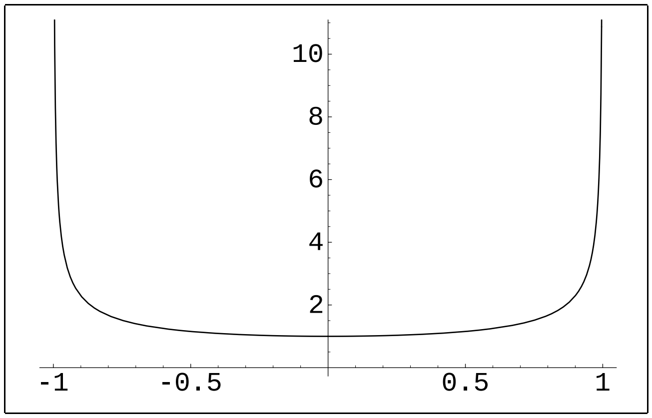

FIGURE 51: *Graph of the β-dependence of $\gamma \equiv 1/\sqrt{1-\beta^2}$, as $\beta \equiv v/c$ ranges on the physical interval $-1 < \beta < +1$. Outside that interval γ becomes imaginary.*

Heretofore we have been content to share our profession with a zippy population of "superluminal inertial observers" who glide past us with speeds $v > c$. But

$$\mathbb{\Lambda}(\beta) \text{ becomes } \underline{\text{imaginary}} \text{ when } \beta^2 > 1$$

We cannot enter into meaningful dialog with such observers; we therefore strip them of their clocks, frames and PhD's and send them into retirement, denied any further collaboration in the development of our relativistic theory of the world[114]—indispensable though they were to our former Galilean activity. Surprisingly, we can get along very well without them, for

$$\begin{aligned}
&\mathbb{\Lambda}(\beta_2)\mathbb{\Lambda}(\beta_1) = \mathbb{\Lambda}(\beta)\\
&\qquad\qquad \beta = \beta(\beta_1,\beta_2) = \tanh(A_1 + A_2)\\
&\qquad\qquad\qquad\qquad = \frac{\tanh A_1 + \tanh A_2}{1 + \tanh A_1 \tanh A_2}\\
&\qquad\qquad\qquad\qquad = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}
\end{aligned} \tag{204}$$

entails (this is immediately evident in Figure 52) that

if $v_1 < c$ and $v_2 < c$ then so also is $v(v_1, v_2) < c$:
one cannot leapfrog into the superluminal domain

The function $\beta(\beta_1,\beta_2)$ plays in (2-dimensional) relativity a role precisely analogous to a "group table" in the theory of finite groups: it describes <u>how Lorentz transformations compose</u>, and possess many wonderful properties, of

[114] This, however, does *not*, of itself, deny any conceivable role to superluminal *signals* or *particles* in a relativistic physics!

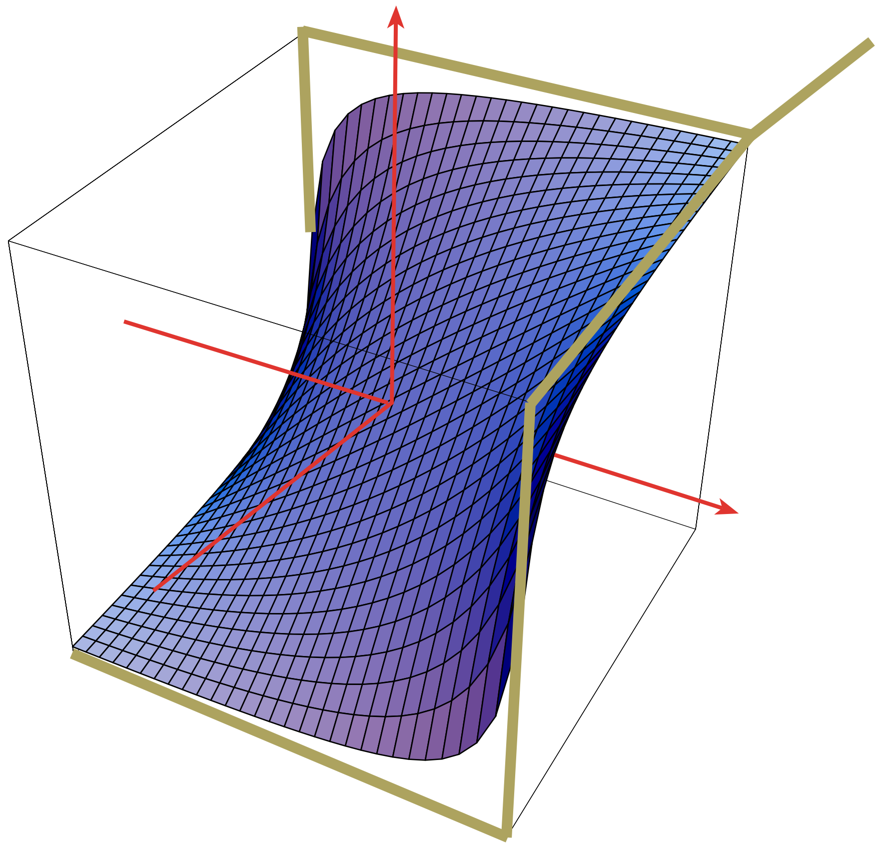

FIGURE 52: *Graph of the function* $\beta(\beta_1, \beta_2)$. *The vertices of the frame stand at the points* $\{\pm 1, \pm 1, \pm 1\}$ *in 3-dimensional* β*-space. If we write* $\beta_3 = -\beta(\beta_1, \beta_2)$ *then (204) assumes the high symmetry*

$$\beta_1 + \beta_2 + \beta_3 + \beta_1\beta_2\beta_3 = 0$$

clearly evident in the figure. The "β*-surface" looks rather like a soap film spanning the 6-sided frame that results when the six untouched edges of the cube are discarded.*

which I list here only a few:

$$\begin{aligned}
\beta(\beta_1, \beta_2) &= \beta(\beta_2, \beta_1) \\
\beta(\beta_1, \beta_2) &= 0 \quad \text{if} \quad \beta_2 = -\beta_1 \\
\beta(1, 1) &= 1
\end{aligned}$$

To this list our forcibly retired superluminal friends might add the following:

$$\beta(\beta_1, \beta_2) = \beta(\tfrac{1}{\beta_1}, \tfrac{1}{\beta_2})$$

If β is subluminal then $\frac{1}{\beta}$ is superluminal. So we have here the statement that the compose of two superluminal Lorentz transformations is subluminal (the i's have combined to become real). Moreover, every subluminal Lorentz transformation can be displayed as such a compose (in many ways). Curious!

Equation (204) is often presented as "relativistic velocity addition formula"

$$\begin{aligned}
v &= \frac{v_1 + v_2}{1 + v_1 v_2/c^2}\\
&= (v_1 + v_2)\cdot\Big[1 - \Big(\frac{v_1 v_2}{c^2}\Big) + \Big(\frac{v_1 v_2}{c^2}\Big)^2 - \Big(\frac{v_1 v_2}{c^2}\Big)^3 + \cdots\Big]\\
&= (\text{Galilean formula})\cdot\Big[\text{relativistic correction factor}\Big]
\end{aligned}$$

but that portrayal of the situation—though sometimes useful—seems to me to miss (or to entail risk of missing) the simple origin and essential significance of (204): the tradition that has, for now nearly a century, presented relativity as a source of endless paradox (and which has, during all that time, contributed little or nothing to understanding—paradox being, as it is, a symptom of imperfect understanding) should be allowed to wither.

In applications we will have need also of $\gamma(\beta_1, \beta_2) \equiv [1 - \beta^2(\beta_1, \beta_2)]^{-\frac{1}{2}}$, the structure of which is developed most easily as follows:

$$\begin{aligned}
\gamma &= \cosh(A_1 + A_2)\\
&= \cosh A_1 \cosh A_2 \big[1 + \tanh A_1 \tanh A_2\big]\\
&= \gamma_1\gamma_2\big[1 + \beta_1\beta_2\big]
\end{aligned}\tag{205}$$

This "γ-composition law"—in which we might (though it is seldom useful) use

$$\beta = \sqrt{1 - \gamma^{-2}} = \frac{\sqrt{(\gamma+1)(\gamma-1)}}{\gamma}$$

to eliminate the surviving β's—will acquire importance when we come to the theory of radiation.

5. Geometric considerations. Our recent work has been algebraic. The following remarks emphasize the geometrical aspects of the situation, and are intended to provide a more vivid sense of what Lorentz transformations are all about. By way of preparation: In Euclidean 3-space the equation $\boldsymbol{x}^{\mathsf{T}}\boldsymbol{x} = r^2$ defines a sphere (concentric about the origin, of radius r) which—consisting as it does of points all of which lie at the *same (Euclidean) distance from the origin*—we may reasonably call an "isometric surface." Rotations ($\boldsymbol{x} \to \boldsymbol{x} = \mathbb{R}\,\boldsymbol{x}$ with $\mathbb{R}^{\mathsf{T}}\mathbb{R} = \mathbb{I}$) cause the points of 3-space to shift about, but by a linear rule (straight lines remain straight) that <u>maps isometric spheres onto themselves</u>: such surfaces are, in short, "$\mathbb{R}$-invariant." Similarly . . .

In spacetime the σ-parameterized equations

$$\boldsymbol{x}^{\mathsf{T}}\!g\,\boldsymbol{x} = \sigma$$

define a population of Lorentz-invariant isometric surfaces Σ_σ. The surfaces that in 3-dimensional spacetime arise from

$$(x^0)^2 - (x^1)^2 - (x^2)^2 = \sigma$$

which describes a

- hyperboloid of two sheets in the case $\sigma > 0$
- cone in the case $\sigma = 0$
- hyperboloid of one sheet in the case $\sigma < 0$

are shown in Figure 53. The analogous construction in 2-dimensional spacetime (Figure 54) is easier to sketch, and serves most purposes well enough, but is misleading in one important respect: it fails to indicate the profound distinction between one-sheeted and two-sheeted hyperboloids. On the former one can move continuously from any point to any other (one can, in particular, get from one to the other by Lorentz transformation), but passage from one sheet to the other is necessarily *dis*continuous (requires "time reflection," can might be symbolized

$$\text{future} \rightleftarrows \text{past}$$

and cannot be executed "a little bit at a time").

How—within the geometric framework just described—is one to represent the action $x \longrightarrow x = \mathbb{\Lambda} x$ of $\mathbb{\Lambda}(\beta)$? I find it advantageous to approach the question somewhat obliquely: Suppose O to be thinking about the points (events)

$$\begin{pmatrix} +1 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ +1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

that mark the vertices of a "unit square" on her spacetime diagram. By quick calculation

$$\left.\begin{array}{l} \begin{pmatrix} +1 \\ +1 \end{pmatrix} \longrightarrow K^+(\beta)\begin{pmatrix} +1 \\ +1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix} \longrightarrow K^+(\beta)\begin{pmatrix} -1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} +1 \\ -1 \end{pmatrix} \longrightarrow K^-(\beta)\begin{pmatrix} +1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ +1 \end{pmatrix} \longrightarrow K^-(\beta)\begin{pmatrix} -1 \\ +1 \end{pmatrix} \end{array}\right\} \tag{206}$$

where

$$K^+(\beta) \equiv \sqrt{\tfrac{1+\beta}{1-\beta}} \quad \text{and} \quad K^-(\beta) \equiv \sqrt{\tfrac{1-\beta}{1+\beta}} \tag{207}$$

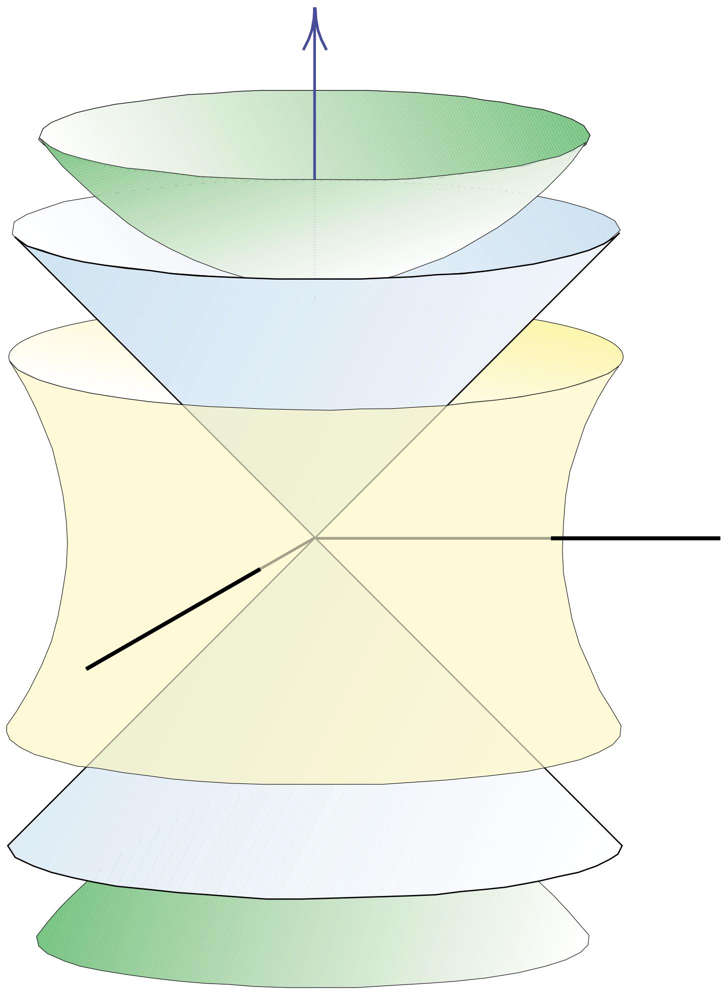

Figure 53: *Isometric surfaces in 3-dimensional spacetime. The arrow is "the arrow of time." Points on the blue "null cone" (or "light cone") are defined by the condition* $\sigma = 0$*: the interval separating such points from the origin has zero squared length (in the Lorentzian sense). Points on the green cup (which is interior to the forward cone) lie in the "future" of the origin, while points on the green cap (interior to the backward cone) lie in the "past:" in both cases* $\sigma > 0$*. Points on the yellow girdle (exterior to the cone) arise from* $\sigma < 0$*: they are separated from the origin by intervals of negative squared length, and are said to lie "elsewhere." In physical (4-dimensional) spacetime the circular cross sections (cut by "time-slices") become spherical. Special relativity acquires many of its most distinctive features from the circumstance that the isometric surfaces* Σ_σ *are* <u>*hyperboloidal*</u>.

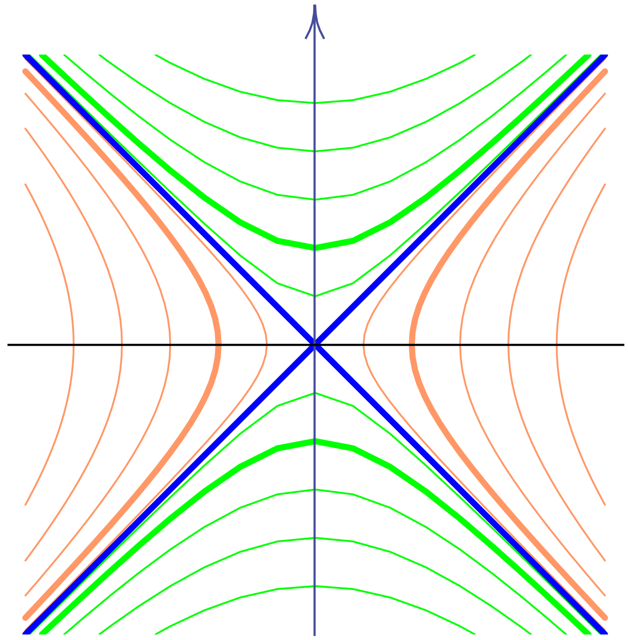

FIGURE 54: *The isometric surfaces shown in the preceding figure become isometric* curves *in 2-dimensional spacetime, where all hyperbolas have two branches. We see that*

$$\begin{pmatrix}1\\0\end{pmatrix} \text{ gives } \sigma = 1^2 - 0^2 = +1, \text{ typical of points with timelike}$$

$$\begin{pmatrix}1\\1\end{pmatrix} \text{ gives } \sigma = 1^2 - 1^2 = 0, \text{ typical of points with lightlike}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \text{ gives } \sigma = 0^2 - 1^2 = -1, \text{ typical of points with spacelike}$$

separation from the origin. And that—since the figure maps to itself under the Lorentz transformations that

- *describe the symmetry structure of spacetime*
- *describe the relationships among inertial observers*

—these classifications are Lorentz-invariant, shared by all *inertial observers.*

Calculation would establish what is in fact made obvious already at (206): the $K^\pm(\beta)$ are precisely the *eigenvalues* of $\mathbb{\Lambda}(\beta)$.[115] Nor are we surprised that the associated eigenvectors are *null* vectors, since

$$(x, x) \to (Kx, Kx) = (x, x) \quad \text{entails} \quad (x, x) = 0$$

[115] We note in passing that $K^-(\beta) = [K^+(\beta)]^{-1} = K^+(-\beta)$.

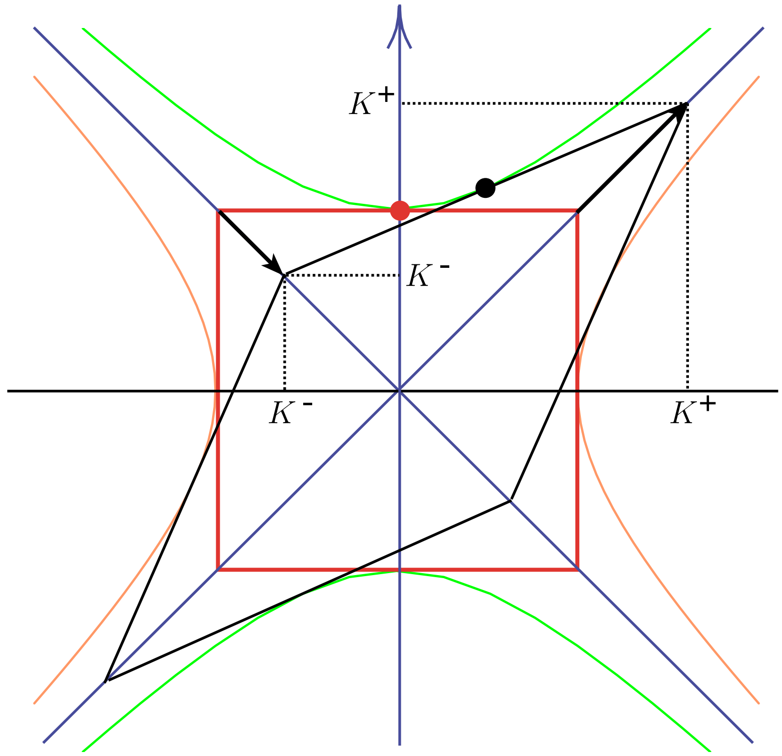

FIGURE 55: *Inertial observer O inscribes a "unit square" $\square$, with lightlike vertices, on her spacetime diagram. $\mathbb{\Lambda}(\beta)$ stretches one diagonal by the factor K^+, and shrinks the other by the factor K^-. That individual points "slide along isometric curves" is illustrated here by the motion $\bullet \to \bullet$ of a point of tangency. Corresponding sides of $\square$ and its transform have different Euclidean lengths, but identical Lorentzian lengths. Curiously, it follows from $K^+K^- = 1$ that $\square$ and its transform have identical Euclidean* areas.[116,117]

The upshot of preceding remarks is illustrated above, and elaborated in the figure on the next page, where I have stated in the caption but here emphasize once again that *such figures, though drawn on the Euclidean page, are to be read as inscriptions on 2-dimensional spacetime*. The distinction becomes especially clear when one examines Figure 57.

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[116] PROBLEM 43.

[117] Some authors stress the utility in special relativity of what they call the "k-calculus:" see, for example, Hermann Bondi, *Relativity and Common Sense: A New Approach to Einstein* (1962), pages 88–121 and occasional papers in the American Journal of Physics. My K-notation is intended to establish contact with that obscure tradition.

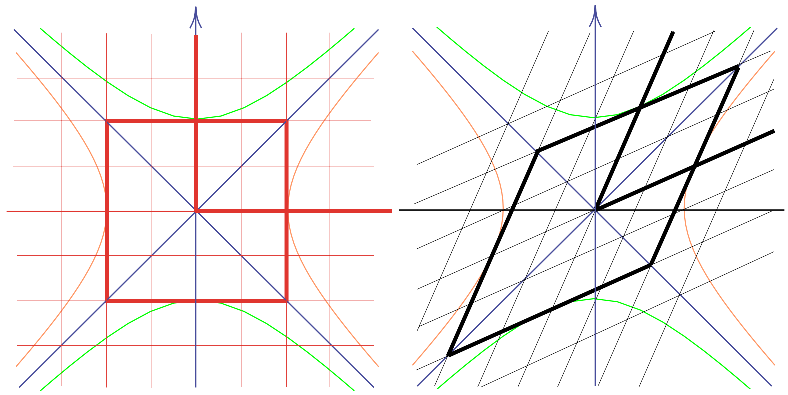

FIGURE 56: *Elaboration of the preceding figure. O has inscribed a Cartesian gridwork on spacetime. On the right is shown the Lorentz transform of that coordinate grid. Misner, Thorne & Wheeler* (Gravitation (1973), page 11) *have referred in this connection to the "collapse of the egg crate," though that picturesque terminology is somewhat misleading: egg crates preserve side-length when they collapse, while the present mode of collapse preserves Euclidean area. Orthogonality, though obviously violated in the Euclidean sense, is preserved in the Lorentzian sense . . . which is, in fact, the only relevant sense, since <u>the figure is inscribed not on the Euclidean plane but on 2-dimensional spacetime</u>. Notice that tangents to isometric curves remain in each case tangent to the same such curve. The entire population of isometric curves (see again Figure 54) can be recovered as the population of envelopes of the grid lines, as generated by allowing β to range over all allowed values* $(-1 < \beta < +1)$.

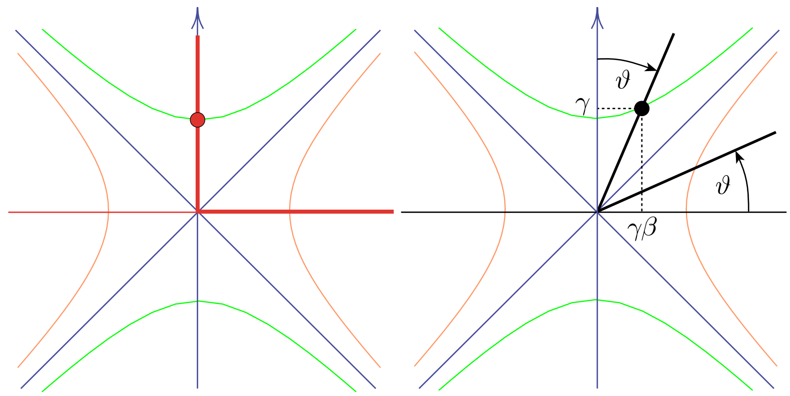

FIGURE 57: *O writes $(ct, 0)$ to describe the "t^{th} tick of her clock." Working from (201) we find that O assigns coordinates $(\gamma t, \gamma\beta t)$ to that same event. The implication is that the (Euclidean) angle ϑ subtended by*

- *O's time axis and*
- *O's representation of O's time axis*

can be described

$$\tan\vartheta = \beta$$

The same angle, by a similar argument, arises when one looks to O's representation of O's space axis. One could, with this information, construct the instance of Figure 56 which is appropriate to any prescribed β-value. Again I emphasize that—their Euclidean appearance notwithstanding—O and O are in agreement that O's coordinate axes are normal in the Lorentzian sense.[118]

We are in position now to **four points of fundamental physical significance**, of which three are temporal, and one spatial. The points I have in mind will be presented in a series of figures, and developed in the captions:

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[118] PROBLEM 44.

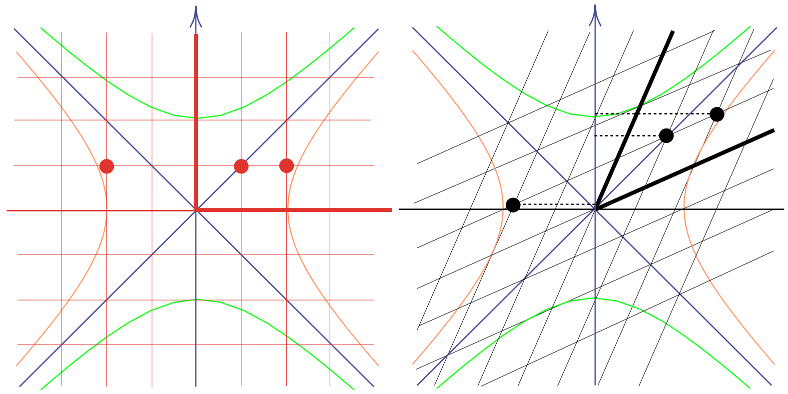

FIGURE 58: **Breakdown of non-local simultaneity.** *O sees three spatially-separated events to be simultaneous. O, on the other hand, assigns distinct x^0-coordinates to those same events (see the figure on the right), which he considers to be non-simultaneous/sequential. It makes relativistic good sense to use the word "simultaneous" only in reference to events which (like the birth of twins) occur at the same moment and at the same spatial point. The Newtonian concept of* "instantaneous action at a distance"*—central to his "Universal Law of Gravitation" but which, on philosophical grounds, bothered not only Newton's contemporaries but also Newton himself —has been rendered relativistically untenable: interactions, in any relativistically coherent physics, have become necessarily local, dominated by what philosophers call the "Principle of Contiguity." They have, in short, become collision-like events, the effects of which propagate like a contagion: neighbor infects neighbor. If "particles" are to participate in collisions they must necessarily be held to be pointlike in the mathematical sense (a hard idealization to swallow), lest one acquire an obligation to develop a physics of processes* interior *to the particle. The language most natural to physics has become field theory—a theory in which all interactions are local field-field interactions, described by partial differential equations.*

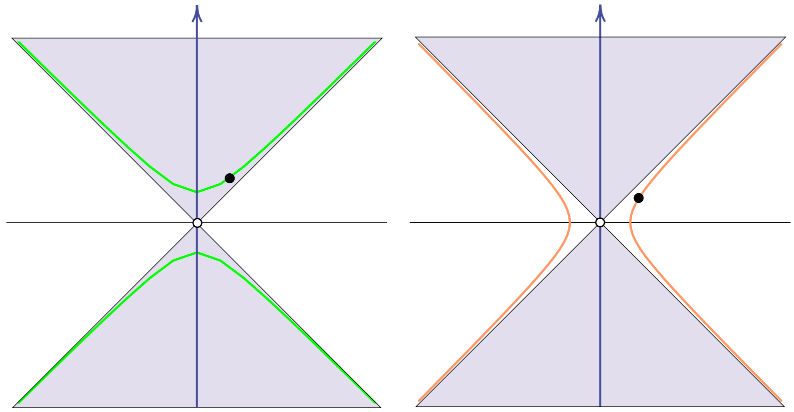

FIGURE 59: **Conditional covariance of causal sequence.** *At left: diverse inertial observers all place the event • on a sheet of the isometric hyperboloid that is confined to the interior of the forward lightcone, and all agree that • lies "in the future" of the origin ◦. But if (as at the right) • is separated from ◦ by a spacelike interval; i.e., if • lies outside the lightcone at ◦, then some observers see • to lie in the future of ◦, while other observers see • to lie in its past. In the latter circumstance it is impossible to develop an agreed-upon sense of causal sequence. Generally: physical events at a point* **p** *can be said to have been "caused" only by events that lie in/on the lightcone that extends backward from* **p**, *and can themselves influence only events that lie in/on the lightcone that extends forward from* **p**. *In electrodynamics it will emerge that (owing to the absence of "photon mass terms") effects propagate* <u>on</u> *the lightcone. Recent quantum mechanical experiments (motivated by the "EPR paradox") are of great interest because they have yielded results that appear to be "acausal" in the sense implied by preceding remarks: the outcome of a quantum coin-flip at* **p** *predetermines the result of a similar measuremennt at* **q** even though the interval separating **q** from **p** is spacelike.

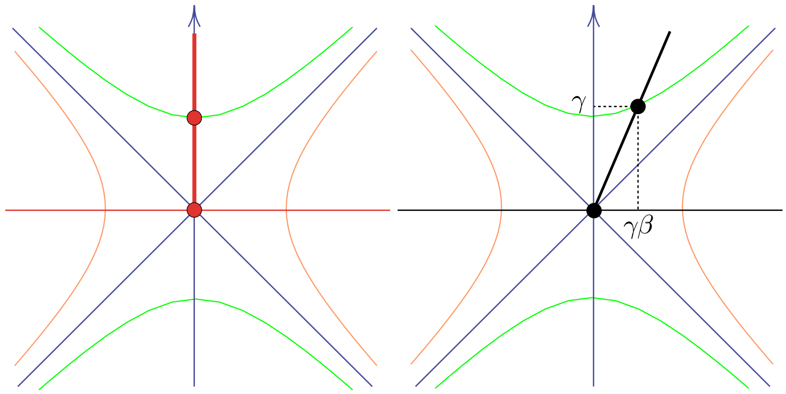

FIGURE 60: **Time dilation.** *Inertial observer O assigns duration x^0 to the interval separating "successive ticks $\bullet \ldots \bullet$ of her clock." A second observer O, in motion relative to O, assigns to those same events (see again Figure 57) the coordinates*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} = \begin{pmatrix} \gamma\, x^0 \\ \gamma\beta\, x^0 \end{pmatrix}$$

He assigns the same Lorentzian value to the squared length of the spacetime interval $\bullet \ldots \bullet$ that O assigned to $\bullet \ldots \bullet$

$$(\gamma\, x^0)^2 - (\gamma\beta\, x^0)^2 = (x^0)^2 - (0)^2$$

but reports that the 2^{nd} *tick occurred at time*

$$x^0 = \gamma\, x^0 > x^0$$

In an example discussed in every text (see, e.g., Taylor & Wheeler, Spacetime Physics (1966), §42) *the "ticking" is associated with the lifetime of an unstable particle—typically a muon—which (relative to the tabulated rest-frame value) seems dilated to observers who see the particle to be in motion.*

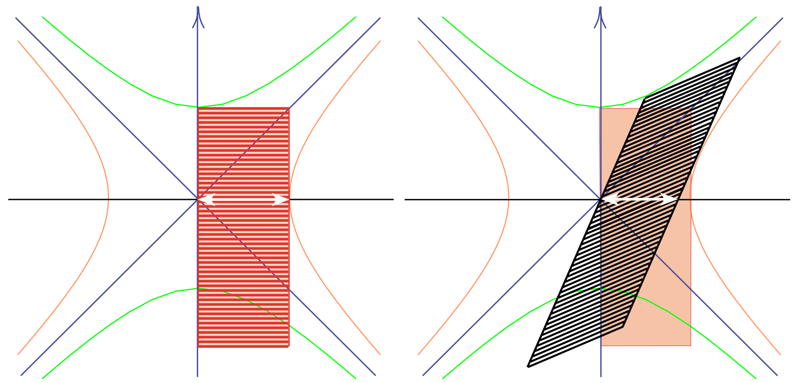

FIGURE 61: **Lorentz contraction.** *This is often looked upon as the flip side of time dilation, but the situation as it pertains to* spatial *intervals is—owing to the fact that <u>metersticks persist, and are therefore not precise analogs of clockticks</u>—a bit more subtle. At left is O's representation of a meterstick sitting there, sitting there, sitting there . . . and at right is O's representation of that same construction. The white arrows indicate that while O and O have the same thought in mind when they talk about the "length of the meterstick" (length of the spatial interval that separates one end from the other <u>at an instant</u>) they are—because they assign distinct meanings to "at an instant"—actually talking about different things. Detailed implications are developed in the following figure.*

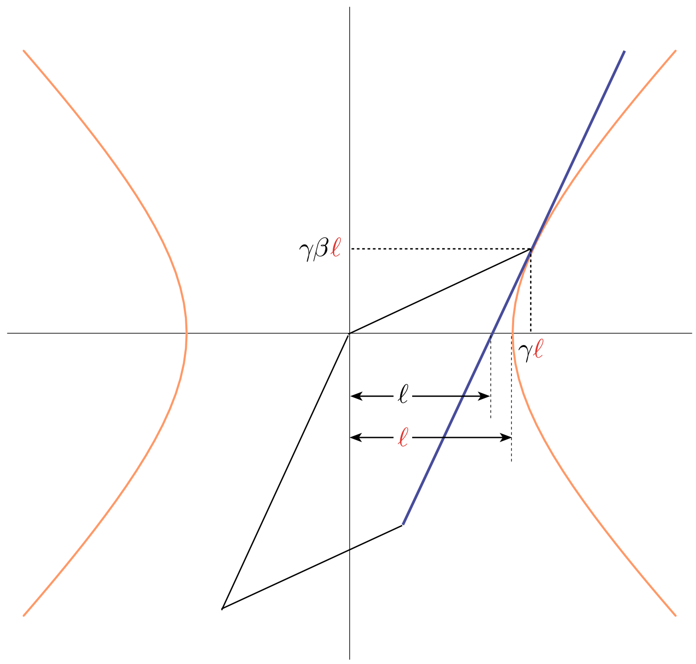

FIGURE 62: **Lorentz contraction (continued).** *When observers speak of the "length of a meterstick" they are really talking about what they perceive to be the width of the "ribbon" which such an extended object inscribes on spacetime. This expanded detail from the preceding figure shows how it comes about that the meterstick which* O *sees to be at rest, and to which she assigns length* ℓ*, is assigned length*

$$\ell = \gamma^{-1}\ell < \ell$$

by O*, who sees the meterstick to be in uniform motion. This familiar result poses, by the way, a problem which did not escape Einstein's attention, and which contributed to the development of general relativity*: The circumference of a rigidly rotating disk has become too short to go all the way around![119]

Prior to Einstein's appearance on the scene (1905) it was universally held that time dilation and "Lorentz-FitzGerald contraction" were *physical* effects, postulated to account for the null result of the Michelson-Morley experiment, and attributed to the interaction of physical clocks and physical metersticks with the physical "æther" through which they were being transported. Einstein

[119] See J. Stachel, "Einstein and the rigidly rotating disk" in A. Held (editor), *General Relativity & Gravitation* (1980), Volume 1, page 1. H. Arzeliès, in *Relativistic Kinematics* (1966), devotes an entire chapter to the disk problem and its relatives.

(with his trains and lanterns) argued that such effects are not "physical," in the sense that they have to do with the properties of "stuff". . . but "metaphysical" (or should one say: *pre*-physical?)—artifacts of the *operational procedures by which one assigns meaning* to lengths and times. In preceding pages I have, in the tradition established by Minkowski, espoused a third view: I have represented all such effects are reflections of the circumstance (brought first to our attention by electrodynamics) that *the hyperbolic geometry of spacetime is a primitive fact of the world, embraced by all inertial observers . . . and written into the design of all possible physics*.

> REMARK: It would be nice if things were so simple (which in leading approximation they are), but when we dismissed Newton's Law of Universal Gravitation as "relativistically untenable" we acquired a question ("How did the Newtonian theory manage to serve so well for so long?") and an obligation—the development of a "field theory of gravitation." The latter assignment, as discharged by Einstein himself, culminated in the invention of "general relativity" and the realization that it is—except in the approximation that gravitational effects can be disregarded—*incorrect* to speak with global intent about the "hyperbolic geometry of spacetime." The "geometry of spacetime" is "hyperbolic" only in the same approximate/tangential sense that vanishingly small regions inscribed on (say) the unit sphere become "Euclidean."

6. Lorentz transformations in 4-dimensional spacetime. The transition from toy 2-dimensional spacetime to physical 4-dimensional spacetime poses an enriched algebraic problem

$$\Lambda = \exp\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \tag{196.1}$$

$$\big\downarrow$$

$$\Lambda = \exp\begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ A_1 & 0 & a_3 & -a_2 \\ A_2 & -a_3 & 0 & a_1 \\ A_3 & a_2 & -a_1 & 0 \end{pmatrix} \tag{193}$$

and brings to light a physically-important point or two which were overlooked by Einstein himself. The algebraic details are, if addressed with a measure of elegance, of some intrinsic interest[120]. . . but I must here be content merely to outline the most basic facts, and to indicate their most characteristic kinematic/physical consequences. Consider first the

[120] See ELEMENTS OF RELATIVITY (1966).

$\boxed{\text{CASE } A_1 = A_2 = A_3 = 0}$ in which $\mathbb{\Lambda}$ possesses only space/space generators.[121] Then

$$\mathbb{\Lambda} = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathbb{A} & \\ 0 & & & \end{pmatrix}$$

where

$$\mathbb{A} \equiv \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \text{ is real and antisymmetric}$$

It follows quite easily that

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathbb{R} & \\ 0 & & & \end{pmatrix} \tag{208}$$

where $\mathbb{R} \equiv e^{\mathbb{A}}$ is a 3×3 *rotation matrix*. The action of such a $\mathbb{\Lambda}$ can be described

$$\begin{pmatrix} x^0 \\ \boldsymbol{x} \end{pmatrix} \longrightarrow \begin{pmatrix} x^0 \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} x^0 \\ \mathbb{R}\boldsymbol{x} \end{pmatrix}$$

as a spatial rotation that leaves time coordinates unchanged. Look to the case $a_1 = a_2 = 0,\ a_3 = \phi$ and use the *Mathematica* command `MatrixExp[Λ]` to obtain

$$\mathbb{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with the evident implication that in the general case such a Lorentz matrix describes a lefthanded rotation through angle $\phi = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$ about the unit vector $\boldsymbol{\lambda} \equiv \hat{\boldsymbol{a}}$.[122] Such Lorentz transformations contain no allusion to $\boldsymbol{v}$ and have *no properly kinematic significance*: O simply stands beside us, using her clock (indistinguishable from ours) and her rotated Cartesian frame to "do physics." What we have learned is that

Spatial rotations are Lorentz transformations

of a special type (a type for which the 2-dimensional theory is too impoverished to make provision). The associated Lorentz matrices will be notated $\mathbb{R}(\phi, \boldsymbol{\lambda})$.

Look next to the complementary . . .

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[121] "Time/time" means ^{0} appears twice, "time/space" and "space/time" mean that ^{0} appears once, "space/space" means that ^{0} is absent.

[122] See CLASSICAL DYNAMICS (1964/65), Chapter 1, pages 83–89 for a simple account of the detailed argument.

$\boxed{\text{CASE } a_1 = a_2 = a_3 = 0}$ in which $\mathbb{A}$ possesses only time/space generators. Here (as it turns out) $\mathbb{A}$ *does* possess kinematic significance. The argument which (on page 139) gave

$$A = \tanh^{-1}\beta \quad \text{with} \quad \beta = v/c$$

now gives

$$\boldsymbol{A} = \tanh^{-1}\beta \cdot \hat{\boldsymbol{v}}$$

while the argument which (on pages 138–139) gave

$$\mathbb{A} = \exp\left\{\tanh^{-1}\beta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\} = \begin{pmatrix}\gamma & v\gamma/c\\ v\gamma/c & \gamma\end{pmatrix}$$

now gives

$$\mathbb{A} = \exp\left\{\tanh^{-1}\beta\begin{pmatrix}0 & \hat{v}_1 & \hat{v}_2 & \hat{v}_3\\ \hat{v}_1 & 0 & 0 & 0\\ \hat{v}_2 & 0 & 0 & 0\\ \hat{v}_3 & 0 & 0 & 0\end{pmatrix}\right\}$$
$$= \begin{pmatrix}\gamma & v_1\gamma/c & v_2\gamma/c & v_3\gamma/c\\ v_1\gamma/c & 1+(\gamma-1)v_1v_1/v^2 & (\gamma-1)v_1v_2/v^2 & (\gamma-1)v_1v_3/v^2\\ v_2\gamma/c & (\gamma-1)v_2v_1/v^2 & 1+(\gamma-1)v_2v_2/v^2 & (\gamma-1)v_2v_3/v^2\\ v_3\gamma/c & (\gamma-1)v_3v_1/v^2 & (\gamma-1)v_3v_2/v^2 & 1+(\gamma-1)v_3v_3/v^2\end{pmatrix}$$

Such Lorentz matrices will be notated

$$= \mathbb{A}(\boldsymbol{\beta}) \tag{209}$$

$$\boldsymbol{\beta} \equiv \boldsymbol{v}/c$$

They give rise to Lorentz transformations $x \longrightarrow x = \mathbb{A}(\boldsymbol{\beta})x$ which are "pure" (in the sense "rotation-free") and are called "boosts." The construction (208) *looks* complicated, but in fact it possesses *precisely the structure that one might (with a little thought) have anticipated*. For (209) supplies[123]

$$\left.\begin{aligned} t &= \gamma t + (\gamma/c^2)\,\boldsymbol{v}\cdot\boldsymbol{x}\\ \boldsymbol{x} &= \boldsymbol{x} + \left\{\gamma t + (\gamma-1)\,(\boldsymbol{v}\cdot\boldsymbol{x})/v^2\right\}\boldsymbol{v}\end{aligned}\right\} \tag{210.1}$$

and if we resolve $\boldsymbol{x}$ and $\boldsymbol{x}$ into components which are parallel/perpendicular to the velocity $\boldsymbol{v}$ with which O sees O to be gliding by

$$\boldsymbol{x} = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel \quad \text{with} \quad \begin{cases}\boldsymbol{x}_\parallel \equiv (\boldsymbol{x}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}} \equiv x_\parallel\hat{\boldsymbol{v}}\\ \boldsymbol{x}_\perp \equiv \boldsymbol{x} - \boldsymbol{x}_\parallel\end{cases}$$

$$\boldsymbol{x} = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel \quad \text{with} \quad \begin{cases}\boldsymbol{x}_\parallel \equiv (\boldsymbol{x}\cdot\hat{\boldsymbol{v}})\,\hat{\boldsymbol{v}} \equiv x_\parallel\hat{\boldsymbol{v}}\\ \boldsymbol{x}_\perp \equiv \boldsymbol{x} - \boldsymbol{x}_\parallel\end{cases}$$

---

[123] PROBLEM 45, 46.

then (210.1) can be written (compare (203))

$$\left.\begin{aligned}\begin{pmatrix} t \\ x_\parallel \end{pmatrix} &= \gamma\begin{pmatrix} 1 & v/c^2 \\ v & 1 \end{pmatrix}\begin{pmatrix} t \\ x_\parallel \end{pmatrix} \\ \boldsymbol{x}_\perp &= \boldsymbol{x}_\perp \end{aligned}\right\} \tag{210.2}$$

And in the Galilean limit we recover

$$\begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{210.3}$$

GENERAL CASE Having discussed the 3-parameter family of rotations $\mathbb{R}(\phi, \boldsymbol{\lambda})$ and the 3-parameter family of boosts $\Lambda(\boldsymbol{\beta})$ the questions arises: *What can one say in the general 6-parameter case*

$$\Lambda = e^{\mathbb{A}}$$

It is—given the context in which the question was posed—natural to write

$$\mathbb{A} = \mathbb{J} + \mathbb{K}$$

with

$$\mathbb{J} \equiv \begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ A_3 & 0 & 0 & 0 \end{pmatrix} \equiv \sum_{i=1}^{3} A_i\, \mathbb{J}_i$$

$$\mathbb{K} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & -a_2 \\ 0 & -a_3 & 0 & a_1 \\ 0 & a_2 & -a_1 & 0 \end{pmatrix} \equiv \sum_{i=1}^{3} a_i\, \mathbb{K}_i$$

and one might on this basis be tempted to write $\Lambda = e^{\mathbb{K}} \cdot e^{\mathbb{J}}$, giving

$$\Lambda_{\text{general}} = (\text{rotation}) \cdot (\text{boost}) \tag{211}$$

Actually, a representation theorem of the form (211) <u>is</u> available, but the argument which here led us to (211) is incorrect: one can write

$$e^{\,\mathbb{J}+\mathbb{K}} = e^{\,\mathbb{K}} \cdot e^{\,\mathbb{J}} \quad \text{if and only if } \mathbb{J} \text{ and } \mathbb{K} \ \underline{\text{commute}}$$

and in the present instance we (by computation) have

$$\begin{aligned}\big[\mathbb{J}, \mathbb{K}\big] &= -\sum_{i=1}^{3} (\boldsymbol{A} \times \boldsymbol{a})_i\, \mathbb{J}_i \\ &= \mathbb{O} \quad \text{if and only if } \boldsymbol{A} \text{ and } \boldsymbol{a} \text{ are } \underline{\text{parallel}}\end{aligned} \tag{212}$$

More careful analysis (which requires some fairly sophisticated algebraic machinery[124]) leads back again to (211), but shows the boost and rotational factors of $\mathbb{A}$ to be different from those initially contemplated. I resist the temptation to inquire more closely into the correct factorization of $\mathbb{A}$, partly because I have other fish to fry ... but mainly because I have already in hand the facts needed to make my major point, which concerns the *composition of boosts in 4-dimensional spacetime*. It follows immediately from (208) that

$$(\text{rotation})\cdot(\text{rotation}) = (\text{rotation}) \tag{213.1}$$

↑ specific description poses a non-trivial but merely technical (algebraic) problem

It might—on analogical grounds—appear plausible therefore that

$$(\text{boost})\cdot(\text{boost}) = (\text{boost})$$

but (remarkably!) <u>this is not the case</u>: actually

$$= (\underline{\text{rotation}})\cdot(\text{boost}) \tag{213.2}$$

Detailed calculation shows more specifically that

$$\mathbb{A}(\boldsymbol{\beta}_2)\cdot\mathbb{A}(\boldsymbol{\beta}_1) = \mathbb{R}(\phi,\boldsymbol{\lambda})\,\mathbb{A}(\boldsymbol{\beta}) \tag{214.0}$$

where

$$\boldsymbol{\beta} = \frac{\big[1+(\beta_2/\beta_1)(1-\frac{1}{\gamma_1})\cos\omega\big]\boldsymbol{\beta}_1 + \frac{1}{\gamma_1}\boldsymbol{\beta}_2}{1+\beta_1\beta_2\cos\omega} \tag{214.1}$$

$$\boldsymbol{\lambda} = \text{unit vector parallel to } \boldsymbol{\beta}_2\times\boldsymbol{\beta}_1 \tag{214.2}$$

$$\omega = \text{angle between } \boldsymbol{\beta}_1 \text{ and } \boldsymbol{\beta}_2 \tag{214.3}$$

$$\phi = \tan^{-1}\left\{\frac{\epsilon\sin\omega}{1+\epsilon\cos\omega}\right\} \tag{214.4}$$

$$\epsilon = \sqrt{(\gamma_1-1)(\gamma_2-1)/(\gamma_1+1)(\gamma_2+1)} \tag{214.5}$$

and where β_1, β_2, γ_1 and γ_2 have the obvious meanings. One is quite unprepared by 2-dimensional experience for results which are superficially so ugly, and which are undeniably so complex. The following points should be noted:

1. Equation (214.1) is the 4-dimensional *velocity addition formula*. Looking with its aid to $\boldsymbol{\beta}\cdot\boldsymbol{\beta}$ we obtain the *speed addition formula*

$$\beta = \frac{\sqrt{\beta_1^2+\beta_2^2+2\beta_1\beta_2\cos\omega-(\beta_1\beta_2\sin\omega)^2}}{1+\beta_1\beta_2\cos\omega} \tag{215}$$

$$\Downarrow$$

$$\beta \leqslant 1 \quad \text{if } \beta_1\leqslant 1 \text{ and } \beta_2 \leqslant 1$$

according to which (see the following figure) *one cannot, by composing velocities, escape from the c-ball*. Note also that

$$\downarrow$$

$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \quad \text{in the collinear case: } \omega = 0$$

[124] The requisite machinery is developed in elaborate detail in ELEMENTS OF SPECIAL RELATIVITY (1966).

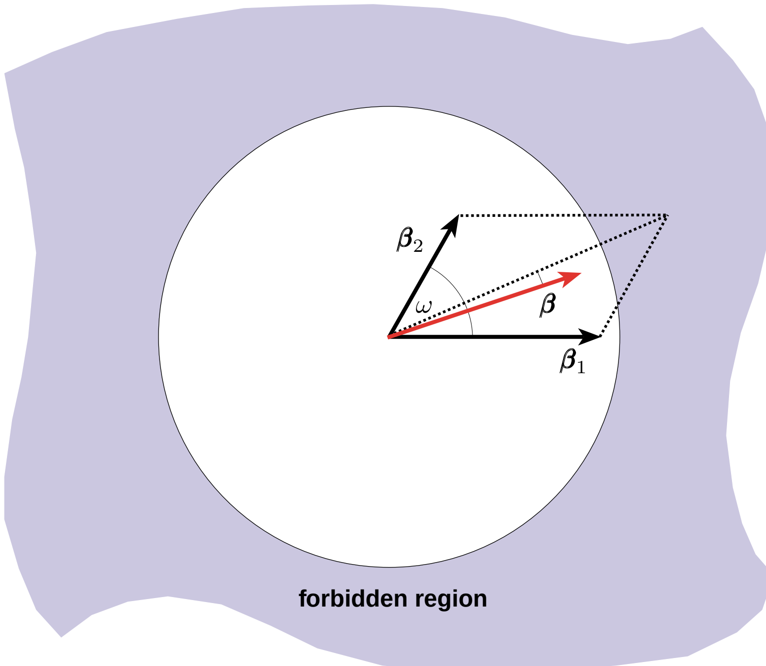

FIGURE 63: *$\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, if not collinear, span a plane in 3-dimensional $\boldsymbol{\beta}$-space. The figure shows the intersection of that plane with what I call the "c-ball," defined by the condition $\beta^2 = 1$. The placement of $\boldsymbol{\beta}$ is given by (214.1). Notice that, while $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$ falls into the forbidden exterior of the c-ball, $\boldsymbol{\beta}$ does not. Notice also that $\boldsymbol{\beta}$ lies on the $\boldsymbol{\beta}_1$-side of $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$, from which it deviates by an angle that turns out to be precisely the ϕ that enters into the design of the rotational factor $\mathbb{R}\,(\phi, \boldsymbol{\lambda})$.*

which is in precise conformity with the familiar 2-dimensional formula (204).

2. It is evident in (214.1) that $\boldsymbol{\beta}$ depends *asymmetrically* upon $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. Not only is $\boldsymbol{\beta} \neq \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$, is its *not even parallel* to $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2$, from which it deviates by an angle that turns out to be precisely the ϕ encountered already—in quite another connection—at (214.4). The asymmetry if the situation might be summed up in the phrase "$\boldsymbol{\beta}_1$ predominates." From this circumstance one acquires interest in the angle Ω between $\boldsymbol{\beta}$ and $\boldsymbol{\beta}_1$: we find

$$
\begin{aligned}
\Omega\ &= \tan^{-1}\left\{\frac{\beta_2 \sin\omega}{\gamma_1(\beta_1 + \beta_2\cos\omega)}\right\} \qquad (216)\\
&\downarrow\\
\Omega_0 &= \tan^{-1}\left\{\frac{\beta_2 \sin\omega}{\beta_1 + \beta_2\cos\omega}\right\} \quad \text{in the non-relativistic limit}
\end{aligned}
$$

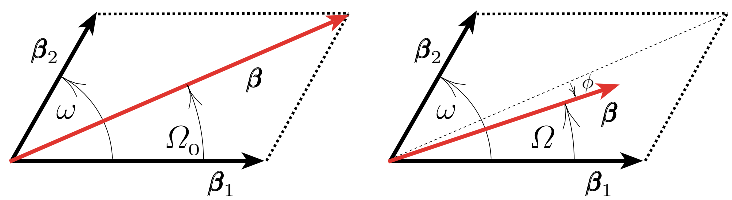

FIGURE 64: *At left: Galilean composition of non-collinear velocities. At right: its Lorentzian counterpart, showing the sense in which "$\boldsymbol{\beta}_1$ predominates." Evidently*

$$\Omega_{\text{relativistic}} = \Omega_0 + \phi \leqslant \Omega_0$$

calculations which are elementary in the Galilean case (see the figure) but become a little tedious in the relativistic case.[125] Asymmetry effects become most pronounced in the ultra-relativistic limit. Suppose, for example, that $\beta_1 = 1$: then $\Omega \downarrow 0$ and

$$\boldsymbol{\beta} \to \boldsymbol{\beta}_1, \text{irrespective of the value assigned to } \boldsymbol{\beta}_2!$$

More physically,[126] suppose $\beta_1 < 1$ but $\beta_2 = 1$: then

$$\Omega = \tan^{-1}\left\{\sqrt{1-\beta_1^2}\,\frac{\sin\omega}{\beta_1 + \cos\omega}\right\}$$

The first occurrence of this formula is in §7 of Einstein's first relativity paper (1905), where it is found to provide the relativistic correction to the classic "law of aberration."[127]

3. It is a corollary of (215) that

$$\gamma = \gamma_1\gamma_2\big[1 + \beta_1\beta_2\cos\omega\big]$$

which gives back (205) in the collinear case.

---

[125] See page 87 in the notes just cited.

[126] I say "more physically" because $\beta = 1$ cannot pertain to an "observer" (though it can pertain to the flight of a massless particle): while it does make sense to ask what an observer in motion (with respect to us) has to say about the lightbeam to which we assign a certain direction of propagation, it makes no sense to ask what the lightbeam has to say about the observer!

[127] "Aberration" is the name given by astronomers to the fact that "fixed stars" are seen to trace small ellipses in the sky, owing to the earth's annual progress along its orbit. See page 17 in W. Pauli's classic *Theory of Relativity* (first published in 1921, when Pauli was only twenty-one years old; reissued with a few additional notes in 1958) or P. G. Bergmann, *Introduction to the Theory of Relativity* (1942), pages 36–38.

4. In the small-velocity approximation (213.1) and (213.4) give

$$\begin{aligned}\boldsymbol{v} &= \boldsymbol{v}_1 + \boldsymbol{v}_2 - \Big[\tfrac{1}{2}\beta_1\beta_2\cos\omega\cdot\boldsymbol{v}_1 + \big(\tfrac{1}{2}\beta_1^2 + \beta_1\beta_2\cos\omega\big)\boldsymbol{v}_2\Big] + \cdots\\ \phi &= \tfrac{1}{4}\beta_1\beta_2\sin\omega + \cdots\end{aligned}$$

according to which all "relativistic correction terms" are of 2^{nd} order.

The presence of the $\mathbb{R}$-factor on the right side of (213)—*i.e.*, the fact that *rotations arise when one composes non-collinear boosts*—can be traced to the following algebraic circumstance:

$$\big[\,\mathbb{J}_1, \mathbb{K}_2\big] = -\mathbb{J}_3 = \big[\,\mathbb{J}_2, \mathbb{K}_1\big] \tag{217.1}$$

$$\big[\mathbb{K}_1, \mathbb{K}_2\big] = -\mathbb{K}_3 \tag{217.2}$$

$$\big[\,\mathbb{J}_1,\ \mathbb{J}_2\big] = +\mathbb{K}_3 \tag{217.3}$$

—each of which remains valid under cyclic index permutation. Equations (217.1) are but a rewrite of (212). The compositional closure (213.1) to the rotations can be attributed to the fact that it is a $\mathbb{K}$ that stands on the right side of (217.2). The fact (213.2) that *the set of boosts is not compositionally closed* arises from the circumstance that it is again a $\mathbb{K}$—not, as one might have expected, a $\mathbb{J}$—that stands on right side of (217.3).

The essential presence of the rotational $\mathbb{R}$-factor on the right side of (214) was discovered by L. H. Thomas (1926: relativity was then already 21 years old), whose motivation was not mathematical/kinematic, but intensely physical: Uhlenbeck & Goudsmit had sought (1925) to derive fine details of the hydrogen spectrum from the assumption that the electron in the Bohr atom possesses intrinsic "spin"... but had obtained results which were invariably off by a factor of 2. Thomas—then a post-doctoral student at the Bohr Institute, and for reasons to which I will return in a moment—speculated that a "relativistic correction" would resolve that problem. Challenged by Bohr to develop the idea (for which neither Bohr nor his associate Kramers held much hope), Thomas "that weekend" argued as follows: (i) A proton •, pinned to the origin of an inertial frame, sees an electron • to be revolving with angular velocity Ω_{orbital} on a circular Bohr orbit of radius R. (ii) Go to the frame of the non-inertial observer who is "riding on the electron" (and therefore sees • to be in circular motion): do this by

> going to the frame of the inertial observer who is instantaneously comoving with • at time $t_0 = 0$, then...
>
> **boosting** to the frame of the inertial observer who is instantaneously comoving with • at time $t_1 = \tau$, then...
>
> **boosting** to the frame of the inertial observer who is instantaneously comoving with • at time $t_2 = 2\tau$, then...
>
> ⋮
>
> **boosting** to the frame of the inertial observer who is instantaneously comoving with • at time $t = N\tau$

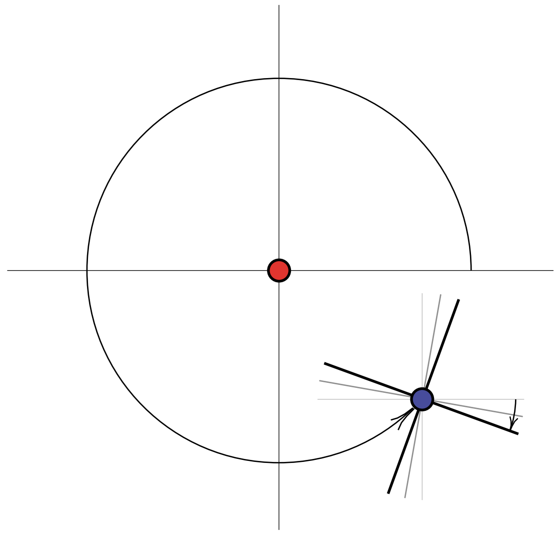

FIGURE 65: *Thomas precession of the non-inertial frame of an observer* • *in circular orbit about an inertial observer* •. *In celestial mechanical applications the effect is typically so small (on the order of seconds of arc per century) as to be obscured by dynamical effects. But in the application to (pre-quantum mechanical) atomic physics that was of interest to Thomas the precession becomes quite brisk (on the order of* $\sim 10^{12}$ *Hz.).*

and by taking that procedure to the limit $\tau \downarrow 0$, $N = t/\tau \uparrow \infty$. One arrives thus at method for *Lorentz transforming to the frame of an accelerated observer*. The curvature of the orbit means, however, that successive boosts are not collinear; rotational factors intrude at each step, and have a cumulative effect which (as detailed analysis[128] shows) can be described

$$\begin{aligned}\tfrac{d\phi}{dt} \equiv \Omega_{\text{Thomas}} &= (\gamma - 1)\Omega_{\text{orbital}} \\ &= \tfrac{1}{2}\beta^2\Omega_{\text{orbital}}\big\{1 + \tfrac{3}{4}\beta^2 + \tfrac{15}{24}\beta^4 + \cdots\big\}\end{aligned}$$

in the counterrotational sense (see the figure). It is important to notice that this *Thomas precessional effect* is of relativistic kinematic origin: it does not

[128] See §103 in E. F. Taylor & J. A. Wheeler, *Spacetime Physics* (1963) or pages 95–116 in the notes previously cited.[122] Thomas' own writing—"The motion of the spinning electron," Nature **117**, 514 (1926); "The kinematics of an electron with an axis," Phil. Mag. **3**, 1 (1927); "Recollections of the discovery of the Thomas precessional frequency" in G. M. Bunce (editor), *High Energy Spin Physics*–1982,AIP Conference Proceedings No. 95 (1983)—have never seemed to me to be particularly clear. See also J. Frenkel, "Die Elektrodynamic des rotierenden Elektrons," Z. für Physik **37**, 243 (1926).

arise from impressed forces. (*iii*) Look now beyond the kinematics to the dynamics: from •'s viewpoint the revolving • is, in effect, a current loop, the generator of a magnetic field $\boldsymbol{B}$. Uhlenbeck & Goudsmit had assumed that the electron possesses a magnetic moment proportional to its postulated spin: such an electron senses the $\boldsymbol{B}$-field, to which it responds by precessing, acquiring precessional energy $\mathcal{E}_{\text{Uhlenbeck \& Goudsmit}}$. Uhlenbeck & Goudsmit worked, however, from a mistaken conception of "•'s viewpoint." The point recognized by Thomas is that when relativistic frame-precession is taken into account[129] one obtains

$$\mathcal{E}_{\text{Thomas}} = \tfrac{1}{2}\mathcal{E}_{\text{Uhlenbeck \& Goudsmit}}$$

—in good agreement with the spectroscopic data. This was a discovery of historic importance, for it silenced those (led by Pauli) who had dismissed as "too classical" the spin idea when it had been put forward by Krönig and again, one year later, by Uhlenbeck & Goudsmit: *"spin" became an accepted/ fundamental attribute of elementary particles*.[130]

So much for the structure and properties of the Lorentz transformations ... to which (following more closely in Minkowski's footsteps than Lorentz') we were led by analysis of the condition

$$\mathbb{\Lambda}^{\mathsf{T}} \mathbb{g}\, \mathbb{\Lambda} = \mathbb{g} \quad \text{everywhere} \tag{182}$$

which arose from <u>one natural interpretation</u> of the requirement that $\mathcal{X} \rightarrow \mathcal{X}$ preserve the form of Maxwell's equations ... but to which Einstein himself was led by quite other considerations: Einstein—recall his trains/clocks/rods and lanterns—proceeded by operational/epistemological analysis of how inertial observers O and O, consistently with the most primitive principles of an idealized macroscopic physics, would *establish* the relationship between their coordinate systems. Einstein's argument was wonderfully original, and lent an air of "inescapability" to his conclusions ... but (in my view) *must today be dismissed as irrelevant*, for <u>special relativity appears to remain effective in the</u>

---

[129] See pages 116–122 in ELEMENTS OF RELATIVITY (1966).

[130] Thomas precession is a relativistic effect which 2-dimensional theory is too impoverished to expose. Einstein himself missed it, and—so far as I am aware—never commented in print upon Thomas' discovery. Nor is it mentioned in Pauli/s otherwise wonderfully complete *Theory of Relativity*.[125] In 1969 I had an opportunity to ask Thomas himself how he had come upon his essential insight. He responded "Nothing is ever really new. I learned about the subject from Eddington's discussion [Eddington was in fact one of Thomas' teachers] of the relativistic dynamics of the moon—somewhere in his relativity book, which was then new. I'm sure the whole business—except for the application to Bohr's atom—was known to Eddington by 1922. Eddington was a smart man." Arthur Stanley Eddington's *The Mathematical Theory of Relativity* (1922) provided the first English-language account of general relativity. The passage to which Thomas evidently referred occurs in the middle of page 99 in the 2nd edition (1954), and apparently was based upon then-recent work by W. De Sitter.

<u>deep microscopic realm</u> where Einstein's operational devices/procedures (his "trains and lanterns") are—for quantum mechanical reasons—meaningless. Einstein built better than he knew—or could know . . . but I'm ahead of my story. <u>The Lorentz transformations enter into the statement of—but do not in and of themselves comprise—special relativity</u>. The "meaning of relativity" is a topic to which I will return in §8.

7. Conformal transformations in N-dimensional spacetime.* We have seen that a second—and hardly less natural—interpretation of "Lorentz' question" gives rise not to (182) but to a condition of the form

$$\mathbb{W}^{\mathsf{T}}\mathscr{g}\,\mathbb{W} = \Omega\,\mathscr{g} \quad \text{everywhere} \tag{185.2}$$

where (as before)

$$\mathscr{g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

My objective here is to describe the specific structure of the transformations $\mathcal{X} \rightarrow \mathcal{X}$ which arise from (185.2).

We begin as we began on page 132 (though the argument will not *not* lead to a proof of enforced linearity). If (185.2) is written

$$g_{\alpha\beta}W^\alpha{}_\mu W^\beta{}_\nu = g_{\mu\nu} \tag{218}$$

then (since the elements of $\mathscr{g}$ are constants) application of ∂_λ gives

$$g_{\alpha\beta}W^\alpha{}_{\lambda\mu} W^\beta{}_\nu + g_{\alpha\beta}W^\alpha{}_\mu W^\beta{}_{\nu\lambda} = g_{\mu\nu}\Omega_\lambda \tag{219}$$

where $W^\alpha{}_{\lambda\mu} \equiv \partial_\lambda W^\alpha{}_\mu = \partial^2 x^\alpha/\partial x^\lambda \partial x^\mu$ and $\Omega_\lambda \equiv \partial_\lambda\Omega$. Let functions $\Gamma_{\mu\nu\lambda}$ and $\varphi_\lambda \equiv \partial_\lambda\varphi$ be defined—deviously—as follows:

$$\Omega_\lambda \equiv 2\Omega\varphi_\lambda \tag{220}$$

$$g_{\alpha\beta}W^\alpha{}_\mu W^\beta{}_{\nu\lambda} \equiv \Omega\Gamma_{\mu\nu\lambda} \quad : \quad \nu\lambda\text{-symmetric} \tag{221}$$

Then (since the stipulated invertibility of $\mathcal{X} \rightarrow \mathcal{X}$ entails $\Omega = \sqrt{W} \neq 0$) equation (219) becomes

$$\Gamma_{\mu\nu\lambda} + \Gamma_{\nu\lambda\mu} = 2g_{\mu\nu}\varphi_\lambda$$

which by the "cyclic permutation argument" encountered on page 132 gives

$$\Gamma_{\lambda\mu\nu} = g_{\lambda\mu}\varphi_\nu + g_{\lambda\nu}\varphi_\mu - g_{\mu\nu}\varphi_\lambda \tag{222}$$

---

* It is the logic of the overall argument—certainly not pedagogical good sense!—that has motivated me to introduce this material (which will *not* be treated in lecture). First-time readers should skip directly to §7.

Now

$$W^{\alpha}{}_{\mu\nu} = \Gamma_{\lambda\mu\nu}\cdot\underbrace{\Omega M^{\lambda}{}_{\beta}g^{\beta\alpha}}_{=\,g^{\lambda\kappa}W^{\alpha}{}_{\kappa}} \qquad \begin{array}{l}\text{by (221)}\\ \text{by (218)}\end{array}$$

so by (222)

$$= \varphi_{\mu}W^{\alpha}{}_{\nu} + \varphi_{\nu}W^{\alpha}{}_{\mu} - g_{\mu\nu}\cdot g^{\lambda\kappa}\varphi_{\lambda}W^{\alpha}{}_{\kappa} \tag{223}$$

where the $\mu\nu$-symmetry is manifest. More compactly

$$= \Gamma^{\kappa}{}_{\mu\nu}W^{\alpha}{}_{\kappa} \tag{224}$$

where

$$\Gamma^{\kappa}{}_{\mu\nu} \equiv g^{\kappa\lambda}\Gamma_{\lambda\mu\nu}$$

Application of ∂_{λ} to (224) gives $W^{\alpha}{}_{\lambda\mu\nu} = \dfrac{\partial\Gamma^{\kappa}{}_{\mu\nu}}{\partial x^{\lambda}}W^{\alpha}{}_{\kappa} + \Gamma^{\kappa}{}_{\mu\nu}W^{\alpha}{}_{\lambda\kappa}$ which (since $W^{\bullet}{}_{\bullet\bullet}$, $W^{\bullet}{}_{\bullet\bullet\bullet}$ and $\Gamma^{\bullet}{}_{\bullet\bullet}$ are symmetric in their subscripts, and after relabling some indices) can be written

$$\begin{aligned} W^{\alpha}{}_{\lambda\mu\nu} &= \frac{\partial\Gamma^{\beta}{}_{\lambda\nu}}{\partial x^{\mu}}W^{\alpha}{}_{\beta} + \Gamma^{\kappa}{}_{\nu\lambda}\underbrace{W^{\alpha}{}_{\kappa\mu}}_{=\,\Gamma^{\beta}{}_{\kappa\mu}W^{\alpha}{}_{\beta}\quad\text{by (224)}} \\ &= \left\{\frac{\partial\Gamma^{\beta}{}_{\lambda\nu}}{\partial x^{\mu}} + \Gamma^{\beta}{}_{\kappa\mu}\Gamma^{\kappa}{}_{\nu\lambda}\right\}W^{\alpha}{}_{\beta} \end{aligned}$$

from which it follows in particular that

$$\begin{aligned} W^{\alpha}{}_{\lambda\mu\nu} - W^{\alpha}{}_{\lambda\nu\mu} &= \left\{\frac{\partial\Gamma^{\beta}{}_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial\Gamma^{\beta}{}_{\lambda\mu}}{\partial x^{\nu}} + \Gamma^{\beta}{}_{\kappa\mu}\Gamma^{\kappa}{}_{\nu\lambda} - \Gamma^{\beta}{}_{\kappa\nu}\Gamma^{\kappa}{}_{\mu\lambda}\right\}W^{\alpha}{}_{\beta} \\ &\equiv R^{\beta}{}_{\lambda\mu\nu}W^{\alpha}{}_{\beta} \end{aligned} \tag{225}$$

The preceding sequence of manipulations will, I fear, strike naive readers as an unmotivated jumble. But those with some familiarity with patterns of argument standard to differential geometry will have recognized that

- the quantities $W^{\alpha}{}_{\mu}$ transform as components of an α-parameterized set of covariant vectors;
- the quantities $\Gamma^{\kappa}{}_{\mu\nu}$ are components of[131] an *affine connection* to which (222) assigns a specialized structure;
- the α-parameterized equations (224) can be notated

  $$D_{\nu}W^{\alpha}{}_{\mu} \equiv \partial_{\nu}W^{\alpha}{}_{\mu} - W^{\alpha}{}_{\kappa}\Gamma^{\kappa}{}_{\mu\nu} = 0$$

  according to which each of the vectors $W^{\alpha}{}_{\mu}$ has the property that its *covariant derivative*[129] vanishes;
- the 4th rank tensor $R^{\beta}{}_{\lambda\mu\nu}$ defined at (225) is just the *Riemann-Christoffel curvature tensor*,[129] to which a specialized structure has in this instance been assigned by (222).

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[131] See again page 123.

But of differential geometry I will make explicit use only in the following—independently verifiable—facts: let

$$R_{\kappa\lambda\mu\nu} \equiv g_{\kappa\beta}R^{\beta}{}_{\lambda\mu\nu}$$

Then—owing entirely to (*i*) the definition of $R^{\beta}{}_{\lambda\mu\nu}$ and (*ii*) the $\mu\nu$-symmetry of $\Gamma^{\beta}{}_{\mu\nu}$—the tensor $R_{\kappa\lambda\mu\nu}$ possess the following *symmetry properties*:

$$\begin{aligned} R_{\kappa\lambda\mu\nu} &= -R_{\kappa\lambda\nu\mu} &&: \quad \text{antisymmetry on the last pair of indices}\\ &= -R_{\lambda\kappa\mu\nu} &&: \quad \text{antisymmetry on the first pair of indices}\\ &= +R_{\mu\nu\kappa\lambda} &&: \quad \text{supersymmetry}\\ R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} &= 0 &&: \quad \text{windmill symmetry} \end{aligned}$$

These serve to reduce the number of independent components from N^4 to $\frac{1}{12}N^2(N^2-1)$:

N	N^4	$\frac{1}{12}N^2(N^2-1)$
1	1	0
2	16	1
3	81	6
4	256	20
5	625	50
6	1296	105
$\vdots$	$\vdots$	$\vdots$

We will, in particular, need to know that in the 2-dimensional case the only non-vanishing components of $R_{\kappa\lambda\mu\nu}$ are

$$R_{0101} = -R_{0110} = -R_{1001} = +R_{1010}$$

Returning now to the analytical mainstream...

The left side of (225) vanishes automatically, and from the invertibility of $\mathbb{W}$ we infer that

$$R_{\kappa\lambda\mu\nu} = 0 \tag{226}$$

Introducing (222) into (225) we find (after some calculation marked by a great deal of cancellation) that $R_{\kappa\lambda\mu\nu}$ has the correspondingly specialized structure

$$R_{\kappa\lambda\mu\nu} = g_{\kappa\nu}\Phi_{\lambda\mu} - g_{\kappa\mu}\Phi_{\lambda\nu} - g_{\lambda\nu}\Phi_{\kappa\mu} + g_{\lambda\mu}\Phi_{\kappa\nu} \tag{227}$$

where

$$\begin{aligned} \Phi_{\lambda\mu} &\equiv \varphi_{\lambda\mu} - \varphi_\lambda\varphi_\mu + \tfrac{1}{2}g_{\lambda\mu}\cdot(g^{\alpha\beta}\varphi_\alpha\varphi_\beta) \\ \varphi_{\lambda\mu} &\equiv \partial\varphi_\lambda/\partial x^\mu = \partial^2\varphi/\partial x^\lambda\partial x^\mu = \varphi_{\mu\lambda} \end{aligned} \tag{228}$$

entail $\varPhi_{\lambda\mu} = \varPhi_{\mu\lambda}$. It follows now from (227) that

$$R_{\lambda\mu} \equiv R^{\alpha}{}_{\lambda\mu\alpha} = \ (N-2)\,\varPhi_{\lambda\mu} + g_{\lambda\mu}\cdot(g^{\alpha\beta}\varPhi_{\alpha\beta}) \tag{229.1}$$

$$R \equiv R^{\beta}{}_{\beta} = 2(N-1)\cdot g^{\alpha\beta}\varPhi_{\alpha\beta} \tag{229.2}$$

must—in consequence of (226)—both vanish:

$$R_{\lambda\mu} = 0 \tag{230.1}$$

$$R = 0 \tag{230.2}$$

In the $\boxed{\text{CASE } N=2}$ the equations (230) are seen to reduce to a solitary condition

$$g^{\alpha\beta}\varPhi_{\alpha\beta} = 0 \tag{231}$$

which in $\boxed{\text{CASES } N>2}$ becomes a corollary of the stronger condition

$$\varPhi_{\alpha\beta} = 0 \tag{232}$$

This is the **conformality condition** from which we will work. When introduced into (227) it renders (226) automatic.[132]

Note that (220) can be written $\partial_\lambda\varphi \equiv \varphi_\lambda = \partial_\lambda \log\sqrt{\Omega}$ and entails

$$\varphi = \log\sqrt{\Omega} + \text{constant}$$

Returning with this information to (228), the conformality condition (232) becomes

$$\frac{\partial^2 \log\sqrt{\Omega}}{\partial x^\mu \partial x^\nu} - \frac{\partial \log\sqrt{\Omega}}{\partial x^\mu}\frac{\partial \log\sqrt{\Omega}}{\partial x^\nu} + \tfrac{1}{2}g_{\mu\nu}\cdot g^{\alpha\beta}\frac{\partial \log\sqrt{\Omega}}{\partial x^\alpha}\frac{\partial \log\sqrt{\Omega}}{\partial x^\beta} = 0$$

which—if we introduce

$$F \equiv \frac{1}{\sqrt{\Omega}} \tag{233}$$

—can be written

$$\frac{\partial^2 \log F}{\partial x^\mu \partial x^\nu} = \tfrac{1}{2}g_{\mu\nu}\cdot g^{\alpha\beta}\frac{\partial \log F}{\partial x^\alpha}\frac{\partial \log F}{\partial x^\beta} - \frac{\partial \log F}{\partial x^\mu}\frac{\partial \log F}{\partial x^\nu}$$

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[132] When $N=2$ one must, on the other hand, proceed from (231). It is therefore of interest that (231) and (226) are—uniquely in the $\boxed{\text{CASE } N=2}$ —notational variants of the *same* statement . . . for

$$\begin{aligned} R_{0101} &= \text{only independent element}\\ &= g_{01}\varPhi_{10} - g_{00}\varPhi_{11} - g_{11}\varPhi_{00} + g_{10}\varPhi_{01} \quad \text{by (227)}\\ &= -g\cdot g^{\alpha\beta}\varPhi_{\alpha\beta} \quad \text{by} \quad \begin{pmatrix} g^{00} & g^{01}\\ g^{10} & g^{11}\end{pmatrix} = g^{-1}\cdot\begin{pmatrix} g_{11} & -g_{01}\\ -g_{10} & g_{00}\end{pmatrix}\end{aligned}$$

We write out the derivatives and obtain these simpler-looking statements

$$F_{\mu\nu} = g_{\mu\nu} \cdot \frac{g^{\alpha\beta}F_\alpha F_\beta}{2F} \tag{234}$$

where $F_\mu \equiv \partial_\mu F$ and $F_{\mu\nu} \equiv \partial_\mu\partial_\nu F$. The implication is that

$$\begin{aligned} \partial_\nu(g^{\lambda\mu}F_\mu) &= g^{\lambda\mu}F_{\mu\nu} \\ &= \left\{\tfrac{1}{2}\frac{g^{\alpha\beta}F_\alpha F_\beta}{F}\right\}\delta^\lambda{}_\nu \quad : \quad \text{vanishes unless } \nu = \lambda \end{aligned}$$

which is to say: $g^{\lambda\mu}F_\mu$ is a function only of x^λ. But $\mathbb{g}$ is, by initial assumption, a *constant diagonal* matrix, so we have

$$F_\mu \text{ is a function only of } x^\mu\text{, and so are all of its derivatives } F_{\mu\nu}$$

Returning with this information to (233), we are brought to the conclusion that the expression $\{\text{etc.}\}$ is a function only of x^0, only of x^1, . . . ; that it is, in short, a *constant* (call it $2C$), and that (233) can be written

$$F_{\mu\nu} = 2Cg_{\mu\nu}$$

giving

$$\begin{aligned} F &= Cg_{\alpha\beta}x^\alpha x^\beta - 2b_\alpha x^\beta + A \\ &= C\cdot(x,x) - 2(b,x) + A \end{aligned} \tag{235}$$

where b_α and A are constants of integration. Returning with this information to (234) we obtain

$$\begin{aligned} 4CF = g^{\alpha\beta}F_\alpha F_\beta &= g^{\alpha\beta}(2Cx_\alpha - 2b_\alpha)(2Cx_\beta - 2b_\beta) \\ &= 4C\left[C\cdot(x,x) - 2(b,x) + \frac{(b,b)}{C}\right] \end{aligned}$$

the effect of which, upon comparison with (235), is to constrain the constants $\{A, b_\alpha, C\}$ to satisfy

$$AC = (b,b)$$

This we accomplish by setting $C = (b,b)/A$, giving

$$F = A - 2(b,x) + \frac{(b,b)(x,x)}{A} \quad : \quad A \text{ and } b_\alpha \text{ now } \textit{un}\text{constrained}$$

Finally we introduce $a_\alpha \equiv b_\alpha/A$ to obtain the pretty result

$$F = A\big[1 - 2(a,x) + (a,a)(x,x)\big] \tag{236}$$

The conformal transformations $\mathfrak{X} \leftarrow \mathfrak{X}$ have yet to be described, but we now know this about W, the *Jacobian* of such a transformation:

$$\Omega = W^{\frac{2}{N}} = \frac{1}{F^2} = \frac{1}{A^2[1 - 2(b, x) + (a, a)(x, x)]^2} \tag{237}$$

Clearly, *tensor weight distinctions do <u>not</u> become moot in the context provided by the conformal group*, as they did (to within signs) in connection with the Lorentz group.

To get a handle on the functions $x^\alpha(x)$ that describe specific conformal transformations $\mathfrak{X} \leftarrow \mathfrak{X}$ we introduce

$$\partial_\mu \varphi \equiv \varphi_\mu = \partial_\mu \log \sqrt{\Omega} = -\partial_\mu \log F = -\frac{1}{F} F_\mu$$

into (223) to obtain

$$F W^\alpha{}_{\mu\nu} + F_\mu W^\alpha{}_\nu + F_\nu W^\alpha{}_\mu = g_{\mu\nu} \cdot g^{\lambda\kappa} F_\lambda W^\alpha{}_\kappa$$

or again (use $W^\alpha{}_\mu = \partial x^\alpha / \partial x^\mu$)

$$(F x^\alpha)_{\mu\nu} = F_{\mu\nu} x^\alpha + g_{\mu\nu} \cdot g^{\lambda\kappa} F_\lambda W^\alpha{}_\kappa \tag{238}$$

To eliminate some subsequent clutter we agree translate from x-coordinates to y-coordinates whose origin coincides with that of the x-coordinate system: we write

$$x^\alpha(x) = y^\alpha(x) + K t^\alpha \quad \text{with} \quad K \equiv A^{-1}$$

and achieve $y^\alpha(0) = 0$ by setting $K t^\alpha \equiv x^\alpha(0)$. Clearly, if the functions $x^\alpha(x)$ satisfy (238) then so also do the functions $y^\alpha(x)$, and conversely. We change dependent variables now once *again*, writing

$$F y^\alpha \equiv z^\alpha$$

Then $y^\alpha{}_\mu = -\frac{1}{F^2} F_\mu x^\alpha + \frac{1}{F} z^\alpha{}_\mu$ and (238) assumes the form

$$z^\alpha{}_{\mu\nu} = \frac{1}{F}\left\{ \underbrace{\left(F_{\mu\nu} - g_{\mu\nu} \cdot \frac{g^{\lambda\kappa} F_\lambda F_\kappa}{F} \right)} z^\alpha + g_{\mu\nu} \cdot g^{\lambda\kappa} F_\lambda z^\alpha{}_\kappa \right\}$$

It follows, however, from the previously established structure of F that

$$= -F_{\mu\nu} = -2C g_{\mu\nu}$$

so

$$= g_{\mu\nu} \cdot \frac{1}{F}\left\{ -2C z^\alpha + g^{\lambda\kappa} F_\lambda z^\alpha{}_\kappa \right\} \tag{239}$$

Each of these α-parameterized equations is structurally analogous to (234), and the argument that gave (235) no gives

$$z^\alpha(x) = P^\alpha \cdot (x, x) + \Lambda^\alpha{}_\beta x^\beta + \left[\begin{array}{l}\text{now no } x\text{-independent term} \\ \text{because } y(0) = 0 \ \Rightarrow\ z(0) = 0\end{array}\right]$$

Returning with this population of results to (239) we obtain

$$2P^\alpha\big[C(x,x)-2(b,x)+A\big] = -2C\big[P^\alpha(x,x)+\Lambda^\alpha{}_\beta x^\beta\big] \\ +\big[2Cx^\beta-2b^\beta\big]\big[2P^\alpha x_\beta+\Lambda^\alpha{}_\beta\big]$$

—the effect of which (after much cancellation) is to constrain the constants P^α and $\Lambda^\alpha{}_\beta$ to satisfy $P^\alpha = -\frac{1}{A}\Lambda^\alpha{}_\beta b^\beta = -\Lambda^\alpha{}_\beta a^\beta$. Therefore

$$z^\alpha(x) = \Lambda^\alpha{}_\beta\big\{x^\beta-(x,x)a^\beta\big\}$$

Reverting to y-variables this becomes

$$y^\alpha(x) = K\frac{\Lambda^\alpha{}_\beta\big\{x^\beta-(x,x)a^\beta\big\}}{1-2(a,x)+(a,a)(x,x)}$$

so in x-variables—the variables of primary interest—we have

$$x^\alpha(x) = K\left[t^\alpha+\frac{\Lambda^\alpha{}_\beta\big\{x^\beta-(x,x)a^\beta\big\}}{1-2(a,x)+(a,a)(x,x)}\right] \tag{240}$$

Finally we set $K = 1$ and $a^\alpha = 0$ (all α) which by (237) serve to establish $\Omega = 1$. But in that circumstance (240) assumes the simple form

$$\downarrow$$
$$= \Lambda^\alpha{}_\beta x^\beta$$

and the equation (185.2) that served as our point of departure becomes $\mathbb{\Lambda}^\mathsf{T}\mathbb{g}\mathbb{\Lambda} = \mathbb{g}$, from which we learn that the $\Lambda^\alpha{}_\beta$ must be elements of a Lorentz matrix.

Transformations of the form (240) have been of interest to mathematicians since the latter part of the 19th Century. Details relating to the derivation of (240) by iteration of *infinitesimal conformal transformations* were worked out by S. Lie, and are outlined on pages 28–32 of J. E. Campbell's *Theory of Continuous Groups* (1903). The finitistic argument given above—though in a technical sense "elementary"—shows the toolmarks of a master's hand, and is in fact due (in essential outline) to H. Weyl (1923). I have borrowed most directly from V. Fock, *The Theory of Space, Time & Gravitation* (1959), Appendix A: "On the derivation of the Lorentz transformations."

Equation (240) describes—for $N \neq 2$—the most general N-dimensional conformal transformation, and can evidently be considered to arise by composition from the following:

Lorentz transformation	:	$x \to x = \mathbb{\Lambda}x$	(241.1)
Translation	:	$x \to x = x + t$	(241.2)
Dilation	:	$x \to x = Kx$	(241.3)
Möbius transformation	:	$x \to x = \dfrac{x-(x,x)a}{1-2(a,x)+(a,a)(x,x)}$	(241.4)

To specify such a transformation one must assign values to

$$\tfrac{1}{2}N(N-1)+N+1+N=\tfrac{1}{2}(N+2)(N+2)$$

adjustable parameters $\{t^\alpha, K, a^\alpha \text{ and the elements of } \log\mathbb{\Lambda}\}$, the physical dimensionalities of which are diverse but obvious. The associated numerology is summarized below:

N	$\tfrac{1}{2}(N+2)(N+1)$
1	3
2	$6+\infty$
3	10
4	15
5	21
6	28
$\vdots$	$\vdots$

Concerning the entry at $N=2$: equation (240) makes perfect sense in the $\boxed{\text{CASE } N=2}$, and that case provides a diagramatically convenient context within which to study the meaning of (240) in the general case. But (240) was derived from (232), which was seen on page 167 to be stronger that the condition (231) appropriate to the 2-dimensional case. The weakened condition requires alternative analysis,[133] and admits of more possibilities—actually *infinitely many more*, corresponding roughly to the infinitely many ways of selecting $f(z)$ in the theory of conformal transformations as it is encountered in complex function theory.[134] I do not pursue the topic because the physics of interest to us is inscribed (as are we) on 4-dimensional spacetime.

Some of the mystery which surrounds the Möbius transformations—which are remarkable for their *nonlinearity*—is removed by the remark that they can be assembled from translations and "inversions," where the latter are defined as follows:

$$\textbf{Inversion} \quad : \quad x \rightarrow x=\mu^2\frac{x}{(x,x)} \tag{241.5}$$

Here μ^2 is a constant of arbitrary value, introduced mainly for dimensional reasons. The proof is by construction:

$$\left.\begin{aligned} x &\xrightarrow[\text{inversion}]{\hspace{3cm}} x=\mu^2x/(x,x)\\ &\xrightarrow[\text{translation with } t=-\mu^2a]{\hspace{3cm}} x=x-\mu^2a\\ &\xrightarrow[\text{inversion}]{\hspace{3cm}} x=\mu^2x/(x,x)\\ &\hspace{3.5cm} =\frac{x-(x,x)a}{1-2(a,x)+(a,a)(x,x)} \end{aligned}\right\} \tag{242}$$

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[133] The problem is discussed in my TRANSFORMATIONAL PHYSICS OF WAVES (1979–1981).

[134] See again page 129.

Inversion—which

- admits readily of geometrical interpretation (as a kind of "radial reflection" in the isometric surface $(x, x) = \mu^2$)
- can be looked upon as the ultimate *source of the nonlinearity* which is perhaps the most striking feature of the conformal transformations (240)

—is one of the sharpest tools available to the conformal theorist, so I digress to examine some of its properties:

We have, in effect, already shown (at (242): set $a = 0$) that *inversion is—like every kind of "reflection"—<u>self-reciprocal</u>*:

$$(\text{inversion}) \cdot (\text{inversion}) = \text{identity} \tag{243}$$

That inversion is conformal in the sense "angle-preserving" can be established as follows: let x and y be the inversive images of $\color{red}{x}$ and $\color{red}{y}$. Then

$$(x, y) = \mu^4 \frac{(\color{red}{x}, \color{red}{y})}{(\color{red}{x}, \color{red}{x})(\color{red}{y}, \color{red}{y})}$$

shows that inversion does not preserve inner products. But immediately

$$\frac{(x, y)}{\sqrt{(x, x)(y, y)}} = \frac{(\color{red}{x}, \color{red}{y})}{\sqrt{(\color{red}{x}, \color{red}{x})(\color{red}{y}, \color{red}{y})}} \tag{244}$$

which is to say:

$$\text{angle} = \color{red}{\text{angle}}$$

Inversion, since conformal, must be describable in terms of the primitive transformations listed at (241). How is that to be accomplished? We notice that each of those transformations—with the sole exception of the improper Lorentz transformations—is *continuous with the identity* (which arises at $\mathbb{\Lambda} = \mathbb{I}$, at $t = 0$, at $K = 1$, at $a = 0$). Evidently improper Lorentz transformations—in a word: reflections—must enter critically into the fabrication of inversion, and it is this observation that motivates the following short digression: For arbitrary non-null a^μ we can always write

$$x = \left[x - \frac{(x, a)}{(a, a)}\,a\right] + \frac{(x, a)}{(a, a)}\,a \equiv x_\parallel + x_\perp$$

which serves to resolve x^μ into components parallel/normal to a^μ. It becomes in this light natural to define

$$\textbf{a-reflection} \quad : \quad \begin{aligned} x &= x_\perp + x_\parallel \\ &\downarrow \\ \hat{x} &= x_\perp - x_\parallel = x - 2\frac{(x, a)}{(a, a)}\,a \end{aligned} \tag{245}$$

and to notice that (by quick calculation)

$$(\hat{x}, \hat{y}) = (x, y) \quad : \quad a\text{-reflection is inner-product preserving}$$

This simple fact leads us to notice that (245) can be written

$$\hat{x} = \mathbb{\Lambda}\,x \quad \text{with} \quad \mathbb{\Lambda} \equiv \|\Lambda^\mu{}_\nu\| = \|\delta^\mu{}_\nu - 2(a,a)^{-1}a^\mu a_\nu\|$$

where a brief calculation (examine $\Lambda^\alpha{}_\mu g_{\alpha\beta}\Lambda^\beta{}_\nu$) establishes that $\mathbb{\Lambda}$ is a Lorentz matrix with (according to *Mathematica*) $\det\mathbb{\Lambda} = -1$. In short:

$$\text{a-reflections are improper Lorentz transformations} \tag{246}$$

Thus prepared, we are led after a little exploratory tinkering to the following sequence of transformations:

$$\begin{aligned}
x \xrightarrow[\text{translation}]{\hspace{6em}} x &= x - \frac{1}{(a,a)}\,a \\
\xrightarrow[\text{reflection}]{\hspace{6em}} x &= x - 2\frac{(x,a)}{(a,a)}\,a \\
\xrightarrow[\text{Möbius}]{\hspace{6em}} x &= \frac{x - (x,x)a}{1 - 2(a,x) + (a,a)(x,x)} \\
&\vdots \ \text{algebraic simplification} \\
&= \frac{1}{(a,a)}\left\{\frac{x}{(x,x)} - a\right\} \\
\xrightarrow[\text{reverse translation}]{\hspace{6em}} x &= x + \frac{1}{(a,a)}a \\
&= \mu^2\frac{x}{(x,x)} \quad \text{with} \quad \mu^2 \equiv (a,a)^{-1}
\end{aligned}$$

The preceding equations make precise the sense in which

$$\text{inversion} = (\text{translation})^{-1}\cdot(\text{Möbius})\cdot(\text{reflection})\cdot(\text{translation}) \tag{247}$$

and confirm the conclusion reached already at (244): inversion is conformal. Finally, if one were to attempt direct evaluation of the Jacobian W of the general conformal transformation (240)—thus to confirm the upshot

$$W = \pm K^N\left[\frac{1}{1 - 2(a,x) + (a,a)(x,x)}\right]^N$$

of (237)—one would discover soon enough that one had a job on one's hands! But the result in question can be obtained as an easy consequence of the following readily-established statements:

$$W_{\text{inversion}} = -\mu^{2N}\frac{1}{(x,x)^N} \tag{248.1}$$
$$W_{\text{Lorentz}} = \pm 1 \tag{248.2}$$
$$W_{\text{translation}} = 1 \tag{248.3}$$
$$W_{\text{dilation}} = K^N \tag{248.4}$$

It follows in particular from (242) that

$$\begin{aligned} W_{\text{Möbius}} &= (-)^2\mu^{2N}\frac{1}{(x,x)^N}\cdot 1\cdot \mu^{2N}\frac{1}{(x,x)^N} \quad \text{with} \quad x = \mu^2\Big[\frac{x}{(x,x)} - a\Big] \\ &= \Big[\frac{1}{1-2(a,x)+(a,a)(x,x)}\Big]^N \end{aligned} \tag{248.5}$$

We are familiar with the fact that specialized Lorentz transformations serve to boost one to the frame of an observer O in uniform motion. I discuss now a related fact with curious electrodynamic implications: *specialized Möbius transformations serve to boost one to the frame of a uniformly accelerated observer*. From (241.4) we infer that a_μ has the dimensionality of reciprocal length, so

$$\tfrac{1}{2}g_\mu \equiv c^2 a_\mu \quad \text{is dimensionally an “acceleration”}$$

and in this notation (241.4) reads

$$x^\mu \rightarrow x^\mu = \frac{x^\mu - \frac{1}{2c^2}(x,x)g^\mu}{1-\frac{1}{c^2}(g,x)+\frac{1}{4c^4}(g,g)(x,x)} \tag{249}$$

We concentrate now on implications of the assumption that g_μ possesses the specialized structure

$$\begin{pmatrix} g_0\\ g_1\\ g_2\\ g_3\end{pmatrix} = \begin{pmatrix} 0\\ \boldsymbol{g}\end{pmatrix}$$

that results from setting $g_0 = 0$. To describe (compare page 139) the “successive ticks of the clock at his origin” O writes

$$\begin{pmatrix} ct\\ \boldsymbol{0}\end{pmatrix}$$

which to describe those same events we write

$$\begin{pmatrix} ct\\ \boldsymbol{x}\end{pmatrix} = \frac{1}{1-(gt/2c)^2}\begin{pmatrix} ct\\ \boldsymbol{0}+\frac{1}{2}\boldsymbol{g}t^2\end{pmatrix}$$

where $g \equiv \sqrt{\boldsymbol{g}\cdot\boldsymbol{g}}$ and the $+$ intruded because we are talking here about g^μ; *i.e.*, because we raised the index. In the *non-relativistic limit* this gives

$$\begin{pmatrix} t\\ \boldsymbol{x}\end{pmatrix} = \begin{pmatrix} t\\ \frac{1}{2}\boldsymbol{g}t^2\end{pmatrix} \tag{250}$$

which shows clearly the sense in which <u>we see O to be in a state of uniform acceleration</u>. To simplify more detailed analysis of the situation we (without loss of generality) sharpen our former assumption, writing

$$\boldsymbol{g} = \begin{pmatrix} g \\ 0 \\ 0 \end{pmatrix}$$

Then

$$1 - \tfrac{1}{c^2}(g, x) + \tfrac{1}{4c^4}(g, g)(x, x) = \frac{\big[(x-\lambda)^2 + y^2 + x^2 - c^2 t^2\big]}{\lambda^2}$$

$$\lambda \equiv \frac{2c^2}{g} \text{ is a "length"}$$

and (249) becomes

$$\left.\begin{aligned} t &= \frac{\lambda^2}{[\text{etc.}]}\cdot t \\ x &= \frac{\lambda^2}{[\text{etc.}]}\cdot\big\{x + \lambda^{-1}(c^2 t^2 - x^2)\big\} \\ &= \frac{\lambda}{[\text{etc.}]}\cdot\Big\{c^2 t^2 - \big(x - \tfrac{1}{2}\lambda\big)^2 + \big(\tfrac{1}{2}\lambda\big)^2\Big\} \\ &= \frac{\lambda}{[\text{etc.}]}\cdot\Big\{ - \big[\text{etc.}\big] - \lambda(x-\lambda)\Big\} \\ y &= \frac{\lambda^2}{[\text{etc.}]}\cdot y \\ z &= \frac{\lambda^2}{[\text{etc.}]}\cdot z \end{aligned}\right\} \tag{251}$$

It is evident that $\big[\text{etc.}\big]$ vanishes—and the transformation (251) becomes therefore *singular*—on the lightcone $c^2 t^2 - (x-\lambda)^2 - y^2 - x^2 = 0$ whose vertex is situated at $\big\{t, x, y, z\big\} = \big\{0, \lambda, 0, 0\big\}$. It is to gain a diagramatic advantage that we now set $y = z = 0$ and study what (251) has to say about how t and x depend upon t and x. We have

$$t = \frac{\lambda^2}{[(x-\lambda)^2 - c^2 t^2]}\cdot t \tag{252.1}$$

$$(x+\lambda) = -\frac{\lambda^2}{[(x-\lambda)^2 - c^2 t^2]}\cdot(x-\lambda) \tag{252.2}$$

which jointly entail

$$\big[c^2t^2 - (x+\lambda)^2\big]\big[c^2 t^2 - (x-\lambda)^2\big] = \lambda^4 \tag{253}$$

But equations (252) can be written

$$\big[c^2 t^2 - (x-\lambda)^2\big] = -\lambda^2\,\frac{t}{t} \tag{254.1}$$

$$= \quad \lambda^2\,\frac{x-\lambda}{x+\lambda} \tag{254.2}$$

and when we return with the latter to (253) we find

$$\big[c^2t^2-(x+\lambda)^2\big]=\lambda^2\,\frac{x+\lambda}{x-\lambda}$$

from which t has been eliminated: complete the square and obtain

$$\Big(x+\lambda\,\frac{x-\frac{1}{2}\lambda}{x-\lambda}\Big)^2-(ct)^2=\Big(\frac{\lambda^2}{2(x-\lambda)}\Big)^2 \tag{255.1}$$

which is seen to describe a x-parmeterized family of hyperbolas inscribed on the (t,x)-plane. These are *Möbius transforms of the lines of constant x inscribed on the (t,x)-plane*. Proceeding similarly to the elimination of $(x-\lambda)$ we find

$$\big[c^2t^2-(x+\lambda)^2\big]=-\lambda^2\,\frac{t}{t}$$

giving

$$\Big(ct+\frac{\lambda^2}{2ct}\Big)^2-(x+\lambda)^2=\Big(\frac{\lambda^2}{2ct}\Big)^2 \tag{255.2}$$

which describes a t-parameterized family of hyperbolas—*Möbius transforms of the "time-slices" or lines of constant t inscribed on the (t,x)-plane*. The following remarks proceed from the results now in hand:

- O, by (252), assigns to O's origin the coordinates $t_0=0$, $x_0=0$; their origins, in short, coincide.
- In (255.1) set $x=0$ and find that O writes

  $$(x+\tfrac{1}{2}\lambda)^2-(ct)^2=(\tfrac{1}{2}\lambda)^2$$

  to describe O's worldline, which O sees to be hyperbolic, with x-intercepts at $x=0$ and $x=-\lambda$ and asymptotes $ct=\pm(x+\frac{1}{2}\lambda)$ that intersect at $t=0$, $x=-\frac{1}{2}\lambda$.
- If, in (252), we set $x=0$ we obtain

  $$t=\frac{\lambda^2}{[\lambda^2-c^2t^2]}\cdot t$$
  $$x=\frac{\lambda^3}{[\lambda^2-c^2t^2]}-\lambda$$

  which provide O's t-parameterized description of O's worldline. Notice that t and x both become infinite at $t=\lambda/c$, and that t thereafter becomes negative!
- To describe her lightcone O writes $x=\pm ct$. Insert $x=+ct$ into (252.1), (ask *Mathematica* to) solve for t and obtain $ct=\lambda ct/(2ct+\lambda)$. Insert that result and $x=+ct$ into (252.2) and, after simplifications, obtain $x=+ct$. Repeat the procedure taking $x=-ct$ as your starting point: obtain $ct=-\lambda ct/(2ct-\lambda)$ and finally $x=-ct$. The striking implication is that (252) sends

  $$O\text{'s lightcone}\quad\longmapsto\quad O\text{'s lightcone}$$

The conformal group is a wonderfully rich mathematical object, of which I have scarcely scratched the surface.[135] But I have scratched deeply enough to illustrate the point which motivated this long and intricate digression, a point made already on page 126:

The covariance group of a theory depends
in part upon *how the theory is expressed*:

One rendering of Maxwell's equations led us to the Lorentz group, and to special relativity. An almost imperceptibly different rendering committed us, however, to an entirely different line of analysis, and led us to an entirely different place—the conformal group, which contains the Lorentz group as a subgroup, but contains also much else . . . including transformations to the frames of "uniformly *accelerated* observers." Though it was electrodynamics which inspired our interest in the conformal group,[136] if you were to ask an elementary particle theorist about the conformal group you would be told that "the group arises as the covariance group of the wave equation

$$\Box\varphi = 0 \quad : \quad \text{conformally covariant}$$

Conformal covariance is broken (reduced to Lorentz covariance) by the inclusion of a "mass term"

$$(\Box + m^2)\varphi = 0 \quad : \quad \text{conformal covariance is broken}$$

It becomes the dominant symmetry in particle physics because at high energy mass terms can, in good approximation, be neglected

$$\text{rest energy } mc^2 \ll \text{total particle energy}$$

and enters into electrodynamics because the photon has no mass." That the group enters also into the physics of *massy* particles[133] is, in the light of such a remark, somewhat surprising. Surprises are imported also into classical electrodynamics by the occurrence of accelerations within the conformal group, for the question then arises: Does a uniformly accelerated charge radiate?[137]

---

[135] I scratch deeper, and discuss the occurance of the conformal group in connection with a rich variety of physical problems, in APPELL, GALILEAN & CONFORMAL TRANSFORMATIONS IN CLASSICAL/QUANTUM FREE PARTICLE DYNAMICS (1976) and TRANSFORMATIONAL PHYSICS OF WAVES (1979–1981).

[136] In "'Electrodynamics' in 2-dimensional spacetime" (1997) I develop a "toy electrodynamics" that gives full play to the exceptional richness that the conformal group has been seen to acquire in the 2-dimensional case.

[137] This question—first posed by Pauli in §32γ of his *Theory of Relativity*—once was the focus of spirited controversy: see T. Fulton & F. Rohrlich, "Classical radiation from a uniformly accelerated charge," Annals of Physics **9**,

8. Transformation properties of electromagnetic fields. To describe such a field at a spacetime point P we might display the values assumed there by the respective components of the electric and magnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$. Or we might display the values assumed there by the components $F^{\mu\nu}$ of the electromagnetic field tensor. To describe the *same physical facts* a second[138] observer O would display the values assumed by $\boldsymbol{E}$ and $\boldsymbol{B}$, or perhaps by $F^{\mu\nu}$. The *question* is

$$\text{How are } \{\boldsymbol{E}, \boldsymbol{B}\} \text{ and } \{\boldsymbol{E}, \boldsymbol{B}\} \text{ related?}$$

The *answer* has been in our possession ever since (at **A** on page 127, and on the "natural" grounds there stated) we assumed it to be the case that

$$F^{\mu\nu} \text{ transforms as a } \textbf{tensor density of unit weight} \tag{256}$$

But now we know things about the "allowed" coordinate transformations that on page 127 we did not know. Our task, therefore, is <u>to make explicit the detailed mathematical/physical consequences of (256)</u>. We know (see again (186) on page 129) that (256) pertains even when $\mathcal{X} \to \mathcal{X}$ is conformal, but I will restrict my attention to the (clearly less problematic, and apparently more important) case (184) in which

$$\mathcal{X} \to \mathcal{X} \text{ is } \textbf{Lorentzian}$$

The claim, therefore, is that

$$x \to x = \mathbb{\Lambda} x \quad \text{induces} \quad \mathbb{F} \to \mathbb{F} = V\cdot \mathbb{\Lambda}\mathbb{F}\mathbb{\Lambda}^{\mathsf{T}}$$

where $\mathbb{\Lambda}^{\mathsf{T}} \mathbb{g}\, \mathbb{\Lambda} = \mathbb{g}$ entails

$$V \equiv \frac{1}{\det \mathbb{\Lambda}} = \pm 1$$

and $\mathbb{F} = V\cdot \mathbb{\Lambda}\mathbb{F}\mathbb{\Lambda}^{\mathsf{T}}$ means $F^{\mu\nu} = V\Lambda^\mu{}_\alpha F^{\alpha\beta}\Lambda^\nu{}_\beta$. It is known, moreover, that (see again (211) on page 157) $\mathbb{\Lambda}$ can be considered to have this factored structure:

$$\mathbb{\Lambda} = \mathbb{R}\cdot\mathbb{\Lambda}(\boldsymbol{\beta})$$

(continued from the preceding page) 499 (1960); T. Fulton, F. Rohrlich & L. Witten, "Physical consequences of a coordinate transformation to a uniformly accelerated frame," Nuovo Cimento **26**, 652 (1962) and E. L. Hill, "On accelerated coordinate systems in classical and relativistic mechanics," Phys. Rev. **67**, 358 (1945); "On the kinematics of uniformly accelerated motions & classical electromagnetic theory," Phys. Rev. **72**, 143 (1947). The matter is reviewed by R. Peierls in §8.1 of *Surprises in Theoretical Physics* (1979), and was elegantly laid to rest by D. Boulware, "Radiation from a uniformly accelerated charge," Annals of Physics **124**, 169 (1980). For more general discussion see T. Fulton, F. Rohrlich & L. Witten, "Conformal invariance in physics," Rev. Mod. Phys. **34**, 442 (1962) and L. Page, "A new relativity," Phys. Rev. **49**, 254 (1936). Curiously, Boulware (with whom I was in touch earlier today: 30 October 2001) proceeded without explicit reference to the conformal group, of which he apparently was (and remains) ignorant.

[138] In view of the conformal covariance of electrodynamics I hesitate to insert here the adjective "inertial."

This means that we can study separately the <u>response of $\mathbb{F}$ to spatial rotations</u> $\mathbb{R}$ and its <u>response to boosts</u> $\mathbb{\Lambda}(\boldsymbol{\beta})$.

RESPONSE TO ROTATIONS Write out again (159)

$$\mathbb{F} = \mathbb{A}(\boldsymbol{E},\boldsymbol{B}) \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_1 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & -\boldsymbol{E}^{\mathsf{T}} \\ \boldsymbol{E} & \mathbb{B} \end{pmatrix}$$

and (208)

$$\mathbb{R} \equiv \begin{pmatrix} 1 & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0} & \mathbb{R} \end{pmatrix}$$

where

$$\mathbb{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$

is a 3×3 rotation matrix: $\mathbb{R}^{-1} = \mathbb{R}^{\mathsf{T}}$. It will, in a moment, become essential to notice that the latter equation, when spelled out in detail, reads

$$\begin{aligned} \frac{1}{\det\mathbb{R}} &\begin{pmatrix} (R_{22}R_{33} - R_{23}R_{32}) & (R_{13}R_{32} - R_{12}R_{33}) & (R_{12}R_{23} - R_{13}R_{22}) \\ (R_{23}R_{31} - R_{21}R_{33}) & (R_{11}R_{33} - R_{13}R_{31}) & (R_{21}R_{13} - R_{23}R_{11}) \\ (R_{32}R_{21} - R_{31}R_{22}) & (R_{31}R_{12} - R_{32}R_{11}) & (R_{11}R_{22} - R_{12}R_{21}) \end{pmatrix} \\ &= \begin{pmatrix} R_{11} & R_{21} & R_{31} \\ R_{12} & R_{22} & R_{32} \\ R_{13} & R_{23} & R_{33} \end{pmatrix} \end{aligned} \tag{257}$$

where

$$\frac{1}{\det\mathbb{R}} = \pm1 \quad \text{according as } \mathbb{R} \text{ is proper/improper}$$

Our task now is the essentially elementary one of evaluating

$$\begin{aligned} \mathbb{F} &= \frac{1}{\det\mathbb{R}} \begin{pmatrix} 1 & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0} & \mathbb{R} \end{pmatrix} \begin{pmatrix} 0 & -\boldsymbol{E}^{\mathsf{T}} \\ \boldsymbol{E} & \mathbb{B} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0}^{\mathsf{T}} \\ \mathbf{0} & \mathbb{R}^{\mathsf{T}} \end{pmatrix} \\ &= \frac{1}{\det\mathbb{R}} \begin{pmatrix} 0 & -(\mathbb{R}\boldsymbol{E})^{\mathsf{T}} \\ \mathbb{R}\boldsymbol{E} & \mathbb{R}\mathbb{B}\mathbb{R}^{\mathsf{T}} \end{pmatrix} \end{aligned}$$

which supplies

$$\boldsymbol{E} = (\det\mathbb{R})^{-1}\cdot\mathbb{R}\boldsymbol{E} \tag{258.1}$$

$$\mathbb{B} = (\det\mathbb{R})^{-1}\cdot\mathbb{R}\mathbb{B}\mathbb{R}^{\mathsf{T}} \tag{258.2}$$

The latter shows clearly how the antisymmetry of $\mathbb{B}$ comes to be inherited by $\mathbb{B}$, but does not much resemble its companion. HOWEVER . . . if we[139] first spell out

[139] PROBLEM 47.

the meaning of (258.2)

$$\begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix} = (\det \mathbb{R})^{-1} \cdot \mathbb{R} \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix} \mathbb{R}^{\mathsf{T}} \tag{259.1}$$

then (on a large sheet of paper) construct a detailed description of the matrix on the right, and finally make simplifications based on the rotational identity (257) . . . we find that (258.1) is precisely equivalent to (which is to say: simply a notational variant of) the statement[140]

$$\Updownarrow$$

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \mathbb{R} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \tag{259.2}$$

Equations (258) can therefore be expressed

$$\boldsymbol{E} = (\det \mathbb{R})^{-1} \cdot \mathbb{R}\boldsymbol{E} \tag{260.1}$$

$$\boldsymbol{B} = \mathbb{R}\boldsymbol{B} \tag{260.2}$$

> REMARK: In the conventional language of 3-dimensional physics, objects $\boldsymbol{A}$ that respond to rotation $\boldsymbol{x} \to \boldsymbol{x} = \mathbb{R}\boldsymbol{x}$ by the rule
>
> $$\boldsymbol{A} \to \boldsymbol{A} = \mathbb{R}\boldsymbol{A}$$
>
> are said to transform as **vectors** (or "polar vectors"), which objects that transform by the rule
>
> $$\boldsymbol{A} \to \boldsymbol{A} = (\det \mathbb{R}) \cdot \mathbb{R}\boldsymbol{A}$$
>
> are said to transform as **pseudovectors** (or "axial vectors"). Vectors and pseudovectors respond identically to proper rotations, but the latter respond to reflections (improper rotations) by acquisition of a minus sign. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both vectors (or both pseudovectors) then $\boldsymbol{C} \equiv \boldsymbol{A}\times\boldsymbol{B}$ provides the standard example of a pseudovector . . . for reasons that become evident when one considers what mirrors do to the "righthand rule."

The assumption[141] that

$F^{\mu\nu}$ transforms as a **tensor density of unit weight**

---

[140] For a more elegant approach to the proof of this important lemma see pages 22–22 in CLASSICAL GYRODYNAMICS (1976).

[141] See again the FIRST POINT OF VIEW, page 126.

was seen at (260) to carry the implication that

$$\left.\begin{array}{l}\boldsymbol{E}\text{ responds to rotation as a } \textbf{pseudovector}\\ \boldsymbol{B}\text{ responds to rotation as a } \textbf{vector}\end{array}\right\} \tag{261.1}$$

If we were, on the other hand, to assume[142] that

$$F^{\mu\nu}\text{ transforms as a } \textbf{weightless}\text{ tensor}$$

then the $(\det\mathbb{R})^{-1}$ factors would disappear from the right side of (258), and we would be led to the opposite conclusion:

$$\left.\begin{array}{l}\boldsymbol{E}\text{ responds to rotation as a } \textbf{vector}\\ \boldsymbol{B}\text{ responds to rotation as a } \textbf{pseudovector}\end{array}\right\} \tag{261.2}$$

The transformation properties of $\boldsymbol{E}$ and $\boldsymbol{B}$ are in either case "opposite,"[143] and it is from $\boldsymbol{E}$ that the transformation properties of ρ and $\boldsymbol{j}$ are inherited. The mirror image of the Coulombic field of a positive charge looks

- like the Coulombic field of a *negative* charge according to (261.1), but
- like the Coulombic field of a *positive* charge according to (261.2).

Perhaps it is for this reason (supported by no compelling physical argument) that (261.2) describes the tacitly-adopted *convention standard to the relativistic electrodynamical literature*. The factors that distinguish tensor densities from weightless tensors are, in special relativity, so nearly trivial ($\det\Lambda = \pm 1$) that many authors successfully contrive to neglect the distinction altogether.

RESPONSE TO BOOSTS All boosts are proper. Our task, therefore, is to evaluate

$$\mathbb{A}(\boldsymbol{E},\boldsymbol{B}) = \Lambda(\boldsymbol{\beta})\,\mathbb{A}(\boldsymbol{E},\boldsymbol{B})\,\Lambda^{\mathsf{T}}(\boldsymbol{\beta}) \tag{262}$$

where $\Lambda(\boldsymbol{\beta})$ has the structure (209) described on page 156. It will serve our exploratory purposes to suppose initially that

$$\boldsymbol{\beta} = \begin{pmatrix}\beta\\0\\0\end{pmatrix}$$

[142] See again the SECOND POINT OF VIEW, page 128.
[143] This fact has been latent ever since—at (67)—we alluded to the "$\boldsymbol{E}$-like character" of $\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}$, since

$$\text{vector}\times\begin{cases}\text{vector}\\ \text{pseudovector}\end{cases} = \begin{cases}\text{pseudovector}\\ \text{vector}\end{cases}$$

—*i.e.*, that we are boosting along the x-axis: then

$$\mathbb{A}(\boldsymbol{\beta}) = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and it follows from (262) by quick calculation that

$$\mathbb{A}(\boldsymbol{E},\boldsymbol{B}) = \begin{pmatrix} 0 & -E_1 & -\gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_2 & -\gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_3 \\ E_1 & 0 & -\gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_3 & +\gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_2 \\ \gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_2 & +\gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_3 & 0 & -B_1 \\ \gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_3 & -\gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_2 & +B_1 & 0 \end{pmatrix}$$

Noting that

$$\begin{aligned} E_1 &= (\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_1 \quad \text{because} \quad (\boldsymbol{\beta}\times\boldsymbol{B}) \perp \boldsymbol{\beta} \\ B_1 &= (\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_1 \quad \text{because} \quad (\boldsymbol{\beta}\times\boldsymbol{E}) \perp \boldsymbol{\beta} \end{aligned}$$

we infer that

$$\left.\begin{aligned} \boldsymbol{E} &= (\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\parallel + \gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\perp \\ \boldsymbol{B} &= (\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\parallel + \gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\perp \end{aligned}\right\} \tag{263}$$

where components $\parallel$ and $\perp$ to $\boldsymbol{\beta}$ are defined in the usual way: generically

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A}_\parallel + \boldsymbol{A}_\perp \\ \boldsymbol{A}_\parallel &\equiv (\boldsymbol{A}\cdot\hat{\boldsymbol{\beta}})\hat{\boldsymbol{\beta}} = \underbrace{\frac{1}{\beta^2}\begin{pmatrix} \beta_1\beta_1 & \beta_1\beta_2 & \beta_1\beta_3 \\ \beta_2\beta_1 & \beta_2\beta_2 & \beta_2\beta_3 \\ \beta_3\beta_1 & \beta_3\beta_2 & \beta_3\beta_3 \end{pmatrix}}_{\text{projects onto } \boldsymbol{\beta}} \boldsymbol{A} \end{aligned}$$

Several comments are now in order:

1. We had already on page 46 (when we are arguing from *Galilean* relativity) reason to suspect that "$\boldsymbol{E}$ & $\boldsymbol{B}$ fields transform in a funny, interdependent way." Equations (263) first appear—somewhat disguised—in §4 of Lorentz (1904).[78] They appear also in §6 of Einstein (1905).[78] They were, in particular, unknown to Maxwell.

2. Equations (263) are ugly enough that they invite reformulation, and can in fact be formulated in a great variety of (superficially diverse) ways . . . some obvious—in the 6-vector formalism[86] one writes

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} = \mathbb{M}(\boldsymbol{\beta}) \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix}$$

where $\mathbb{M}(\boldsymbol{\beta})$ is a 6×6 matrix whose elements can be read off from (263)—and some not so obvious. I would pursue this topic in response to some specific formal need, but none will arise.

3. The following statements are equivalent:

$$\left.\begin{array}{l}
\text{Maxwell's equations}\\[6pt]
\qquad\qquad\begin{aligned}
\nabla\!\cdot\!\boldsymbol{E} &= \rho\\
\nabla\!\times\!\boldsymbol{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} &= \tfrac{1}{c}\boldsymbol{j}\\
\nabla\!\cdot\!\boldsymbol{B} &= 0\\
\nabla\!\times\!\boldsymbol{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} &= \boldsymbol{0}
\end{aligned}\\[6pt]
\text{simply “turn black” in response to}\\[6pt]
\qquad\begin{aligned}
t &= \gamma t + \tfrac{1}{c^2}\gamma\boldsymbol{v}\!\cdot\!\boldsymbol{x}\\
\boldsymbol{x} &= \boldsymbol{x} + \big\{\gamma t + (\gamma-1)\tfrac{1}{v^2}\boldsymbol{v}\!\cdot\!\boldsymbol{x}\big\}\boldsymbol{v}\\
\rho &= \gamma\rho + \tfrac{1}{c^2}\gamma\boldsymbol{v}\!\cdot\!\boldsymbol{j}\\
\boldsymbol{j} &= \boldsymbol{j} + \big\{\gamma\rho + (\gamma-1)\tfrac{1}{v^2}\boldsymbol{v}\!\cdot\!\boldsymbol{j}\big\}\boldsymbol{v}\\
\boldsymbol{E} &= (\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\parallel + \gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\perp\\
\boldsymbol{B} &= (\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\parallel + \gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\perp
\end{aligned}
\end{array}\right\}\tag{264.1}$$

$$\left.\begin{array}{l}
\text{Maxwell's equations}\\[6pt]
\qquad\qquad\begin{aligned}
\partial_\mu F^{\mu\nu} &= \tfrac{1}{c}j^\nu\\
\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} &= 0
\end{aligned}\\[6pt]
\text{simply “turn black” in response to}\\[6pt]
\qquad\qquad\begin{aligned}
x^\mu &= \Lambda^\mu{}_\alpha x^\alpha\\
j^\nu &= \Lambda^\nu{}_\beta j^\beta\\
F^{\mu\nu} &= \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta}
\end{aligned}
\end{array}\right\}\tag{264.2}$$

and provide detailed statements of what one means when one refers to the “Lorentz covariance of Maxwellian electrodynamics.” Note that it is not enough to know how Lorentz transformations act on spacetime coordinates: one must know also how they act on fields and sources. The contrast in the formal appearance of (264.1: Lorentz & Einstein) and (264.2: Minkowski) is striking, and motivates me to remark that

- it is traditional in textbooks to view (264.1) as “working equations,” and to regard (264.2) as “cleaned-up curiosities,” to be written down and admired as a kind of afterthought . . . but
- my own exposition has been designed to emphasize the *practical utility* of (264.2): I view (264.1) as “elaborated commentary” upon (264.2)—too complicated to work with except in some specialized applications.

4. We know now how to translate electrodynamical statements from one inertial frame to another. But we do *not* at present possess answers to questions such as the following:

- How do electromagnetic fields and/or Maxwell's equations look to an observer in a *rotating* frame?
- How—when Thomas precession is taken into account—does the nuclear Coulomb field look to an observer sitting on an electron in Bohr orbit?
- How do electromagnetic fields and the field equations look to an *arbitrarily accelerated* observer?

We are, however, *in position now to attack* such problems, should physical motivation arise.

5. Suppose O sees a pure $\boldsymbol{E}$-field: $\boldsymbol{B}(\boldsymbol{x}) = 0$ (all $\boldsymbol{x}$). It follows from (263) that *we* would see and electro*magnetic* field of the form

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{E}_\parallel + \gamma\boldsymbol{E}_\perp &&= \gamma\boldsymbol{E} + (1-\gamma)\tfrac{1}{v^2}(\boldsymbol{v}\cdot\boldsymbol{E})\,\boldsymbol{v} \\ \boldsymbol{B} &= \qquad\gamma(\boldsymbol{\beta}\times\boldsymbol{E}) &&= \tfrac{1}{c}\gamma(\boldsymbol{v}\times\boldsymbol{E}) \end{aligned}$$

Our $\boldsymbol{B}$-field is, however, <u>structurally atypical</u>: it has a specialized ancestory, and (go to O's frame) *can be transformed away—globally*. In general it is not possible by Lorentz transformation to kill $\boldsymbol{B}$ (or $\boldsymbol{E}$) even locally, for to do so would be (unless $\boldsymbol{E}\perp\boldsymbol{B}$ at the spacetime point in question) to stand in violation of the second of the following remarkable equations[144]

$$\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B} = \boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B} \tag{265.1}$$

$$\boldsymbol{E}\cdot\boldsymbol{B} = \boldsymbol{E}\cdot\boldsymbol{B} \tag{265.2}$$

The preceding remark makes vividly clear, by the way, why it is that attempts to "derive" electrodynamics from "Coulomb's law + special relativity" are doomed to fail: with only that material to work with one cannot escape from the force of the special/atypical condition $\boldsymbol{E}\cdot\boldsymbol{B} = 0$.

6. We do *not* have in hand the statements analogous to (264) that serve to lend detailed meaning to the "<u>conformal</u> covariance of Maxwellian electrodynamics." To gain a sense of the most characteristic features of the enriched theory it would be sufficient to describe how electromagnetic fields and sources respond to dilations and inversions.

7. An uncharged copper rod is transported with velocity $\boldsymbol{v}$ in the presence of a homogeneous magnetic field $\boldsymbol{B}$. We see a *charge separation* to take place (one end of the rod becomes positively charge, the other negatively: see Figure 66), which we attribute the presence $q(\boldsymbol{v}\times\boldsymbol{B})$-forces. But an observer O co-moving with the rod sees no such forces (since $\boldsymbol{v} = \boldsymbol{0}$), and must attribute the charge separation phenomenon to the presence of an *electric* field $\boldsymbol{E}$. <u>It was to account for such seeming "explanatory asymmetry" that Einstein *invented* the theory of relativity</u>. I quote from the beginning of his 1905 paper:

[144] PROBLEM 48.

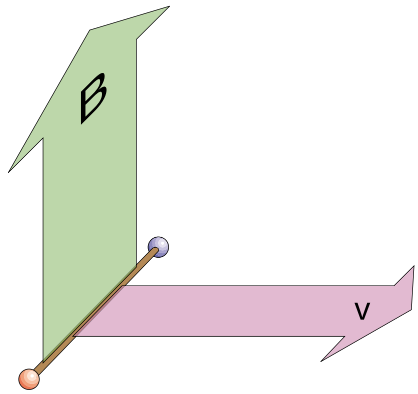

FIGURE 66: *A copper rod is transported with constant velocity* $\boldsymbol{v}$ *in a homogeneous magnetic field. Charge separation is observed to occur in the rod. Observers in relative motion explain the phenomenon in—unaccountably, prior to the invention of special relativity—quite different ways.*

ON THE ELECTRODYNAMICS OF MOVING BODIES

A. EINSTEIN

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighborhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an electrtomotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to elecric currents of the same path and intensity as those produced by the electric forces in the former case.

Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.

After sixteen pages of inspired argument Einstein arrives at equations (263), from which he concludes that

> . . . electric and magnetic forces do not exist independently of the state of motion of the system of coordinates.
>
> Furthermore it is clear that the asymmetry mentioned in the introduction as arising when we consider the currents produced by the relative motion of a magnet and a conductor now disappears.

He comes to the latter conclusion by arguing that to determine the force $\boldsymbol{F}$ experienced by a moving charge q in an electromagnetic field $\{\boldsymbol{E},\boldsymbol{B}\}$ a typical inertial observer should

i) transform $\{\boldsymbol{E},\boldsymbol{B}\} \rightarrow \{\boldsymbol{E}_0,\boldsymbol{B}_0\}$ to the instantaneous rest frame of the charge;

ii) write $\boldsymbol{F}_0 = q\boldsymbol{E}_0$;

iii) transform back again to his own reference frame: $\boldsymbol{F} \leftarrow \boldsymbol{F}_0$.

We don't, as yet, know how to carry out the last step (because we have yet to study relativistic mechanics). It is already clear, however, that Einstein's program *eliminates asymmetry because it issues identical instructions to every inertial observer*. Note, moreover, that it contains no reference to "the" velocity . . . but refers only to the *relative* velocity (of charge and observer, of observer and observer).

The field-transformation equations (263) lie, therefore, at the motivating heart of Einstein's 1905 paper. All the rest can be read as "technical support"—evidence of the extraordinary surgery Einstein was willing to perform to remove a merely *aesthetic blemish* from a theory (Maxwellean electrodynamics) which—after all—worked perfectly well as it was! Several morals could be drawn. Most are too obvious to state . . . and all are too important for the creative physicist to ignore.

9. Principle of relativity. The arguments which led Einstein to the Lorentz transformations differ profoundly from those which (unbeknownst to Einstein) had led Lorentz to the same result. Lorentz argued (as we have seen . . . and done) from the structure of Maxwell's equations. Einstein, on the other hand (and though he had an electrodynamic problem in mind), extracted the Lorentz transformations from an unprecedented *operational analysis*: his argument assumed very little . . . and he had, therefore, correspondingly greater confidence in the inevitability and generality of his conclusions. His argument was, in particular, entirely free from any reference to Maxwell's equations, so his conclusion—that inertial observers are interrelated by Lorentz transformations —*could not be specific to Maxwellean electrodynamics*. It was this insight—and the firmness[145] with which he adhered to it—which distinguished Einstein's thought from that of his contemporaries (Lorentz, Poincaré). It led him to

[145] I have indicated on page 163 why, in the light of subsequent developments, Einstein's "firmness" can be argued to have been *inappropriately* strong.

propose, at the beginning of his §2, two principles . . . which amount, in effect, to this, the

> **Principle of Relativity:** The concepts, statements and formulæ of physics—whatever the phenomenology to which they specifically pertain—must *preserve their structure* when subjected to Lorentz transformation. (266)

The principle of relativity functions as a "syntactical constraint" on the "statements that physicists may properly utter"—at least when they are doing *local* physics. Concepts/statements/theories which fail to pass the (quite stringent) "Lorentz covariance test" can, according to the principle of relativity, be dismissed out of hand as ill-formed, inconsistent with the grammar of physics . . . and therefore physically untenable. Theories that pass the test are said to be "relativistic," "Lorentz invariant" or (more properly) *Lorentz covariant*. The *physical correctness* of such a theory is, of course, not guaranteed. What is guaranteed is the ultimate physical incorrectness of any theory—whatever may be its utility in circumscribed contexts (think of non-relativistic classical and quantum mechanics!)—that stands in violation of the principle of relativity.[146]

Some theories—such as the version of Maxwellean electrodynamics that was summarized at (264.1)—conform to the principle of relativity, but do so "non-obviously." Other theories—see again (264.2)—conform more obviously. Theories of the latter type are said to be "*manifestly Lorentz covariant*." Manifest is, for obvious reasons, a very useful formal attribute for a physical theory to possess. Much attention has been given therefore to the cultivation of principles and analytical techniques which sharpen one's ability to generate manifestly covariant theories "automatically." Whence the importance which theoretical physicists nowadays attach to *variational principles, tensor analysis, group representation theory,* . . . (Einstein did without them all!).

Clearly, the principle of relativity involves much besides the simple "theory of Lorentz transformations" (it involves, in short, all of physics!) . . . but one must have a good command of the latter subject in order to *implement* the principle. If in (266) one substitutes for the word "Lorentz" the words "Galilean," "conformal," . . . one obtains the "**principle of Galilean relativity**," the "**principle of conformal relativity**," etc. These do have some physically illuminating formal consequences, but appear to pertain only *approximately* to the world-as-we-find-it . . . while the principle announced by Einstein pertains "exactly/universally."

I have several times emphasized the <u>universal applicability</u> of the principle

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[146] But *every* physical theory is *ultimately* incorrect! So the question that confronts physicists in individual cases is this: Is Lorentz non-covariance the *principal* defect of the theory in question, the defect worth of my corrective attention? Much more often than not, the answer is clearly "No."

of relativity. It is, therefore, *by way of illustrative application* that in Part II of his paper Einstein turns to the specific physics which had served initially to motivate his research—Maxwellean electrodynamics. It is frequently stated that "electrodynamics was *already* relativistic (while Newtonian dynamics had to be deformed to conform)." But this is not quite correct. The electrodynamics inherited by Einstein contained field equations, but it *contained no allusion to a field transformation law*. Einstein *produced* such a law—namely (263)—by *insisting that Maxwell's field equations conform to the principle of relativity*. Einstein *derived* (from Maxwell's equations + relativity, including prior knowledge of the Lorentz transformations) a result—effectively: that the $F^{\mu\nu}$ transform tensorially—which we were content (on page 127) to *assume*. We, on the other hand, used Maxwell's equations + tensoriality to *deduce the design of the Lorentz transformations*. Our approach—which is effectively Lorentz'—is efficient (also free of allusions to trains & lanterns), but might be criticized on the ground that it is excessively "parochial," too much rooted in specifics of electrodynamics. It is not at all clear that our approach would have inspired anyone to utter a generalization so audacious as Einstein's (266). Historically it didn't: both Lorentz and Poincaré were in possession of the technical rudiments of relativity already in 1904, yet both—for distinct reasons—failed to recognize the revolutionary force of the idea encapsulated at $\boxed{}$. Einstein was, in this respect, well served by his trains and lanterns. But it was not Einstein but Minkowski who first appreciated that at $\boxed{}$ Einstein had in effect prescribed that

> The physics inscribed on spacetime must mimic
> the symmetry structure of spacetime itself.

10. Relativistic mechanics of a particle. We possess a Lorentz covariant field dynamics. We want a theory of *fields and (charged) particles in interaction*. Self-consistency alone requires that the associated particle dynamics be Lorentz covariant. So also—irrespective of any reference to electromagnetism—does the principle of relativity.

The discussion which follows will illustrate how non-relativistic theories are "deformed to conform" to the principle of relativity. But it is offered to serve a more explicit and pressing need: my primary goal will be *to develop descriptions of the relativistic analogs of the pre-relativistic concepts of energy, momentum, force*, ... though a number of collateral topics will be treated *en route*.

In Newtonian dynamics the "worldline" of a mass point m is considered to be described by the 3-vector-valued solution $\boldsymbol{x}(t)$ of a differential equation of the form

$$\boldsymbol{F}(t,\boldsymbol{x}) = m\tfrac{d^2}{dt^2}\boldsymbol{x}(t) \tag{267}$$

This equation conforms to the **principle of Galilean covariance** (and it was from this circumstance that historically we acquired our interest in the "population

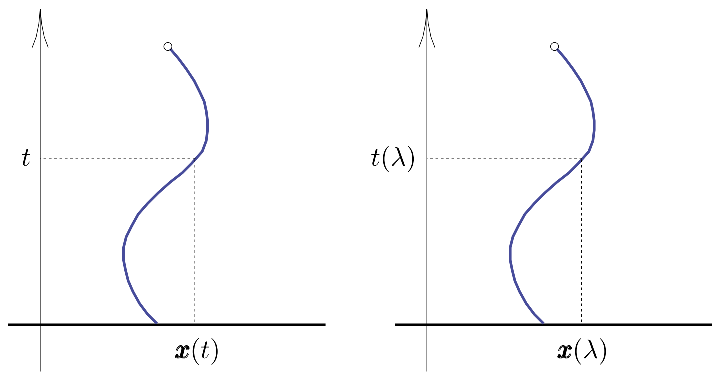

FIGURE 67: *At left: the time-parameterized flight of a particle, standard to Newtonian mechanics, where t is assigned the status of an independent variable and $\boldsymbol{x}$ is a set of dependent variables. At right: arbitrarily parameterization permits t to join the list of dependent variables; i.e., to be treated co-equally with $\boldsymbol{x}$.*

of inertial observers"), but *its Lorentz non-covariance is manifest* ...for the equations treats t and $\boldsymbol{x}$ with a distinctness which the Lorentz transformations do not allow because they do not preserve. We confront therefore this problem: <u>How to describe a worldline in conformity with the requirement that space and time coordinates be treated *co-equally*</u>? One's first impulse it to give up t-parameterization in favor of an *arbitary* parameterization of the worldline (Figure 67), writing $x^\mu(\lambda)$. This at least treats space and time co-equally ...but leaves every inertial observer to his own devices: the resulting theory (kinematics) would be too sloppy to support sharp physics. The "slop" would, however, disappear if λ could be assigned a "natural" meaning—a meaning which *stands in the same relationship to all inertial observers*. Einstein's idea—foreshadowed already on page 186—was to assign to λ the meaning/value of "time as measured by a comoving clock." The idea is implemented as follows (see Figure 68): Let O write $x(\lambda)$ to describe a worldline, and let him write

$$dx(\lambda) \equiv x(\lambda + d\lambda) - x(\lambda) = \begin{pmatrix} cdt \\ \boldsymbol{dx} \end{pmatrix}$$

to describe the interval separating a pair of "neighboring points" (points on the tangent at $x(\lambda)$). If and <u>only if $dx(\lambda)$ is timelike</u> will O be able to *boost to the instantaneous restframe* (*i.e.*, to the frame of an observer O who sees the particle to be *momentarily resting at her origin*):

$$\begin{pmatrix} cdt \\ \boldsymbol{dx} \end{pmatrix} = \mathbb{\Lambda}(\boldsymbol{\beta}) \begin{pmatrix} cd\tau \\ \mathbf{0} \end{pmatrix}$$

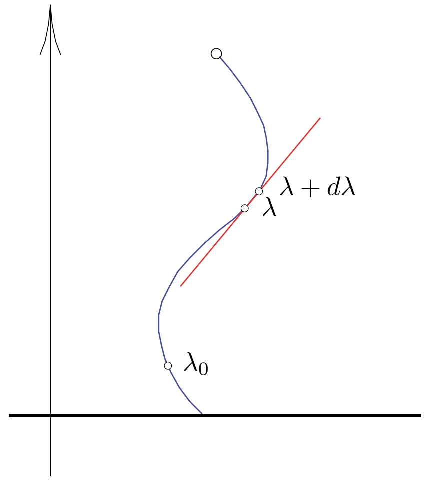

FIGURE 68: *An accelerated observer/particle borrows his/its* proper time *increments $d\tau$ from the wristwatches of momentarily comoving inertial observers*.

where from the boost-invariant structure of spacetime it follows that

$$\begin{aligned} d\tau &= \sqrt{(dt)^2 - \tfrac{1}{c^2}\boldsymbol{dx}\cdot\boldsymbol{dx}} = \sqrt{1-\beta^2(t)}\,dt \\ &\equiv \text{time differential measured by instantaneously comoving clock} \\ &= \tfrac{1}{c}\sqrt{\big(\tfrac{dx^0}{d\lambda}\big)^2 - \big(\tfrac{dx^1}{d\lambda}\big)^2 - \big(\tfrac{dx^2}{d\lambda}\big)^2 - \big(\tfrac{dx^3}{d\lambda}\big)^2}\,d\lambda \\ &= \tfrac{1}{c}ds \end{aligned} \tag{268}$$

The <u>*proper time* τ</u> associated with a *finitely*-separated pair of points is defined

$$\begin{aligned} \tau(\lambda,\lambda_0) &= \tfrac{1}{c}\int_{\lambda_0}^{\lambda}\sqrt{g_{\alpha\beta}\frac{dx^\alpha(\lambda')}{d\lambda'}\frac{dx^\beta(\lambda')}{d\lambda'}}\,d\lambda' = \int d\tau = \frac{\text{arc-length}}{c} \\ &= 0 \text{ at } \lambda = \lambda_0 : \begin{cases} x(\lambda_0) \text{ is the reference point at which} \\ \text{we “start the proper clock”} \end{cases} \end{aligned}$$

Functional inversion gives

$$\lambda = \lambda(\tau,\lambda_0)$$

and in place of $x(\lambda)$ it becomes natural to write

$$x(\tau) \equiv x(\lambda(\tau,\lambda_0)) \quad : \quad \tau\text{-parameterized description of the worldline}$$

Evidently τ-parameterization is equivalent (to within a c-factor) to *arc-length parameterization*—long known by differential geometers to be “most natural” in metric spaces. Two points deserve comment:

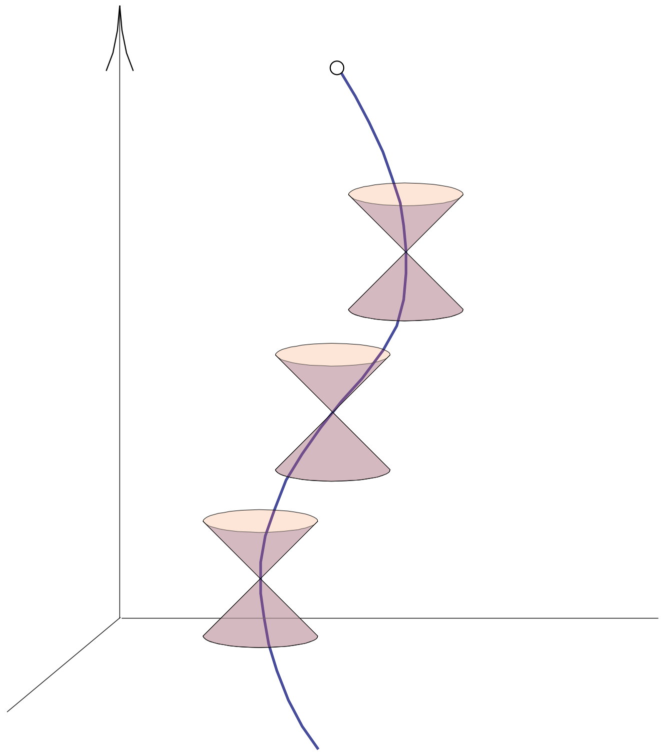

FIGURE 69: *The worldline of a masspoint lies everywhere interior to lightcones with vertices on the worldline. The spacetime interval separating any two points on a worldline is therefore time-like, and the constituent points of the worldline fall into a temporal sequence upon which all inertial observers agree.*

1. Einstein's program works if and only if all tangents to the worldline are *timelike* (Figure 69). One cannot, therefore, τ-parameterize the worldline of a photon. Or of a "tachyon." The reason is that one cannot boost such particles to rest: one cannot Lorentz transform the tangents to such worldlines into local coincidence with the x^0-axis.

2. The $d\tau$'s in $\int d\tau$ refer to a *population of osculating inertial observers*. It is a big step—a step which Einstein (and also L. H. Thomas) considered quite "natural," but a big step nonetheless—to suppose that τ has anything literally to do with "time as measured by a comoving (which in the general case means an accelerating) clock." The relativistic dynamics of particles is, in fact, *independent* of whether attaches literal meaning to the preceding phrase. Close reading of Einstein's paper shows, however, that he *did* intend to be understood literally (even though—patent clerk that he was—he would not have expected his mantle clock to keep good time if jerked about). Experimental evidence supportive of Einstein's view derives from the *decay of accelerated radioactive* *particles* and from recent observations pertaining to the so-called *twin paradox* (see below).

Given a τ-parameterized (whence everywhere timelike) worldline $x(\tau)$, we define by

$$u(\tau) \equiv \tfrac{d}{d\tau}x(\tau) = \tfrac{dt}{d\tau}\tfrac{d}{dt}\begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \gamma\begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix} \tag{269}$$

the *4-velocity* $u^\mu(\tau)$, and by

$$\begin{aligned} a(\tau) &\equiv \tfrac{d^2}{d\tau^2}x(\tau) \\ &= \tfrac{d}{d\tau}u(\tau) = \tfrac{dt}{d\tau}\tfrac{d}{dt}\gamma\begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{c}\gamma^4(\boldsymbol{a}\cdot\boldsymbol{v}) \\ \gamma^2\boldsymbol{a} + \frac{1}{c^2}\gamma^4(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{v} \end{pmatrix} \end{aligned} \tag{270}$$

the *4-acceleration* $a^\mu(\tau)$. These are equations written by inertial observer O: $\boldsymbol{v}$ refers to O's perception of the particle's instantaneous velocity $\boldsymbol{v}(t)$, and $\gamma \equiv \big[1 - \frac{1}{c^2}\boldsymbol{v}\cdot\boldsymbol{v}\big]^{-\frac{1}{2}}$.[147] Structurally similar equations (but with everything turned red) would be written by a second observer O. In developing this aspect of the subject one must be very careful to distinguish—both notationally and conceptually—the following:

O's perception of the instantaneous particle velocity $\boldsymbol{v}$
O's perception of O's velocity $\boldsymbol{s}$
O's perception of the instantaneous particle velocity $\boldsymbol{v}$

Supposing O and O to be boost-equivalent (no frame rotation)

$$x = \mathbb{\Lambda}(\boldsymbol{s}/c)\,x$$

we have

$$u = \mathbb{\Lambda}(\boldsymbol{s}/c)\,u \tag{271.1}$$
$$a = \mathbb{\Lambda}(\boldsymbol{s}/c)\,a \tag{271.2}$$

These equations look simple enough, but their explcit meaning is—owing to the complexity of $\mathbb{\Lambda}(\boldsymbol{s}/c)$, of u^μ and particularly of a^μ—actually quite complex. I will develop the detail only when forced by explicit need.[148]

It follows from (269) that

$$(u,u) = g_{\alpha\beta}u^\alpha u^\beta = \gamma^2(c^2 - v^2) = c^2\cdot\gamma^2(1-\beta^2) = c^2 \tag{272}$$

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[147] PROBLEM 49.
[148] In the meantime, see my ELECTRODYNAMICS (1972/73), pages 202–205.

according to which <u>all velocity 4-vectors have the same Lorentzian length</u>. All are, in particular (since $(u,u) = c^2 > 0$), *timelike*. Differentiating (272) with respect to τ we obtain

$$\tfrac{d}{d\tau}(u,u) = 2(u,a) = 0 \tag{273}$$

according to which <u>it is invariably the case that $u \perp a$ in the Lorentzian sense</u>. It follows now from the timelike character of u that all acceleration 4-vectors are *spacelike*. Direct verification of these statements could be extracted from (269) and (270). The statement $(u,u) = c^2$—of which (273) is an immediate corollary—has no precursor in non-relativistic kinematics,[149] but is, as will emerge, absolutely fundamental to relativistic kinematics/dynamics.

Looking "with relativistic eyes" to Newton's 2nd law (267) we write

$$K^\mu = m\tfrac{d^2}{d\tau^2}x^\mu(\tau) \tag{274}$$

This equation would be Lorentz covariant—*manifestly* covariant—if

$$K^\mu \equiv \textit{Minkowski force} \text{ transforms like a 4-vector}$$

and m transforms as an invariant. The Minkowski equation (274) can be reformulated

$$K^\mu = m\tfrac{d}{d\tau}u^\mu = ma^\mu$$

or again

$$= \tfrac{d}{d\tau}p^\mu$$

where

$$p^\mu \equiv mu^\mu = \gamma m\begin{pmatrix} c \\ \boldsymbol{v}\end{pmatrix} \equiv \begin{pmatrix} p^0 \\ \boldsymbol{p}\end{pmatrix} \tag{275}$$

From the γ-expansion (202) we obtain

$$\begin{aligned} p^0 &= \gamma mc && (276.1)\\ &= \big(1 + \tfrac{1}{2}\beta^2 + \tfrac{3}{8}\beta^4 + \cdots\big)mc \\ &= \tfrac{1}{c}\big(mc^2 + \tfrac{1}{2}mv^2 + \cdots\big) \end{aligned}$$

familiar from non-relativistic dynamics as <u>kinetic energy</u>

$$\begin{aligned} \boldsymbol{p} &= \gamma m\boldsymbol{v} && (276.2)\\ &= m\boldsymbol{v} + \cdots \end{aligned}$$

familiar from non-relativistic dynamics as <u>linear momentum</u>

It becomes in this light reasonable to call p^μ the *energy-momentum* 4-vector.

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[149] The constant speed condition

$$\boldsymbol{v}\cdot\boldsymbol{v} = \text{constant}$$

is sometimes encountered, but has no claim to "universality" in non-relativistic physics: *when* encountered (as in uniform circular motion), it entails $\boldsymbol{v} \perp \boldsymbol{a}$.

Looking to the finer details of standard relativistic terminology . . . one writes

$$p^0 = \tfrac{1}{c}E \tag{277}$$

and calls
$$E = \gamma mc^2 = mc^2 + \tfrac{1}{2}mv^2 + \cdots$$

the *relativistic energy*. More particularly

$$\begin{aligned} E_0 &\equiv mc^2 && \text{is the } \textit{rest energy} \\ T &\equiv E - E_0 && \text{is the } \textit{relativistic kinetic energy} \end{aligned} \tag{278}$$

In terms of the $\boldsymbol{v}$-dependent “relativistic mass” defined[150]

$$M \equiv \gamma m = \frac{m}{\sqrt{1 - v^2/c^2}} \tag{279}$$

we have

$$E = Mc^2 \tag{280.1}$$

and
$$T = (M - m)c^2 = \left\{ \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right\} mc^2$$

The *relativistic momentum* can in this notation be described

$$\boldsymbol{p} = M\boldsymbol{v} \tag{280.2}$$

It is—so far as I can tell—the “non-relativistic familiarity” of (280.2) that tempts some people[151] to view (283) as the fruit of an astounding “empirical discovery,” lying (they would have us believe) near the physical heart of special relativity. But (283) is, I insist, a *definition*—an occasional convenience, nothing more—one incidental detail among many in a coherent theory. It is naive to repeat the tired claim that “in relativity mass becomes velocity dependent:” it is profoundly wrongheaded to attempt to force relativistic dynamics to look less relativistic than it is.

We have
$$p = \begin{pmatrix} \frac{1}{c}E \\ \boldsymbol{p} \end{pmatrix} = mu$$

and from (272) it follows that

$$(p, p) = (E/c)^2 - \boldsymbol{p}\cdot\boldsymbol{p} - m^2c^2 \tag{281}$$

This means that p lies always on a certain m-determined hyperboloid (called the “*mass shell*”: see Figure 70) in 4-dimensional energy-momentum space.

[150] It becomes natural in the following context to call m the *rest mass*, though in grown-up relativistic physics there is *really no other kind*. Those who write m when they mean M are obliged to write m_0 to distinguish the rest mass.
[151] See, for example, A. P. French, *Special Relativity: The MIT Introductory Physics Series* (1968), page 23.

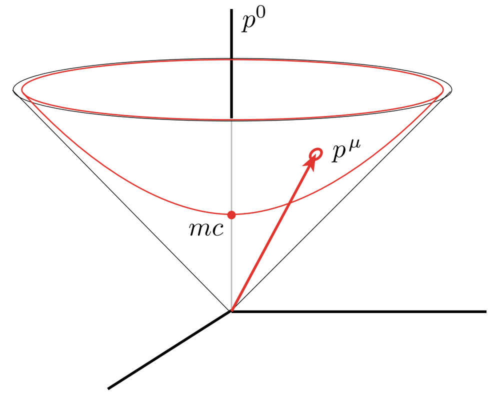

FIGURE 70: *The hyperboloidal mass shell, based upon (281) and drawn in energy-momentum space. The p^0-axis (energy axis) runs up. The mass shell intersects the p^0-axis at a point determined by the value of m:*

$$p^0 = mc \quad i.e., \quad E = mc^2$$

The figure remains meaningful (though the hyperboloid becomes a cone) even in the limit $m \downarrow 0$, which provides first indication that relativistic mechanics supports a theory of massless particles.

From (281) we obtain

$$\begin{aligned} E &= \pm c\sqrt{\boldsymbol{p}\cdot\boldsymbol{p} + (mc)^2} \\ &= \pm\Big\{mc^2 + \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p} + \cdots\Big\} \end{aligned} \tag{282}$$

which for a relativistic particle describes the $\boldsymbol{p}$-dependence of the energy E, and should be compared with its non-relativistic free-particle counterpart

$$E = \tfrac{1}{2m}\boldsymbol{p}\cdot\boldsymbol{p}$$

The $\pm$ assumes major importance in relativistic *quantum* mechanics (where it must be explained away lest it provide a rathole that would de-stabilize the world!), but in relativistic classical mechanics one simply *abandons* the minus sign—dismisses it as an algebraic artifact.

Looking next to the structure of K^μ ...ot follows from the Minkowski equation $K = ma$ by $(u, a) = 0$ that

$$(K, u) = 0 \quad : \quad K \perp u \text{ in the Lorentzian sense} \tag{283}$$

We infer that the 4-vectors that describe Minkowski forces are *invariably spacelike*. It follows moreover from (283) that as $p \sim u$ moves around the K-vector must move in concert, contriving always to be $\perp$ to u: <u>in relativistic</u>

dynamics all forces are velocity-dependent. What was fairly exceptional in non-relativistic dynamics (where $\boldsymbol{F}_{\text{damping}} = -b\,\boldsymbol{v}$ and $\boldsymbol{F}_{\text{magnetic}} = (q/c)\,\boldsymbol{v}\times\boldsymbol{B}$ are the only vecocity-dependent forces that come readily to mind) is in relativistic dynamics *universal*. Symbolically

$$K = K(u,\ldots)$$

where the dots signify such other variables as may in particular cases enter into the construction of K. The simplest case—which is, as we shall see, the case of electrodynamical interest—arises when K depends *linearly* on u:

$$K_\mu = A_{\mu\nu}u^\nu \tag{284.1}$$

where $(K,u) = A_{\mu\nu}u^\mu u^\nu = 0$ forces the quantities $A_{\mu\nu}(\ldots)$ to satisfy the

$$\text{antisymmetry condition} \quad : \quad A_{\mu\nu} = -A_{\nu\mu} \tag{284.2}$$

K-vectors that depend *quadratically* upon u exist in much greater variety: the following example

$$K_\mu = \phi^\alpha(x)\big[c^2 g_{\alpha\mu} - u_\alpha u_\mu\big]$$

figured prominently in early (unsuccessful) efforts to construct a special relativistic theory of gravitation.[152,153]

If K is notated

$$K = \begin{pmatrix} K^0 \\ \boldsymbol{K} \end{pmatrix} \tag{285}$$

then (283)—written $\gamma(K^0 c - \boldsymbol{K}\cdot\boldsymbol{v}) = 0$—entails

$$K^0 = \tfrac{1}{c}\boldsymbol{K}\cdot\boldsymbol{v} \quad : \quad \text{knowledge of } \boldsymbol{K} \text{ determines } K^0 \tag{286}$$

It follows in particular that

$$K^0 = 0 \quad \text{in the (momentary) } \textit{rest frame} \tag{287}$$

It is, of course, the *non*-zero value of $\boldsymbol{K}$ that causes the particle to take leave of (what a moment ago was) the rest frame. Borrowing notation from (275) and

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[152] This work (∼1912) is associated mainly with the name of G. Nordström, but for a brief period engaged the enthusiastic attention of Einstein himself: see page 144 in Pauli,[135] and also A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (1965), page 56; A. Pais, *Subtle is the Lord: The Science and Life of Albert Einstein* (1982), page 232.

[153] For further discussion of the "general theory of K-construction" see my RELATIVISTIC DYNAMICS (1967), pages 13–22.

(285), the Minkowski equation (274) becomes

$$\begin{pmatrix} K^0 \\ \boldsymbol{K} \end{pmatrix} = \gamma\tfrac{d}{dt}\begin{pmatrix} \gamma mc \\ \gamma m\boldsymbol{v} \end{pmatrix} \tag{288}$$

where use has been made once again of $\frac{d}{d\tau} = \gamma\frac{d}{dt}$. In the *non-relativistic limit*

$$\begin{pmatrix} 0 \\ \boldsymbol{F} \end{pmatrix} \overset{\downarrow}{=} \begin{pmatrix} 0 \\ \frac{d}{dt}m\boldsymbol{v} \end{pmatrix} \longleftarrow \text{Newtonian!}$$

where we have written

$$\boldsymbol{F} \equiv \lim_{c\uparrow\infty} \boldsymbol{K} \tag{289}$$

to account for such c-factors as may lurk in the construction of $\boldsymbol{K}$. We are used to thinking of the "non-relativistic limit" as an *approximiation* to relativistic physics, but at this point it becomes appropriate to remark that

> In fully relativistic particle dynamics the "non-relativistic limit" becomes literally effective in the momentary rest frame.

The implication is that *if we knew* the force $\boldsymbol{F}$ experienced by a particle *at rest* then we could by Lorentz transformation *obtain the Minkowski force K active upon a moving particle*:

$$\begin{pmatrix} K^0 \\ \boldsymbol{K} \end{pmatrix} = \mathbb{\Lambda}(\boldsymbol{\beta})\begin{pmatrix} 0 \\ \boldsymbol{F} \end{pmatrix} \tag{290}$$

Reading from (210.1) it follows more particularly that

$$\left.\begin{aligned} K^0 &= \gamma\tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{F} \\ \boldsymbol{K} &= \boldsymbol{F} + \big\{(\gamma-1)(\boldsymbol{v}\cdot\boldsymbol{F})/v^2\big\}\,\boldsymbol{v} = \boldsymbol{F}_\perp + \gamma\boldsymbol{F}_\parallel \end{aligned}\right\} \tag{291}$$

from which, it is gratifying to observe, one can recover both (289) and (286).

We stand not (at last) in position to trace the details of the program proposed[154] in a specifically electrodynamical setting by Einstein. Suppose that a charged particle experiences a force

$$\boldsymbol{F} = q\boldsymbol{E} \;\;:\;\; \boldsymbol{E} \equiv \text{electrical field in the particle's rest frame}$$

Then

$$\boldsymbol{K} = q(\boldsymbol{E}_\perp + \gamma\boldsymbol{E}_\parallel)$$

But from the field transformation equations (263) it follows that

$$\begin{aligned} \boldsymbol{E}_\perp &= \gamma(\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})_\perp \\ \boldsymbol{E}_\parallel &= (\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B})_\parallel \end{aligned}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ refer to *our* perception of the electric and magnetic fields at the particle's location, and $\boldsymbol{\beta}$ to our perception of the particle's velocity. So (because the γ-factors interdigitate so sweetly) we have

$$\boldsymbol{K} = \gamma q(\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{B}) \tag{292}$$

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[154] See again page 186.

But (288) supplies $\boldsymbol{K} = \gamma\frac{d}{dt}(\gamma m\boldsymbol{v})$, so (dropping the γ-factors on left and right) we have[155]

$$q(\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{B}) = \tfrac{d}{dt}(\gamma m\boldsymbol{v}) \tag{293}$$

This famous equation describes the <u>relativistic motion of a charged particle in an impressed electromagnetic field</u> (no radiation or radiative reaction), and is the upshot of[156] the *Lorentz force law*—obtained here not as an it ad hoc assumption, but as a forced consequence of

- some general features of relativistic particle dynamics
- the transformation properties of electromagnetic fields
- the operational definition of $\boldsymbol{E}$ ... all fitted into
- Einstein's "go to the frame of the particle" program (pages 186 & 189).

Returning with (292) to (286) we obtain

$$K^0 = \tfrac{1}{c}\gamma q\boldsymbol{E}\cdot\boldsymbol{v} \tag{294}$$

so the *Minkowski 4-force experienced by a charged particle in an impressed elecromagnetic field* becomes

$$\begin{aligned} K = \begin{pmatrix} K^0 \\ \boldsymbol{K} \end{pmatrix} &= \gamma q \begin{pmatrix} \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{v} \\ \boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B} \end{pmatrix} \\ &= (q/c)\begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}\begin{pmatrix} \gamma c \\ \gamma v_1 \\ \gamma v_2 \\ \gamma v_3 \end{pmatrix} \\ &\downarrow \\ K^\mu &= (q/c)F^\mu{}_\nu u^\nu \end{aligned} \tag{295}$$

We are brought thus to the striking conclusion that *the electromagnetic Minkowski force is, in the described at (284), <u>simplest possible</u>*.

The theory in hand descends from $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$, and might plausibly be called "relativistic Newtonian dynamics." Were we to continue this discussion we might expect to busy ourselves with the construction of

- a "relativistic Lagrangian dynamics"
- a "relativistic Hamiltonian dynamics"
- a "relativistic Hamilton-Jacobi formalism"
- "relativistic variational principles," *etc.*

—all in an effort to produce a full-blown "relativistic dynamics of particles." The subject[157] is, however, a minefield, and must be persued with much greater delicacy than the standard texts suggest. Relativistic particle mechanics

[155] PROBLEM 50.

[156] See again equation (67) on page 35.

[157] See the notes[153] already cited.

remains in a relatively primitive state of development because many of the concepts central to non-relativistic mechanics are—for reasons having mainly to do with the breakdown of non-local simultaneity—in conflict with the principle of relativity. But while the relativistic theory of interacting particles presents awkwardnesses at every turn, the relativistic theory of interacting fields unfolds with great ease and naturalness: *it appears to be a lesson of relativity that we should adopt a field-theoretic view of the world*.

We have already in hand a relativistic particle mechanics which, though rudimentary, is sufficient to our electrodynamic needs. Were we to pursue this subject we would want to look to the problem of *solving* Minkowski's equation of motion (274) isn illustrative special cases . . . any short list of which would include

- the relativistic harmonic oscillator
- the relativistic Kepler problem
- motion in a (spatially/temporally) constant electromagnetic field.

This I do on pages 245–275 of ELECTRODYNAMICS (1972/73), where I give also many references. The most significant point to emerge from that discussion is that *distinct relativistic systems can have the same non-relativistic limit; i.e., that constructing the relativistic generalization of a non-relativistic system is an inherently ambiguous process*. For the present I must be content to examine two physical questions that have come already to the periphery of our attention.

$\boxed{\text{HYPERBOLIC MOTION: THE “TWIN PARADOX”}}$ We—who call ourselves O—are inertial. A second observer Q sits on a mass point m which we see to be moving with (some dynamically possible but otherwise) arbitrary motion along our x-axis. I am tempted to say that Q rides in a little rocket, but that would entail (on physical grounds extraneous to my main intent) the temporal variability of m: let us suppose therefore that Q moves (accelerates) because m is acted on by impressed forces. In any event, we imagine Q to be equipped with

- a **clock** which—since co-moving—measures proper time τ
- an **accelerometer**, with output g. If Q were merely a passenger then $g(\tau)$ would constitute a king of log. But if Q were a rocket captain then $g(\tau)$ might describe his flight instructions, his prescribed "throttle function."

Finally, let O_τ designate the *inertial* observer who at proper time τ sees O_τ to be instantaneously at rest: spacetime points to which we assign coordinates x are by O_τ assigned coordinates x_τ. Our interest attaches initially to questions such as the following: Given the throttle function $g(\tau)$,

1) What is the boost $\Lambda(\tau)$ associated with $O \leftarrow O_\tau$?
2) What is the functional relationship between t and τ?
3) What are the functions
   $x(t)$ that describes our sense of Q's position at time t
   $\beta(t)$ that describes our sense of Q's velocity at time t
   $a(t)$ that describes our sense of Q's acceleration at time t?

Since O_τ sees Q to be momentarily resting at O_τ's origin we have

$$\begin{aligned} u(\tau) &= \mathbb{\Lambda}(\tau)\begin{pmatrix} c \\ 0 \end{pmatrix} && \text{by (269)} \\ a(\tau) &= \mathbb{\Lambda}(\tau)\begin{pmatrix} 0 \\ g(\tau) \end{pmatrix} && \text{by (269)} \end{aligned} \tag{296}$$

But

$$= \frac{du(\tau)}{d\tau} = \frac{d\mathbb{\Lambda}(\tau)}{d\tau}\begin{pmatrix} c \\ 0 \end{pmatrix}$$

We know, moreover, that[158]

$$\mathbb{\Lambda}(\tau) = e^{A(\tau)\mathbb{J}} \qquad \text{with} \quad \mathbb{J} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; A(\tau) = \tanh^{-1}\beta(\tau)$$

so
$$\begin{aligned} \frac{d\mathbb{\Lambda}(\tau)}{d\tau} &= \mathbb{\Lambda}(\tau)\cdot\frac{dA(\tau)}{d\tau}\mathbb{J} \\ \frac{dA(\tau)}{d\tau} &= \frac{1}{1-\beta^2}\frac{d\beta}{d\tau} \end{aligned}$$

Returning with this information to (296) we obtain

$$\frac{1}{1-\beta^2}\frac{d\beta}{d\tau} = \tfrac{1}{c}g(\tau)$$

where integration of $dt/d\tau = \gamma$ supplies

$$\tau = \int^t \sqrt{1-\beta^2(t')}\,dt' \tag{297}$$

Given $g(\bullet)$, our assignment therefore is to solve

$$\Big[\frac{1}{1-\beta^2(t)}\Big]^{\frac{3}{2}}\frac{d\beta(t)}{dt} = \tfrac{1}{c}g\big(\int^t \sqrt{1-\beta^2(t')}\,dt'\big) \tag{298}$$

for $\beta(t)$: a final integration would then supply the $x(t)$ that describes our perception of Q's worldline. The problem presented by (298) appears in the general case to be hopeless ... but let us at this point **assume** that the throttle function has the simple structure

$$g(\tau) = g \quad : \quad \textit{constant}$$

[158] See again pages 138 and 139.

The integrodifferential equation (298) then becomes a differential equation which integrates at once: assuming $\beta(0) = 0$ we obtain $\beta/\sqrt{1-\beta^2} = (g/c)t$ giving

$$\beta(t) = \frac{t}{\sqrt{(c/g)^2 + t^2}} \tag{299.1}$$

By integration we therefore have[159]

$$\big[x(t) - x(0) + (c^2/g)\big]^2 - (ct)^2 = (c^2/g)^2 \tag{299.2}$$

and

$$\tau(t) = (c/g)\sinh^{-1} gt/c \tag{299.3}$$

while expansion in powers of gt/c (which presumes $gt \ll c$) gives

$$\begin{aligned} v(t) &= \underbrace{\phantom{x(0) + \tfrac{1}{2}}gt}\big[1 - \tfrac{1}{2}(gt/c)^2 + \cdots\big] \\ x(t) &= x(0) + \tfrac{1}{2}gt^2\big[1 - \tfrac{1}{4}(gt/c)^2 + \cdots\big] \\ \tau(t) &= \underbrace{\phantom{x(0) + \tfrac{1}{2}g}t\big[1}_{\text{conform to non-relativistic experience}} - \tfrac{1}{6}(gt/c)^2 + \cdots\big] \end{aligned} \tag{300}$$

According to (299.2) we see Q to trace out (not a parabolic worldline, as in non-relativistic physics, but) a *hyperbolic worldline*, as shown in Figure 71.

The results now in hand place us in position to construct concrete illustrations of several points that have been discussed thus far only as vague generalities:

1. Equation (299.1) entails

$$\gamma(t) = \sqrt{1 + (gt/c)^2}$$

which places us in position to construct an explicit description

$$\underset{\substack{\uparrow \\ t\,=\,(c/g)\sinh g\tau/c\,,\ \text{by (299.3)}}}{\mathbb{A}(t)} = \gamma(t)\begin{pmatrix} 1 & \beta(t) \\ \beta(t) & 1 \end{pmatrix} \quad : \quad \text{recall (201)}$$

of the Lorentz matrix that achieves $O \leftarrow O_\tau$, and thus to answer a question posed on page 199. We can use that information to (for example) write

$$K(t) = ma(t) = \mathbb{A}(t)\begin{pmatrix} 0 \\ mg \end{pmatrix}$$

to describe the relationship between

$$K(t) \equiv \text{our perception of the Minkowski force impressed upon } m$$
$$\begin{pmatrix} 0 \\ mg \end{pmatrix} \equiv O\text{'s perception of that Minkowski force}$$

[159] PROBLEM 51.

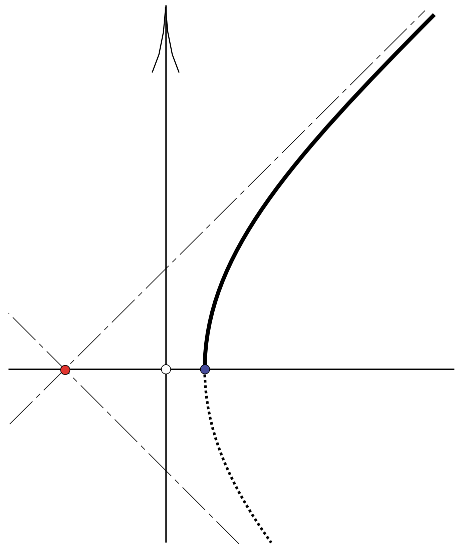

FIGURE 71: *Our (inertial) representation of the hyperbolic worldline of a particle which initially rests at the point $x(0)$ but moves off with (in its own estimation) constant acceleration g. With characteristic time c/g it approaches (and in Galilean physics would actually achieve) the speed of light. If we assign to g the comfortable value 9.8 meters/second^2 we find $c/g = 354.308$ days.*

2.* In (299.2) set $x(0) = 0$. The resulting spacetime hyperbola is, by notational adjustment $\frac{1}{2}\lambda \mapsto c^2/g$, identical to that encountered at the middle of page 176: our perception of Q's worldline is a *conformal transform* Q's own perception of her (from her point of view trivial) worldline. If Q elected to pass her time doing electrodynamics she would—though non-inertial—use equations that are structurally identical to the (conformally covariant) equations that we might use to describe those same electrodynamical events.

3. O is inertial, content to sit home at $x = 0$. Q—O's twin—is an astronaut, who at time $t = 0$ gives her brother a kiss and sets off on a flight along the x-axis, on which her instruction is to execute the following throttle function:

$$g(\tau) = \begin{cases} +g & : \quad 0 < \tau < \frac{1}{4}\mathfrak{T} \\ -g & : \quad \frac{1}{4}\mathfrak{T} < \tau < \frac{3}{4}\mathfrak{T} \\ +g & : \quad \frac{3}{4}\mathfrak{T} < \tau < \mathfrak{T} \end{cases}$$

Pretty clearly, O's representation of Q's worldline will be assembled from four hyperbolic segments (Figure 72), each of duration $(c/g)\sinh g\mathfrak{T}/4c$. At

* This remark will be intelligible only to those brave readers who ignored my recommendation that they skip §6.

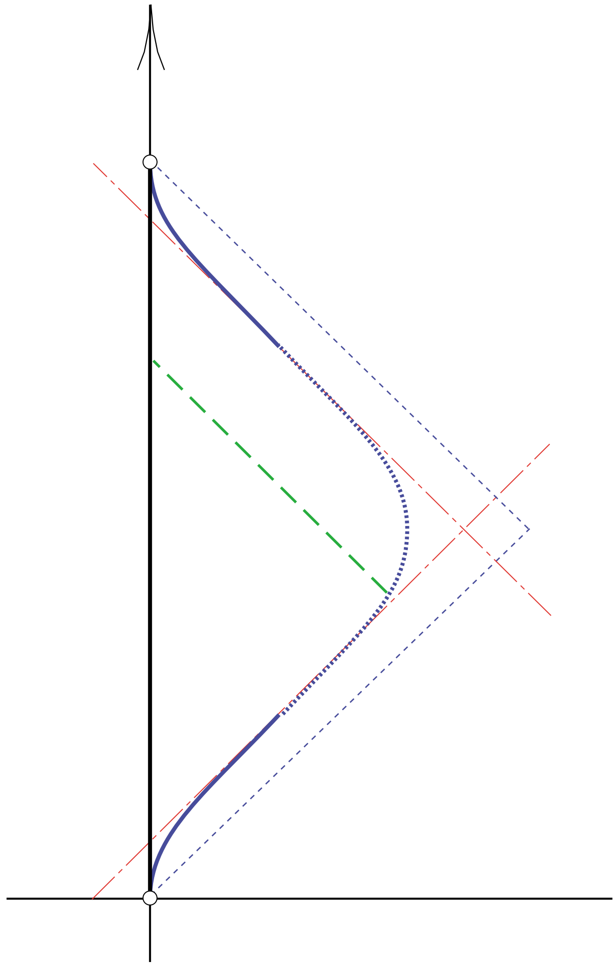

FIGURE 72: *Inertial observer O's representation of the rocket flight of his twin sister Q. If* $\mathfrak{T} \gg 4c/g$ *then O will see Q to be moving much of the time at nearly the speed of light (hyperbola approaches its asymptote). The dashed curve represents the flight of a lightbeam that departs/returns simultaneously with Q.*

the moment of her return the clock on Q's control panel will read $\mathfrak{T}$, but according to O's clock the

$$\text{return time} = \mathfrak{T}\cdot(4c/g\mathfrak{T})\sinh g\mathfrak{T}/4c = \begin{cases} > \mathfrak{T} \\ \sim \mathfrak{T} \quad \text{only if} \quad \mathfrak{T} \ll 4c/g \end{cases} \tag{301.1}$$

and Q's adventure will have taken her to a turn-around point lying[160] a

[160] Work from (299.2).

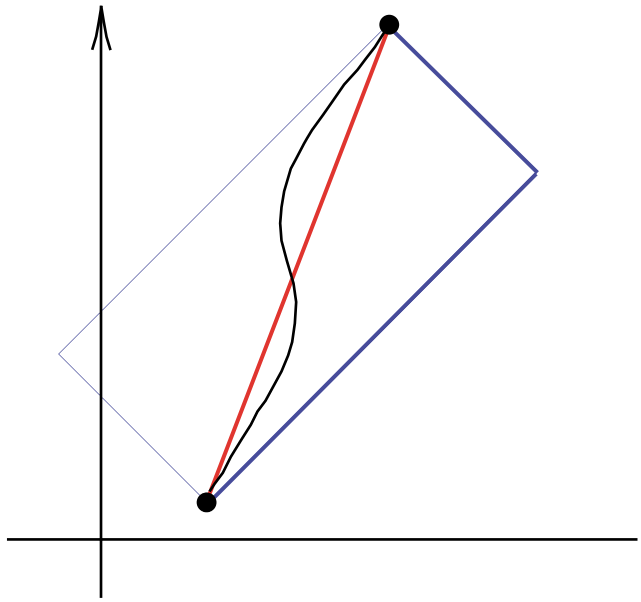

FIGURE 73: *Particle worldlines* $\bullet \to \bullet$ *all lie within the confines of the blue box (interior of the spacetime region bounded by the lightcones that extend forward form the lower vertex, and backward from the later vertex). The red trajectory—though shortest-possible in the Euclidean sense—is longest-possible in Minkowski's sense, while the blue trajectory has zero length. The "twin paradox" hinges on the latter fact. The acceleration experienced by the rocket-borne observer Q is, however, not abrupt (as at the kink in the blue trajectory) but evenly distributed.*

$$\text{distance} = 2\Big\{\sqrt{(ct)^2 + (c^2/g)^2} - c^2/g\Big\} \qquad (301.2)$$
$$t = \tfrac{1}{4}(\text{return time})$$

away. For brief trips we therefore have

$$\text{distance} = 2(c^2/g)\Big\{\sqrt{1+(gt/c)^2} - 1\Big\} = 2\cdot\Big\{\tfrac{1}{2}gt^2 + \cdots\Big\}$$

while for long trips

$$\text{distance} = 2ct\underbrace{\Big\{\sqrt{1+(c/gt)^2} - (c/gt)\Big\}}_{\text{this factor is always positive, always }<1,\text{ and approaches unity as }t\uparrow\infty}$$

—both of which make good intuitive sense.[161] Notice (as Einstein—at the end of §4 in his first relativity paper—was the first to do) that

$$Q \text{ is younger than } O \text{ upon her return}$$

and that this surprising fact can be attributed to a basic metric property of spacetime (Figure 73).[162] The so-called **twin paradox** arises when one argues that from Q's point of view it is O who has been doing the accelerating, and who should return younger ... and they can't *both* be younger! But those who pose the "paradox" misconstrue the meaning of the "relativity of motion." Only O remained inertial throughout the preceding exercise, and only Q had to purchase rocket fuel ... and those facts break the supposed "symmetry" of the situation. The issue becomes more interesting with the observation that *we* have spent our lives in (relative to the inertial frames falling through the floor) "a rocket accelerating upward with acceleration g" (but have managed to do so without an investment in "fuel"). <u>Why does our predicament not more nearly resemble the the predicament of Q than of O?</u>[163]

CURRENT-CHARGE INTERACTION FROM TWO POINTS OF VIEW We possess a command of relativistic electrodynamics/particle dynamics that is now so complete that we can contemplate detailed analysis of the "asymmetries" that served to motivate Einstein's initial relativistic work. The outline of the illustrative discussion which follows was brought to my attention by Richard Crandall.[164] The discussion involves rather more than mere "asymmetry:" on its face it involves a "paradox." The system of interest, and the problem it presents, are described in Figure 74. The observer O who is *at rest with respect to the wire* sees an electromagnetic field which (at points exterior to the wire) can be described

$$\boldsymbol{E} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 0 \\ -Bz/R \\ +By/R \end{pmatrix}$$

where $B = I/2\pi cR$ and $R = \sqrt{y^2 + z^2}$. The Minkowski 4-force experienced by q therefore becomes (see again (295))

$$\begin{pmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \end{pmatrix} = (q/c) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & By/R & Bz/R \\ 0 & -By/R & 0 & 0 \\ 0 & -Bz/R & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma c \\ \gamma v \\ 0 \\ 0 \end{pmatrix}$$

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[161] PROBLEM 52.

[162] PROBLEM 53.

[163] See at this point C. W. Sherwin, "Some recent experimental tests of the clock paradox," Phys. Rev. **120**, 17 (1960).

[164] For parallel remarks see §5.9 in E. M. Purcell's *Electricity & Magnetism: Berkeley Physics Course–Volume 2* (1965) and §13.6 of *The Feynman Lectures on Physics–Volume 2* (1964).

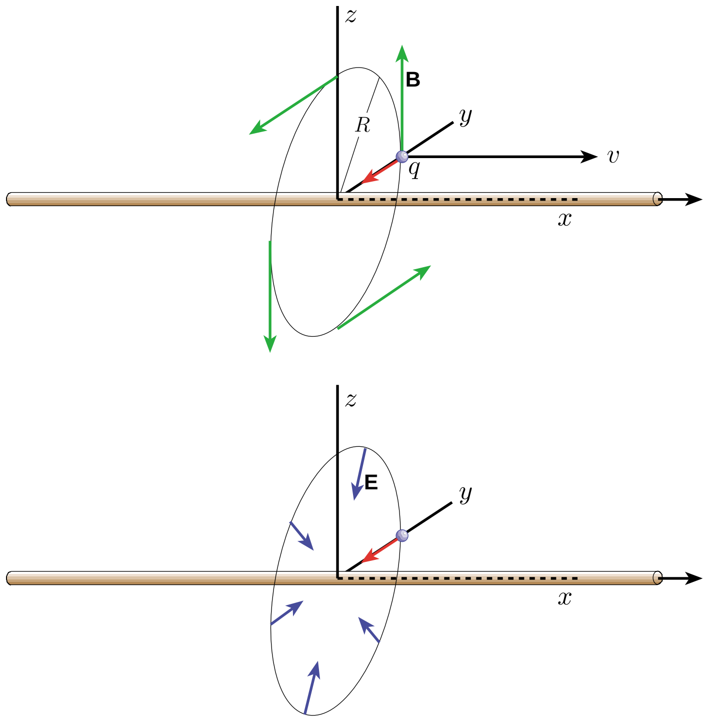

FIGURE 74: *At top: O's view of the system of interest . . . and at bottom: O's view. O—at rest with respect to a cylindrical conductor carrying current I—sees a charge q whose initial motion is parallel to the wire. He argues that the wire is wrapped round by a solenoidal magnetic field, so the moving charge experiences a* $(\boldsymbol{v}\times\boldsymbol{B})$*-force directed toward the wire, to which the particle responds by veering toward and ultimately impacting the wire. O is (initially) at rest with respect to the particle, so must attribute the impact an electrical force. But electrical forces arise (in the absence of time-dependent magnetic fields) only from charges. The nub of the problem*: How do uncharged current-carrying wires manage to appear charged to moving observers?

So we have

$$\begin{pmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -(\gamma qBv/c)y/R \\ -(\gamma qBv/c)z/R \end{pmatrix} = \begin{pmatrix} K^0 \\ \boldsymbol{K} \end{pmatrix}$$

according to which $\boldsymbol{K}$ is directed radially toward the wire. To describe this same physics O—who sees O to be moving *to the left* with speed v—writes

$$K = \Lambda K = (q/c)\cdot\underbrace{\Lambda\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & By/R & Bz/R\\ 0 & -By/R & 0 & 0\\ 0 & -Bz/R & 0 & 0\end{pmatrix}\Lambda^{-1}}_{\mathbb{F}}\cdot\underbrace{\Lambda\begin{pmatrix}\gamma c\\ \gamma v\\ 0\\ 0\end{pmatrix}}_{u}$$

with

$$\Lambda = \gamma\begin{pmatrix}1 & -\beta & 0 & 0\\ -\beta & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Straightforward computation supplies

$$= (q/c)\cdot\begin{pmatrix}0 & 0 & -\beta\gamma By/R & -\beta\gamma By/R\\ 0 & 0 & +\ \gamma By/R & +\ \gamma Bz/R\\ -\beta\gamma By/R & \gamma By/R & 0 & 0\\ -\beta\gamma Bz/R & \gamma Bz/R & 0 & 0\end{pmatrix}\begin{pmatrix}c\\ 0\\ 0\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}0\\ 0\\ -(\gamma qBv/c)y/R\\ -(\gamma qBv/c)z/R\end{pmatrix} = \begin{pmatrix}K^0\\ \boldsymbol{K}\end{pmatrix}$$

While O saw only a $\boldsymbol{B}$-field, it is clear from the computed structure of $\mathbb{F}$ that O sees both a $\boldsymbol{B}$-field (γ times stronger that O's) and an $\boldsymbol{E}$-field. We have known since (210.2) that

$$\text{(spatial part of any 4-vector)}_\perp \text{ boosts by invariance}$$

so (since $\boldsymbol{K}\perp\boldsymbol{v}$) are not surprised to discover that

$$\boldsymbol{K} = \boldsymbol{K},\ \text{ but observe that } \begin{cases} O \text{ considers } \boldsymbol{K} \text{ to a magnetic effect}\\ O \text{ considers } \boldsymbol{K} \text{ to an electric effect}\end{cases}$$

More specifically, O sees (Figure 74) a centrally-directed electric field of just the strength

$$E = \beta\gamma B = \beta\gamma I/2\pi cR$$

that would arise from an infinite line charge linear density

$$\lambda = -\beta\gamma I/c$$

The question now before us: *How* does the current-carrying wire acquire, in O's estimation, a net charge? An answer of sorts can be obtained as follows: Assume (in the interest merely of simplicity) that the current is uniformly distributed on the wire's cross-section:

$$I = ja \quad \text{where } a \equiv \pi r^2 = \text{cross-sectional area}$$

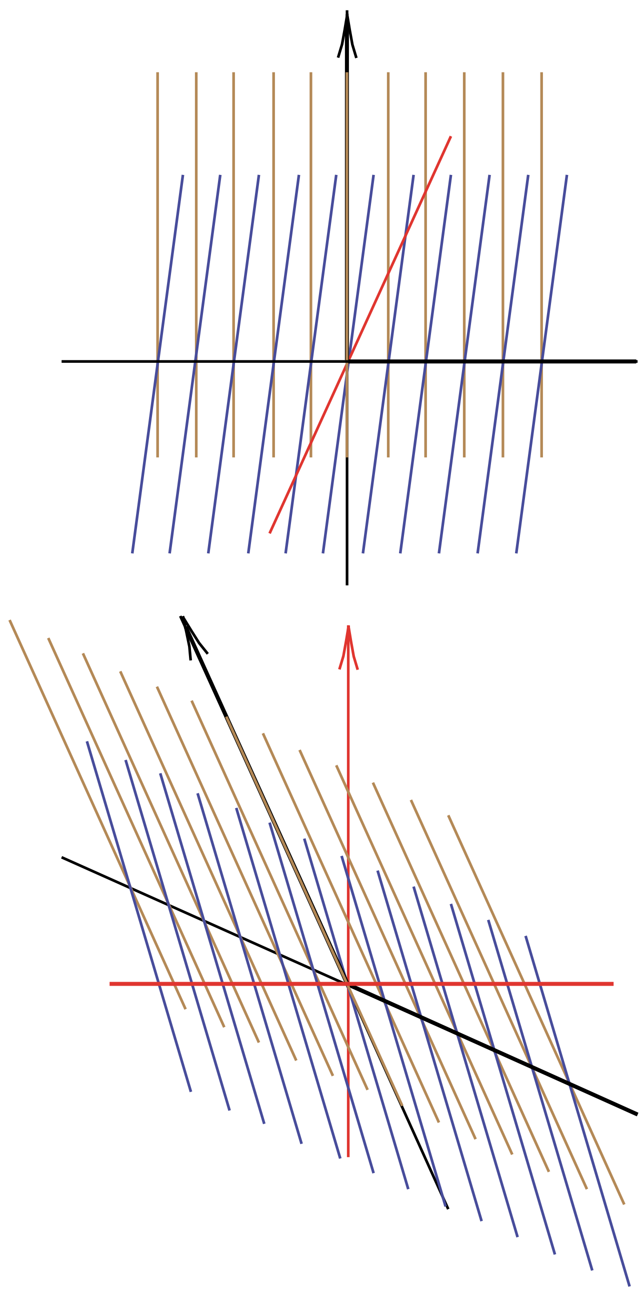

FIGURE 75: *O's representation of current flow in a stationary wire and (below) the result of Lorentz transforming that diagram to the frame of the passing charge q. For interpretive commentary see the text.*

To describe the *current 4-vector interior to the wire* O therefore writes

$$j = \begin{pmatrix} 0 \\ I/a \\ 0 \\ 0 \end{pmatrix}$$

O, on the other hand, writes the Lorentz transform of j:

$$j \equiv \begin{pmatrix} c\rho \\ \boldsymbol{j} \end{pmatrix} = \Lambda\, j = \begin{pmatrix} -\beta\gamma I/a \\ \gamma I/a \\ 0 \\ 0 \end{pmatrix} \implies \rho = -\beta\gamma I/ac$$

O and O assign identical values to the cross-sectional area

$$a = a \quad \text{because cross-section} \perp \boldsymbol{v}$$

so O obtains

$$\begin{aligned} \lambda \equiv \text{charge per unit length} &= \rho a \\ &= -\beta\gamma I/c \end{aligned}$$

—in precise agreement with the result deduced previously. Sharpened insight into the mechanism that lies at the heart of this counterintuitive result can be gained from a comparison of the spacetime diagrams presented in Figure 75. At top we see O's representation of current in a stationary wire: negatively ionized atoms stand in place, positive charges drift in the direction of current flow.[165] In the lower figure we see how the situation presents itself to an observer O who is moving with speed v in a direction parallel to the current flow. At any instant of time (look, for example, to his $x^0 = 0$ timeslice, drawn in red) O sees ions and charge carriers to have distinct linear densities . . . the reason being that she sees ions and charge carriers to be moving with distinct speeds, and the intervals separating one ion from the next, one charge carrier from the next to be Lorentz contracted by distinct amounts. O's charged wire is, therefore, a *differential Lorentz contraction* effect. That such a small velocity differential

$$\text{drift velocity relative to ions} \sim 10^{-11}c$$

can, from O's perspective, give rise to a measureable net charge is no more surprising than that it can, from O's perspective, give rise to a measureable net current: both can be attributed to the fact that an awful *lot* of charges participate in the drift.

[165] O knows perfectly well that in point of physical fact the ionized atoms are positively charged, the current carriers negatively charged, and their drift opposite to the direction of current flow: the problem is that Benjamin Franklin did not know that. But the logic of the argument is unaffected by this detail.

Just about *any* electro-mechanical system would yield similar asymmetries/ “paradoxes” when analysed by alternative inertial observers O and $\mathcal{O}$. The preceding discussion is in all respects typical, and serves to illustrate two points of general methodological significance:

- The formal mechanisms of (manifestly covariant) relativistic physics are so powerful that they tend to lead one *automatically* past conceptual difficulties of the sort that initially so bothered Einstein, and (for that very reason) . . .
- They tend, when routinely applied, to divert one’s attention from certain (potentially quite useful) physical insights: there exist points of physical principle which relativistic physics illuminates only when explicitly interrogated.

When using powerful tools one should always wear goggles.

3

MECHANICAL PROPERTIES OF THE ELECTROMAGNETIC FIELD

Densities, fluxes & conservation laws

Introduction. Energy, momentum, angular momentum, center of mass, moments of inertia . . . these are concepts which derive historically from the mechanics of particles. And it is from particle mechanics that—for reasons that are interesting to contemplate—they derive their intuitive force. But these are concepts which are now recognized to pertain, if in varying degrees, to the *totality* of physics. My objective here will be to review how the mechanical concepts listed above pertain, in particular, to the electromagnetic field. The topic is of great practical importance. But it is also of some philosophical importance . . . for it supplies the evidence on which we would assess the ontological question: *Is the electromagnetic field "real"*?

How to proceed? Observe that in particle mechanics the concepts in question arise not as "new physics" but as *natural artifacts implicit in the design of the equations of motion*. We may infer that the definitions we seek

i) will arise as "natural artifacts" from Maxwell's equations

ii) must mesh smoothly with their particulate counterparts.

But again: how—within those guidelines—to proceed? The literature provides many alternative lines of argument, the most powerful of which lie presently beyond our reach.[166] In these pages I will outline two complementary

[166] I am thinking here of the Lagrangian formulation of the classical theory of fields, which is usually/best studied as an antonomous subject, then *applied* to electrodynamics as a (rather delicate) special case.

approaches to the electrodynamical concepts of *energy* and *momentum*. The first approach is inductive, informal. The second is deductive, and involves formalism of a relatively high order. Both approaches (unlike some others) draw explicitly on the spirit and detailed substance of relativity. The discussion will then be extended to embrace *angular momentum* and certain more esoteric notions.

1. Electromagnetic energy/momentum: first approach. We know from prior work of an elementary nature[167] that it makes a certain kind of sense to write

$$\left.\begin{aligned}\tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E} &= \text{energy density of an electrostatic field}\\ \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} &= \text{energy density of a magnetostatic field}\end{aligned}\right\}\tag{302}$$

But what should we write to describe the energy density $\mathcal{E}$ of an unspecialized electro*dynamical* field? Relativity suggests that we should consider this question in intimate association with a second question: What should we write to describe the *momentum* density $\boldsymbol{\mathcal{P}}$ of an arbitrary electromagnetic field? We are led thus to anticipate[168] the theoretical importance of a *quartet of densities*

$$\mathcal{P} = \begin{pmatrix}\mathcal{P}^0\\ \mathcal{P}^1\\ \mathcal{P}^2\\ \mathcal{P}^3\end{pmatrix} \qquad \text{with } \mathcal{P}^0 \equiv \tfrac{1}{c}\mathcal{E}\tag{303}$$

where $[\mathcal{P}^\mu] = \text{momentum}/3\text{-volume}$.

Intuitively we expect *changes* in the energy/momentum at a spacetime point to arise from a combination of

1) the corresponding fluxes (or energy/momentum "currents")
2) the local action of charges (or "sources")

so *at source-free points* we expect[169] to have

$$\begin{aligned}\tfrac{\partial}{\partial t}\mathcal{E} + \boldsymbol{\nabla}\cdot(\text{energy flux vector}) &= 0\\ \tfrac{\partial}{\partial t}\mathcal{P}^1 + \boldsymbol{\nabla}\cdot(\text{flux vector associated with } 1^{\text{st}} \text{ component of momentum}) &= 0\\ \tfrac{\partial}{\partial t}\mathcal{P}^2 + \boldsymbol{\nabla}\cdot(\text{flux vector associated with } 2^{\text{nd}} \text{ component of momentum}) &= 0\\ \tfrac{\partial}{\partial t}\mathcal{P}^3 + \boldsymbol{\nabla}\cdot(\text{flux vector associated with } 3^{\text{rd}} \text{ component of momentum}) &= 0\end{aligned}$$

This quartet of conservation laws would be expressed quite simply

$$\partial_\mu S^{\mu\nu} = 0 \quad : \quad (\nu = 0,1,2,3)\tag{304}$$

[167] The argument proceeded from elementary mechanics in the electrostatic case (pages 19–24), but was more formal/tentative (page 60) and ultimately more intricate (pages 97–98) in the magnetostatic case.
[168] See again pages 193 and 194.
[169] See again pages 36–37.

if we were to set (here the Roman indices i and j range on $\{1,2,3\}$)

$$\left.\begin{aligned} \mathcal{E}\equiv c\mathcal{P}^0 \equiv S^{00} &\equiv \text{energy density}\\ S^{i\,0} &\equiv \tfrac{1}{c}(i^{\text{th}}\text{ component of the energy flux vector})\\ c\mathcal{P}^j \equiv S^{0j} &\equiv c\,(j^{\text{th}}\text{-component-of-momentum density})\\ S^{i\,j} &\equiv \quad (i^{\text{th}}\text{ component of the }\mathcal{P}^j\text{ flux vector}) \end{aligned}\right\}\tag{305}$$

where c-factors have been introduced to insure that the $S^{\mu\nu}$ all have the same dimensionality—namely that of $\mathcal{E}$.

Not only are equations (304) wonderfully compact, they seem on their face to be "relativistically congenial." They become in fact *manifestly Lorentz covariant* if it is assumed that

$$S^{\mu\nu}\text{ transforms as a }2^{\text{nd}}\text{ rank tensor}\tag{306}$$

of presently unspecified weight. This natural assumption carries with it the notable consequence that

$$\text{The }\mathcal{P}^\mu\equiv\tfrac{1}{c}S^{0\mu}\text{ do }\textit{not}\text{ transform as components of a 4-vector}$$

or even (as might have seemed more likely) as components of a 4-vector *density*.

The question from which we proceeded—How to describe $\mathcal{E}$ as a function of the dynamical field variables?—has now become sixteen questions: *How to describe* $S^{\mu\nu}$? But our problem is not on this account sixteen times harder, for (304) and (306) provide powerful guidance. Had we proceeded naively (*i.e.*, without reference to relativity) then we might have been led from the structure of (302) to the conjecture that $\mathcal{E}$ depends in the general case upon $\boldsymbol{E}\cdot\boldsymbol{E}$, $\boldsymbol{B}\cdot\boldsymbol{B}$, maybe $\boldsymbol{E}\cdot\boldsymbol{B}$ and upon scalars formed from $\dot{\boldsymbol{E}}$ and $\dot{\boldsymbol{B}}$ (terms that we would not see in static cases). Relativity suggests that $\mathcal{E}$ should then depend also upon $\boldsymbol{\nabla}\cdot\boldsymbol{E}$, $\boldsymbol{\nabla}\cdot\boldsymbol{B}$, $\boldsymbol{\nabla}\times\boldsymbol{E}$, $\boldsymbol{\nabla}\times\boldsymbol{B}$, ... but such terms are—surprisingly—absent from (302). Equations (304) and (306) enable us to recast this line of speculation ... as follows:

1) We expect $S^{\mu\nu}$ to be a tensor-valued function of $g_{\mu\nu}$, $F_{\mu\nu}$, $F^{\star}_{\mu\nu}$ and possibly of $\partial_\alpha F_{\mu\nu}$, $\partial_\alpha\partial_\beta F_{\mu\nu}$, ... with the property that
2) S^{00} gives back (302) in the electrostatic and magnetostatic cases. We require, moreover, that
3) In source-free regions it shall be the case that Maxwell's equations

$$\partial_\mu F^{\mu\nu}=0\text{ and }\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}+\partial_\lambda F_{\mu\nu}=0 \quad\Longrightarrow\quad \partial_\mu S^{\mu\nu}=0$$

Two further points merit attention:

4) Dimensionally $[S^{\mu\nu}]=[F^{\mu\nu}]^2$: $S^{\mu\nu}$ is in this sense a *quadratic* function of $F^{\mu\nu}$.
5) Source-free electrodyanmics contains but a single physical constant, namely c: it contains in particular *no natural length*[170]... so one must make do with *ratios* of ∂F-terms, which are transformationally unnatural.

[170] That's a symptom of the *conformal covariance* of the theory.

Motivated now by the 2nd and 4th of those points, we look to the explicit descriptions (159) and (161) of $\|F^{\mu\nu}\|$ and $\|G^{\mu\nu}\|$ and observe that by direct computation[171]

$$\left.\begin{aligned}
\|F^\mu{}_\alpha F^{\alpha\nu}\| &= \begin{pmatrix} \boldsymbol{E}\cdot\boldsymbol{E} & (\boldsymbol{E}\times\boldsymbol{B})_1 & (\boldsymbol{E}\times\boldsymbol{B})_2 & (\boldsymbol{E}\times\boldsymbol{B})_3 \\ (\boldsymbol{E}\times\boldsymbol{B})_1 & C_{11}+\boldsymbol{B}\cdot\boldsymbol{B} & C_{12} & C_{13} \\ (\boldsymbol{E}\times\boldsymbol{B})_2 & C_{21} & C_{22}+\boldsymbol{B}\cdot\boldsymbol{B} & C_{23} \\ (\boldsymbol{E}\times\boldsymbol{B})_3 & C_{31} & C_{32} & C_{33}+\boldsymbol{B}\cdot\boldsymbol{B} \end{pmatrix} \\
\|F^\mu{}_\alpha G^{\alpha\nu}\| &= -\boldsymbol{E}\cdot\boldsymbol{B}\|g^{\mu\nu}\| \\
\|G^\mu{}_\alpha G^{\alpha\nu}\| &= \begin{pmatrix} \boldsymbol{B}\cdot\boldsymbol{B} & (\boldsymbol{E}\times\boldsymbol{B})_1 & (\boldsymbol{E}\times\boldsymbol{B})_2 & (\boldsymbol{E}\times\boldsymbol{B})_3 \\ (\boldsymbol{E}\times\boldsymbol{B})_1 & C_{11}+\boldsymbol{E}\cdot\boldsymbol{E} & C_{12} & C_{13} \\ (\boldsymbol{E}\times\boldsymbol{B})_2 & C_{21} & C_{22}+\boldsymbol{E}\cdot\boldsymbol{E} & C_{23} \\ (\boldsymbol{E}\times\boldsymbol{B})_3 & C_{31} & C_{32} & C_{33}+\boldsymbol{E}\cdot\boldsymbol{E} \end{pmatrix} \\
&= \|F^\mu{}_\alpha F^{\alpha\nu}\| - (\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B})\cdot\|g^{\mu\nu}\|
\end{aligned}\right\} \tag{307}$$

where $C_{ij} \equiv -E_iE_j - B_iB_j$.[172] The arguments that gave (302) assumed in the first instance that $\boldsymbol{B} = \boldsymbol{0}$ and in the second instance that $\boldsymbol{E} = \boldsymbol{0}$, so provide no evidence whether we should in the general case expect the presence of an $\boldsymbol{E}\cdot\boldsymbol{B}$ term. If we assume tentatively that in the general case

$$S^{00} \equiv \mathcal{E} = \tfrac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E} + \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{B} + \lambda\boldsymbol{E}\cdot\boldsymbol{B} \quad : \quad \lambda \text{ an adjustable constant}$$

then we are led by (307) to write

$$\begin{aligned}
S^{\mu\nu} &= \tfrac{1}{2}F^\mu{}_\alpha F^{\alpha\nu} + \tfrac{1}{2}G^\mu{}_\alpha G^{\alpha\nu} - \lambda F^\mu{}_\alpha G^{\alpha\nu} \\
&= \tfrac{1}{2}F^\mu{}_\alpha F^{\alpha\nu} + \tfrac{1}{2}\big[F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{2}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu}\big] - \lambda\tfrac{1}{4}(F^{\alpha\beta}G_{\beta\alpha})g^{\mu\nu} \\
&= F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}F^{\alpha\beta}(F_{\beta\alpha} + \lambda G_{\beta\alpha})g^{\mu\nu}
\end{aligned} \tag{308}$$

We come now to the critical question: <u>Does the $S^{\mu\nu}$ of (308) satisfy (304)</u>? The answer can be discovered only by computation: we have

$$\partial_\mu S^\mu{}_\nu = \underbrace{(\partial_\mu F^{\mu\alpha})F_{\alpha\nu}}_{\textbf{a}} + \underbrace{F^{\mu\alpha}\partial_\mu F_{\alpha\nu}}_{\textbf{b}} - \underbrace{\tfrac{1}{4}\partial_\nu(F^{\alpha\beta}F_{\beta\alpha})}_{\textbf{c}} - \lambda\tfrac{1}{4}\partial_\nu(F^{\alpha\beta}G_{\beta\alpha})$$

---

[171] PROBLEM 54.

[172] Recall in this connection that the Lorentz invariance of

$$\begin{aligned}
\tfrac{1}{2}F^{\alpha\beta}F_{\beta\alpha} &= \boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B} = -\tfrac{1}{2}G^{\alpha\beta}G_{\beta\alpha} \\
\tfrac{1}{4}F^{\alpha\beta}G_{\beta\alpha} &= -\,\boldsymbol{E}\cdot\boldsymbol{B}
\end{aligned}$$

was established already in PROBLEM 48b.

But

$$
\begin{aligned}
\textbf{a} &= 0 \quad \text{by Maxwell: } \partial_\mu F^{\mu\alpha} = \tfrac{1}{c}J^\alpha \text{ and we have assumed } J^\alpha = 0\\
\textbf{b} &= \tfrac{1}{2}F^{\mu\alpha}(\partial_\mu F_{\alpha\nu} - \partial_\alpha F_{\mu\nu}) \quad \text{by antisymmetry of } F^{\mu\alpha}\\
&= \tfrac{1}{2}F^{\mu\alpha}(\partial_\mu F_{\alpha\nu} + \partial_\alpha F_{\nu\mu}) \quad \text{by antisymmetry of } F_{\mu\nu}\\
&= -\tfrac{1}{2}F^{\mu\alpha}\partial_\nu F_{\mu\alpha} \quad \text{by Maxwell: } \partial_\mu F_{\alpha\nu} + \partial_\alpha F_{\nu\mu} + \partial_\nu F_{\mu\alpha} = 0\\
&= \tfrac{1}{4}\partial_\nu(F^{\alpha\beta}F_{\beta\alpha})\\
&= \textbf{c}
\end{aligned}
$$

so we have

$$
\begin{aligned}
\partial_\mu S^\mu{}_\nu &= -\lambda\tfrac{1}{4}\partial_\nu(F^{\alpha\beta}G_{\beta\alpha})\\
&= \lambda\partial_\nu(\boldsymbol{E\cdot B})
\end{aligned}
$$

It is certainly not in general the case that $\boldsymbol{E\cdot B}$ is x-independent (as $\partial_\nu(\boldsymbol{E\cdot B})=0$ would require) so to achieve

$$= 0$$

we are <u>forced to set $\lambda = 0$</u>. Returning with this information to (308) we obtain

$$
\begin{aligned}
S^{\mu\nu} &= \tfrac{1}{2}(F^\mu{}_\alpha F^{\alpha\nu} + G^\mu{}_\alpha G^{\alpha\nu})\\
&= F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu}
\end{aligned}
\tag{309}
$$

. . . which possesses all of the anticipated/required properties (see again the list on page 213), and in addition posses two others: $S^{\mu\nu}$ is *symmetric*

$$S^{\mu\nu} = S^{\nu\mu} \tag{310}$$

and *traceless*

$$S^\alpha{}_\alpha = 0 \tag{311}$$

Equation (309) describes the elements of what is called the *electromagnetic <u>stress-energy tensor</u>*. Reading from (307) we obtain

$$
\|S^{\mu\nu}\| = \begin{pmatrix} \tfrac{1}{2}(E^2 + B^2) & (\boldsymbol{E}\times\boldsymbol{B})^\intercal \\ (\boldsymbol{E}\times\boldsymbol{B}) & \mathbb{T} \end{pmatrix}
$$

where $E^2 \equiv \boldsymbol{E\cdot E}$, $B^2 \equiv \boldsymbol{B\cdot B}$ and where

$$\mathbb{T} \equiv \|(\tfrac{1}{2}E^2\delta_{ij} - E_iE_j) + (\tfrac{1}{2}B^2\delta_{ij} - B_iB_j)\|$$

is the negative of what is—for historical reasons—called the "Maxwell stress tensor" (though it is, with respect to non-rotational elements of the Lorentz group, *not* a tensor!). Writing

$$
= \begin{pmatrix} \mathcal{E} & c\boldsymbol{\mathcal{P}}^\intercal \\ \tfrac{1}{c}\boldsymbol{S} & \mathbb{T} \end{pmatrix}
$$

we conclude (see again page 212) that

> $\mathcal{E} = \frac{1}{2}(E^2 + B^2)$ describes **energy density**. This construction was first studied by W. Thompson (Lord Kelvin) in 1853.
>
> $\boldsymbol{S} = c(\boldsymbol{E}\times\boldsymbol{B})$ describes **energy flux**. This construction was discovered by J. H. Poynting and (independently) by O. Heaviside in 1884. It is called the "Poynting vector" (though it is vectorial only with respect to the rotation group).
>
> $\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ describes **momentum density**, and was discovered by J. J. Thompson in 1893.
>
> The successive columns in $\mathbb{T}$ are **momentum fluxes** associated with the successive elements of $\boldsymbol{\mathcal{P}}$. The "stress tensor" was introduced by Maxwell, but to fill quite a different formal need.[173]

It is remarkable that the individual elements of the stress-energy tensor issued historically from so many famous hands . . . and over such a protracted period of time.

The following comments draw attention to aspects of the specific design (309) of the electromagnetic stress-energy tensor $S^{\mu\nu}$:

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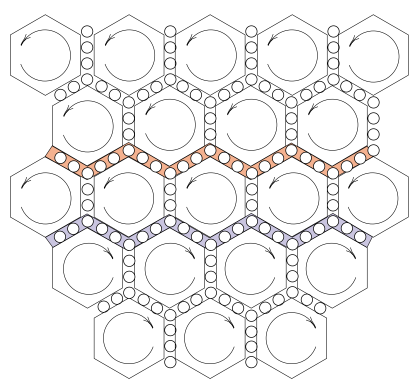
[173] Maxwell considered it to be his job to describe the "mechanical properties of the æther," and so found it natural to borrow concepts from fluid dynamics and the theory of elastic media. The following design—taken from his "On



physical lines of force" (1861)—illustrates how fantastic he allowed his mechanical imagination to become [see R. Tricker, *Contributions of Faraday & Maxwell to Electrical Science* (1966) page 118 or C. Everitt, *James Clerk Maxwell: Physicist & Natural Philosopher* (1975) page 96 for accounts of the idea the figure was intended to convey]. In his *Treatise* Maxwell writes that he was "only following out the conception of Faraday, that lines of force tend to shorten themselves, and that they repel each other when placed side by side: all that we have done is express the value of the tension along the lines, and the pressure at right angles to them, in mathematical language . . . "

1. Though we have already noted (page 213) that—in view of the facts that $[F^{\mu\nu}] = \sqrt{\text{energy density}}$ and electrodynamics supplies no "natural length"—it would be difficult to build ∂F-dependence into the design of $S^{\mu\nu}$, it still seems remarkable that we have achieved success with a design that *depends not at all on the derivatives of the field* ... for elsewhere in physics *energy and momentum typically depend critically upon time-derivatives of the dynamical variables*. It was on account of this electrodynamical quirk that the static arguments that gave (302) led us to an $\mathcal{E}$ found to pertain also to *dynamical* fields.

2. It is gratifying that energy density (and therefore also the integrated total energy) is <u>bounded below</u>:

$$S^{00} \equiv \mathcal{E} \geqslant 0 \quad : \quad \text{vanishes if and only if } F^{\mu\nu} \text{ vanishes}$$

For otherwise the electromagnetic field would be an insatiable energy sink (in short: a "rat hole") and would de-stabilize the universe.

3. From the fact that $S^{\mu\nu}$ is a *quadratic* function of $F^{\mu\nu}$ it follows (see again (45) on page 24) if follows that <u>stress-energy does not superimpose</u>:

$$\begin{aligned} F^{\mu\nu} &= F_1^{\mu\nu} + F_2^{\mu\nu} \quad : \quad \text{superimposed fields} \\ &\downarrow \\ S^{\mu\nu} &= S_1^{\mu\nu} + S_2^{\mu\nu} + (\text{cross term}) \end{aligned}$$

4. From the symmetry of $S^{\mu\nu}$ it follows rather remarkably that

$$\text{energy flux} \sim \text{momentum density} \quad : \quad \boldsymbol{S} = c^2 \boldsymbol{\mathcal{P}}$$

The discussion that led from (302) to (309) can be read as a further example of the "bootstrap method in theoretical physics," but has been intended to illustrate the *theory-shaping power of applied relativity*. With a little physics and a modest amount of relativity one can often go a remarkably long way. In the present instance—taking a conjectured description of S^{00} as our point of departure—we have managed to deduce the design of all fifteen of the other elements of $S^{\mu\nu}$, and to achieve at (309) a highly non-obvious result of fundamental physical importance.

Suppose now we were to **abandon our former assumption** that $F^{\mu\nu}$ moves "freely;" *i.e.*, that $J^\nu = 0$. The argument that led from the bottom of page 214 to (309) then supplies

$$\begin{aligned} \partial_\mu S^\mu{}_\nu &= \tfrac{1}{c} J^\alpha F_{\alpha\nu} + \underbrace{\mathbf{b} - \mathbf{c}}_{0 \text{ by previous argument}} \\ &\downarrow \\ \partial_\mu S^{\mu\nu} &= -\tfrac{1}{c} F^\nu{}_\alpha J^\alpha \end{aligned} \tag{312}$$

The flux components $\bullet$ of the stress-energy tensor

$$\mathbb{S} = \begin{pmatrix} \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

describe how energy/momentum are sloshing about in spacetime, causing local adjustments of the energy/momentum densities $\circ$. It becomes in this light natural to suppose that the expression on the right side of (312)

$$-\tfrac{1}{c}F^{\nu}{}_{\alpha}J^{\alpha}\left\{\begin{array}{l}\text{describes locally the rate at which energy/momentum}\\ \text{are being } exchanged \text{ between the electromagnetic field}\\ F^{\mu\nu} \text{ and the source field } J^{\mu}\end{array}\right.$$

We turn now to a discussion intended to lend substance to that interpretation.

2. Electromagnetic energy/momentum: second approach. We know that in the presence of an *impressed* electromagnetic field $F^{\mu\nu}$ a charged particle feels a Minkowski force given (see again page 198) by

$$K^{\mu} = (q/c)F^{\mu}{}_{\nu}u^{\nu} \tag{295}$$

...to which the particle responds by changing its energy/momentum; *i.e.*, by exchanging energy/momentum with—ultimately—the agent who impressed the field (the field itself acting here as intermediary). I propose to adjust the image—to remove the puppeteer ("agent") and let the puppets themselves (electromagnetic field on the one hand, charged matter on the other) battle it out. For formal reasons—specifically: to avoid the conceptual jangle that tends to arise when *fields* rub elbows with *particles*—it proves advantageous in this context to *consider the source to be spatially distributed*, having the nature of a charged fluid/gas/dust cloud, from which we recover particulate sources as a kind of degenerate limit: "lumpy gas." But to carry out such a program we must have some knowledge of the basic rudiments of fluid mechanics—a subject which was, by the way, well-known to Maxwell,[174] and from which (see again the words quoted in footnote #173) he drew some of his most characteristic images and inspiration.

DIGRESSION: ELEMENTARY ESSENTIALS OF FLUID DYNAMICS

Fluid dynamics is a phenomenological theory, formulated without explicit reference to the underlying microscopic physics.[175] It seeks to develop the $(\boldsymbol{x},t)$-dependence of

- $\rho(\boldsymbol{x},t)$, a scalar field which describes *mass density*, and
- $\boldsymbol{v}(\boldsymbol{x},t)$, a vector field which describes *fluid velocity*.

The product of these admits of two modes of interpretation:

$$\rho\,\boldsymbol{v} \equiv \text{mass current} = \text{momentum density}$$

[174] G. G. Stokes (1819–1903) was twelve years older than Maxwell, and had completed most of his fluid dynamical work by 1850.

[175] ...Imagined by Navier to be "atomic." Stokes, on the other hand, was not yet convinced of the reality of atoms, and contrived to do without the assistance that might be gained from an appeal to the "atomic hypothesis."

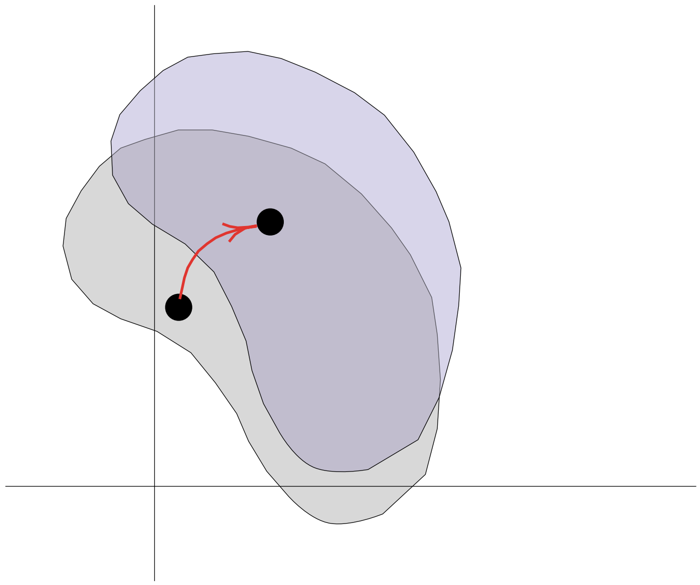


FIGURE 77: *A designated drop of liquid (think of a drop of ink dripped into a glass of water) shown at times t and $t+dt$. Every point in the evolved drop originated as a point in the initial drop. Not shown is the surrounding fluid.*

(in which connection it is instructive to recall that two pages ago we encountered

$$\tfrac{1}{c^2}\boldsymbol{S} = \boldsymbol{\mathcal{P}} \quad : \quad \text{mass flux} \equiv \frac{\text{energy flux}}{c^2} = \text{momentum density}$$

as an expression of the symmetry of a stress-energy tensor). The first of those interpretations supplies

$$\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{v}) = 0 \tag{313}$$

as an expression of mass conservation . . . while from the second interpretation we infer that the i^{th} component of momentum of a designated drop V of fluid can at times t and $t+dt$ be described

$$\underbrace{\int_V \rho(\boldsymbol{x}, t+dt)v_i(\boldsymbol{x}, t+dt)d^3x}_{\mathbf{a}_i} \quad \text{and} \quad \int_V \rho(\boldsymbol{x}, t)v_i(\boldsymbol{x}, t)d^2x$$

The integrals (see Figure 77) range over distinct domains, but can be made to range over the same domain by a change of variables:

$$\mathbf{a}_i = \int_V \rho(\boldsymbol{x}+\boldsymbol{v}\,dt, t+dt)v_i(\boldsymbol{x}+\boldsymbol{v}\,dt, t+dt)\left|\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{x}}\right| d^3x$$

Expanding $\rho(\boldsymbol{x}+\boldsymbol{v}\,dt, t+dt)$, $v_i(\boldsymbol{x}+\boldsymbol{v}\,dt, t+dt)$ and the Jacobian[176]

$$\left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}}\right| = \begin{vmatrix} 1+v_{11}dt & v_{12}dt & v_{13}dt \\ v_{21}dt & 1+v_{22}dt & v_{23}dt \\ v_{31}dt & v_{32}dt & 1+v_{33}dt \end{vmatrix} = 1 + \underbrace{(v_{11}+v_{22}+v_{33})}_{\boldsymbol{\nabla}\cdot\boldsymbol{v}}\,dt + \cdots$$

we obtain

$$\mathsf{a}_i = \int_V \Big\{\rho v_i + \big[\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\rho v_i + \rho v_i \boldsymbol{\nabla}\cdot\boldsymbol{v}\big]\,dt + \cdots\Big\}\,d^3x$$

From this it follows that the *temporal rate of change of the* i^{th} *component of the momentum of our representative drop* can be described

$$\dot{P}_i(\text{drop}) = \int_V \Big[\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\rho v_i + \rho v_i \boldsymbol{\nabla}\cdot\boldsymbol{v}\Big]\,d^3x \tag{314.1}$$

This quantity arises physically from *forces experienced by* our drop, which can be considered to be of two types:

$$\textbf{impressed volume forces} \quad : \quad \int_V f_i(\boldsymbol{x},t)\,d^3x$$
$$\textbf{surface forces} \quad : \quad \int_{\partial V} \boldsymbol{\sigma}_i\cdot\boldsymbol{dS} = \int_V \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_i\,d^3x$$

The latter describe *interaction of the drop with adjacent fluid elements*. So we have

$$= \int_V \Big[f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x^j}\Big]\,d^3x \tag{314.2}$$

where σ_{ij} refers to the j^{th} component of $\boldsymbol{\sigma}_i$. The right sides of equations (314) are equal *for all* V so evidently

$$\underbrace{\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)\rho v_i + \rho v_i \boldsymbol{\nabla}\cdot\boldsymbol{v}} = f_i + \sum_j \frac{\partial \sigma_{ij}}{\partial x^j}$$

These are **Euler's equations of fluid motion**, and can be notated in a great variety of ways: from

$$\begin{aligned} &= \tfrac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j) \\ &= v_i \underbrace{\big[\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{v})\big]}_{0 \text{ by mass conservation (313)}} + \rho\big(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\big)v_i \end{aligned}$$

we see that we can, in particular, write

---

[176] Here $v_{i\,j} \equiv \partial v_i/\partial x^j$.

$$\tfrac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j - \sigma_{ij}) = f_i \tag{315}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ↑ impressed *force density*

. . . but any attempt to *solve* equations (313) and (315) must await <u>structural specification of the "stress tensor"</u> σ_{ij}. It is in this latter connection that specific fluid models are described/distinguished/classified. General considerations (angular momentum conservation) can be shown to force the *symmetry* of the stress tensor ($\sigma_{ij} = \sigma_{ji}$), but still leave the model-builder with a vast amount of freedom. "Newtonian fluids" arise from the assumption

$$\sigma_{ij} = -p\,\delta_{ij} + \sum_{k,l} \mathcal{D}_{ijkl} V_{kl}$$

where $V_{kl} \equiv \frac{1}{2}(\partial_l v_k + \partial_k v_l)$ are components of the so-called "rate of deformation tensor," where the $\mathcal{D}_{ijkl}$ are the so-called "viscosity coefficients" and where p is the "static pressure." Isotropy (the rotational invariance of $\mathcal{D}_{ijkl}$) can be shown to entail $\mathcal{D}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and thus to reduce the number of independently specifiable $\mathcal{D}$-coefficients from 36 to 2, giving

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\sum_k V_{kk} + 2\mu V_{ij}$$

Then $\sum_k \sigma_{kk} = -3p + (3\lambda + 2\mu)\sum_k V_{kk}$ and in the case $3\lambda + 2\mu = 0$ we obtain the stress tensor characteristic of a "Stokes fluid"

$$\sigma_{ij} = -p\delta_{ij} + 2\mu V_{ij} - \tfrac{2}{3}\mu\delta_{ij}\sum_k V_{kk}$$

For an "incompressible Stokes fluid" this simplifies

$$\sigma_{ij} = -p\delta_{ij} + 2\mu V_{ij}$$

and in the absence of viscosity simplifies still further

$$\sigma_{ij} = -p\delta_{ij}$$

At zero pressure we obtain what is technically called **dust**:

$$\sigma_{ij} = 0 \tag{316}$$

We will have need of (313), (315) and (316). Other remarks on this page have been included simply to place what we will be doing in its larger context, to stress that we will be concerned only with the simplest instance of a vast range of structured possibilities—the number of which is increased still further when one endows the fluid with "non-Newtonian," or thermodynamic, or (say) magnetohydrodynamic properties. END OF DIGRESSION

The charges which comprise the "sources" of an electromagnetic field must, for fundamental reasons, satisfy *Lorentz-covariant equations of motion*. We propose to consider the sources to comprise collectively a kind of "fluid." We stand in need, therefore, of a relativistic fluid dynamics. To that end we observe that equations $c\cdot(313)\oplus(315)$ comprise a quartet of equations that can be written

$$\partial_\mu s^{\mu\nu} = f^\nu \tag{316}$$

with

$$\|s^{\mu\nu}\| \equiv \begin{pmatrix} \rho c^2 & \rho c v_1 & \rho c v_2 & \rho c v_3 \\ \rho c v_1 & \rho v_1 v_1 - \sigma_{11} & \rho v_1 v_2 - \sigma_{12} & \rho v_1 v_3 - \sigma_{13} \\ \rho c v_2 & \rho v_2 v_1 - \sigma_{21} & \rho v_2 v_2 - \sigma_{22} & \rho v_2 v_3 - \sigma_{23} \\ \rho c v_3 & \rho v_3 v_1 - \sigma_{31} & \rho v_3 v_2 - \sigma_{32} & \rho v_3 v_3 - \sigma_{33} \end{pmatrix}, \quad \|f^\nu\| \equiv \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

In the instantaneous rest frame of a designated fluid element

$$\|s^{\mu\nu}\| \xrightarrow{} \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ 0 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ 0 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{pmatrix}$$

and for a "non-viscous Newtonian fluid"—a model that is, as will emerge, adequate to our intended application—we obtain

$$\|s^{\mu\nu}\| \xrightarrow{} \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{317}$$

Equation (316) looks a lot more relativistic than (at the moment) it is, but *becomes fully relativistic if it is assumed that*

i) $s^{\mu\nu}$ and f^ν are prescribed in the local rest frame and

ii) respond tensorially to Lorentz transformations.

Thus

$$\begin{aligned} \partial_\mu s^{\mu\nu} &= f^\nu \quad \text{in the rest frame of a fluid element} \\ &\downarrow \\ \partial_\mu s^{\mu\nu} &= k^\nu \quad \text{in the lab frame} \end{aligned} \tag{318.1}$$

where

$$k^\nu \equiv \Lambda^\nu{}_\beta(\boldsymbol{\beta}) f^\beta \equiv \text{“Minkowski force density”} \tag{318.2}$$

$$\boldsymbol{\beta} \equiv \tfrac{1}{c}\cdot\begin{pmatrix} \text{velocity with which we in the lab frame} \\ \text{see the fluid element to be moving} \end{pmatrix}$$

$$s^{\mu\nu} \equiv \Lambda^\mu{}_\alpha(\boldsymbol{\beta})\Lambda^\nu{}_\beta s^{\alpha\beta} \tag{318.3}$$

Details relating to the construction (318.2) of k^ν have been described already at (290/291) on page 197. We look now to details implicit in the construction (318.3) of the "stress-energy tensor $s^{\mu\nu}$ of the relativistic fluid." Notice first that $s^{\mu\nu}$ shares the physical dimensionality of $S^{\mu\nu}$:

$$[s^{\mu\nu}] = \frac{\text{force}}{(\text{length})^2} = \frac{\text{energy}}{\text{3-volume}} = \text{pressure}$$

If we take $s^{\mu\nu}$ to be given by (317) and $\mathbb{\Lambda}(\boldsymbol{\beta})$ to possess the general boost design (209) then a straightforward computation[177] supplies

$$s^{\mu\nu} = (\rho + \tfrac{1}{c^2}p)u^\mu u^\nu - pg^{\mu\nu} \tag{319}$$

where

$$\rho \equiv \text{mass density in the local rest frame}$$
$$u^\mu \equiv \gamma \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix} \equiv \text{4-velocity of the fluid element}$$

At (319) we encounter the stress-energy tensor of a "relativistic non-viscous Newtonian fluid" which plays a major role in relativistic cosmology, where theorists speak of a "fluid" the elements of which are galaxies![178] If in (319) we set $p = 0$ we obtain the **stress-energy tensor of relativistic dust**

$$s^{\mu\nu} = \rho u^\mu u^\nu \tag{320}$$

where $u^\mu(x)$ is the 4-velocity field characteristic of the moving dust, and $\rho(x)$ is the rest mass density. The simplicity of (320) reflects *the absence (in dust) of any direct interparticle interaction*, and has the consequence that (for dust) the fluid dynamical equations

$$\partial_\mu s^{\mu\nu} = k^\nu$$

are but thinly disguised variants of the equations of particulate motion:

$$\begin{aligned} \text{expression on the left} &= u^\nu \cdot \underbrace{\partial_\mu(\rho u^\mu)}_{0 \text{ by mass conservation}} + \rho(u^\mu \partial_\mu)u^\nu \\ &= \rho(\tfrac{d}{d\tau})u^\nu \\ &= k^\nu \quad \text{by Minkowski's equation (275), adapted} \\ &\qquad\quad \text{here to mass/force } \underline{\text{densities}} \end{aligned}$$

For <u>a "dust cloud" which contains but a single particle</u> we expect $s^{\mu\nu}(x)$ to *vanish except on the worldline of the particle*, and are led from (320) to the odd-looking construction

$$s^{\mu\nu}(x) = mc\int_{-\infty}^{+\infty} u^\mu(\tau)u^\nu(\tau)\delta(x - \underset{\llcorner\text{solution of } m\frac{d}{d\tau}u^\nu = K^\nu}{x(\tau)})\,d\tau \tag{321}$$

[177] PROBLEM 55.

[178] See, for example, C. W. Misner, K. S. Thorne & J. A. Wheeler, *Gravitation* (1973), pages 153–154.

where the c-factor arises from dimensional considerations.[179] Equation (321) describes the *stress-energy tensor of a relativistic mass point*,[180] and if, in particular, it is the Lorentz force

$$K^\mu = (q/c)F^\mu{}_\alpha u^\alpha \tag{295}$$

that "steers" the particle then (321) becomes the **stress-energy tensor of a relativistic charged particle**—a concept introduced by Minkowski himself in 1908.

If all the constituent particles in a *charged dust cloud* are of then same species (*i.e.*, if the value of q/m is invariable within the cloud) then

$$\begin{aligned}\rho u^\mu &\equiv \text{mass-current 4-vector field}\\ &= (m/q)\cdot\text{charge-current 4-vector field}\\ &= (m/q)\cdot J^\mu\end{aligned}$$

and (320) becomes

$$s^{\mu\nu}(x) = (m/q)\cdot J^\mu(x)u^\nu(x) \tag{322}$$

This is the <u>stress-energy tensor of a single-species charged dust cloud</u>. For a single charged particle—looked upon as a "degenerate charged dust cloud"—we have

$$J^\mu(x) = qc\int_{-\infty}^{+\infty} u^\mu(\tau)\delta(x - x(\tau))\,d\tau \tag{323}$$

which when introduced into (320) gives back (321).

From (295)—written

$$K^\mu/(\text{unit 3-volume}) = \tfrac{1}{c}F^\mu{}_\alpha(q/\text{unit 3-volume})\,u^\alpha$$

—we infer that the *Lorentz force density experienced by a charged dust cloud* can be described

$$k^\mu = \tfrac{1}{c}F^\mu{}_\alpha J^\alpha \tag{324}$$

which positions us to address the main point of this discussion: I show now how <u>(324) can be used to motivate the definition (309)</u> of the stress-energy tensor $S^{\mu\nu}$ of the electromagnetic field. Most of the work has, in fact, already been

[179] [4-dimensional δ-function]= (4-volume)$^{-1}$ so

$$[c\delta(x - x(x))] = (\text{3-volume})^{-1}$$

[180] PROBLEM 56.

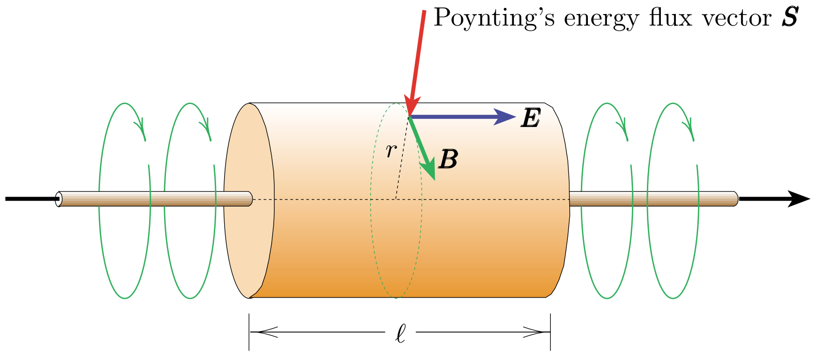


FIGURE 78: *Current I passes through a cylindrical resistor with resistance $R = \rho\ell/\pi r^2$. The potential $V = IR$ implies the existence of an axial electric field $\boldsymbol{E}$ of magnitude $E = V/\ell$, while at the surface of the resistor the magnetic field is solenoidal, of strength $B = I/c2\pi r$. The Poynting vector $\boldsymbol{S} = c(\boldsymbol{E}\times\boldsymbol{B})$ is therefore centrally directed, with magnitude $S = cEB$, which is to say: the field dumps energy into the resistor at the rate given by*

$$\begin{aligned}\text{rate of energy influx} &= S \cdot 2\pi r\ell \\ &= c(IR/\ell)(I/c2\pi r)2\pi r\ell \\ &= I^2 R\end{aligned}$$

The steady field can, from this point of view, be considered to act as a conduit for energy that flows from battery to resistor. The resistor, by this account, heats up not because copper atoms are jostled by conduction electrons, but because it drinks energy dumped on it by the field.

done: we have (drawing only upon Maxwell's equations and the antisymmetry of $F^{\mu\nu}$) at (312) already established that

$$\tfrac{1}{c}F^{\nu}{}_{\alpha}J^{\alpha} \quad \text{can be expressed} \quad -\partial_\mu S^{\mu\nu}$$
$$\text{with} \quad S^{\mu\nu} \equiv F^{\mu}{}_{\alpha}F^{\alpha\nu} - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu}$$

So we have

$$\partial_\mu s^{\mu\nu} = k^\nu = -\partial_\mu S^{\mu\nu}$$

giving

$$\partial_\mu \underbrace{(s^{\mu\nu} + S^{\mu\nu})}_{\text{stress-energy tensor of } \textbf{total system: sources + field}} = 0 \tag{325}$$

This equation provides (compare page 218) a detailed local description of energy/momentum traffic back and forth between the field and its sources, and does so in a way that conforms manifestly to the principle of relativity. We speak with intuitive confidence about the energy and momentum of particulate systems, and of their continuous limits (*e.g.*, fluids), and can on the basis of (325) speak with that same confidence about the "energy & momentum of the electromagnetic field."

The language employed by Maxwell (quoted on page 216) has by this point lost much of its quaintness, for the electromagnetic field has begun to acquire the status of a physical "object"—a sloshy object, but as real as any fluid. The emerging image of "field as dynamical object" acquires even greater plausibility from illustrative applications—such as that presented here as Figure 78—and from the discussion to which we now turn:

3. Electromagnetic angular momentum. If $\boldsymbol{E}$ and $\boldsymbol{B}$ describe the electric and magnetic fields at a point $\boldsymbol{x}$ then (see again page 216) $\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ describes the momentum density at $\boldsymbol{x}$, and it becomes natural to suppose that

$$\boldsymbol{\mathcal{L}} \equiv \boldsymbol{x} \times \boldsymbol{\mathcal{P}} = \tfrac{1}{c}\boldsymbol{x} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{326}$$

describes—relative to the origin—the *angular momentum density* of the field at $\boldsymbol{x}$. From the "triple cross product identity" we infer that

$$\boldsymbol{\mathcal{L}} = \tfrac{1}{c}\big\{(\boldsymbol{x}\cdot\boldsymbol{B})\boldsymbol{E} - (\boldsymbol{x}\cdot\boldsymbol{E})\boldsymbol{B}\big\} \quad \text{lies in the local } (\boldsymbol{E}\cdot\boldsymbol{B})\text{-plane}$$

We expect that the *total angular momentum* resident in the field will be given by an equation of the form

$$\boldsymbol{L} = \int_{\text{all space}} \boldsymbol{\mathcal{L}}\, d^3x$$

. . . that *angular momentum flux vectors* will be associated with each of the components of $\boldsymbol{\mathcal{L}}$ . . . and that there will, in general be *angular momentum exchange between the field and its sources*. All these expectations—modulo some surprises—will be supported by subsequent events. We begin, however, by looking not to formal fundamentals but to the particulars of a tractable special case:

ELECTROMAGNETIC GYROSCOPE WITH NO MOVING PARTS

Suppose—with J. J. Thompson (1904)—that an electric charge e has been glued to one end of a stick of length a, and that a "magnetic charge" g has been glued to the other end. It is immediately evident (see Figure 79) that the superimposed $\boldsymbol{E}$ and $\boldsymbol{B}$-fields that result from such a static charge configuration give rise to a momentum field $\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ that circulates about the axis defined by the stick, so that if you held such a construction in your hand it would feel and act like a gyroscope . . . though it contains *no moving parts*! We wish to quantify that intuitive insight, to calculate the total angular momentum resident within the static electromagnetic field. Taking our notation from the figure, we have

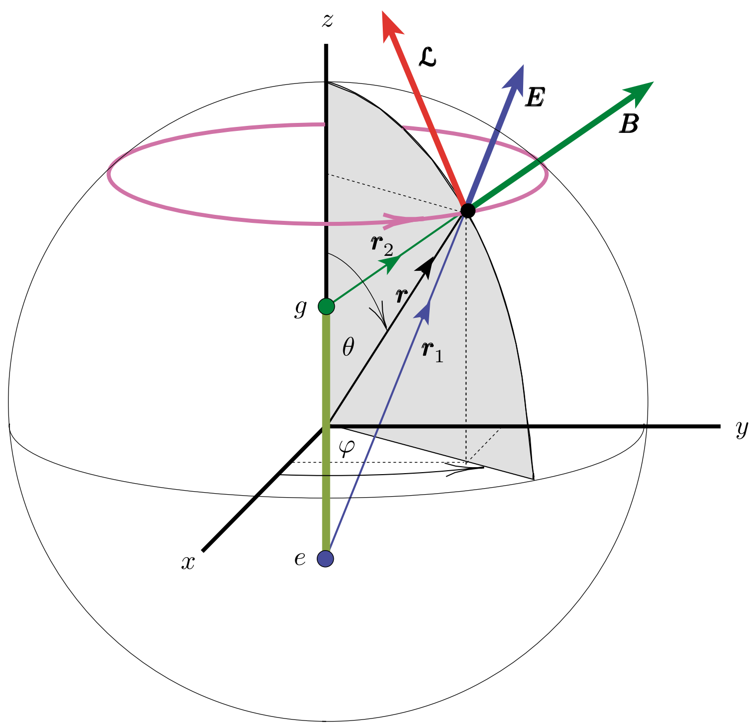


FIGURE 79: *Notations used in analysis of the "Thompson monopole" (or "mixed dipole").* Momentum circulation *is represented by the purple ellipse, and is right-handed with respect to the axis defined by the vector* $\boldsymbol{a}$ *directed from* e *to* g : (• → •). *Momentum circulation gives rise to a local angular momentum density that lies in the local* $(\boldsymbol{E},\boldsymbol{B})$*-plane. Only the axial component of* $\boldsymbol{L}=\int \boldsymbol{\mathfrak{L}}\,d^3x$ *survives the integration process.*

$$\boldsymbol{E}=\frac{e}{4\pi r_1^3}\boldsymbol{r}_1 \quad \text{with} \quad \begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{r}+\tfrac{1}{2}\boldsymbol{a} \\ r_1^2 &= r^2+\boldsymbol{r}\cdot\boldsymbol{a}+\tfrac{1}{4}a^2 \end{aligned}$$

$$\boldsymbol{B}=\frac{g}{4\pi r_1^3}\boldsymbol{r}_2 \quad \text{with} \quad \begin{aligned} \boldsymbol{r}_2 &= \boldsymbol{r}-\tfrac{1}{2}\boldsymbol{a} \\ r_2^2 &= r^2-\boldsymbol{r}\cdot\boldsymbol{a}+\tfrac{1}{4}a^2 \end{aligned}$$

giving

$$\begin{aligned} \boldsymbol{\mathfrak{P}} &= \frac{eg/c}{(4\pi)^2}\frac{1}{r_1^3 r_2^3}\,\boldsymbol{a}\times\boldsymbol{r} \\ \boldsymbol{\mathfrak{L}} &= \frac{eg/c}{(4\pi)^2}\frac{1}{r_1^3 r_2^3}\,\boldsymbol{r}\times(\boldsymbol{a}\times\boldsymbol{r}) \end{aligned}$$

But

$$\boldsymbol{r}\times(\boldsymbol{a}\times\boldsymbol{r}) = r^2\boldsymbol{a}-(\boldsymbol{r}\cdot\boldsymbol{a})\boldsymbol{r} = r^2 a\begin{pmatrix} -\cos\theta\cdot\sin\theta\cos\varphi \\ -\cos\theta\cdot\sin\theta\sin\varphi \\ 1-\cos\theta\cdot\cos\theta \end{pmatrix}$$

The x and y-components are killed by the process $\int_0^{2\pi} d\varphi$, so (as already anticipated) we have

$$\boldsymbol{L} = \begin{pmatrix} 0 \\ 0 \\ L \end{pmatrix}$$

with

$$\begin{aligned} L &= \frac{eg/c}{(4\pi)^2} \iiint \frac{1}{r_1^3 r_2^3}\, r^2 a \sin^2\theta \cdot r^2 \sin\theta \, dr d\theta d\varphi \\ &= 2\pi \frac{eg/c}{(4\pi)^2} \iint \frac{1}{r_1^2 r_2^2} \frac{ra\sin\theta}{r_1 r_2} (r\sin\theta)^2 \cdot r dr d\theta \end{aligned}$$

Write $r = \frac{1}{2}sa$ and obtain

$$= 4\pi \frac{eg/c}{(4\pi)^2} \iint \frac{1}{s_1^2 s_2^2} \frac{s\sin\theta}{s_1 s_2} (s\sin\theta)^2 \cdot s\, ds d\theta \tag{327}$$

$$\begin{aligned} s_1^2 &\equiv s^2 + 1 + 2s\cos\theta \\ s_1^2 &\equiv s^2 + 1 - 2s\cos\theta \end{aligned}$$

from which all reference to the stick-length—the only "natural length" which Thompson's system provides—has disappeared:

> The angular momentum in the field of Thompson's mixed dipole is *independent of stick-length.*

Evaluation of the $\iint$ poses a non-trivial but purely technical problem which has been discussed in detail—from at least six points of view!—by I.Adawi.[181] The argument which follows—due in outline to Adawi—illustrates the power of what might be called "symmetry-adapted integration" and the sometimes indispensable utility of "exotic coordinate systems."

Let (327) be written

$$L = \frac{eg/c}{4\pi} \iint \Big(\frac{w}{s_1 s_2}\Big)^3 d(\text{area}) \tag{328}$$

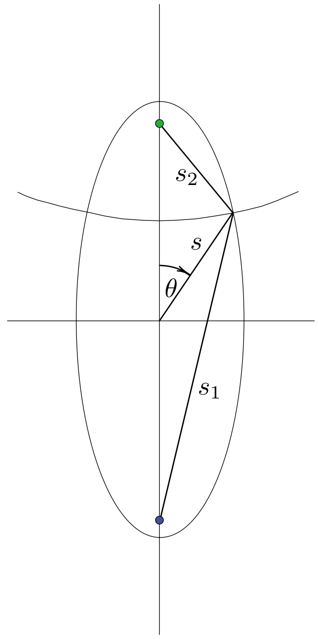
with $w = s\,\sin\theta$ and $d(\text{area}) = s\, ds d\theta$. The dimensionless variables s_1, s_2 and w admit readily of geometric interpretation (see Figure 80). Everyone familiar with the "string construction" knows that

$s_1 + s_2 = 2u$ describes an ellipse with pinned foci

and will be readily convinced that

$s_1 - s_2 = 2v$ describes (one branch of) a hyperbola

[181] "Thompson's monopoles," AJP **44**, 762 (1976). Adawi learned of this problem—as did I—when we were both graduate students of Philip Morrison at Cornell University (1955/56). Adawi was famous among his classmates for his exceptional analytical skill.



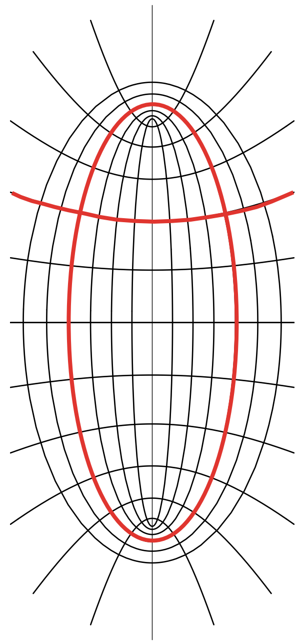


FIGURE 80: *In dimensionless variables*

$$\zeta \equiv s\cos\theta = 2z/a \quad \textit{and} \quad w \equiv s\sin\theta = (2r/a)\sin\theta$$

the electric charge • sits on the ζ-axis at $\zeta = -1$, the magnetic charge • at $\zeta = +1$. The "confocal conic coordinate system," shown at right, simplifies the analysis because it conforms optimally to the symmetry of the system.

It is equally evident on geometrical grounds that the parameters u and v are subject to the constraints indicated in Figure 81 below, and that the (u, v)-parameterized ellipses/hyperbolas are *confocal*. Some tedious but straightforward analytical geometry shows moreover that

$$\begin{aligned} \frac{\zeta^2}{u^2} + \frac{w^2}{u^2 - 1} &= 1 \quad \text{describes the } u\text{-ellipse} \\ \frac{\zeta^2}{v^2} - \frac{w^2}{1 - v^2} &= 1 \quad \text{describes the } v\text{-hyperbola} \end{aligned}$$

Equivalently

$$\begin{aligned} \frac{\zeta^2}{\cosh^2\alpha} + \frac{w^2}{\sinh^2\alpha} &= 1 \quad \text{with} \quad u \equiv \cosh\alpha \\ \frac{\zeta^2}{\cos^2\beta} - \frac{w^2}{\sin^2\beta} &= 1 \quad \text{with} \quad v \equiv \cos\beta \end{aligned}$$

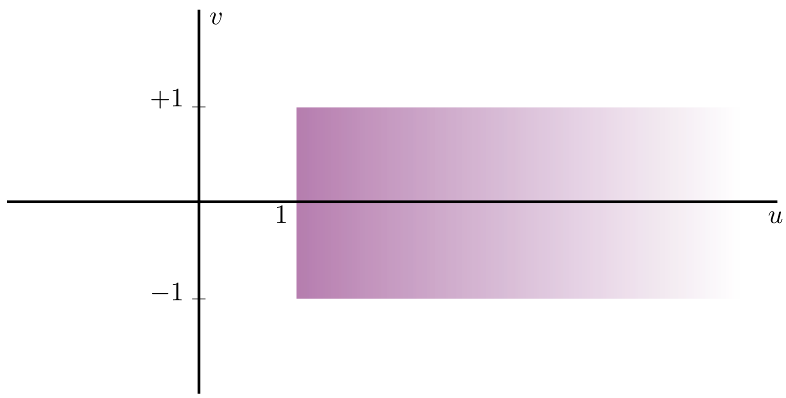


FIGURE 81: *The parameters u and v are subject to the constraints*

$$1 < u < \infty$$
$$-1 < v < +1$$

which is to say: they range on the purple strip.

from which it follows readily that

$$\zeta = \cosh\alpha\cos\beta = u\,v$$
$$w = \sinh\alpha\sin\beta = \sqrt{(u^2-1)(1-v^2)}$$

The last pair of equations describe a coordinate transformation

$$(\zeta, w) \longmapsto (u, v)$$

and it is in the confocal coordinates (u, v) that we propose to evaluate the $\iint$. To that end, we observe that

$$s_1 s_2 = \Big(\frac{s_1+s_2}{2}\Big)^2 - \Big(\frac{s_1-s_2}{2}\Big)^2 = u^2 - v^2$$

and

$$d\zeta dw = J\,dudv$$
$$J = \det\begin{pmatrix} \frac{\partial\zeta}{\partial u} & \frac{\partial\zeta}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v}\end{pmatrix} = \frac{u^2-v^2}{\sqrt{(u^2-1)(1-v^2)}} = \frac{s_1 s_2}{w}$$

Returning with this information to (328) we obtain

$$L = 2\cdot\frac{eg/c}{4\pi}\int_0^1 dv\int_1^\infty \frac{(u^2-1)(1-v^2)}{(u^2-v^2)^2}\,du$$

where the leading 2-factor comes from $\int_{-1}^{+1} = 2\int_0^{+1}$ (because the integrand is an even function of v). Finally write $u = 1/t$ and use $du = -(1/t^2)dt$ to obtain the remarkably symmetric result

$$= 2\cdot\frac{eg/c}{4\pi}\int_0^1\!\!\int_0^1 \frac{(1-t^2)(1-v^2)}{(1-t^2v^2)^2}\,dtdv$$

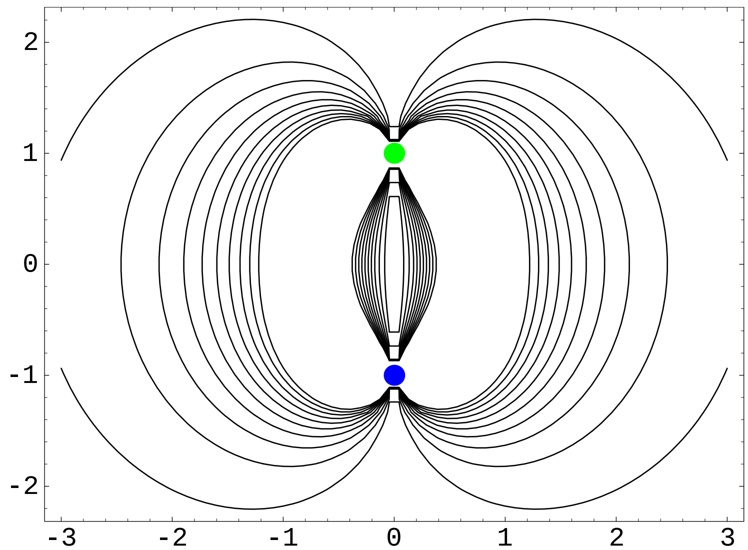


FIGURE 82: *Only the axial component (the component parallel to* • → •*) of* $\boldsymbol{\mathfrak{L}}$ *survives the integration process. From results developed in the text we discover the density of that component to be given in Cartesian coordinates by* $\mathfrak{L}_{\text{axial}} = \frac{1}{4\pi}(eg/c)\cdot f(w,\zeta)$ *with*

$$f(w,\zeta) = \frac{w^2}{[w^2+(\zeta-1)^2]^{\frac{3}{2}}[w^2+(\zeta+1)^2]^{\frac{3}{2}}}$$

of which the figure provides a contour plot. The angular momentum of Thompson's mixed dipole is seen to reside mainly in the "meat of the apple," exclusive of its core.

The double integral yields to a rather pretty direct analysis,[182] but I will on this occasion be content simply to ask *Mathematica*, who supplies

$$\int_0^1\!\!\int_0^1 \frac{(1-t^2)(1-v^2)}{(1-t^2v^2)^2}\,dtdv = \int_0^1 \frac{v^3-v+(1-v^4)\tanh^{-1}v}{2v^3}\,dv = \tfrac{1}{2}$$

So we have Thomson's relation

$$L = \frac{eg}{4\pi c}$$

which in rationalized units $\tilde{e}\equiv e/\sqrt{4\pi}$ and $\tilde{g}\equiv g/\sqrt{4\pi}$ assumes the still simpler

[182] See CLASSICAL ELECTRODYNAMICS (1980), page 319.

form

$$L = \frac{\tilde{e}\tilde{g}}{c} \quad : \quad \text{independently of the “stick length” } a \tag{328}$$

We know (which Thompson did not) that the intrinsic angular momentum (“spin”) of an elementary particle is always an integral multiple of $\frac{1}{2}\hbar$. It becomes attractive therefore to set

$$= n \cdot \tfrac{1}{2}\hbar$$

giving

$$\tilde{e}\tilde{g} = n\tfrac{1}{2}\hbar c$$

But

$$\hbar c = 137\,\tilde{e}^2$$

so on these grounds

$$\tilde{g} = n\tfrac{137}{2}\,\tilde{e} \tag{329}$$

which suggests that *if the universe contained even a single magnetic monopole then we could on this basis understand the observed* <u>*quantization of electric charge*</u>. Magnetic monopoles are, according to (329) “strongly” charged, and therefore should be conspicuous. On the other hand, they should be relatively hard to isolate, for they are bound by forces $(n\frac{137}{2})^2 = 4692n^2$ times stronger than the forces which bind electric monopoles. This line of thought originates in a paper of classic beauty by P. A. M. Dirac (1931), and after seventy years continues to haunt/taunt the imagination of physicists (J. Schwinger, A. O. Barut and many others). For a good review (and basic references) see §6.12 in J. D. Jackson's *Classical Electrodynamics* (3^{rd} edition 1999).

We return now—with our relativistic goggles on—to the more general issues posed on page 226. I ask: *How does* $\boldsymbol{\mathcal{L}}$ *transform*? . . . my double intent being

1) to achieve manifest <u>conformity with the principle of relativity</u>, and
2) to develop formulæ which describe the <u>angular momentum flux vectors</u>.

Here as so often, index play provides the essential clue. If we bring to (326) the recollection (page 216) that

$$\mathcal{P}^i = \tfrac{1}{c}S^{0i} \quad : \quad i = 1, 2, 3$$

we obtain

$$\begin{aligned}
\mathcal{L}_1 &= x^2\mathcal{P}^3 - x^3\mathcal{P}^2 = \tfrac{1}{c}(x^2S^{03} - x^3S^{02}) \equiv \mathcal{L}^{023}\\
\mathcal{L}_2 &= x^3\mathcal{P}^1 - x^1\mathcal{P}^3 = \tfrac{1}{c}(x^3S^{01} - x^1S^{03}) \equiv \mathcal{L}^{031}\\
\mathcal{L}_3 &= x^1\mathcal{P}^2 - x^2\mathcal{P}^1 = \tfrac{1}{c}(x^1S^{02} - x^2S^{01}) \equiv \mathcal{L}^{012}
\end{aligned}$$

From $\mathcal{L}_1 \equiv \mathcal{L}^{023}$ and the experience of pages 212–216 we infer that the equations $\mathcal{L}^{i23} \equiv \frac{1}{c}(x^2S^{i3} - x^3S^{i2})$ may very well describe the components $(i = 1, 2, 3)$ of the $\mathcal{L}_1$-*flux vector*. This is a conjecture which can be confirmed by direct calculation:

$$\begin{aligned}
\partial_\alpha\mathcal{L}^{\alpha 23} &= \tfrac{1}{c}\partial_\alpha(x^2S^{\alpha 3} - x^3S^{\alpha 2})\\
&= \tfrac{1}{c}(\delta^2{}_\alpha S^{\alpha 3} + x^2\partial_\alpha S^{\alpha 3} - \delta^3{}_\alpha S^{\alpha 2} - x^3\partial_\alpha S^{\alpha 2})
\end{aligned}$$

The 2nd and 4th terms on the right vanish individually (in source-free regions) as instances of momentum conservation ($\partial_\alpha S^{\alpha i} = 0$), so

$$\begin{aligned} &= \tfrac{1}{c}(S^{23} - S^{32}) \\ &= 0 \quad \text{by the } \underline{\text{symmetry}} \text{ of } S^{\mu\nu} \end{aligned}$$

Similar remarks pertain to $\mathcal{L}_2$ and $\mathcal{L}_3$. Indeed, the same argument supplies

$$\partial_\alpha \mathcal{L}^{\alpha\mu\nu} = 0 \quad \text{in source-free regions} \;:\; \mu,\nu = 0,1,2,3 \tag{330}$$

where

$$\mathcal{L}^{\alpha\mu\nu} \equiv \tfrac{1}{c}(x^\mu S^{\alpha\nu} - x^\nu S^{\alpha\mu}) \tag{331}$$

has obviously the following *antisymmetry* property:

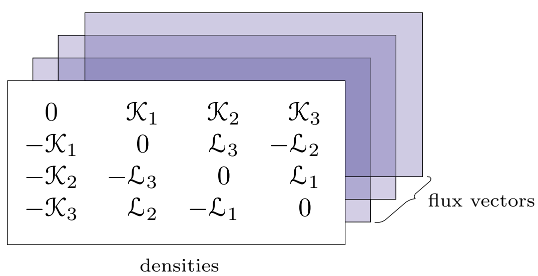
$$\mathcal{L}^{\alpha\mu\nu} = -\mathcal{L}^{\alpha\nu\mu} \tag{332}$$

Starting from the construction (326) of the three components of the angular momentum density vector $\boldsymbol{\mathcal{L}}$, and drawing upon a little bit of relativity ... we have been led

- to explicit descriptions of the associated angular momentum fluxes, and
- to <u>three unanticipated conservation laws</u>:

$$\partial_\alpha \mathcal{K}^{\alpha 1} = \partial_\alpha \mathcal{K}^{\alpha 2} = \partial_\alpha \mathcal{K}^{\alpha 2} = 0 \quad \text{with} \quad \mathcal{K}^{\alpha i} \equiv \mathcal{L}^{\alpha 0 i} \tag{333}$$

We have been led, in short, from an initial trio of field functions to a final total of 24—the components of a $\mu\nu$-antisymmetric third-rank tensor $\mathcal{L}^{\alpha\mu\nu}$



... all of which become intricately (but linearly) intermixed when Lorentz transformed. And an anticipated trio of conservation laws (conservation of angular momentum) have—by force of Lorentz covariance—been joined by an unanticipated second trio. We confront, therefore, this unanticipated question: *What is the <u>physical significance</u> of the conserved vector*

$$\boldsymbol{K} \equiv \int_{\text{all space}} \boldsymbol{\mathcal{K}}\, d^3x \tag{334.1}$$

$$\boldsymbol{\mathcal{K}} \equiv \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \quad \text{with} \quad \mathcal{K}_i \equiv \mathcal{K}^{0i} \equiv \mathcal{L}^{00i} \tag{334.2}$$

4. Motion of the "center of mass" of a free field. Bringing to (334.2) the definition (331) we have

$$\mathcal{K}_i = \mathcal{L}^{00i} = \tfrac{1}{c}(x^0 S^{0i} - x^i S^{00})$$

which in the notations introduced at the bottom of page 215 becomes

$$\begin{aligned} &= \tfrac{1}{c}\Big\{(ct)(c\mathcal{P}^i) - x^i\,\mathcal{E}\Big\} \\ &= c(t\mathcal{P}^i - \mathcal{M}x^i) \end{aligned}$$

$$\mathcal{M} \equiv \mathcal{E}/c^2 \equiv \text{local "mass density" of the field} \tag{335}$$

giving

$$\boldsymbol{\mathcal{K}} = c(t\boldsymbol{\mathcal{P}} - \mathcal{M}\boldsymbol{x}) \tag{336}$$

For free fields

$$\boldsymbol{P} \equiv \int \boldsymbol{\mathcal{P}}\; d^3x = \text{total linear momentum}$$

and

$$\begin{aligned} M \equiv \int \mathcal{M}\, d^3x &= \text{total effective mass} \\ &= \frac{\text{total energy}}{c^2} \end{aligned}$$

are known to be constants of the motion. So writing

$$\begin{aligned} \boldsymbol{K} &\equiv \int \boldsymbol{\mathcal{K}}\, d^3x \\ &= c(t\boldsymbol{P} - \underbrace{\int \mathcal{M}\boldsymbol{x}\, d^3x}_{=\,M\boldsymbol{X}(t)}) \end{aligned}$$

$$\begin{aligned} \boldsymbol{X}(t) &\equiv \tfrac{1}{M}\int \boldsymbol{x}\mathcal{M}d^3x = \tfrac{1}{E}\int \boldsymbol{x}\,\mathcal{E}d^3x \\ &= \textbf{center of mass/energy} \text{ of the free field} \end{aligned} \tag{337}$$

we see that $\boldsymbol{K}$-conservation

$$\tfrac{d}{dt}\boldsymbol{K} = \boldsymbol{0}\,, \quad \text{the upshot of the local conservations laws (333)}$$

amounts simply to the satisfying statement that *the center of mass/energy of a free electromagnetic field* <u>*moves uniformly/rectilinearly*</u>:

$$\tfrac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{P}/M = \text{constant} \tag{338}$$

In this respect a free electromagnetic field is very like a Newtonian free particle! Or more precisely: like an *isolated system* of Newtonian particles.

> IMPORTANT REMARK: Such frequently-encountered (because frequently useful) abstractions as "electromagnetic plane waves" are utterly non-localized. Their total mass/energy/momentum are defined by non-convergent integrals so the definition (337) becomes meaningless: **no center of mass can be assigned** to such idealized solutions of Maxwell's equations. We are led to regard as "physical" only those free fields to which the center of mass concept *does* pertain—fields which (because of the manner in which they "vanish at infinity") can be considered to be "isolated." Fourier analysis is in this respect strange (though no stranger here than in quantum mechanics), for it invites us to display semi-localized physical free fields as wavepacket-like superpositions of idealized *non*-physical free fields.

Distributed quantities—wherever in pure/applied mathematics they may be encountered—are often most usefully described in terms of their *moments* of ascending order. If, for example, $\rho(\boldsymbol{x})$ describes a mass distribution in 3-space then we standardly define

$$\begin{aligned}
0^{\text{th}}\text{ moment } M &\equiv \int \rho(\boldsymbol{x})\,d^3x \equiv \langle 1\rangle = \text{total mass}\\
1^{\text{st}}\text{ moments } M^i &\equiv \int x^i\rho(\boldsymbol{x})\,d^3x \equiv \langle x^i\rangle\\
2^{\text{nd}}\text{ moments } M^{ij} &\equiv \int x^i x^j\rho(\boldsymbol{x})\,d^3x \equiv \langle x^i x^j\rangle\\
&\vdots
\end{aligned}$$

and from those construct such objects as[183]

$$\begin{aligned}
\text{center of mass vector} \quad &: \quad X^i \equiv \frac{\langle x^i\rangle}{\langle 1\rangle}\\
\text{matrix of centered } 2^{\text{nd}} \text{ moments} \quad &: \quad C^{ij} \equiv \langle (x^i - X^i)(x^j - X^j)\rangle\\
\text{moment of inertia matrix} \quad &: \quad I^{ij} \equiv (C^{11} + C^{22} + C^{33})\delta^{ij} - C^{ij}\\
&\vdots
\end{aligned}$$

where C^{ij} provides leading-order information about *how the mass is distributed about* the center of mass, I^{ij} is a construction natural to the dynamics of rigid bodies, etc. The point is that *such objects—defined in reference to a variety of density functions—can be associated with isolated electromagnetic fields*. This is not commonly done, but is an analytical device that has been exploited to good effect by Schwinger.[184] Following (except notationally) in Schwinger's

[183] See CLASSICAL GYRODYNAMICS (1976), pages 9–11.
[184] See J. Schwinger *et al*, *Classical Electrodynamics* (1998), Chapter 3.

footsteps, let us agree to write

$$\langle \boldsymbol{x}\rangle^0 \equiv \frac{1}{E}\int_{\text{pulse}} \boldsymbol{x}\, \mathcal{E}\ d^3x \quad : \quad \mathcal{E}\text{-weighted mean position}$$

$$\langle \boldsymbol{x}\rangle^i \equiv \frac{1}{P^i}\int_{\text{pulse}} \boldsymbol{x}\, \mathcal{P}^i\, d^3x \quad : \quad \mathcal{P}^i\text{-weighted mean position}$$

$\vdots$ and more generally

$$\langle \bullet\rangle^\nu \equiv \left[\int_{\text{pulse}} S^{0\nu}\, d^3x\right]^{-1}\int_{\text{pulse}} \bullet\, S^{0\nu}\, d^3x \quad : \quad S^{0\nu}\text{-weighted mean } \bullet$$

where "pulse" is the term used by Schwinger to emphasize that his results—all of which refer to the <u>motion of moments</u>—pertain only to *isolated* electromagnetic fields. In this notation (338) reads

$$M\tfrac{d}{dt}\langle \boldsymbol{x}\rangle^0 = \boldsymbol{P}$$

which when integrated becomes

$$\begin{aligned}\langle \boldsymbol{x}\rangle^0_t &= \boldsymbol{v}\,t + \langle \boldsymbol{x}\rangle^0_0\\ \boldsymbol{v} &\equiv \tfrac{1}{M}\boldsymbol{P} \equiv \text{constant velocity of the center of energy}\end{aligned} \tag{339.1}$$

A natural companion to the preceding statement arises from Schwinger's (characteristically clever) observation that

$$\begin{aligned}\tfrac{d}{dt}\int_{\text{pulse}} \boldsymbol{x}\cdot\boldsymbol{\mathcal{P}}\, d^3x &= \tfrac{d}{dt}\Big\{P^1\langle x^1\rangle^1 + P^2\langle x^2\rangle^2 + P^3\langle x^3\rangle^3\Big\}\\ &= -\int_{\text{pulse}} x_i\, \underbrace{\partial_0 c\mathcal{P}^i}_{=\,\partial_0 S^{0i}\,=\,-\partial_k S^{ki}\text{ by }\partial_\mu S^{\mu i}=0}\, d^3x\\ &= +\int_{\text{pulse}} \big[\underbrace{\partial_k(x_i S^{ki})}_{\text{contributes a surface term, which vanishes}} - S^{ki}g_{ki}\big]\, d^3x\\ &= E \quad \text{because } S^k{}_k = -S^0{}_0 = -\mathcal{E} \text{ by (311)}\end{aligned}$$

The implication is that if we define

$$\boldsymbol{u} \equiv \tfrac{d}{dt}\boldsymbol{\xi} \quad \text{with} \quad \boldsymbol{\xi} \equiv \underbrace{\begin{pmatrix}\langle x^1\rangle^1\\ \langle x^2\rangle^2\\ \langle x^3\rangle^3\end{pmatrix}}_{\text{an object curiouser than Schwinger would have us believe!}}$$

then

$$E = \boldsymbol{P}\cdot\boldsymbol{u} \tag{340}$$

...which tells us nothing about $\boldsymbol{u}_\perp$ but informs us that

$$\boldsymbol{u}_\parallel = u_\parallel \hat{\boldsymbol{P}} \quad \text{is a constant vector, with} \quad u_\parallel = E/P \tag{339.2}$$

From equations (339) it follows that

$$\boldsymbol{v}\cdot\boldsymbol{u}_\parallel = \tfrac{1}{M}(E/P)\boldsymbol{P}\cdot\hat{\boldsymbol{P}} = c^2$$

Schwinger observes that *if*

i) $\boldsymbol{v}$ refers to the velocity of energy transport

ii) $\boldsymbol{u}_\parallel$ refers to the velocity of momentum transport

iii) and if, moreover, (as would then seem plausible) those are identical

then

$$v = c \quad : \quad \begin{cases} \text{for isolated free fields ("pulses") with} \\ \text{identical energy/momentum transport} \\ \text{velocites } (\boldsymbol{v} = \boldsymbol{u}_\parallel) \text{ the transport speed} \\ \text{is necessarily the speed of light} \end{cases} \tag{341}$$

and (340) becomes

$$E = cP \tag{342}$$

which—interestingly—is of the design assumed by (282) in the massless limit:

$$\begin{aligned} E &= c\sqrt{\boldsymbol{p}\cdot\boldsymbol{p} + (mc)^2} \\ &\downarrow \\ &= cp \quad \text{as} \quad m \downarrow 0 \end{aligned}$$

But this line of argument provides no insight into the (seemingly plausible, but in fact highly specialized) conditions under which Schwinger's hypotheses hold.

Sharpened results can be obtained by looking to motion of the energetic *second* moment $\langle g_{\mu\nu}x^\mu x^\nu\rangle^0$: from local energy conservation $\partial_\alpha S^{\alpha 0} = 0$ it follows trivially that

$$(g_{\mu\nu}x^\mu x^\nu)\partial_\alpha S^{0\alpha} = \partial_\alpha\big[(g_{\mu\nu}x^\mu x^\nu)S^{0\alpha}\big] - 2S^{0\alpha}x_\alpha = 0$$

which can be spelled out

$$\begin{aligned} \tfrac{1}{c}\partial_t\big[(c^2t^2 - \boldsymbol{x}\cdot\boldsymbol{x})\mathcal{E}\big] + \boldsymbol{\nabla}\cdot(\textbf{etc.}) - 2c(\mathcal{E}\,t - \boldsymbol{\mathcal{P}}\cdot\boldsymbol{x}) &= 0 \\ &\Downarrow \\ \tfrac{d}{dt}\int_{\text{pulse}}(c^2t^2 - \boldsymbol{x}\cdot\boldsymbol{x})\mathcal{E}\,d^3x + \text{vanishing surface term} - 2c^2\Big\{Et - \int_{\text{pulse}}\boldsymbol{\mathcal{P}}\cdot\boldsymbol{x}\,d^3x\Big\} &= 0 \end{aligned}$$

But it was established on the preceding page that $\frac{d}{dt}\{\text{etc.}\} = 0$; *i.e.*, that $\{\text{etc.}\}$ is a constant of the motion:

$$E\,t - \int_{\text{pulse}}\boldsymbol{\mathcal{P}}\cdot\boldsymbol{x}\,d^3x = E\,t - \boldsymbol{P}\cdot\boldsymbol{\xi}_t = \text{constant} = -\boldsymbol{P}\cdot\boldsymbol{\xi}_0$$

from which we could recover $E = \boldsymbol{P}\cdot\boldsymbol{u}$ by t-differentiation. So we have

$$\tfrac{d}{dt}\big(c^2t^2E - E\langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0\,\big) + 2c^2\boldsymbol{P}\cdot\boldsymbol{\xi}_0 = 0$$

giving $E\frac{d}{dt}\langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0 = 2c^2(Et+\boldsymbol{P}\cdot\boldsymbol{\xi}_0)$ whence (divide by $E = Mc^2$ and integrate)

$$\langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0_t = c^2t^2 + 2\tfrac{1}{M}\boldsymbol{P}\cdot\boldsymbol{\xi}_0 t + \langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0_0 \tag{343}$$

To gain leading-order information about the *evolving spatial distribution* of the field we introduce the centered second moment with respect to $\mathcal{E}$:

$$\begin{aligned}
\sigma^2 &\equiv \tfrac{1}{E}\int_{\text{pulse}}(\boldsymbol{x}-\langle\boldsymbol{x}\rangle^0)\cdot(\boldsymbol{x}-\langle\boldsymbol{x}\rangle^0)\mathcal{E}\,d^3x\\
&= \langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0 - \langle\boldsymbol{x}\rangle^0\cdot\langle\boldsymbol{x}\rangle^0
\end{aligned}$$

Necessarily $\sigma^2 \geqslant 0$, with equality if and only if the pulse is "point-like." Results in hand now supply

$$\begin{aligned}
\sigma_t^2 &= \Big[c^2t^2 + 2\tfrac{1}{M}\boldsymbol{P}\cdot\boldsymbol{\xi}_0 t + \langle\boldsymbol{x}\cdot\boldsymbol{x}\rangle^0_0\Big] - \Big[\tfrac{1}{M}\boldsymbol{P}t + \langle\boldsymbol{x}\rangle^0_0\Big]\cdot\Big[\tfrac{1}{M}\boldsymbol{P}t + \langle\boldsymbol{x}\rangle^0_0\Big]\\
&= \Big[1 - \frac{P^2}{M^2c^2}\Big](ct)^2 + 2\frac{1}{Mc}\boldsymbol{P}\cdot\big[\boldsymbol{\xi}_0 - \langle\boldsymbol{x}\rangle^0_0\big]ct + \sigma_0^2\\
&\equiv A\,(ct)^2 + 2B\,(ct)^1 + C\,(ct)^0
\end{aligned}\tag{344}$$

...which pertains to *all* isolated fields, and is plotted in Figure 83. The roots of $\sigma_t^2 = 0$ are evidently both *complex*, which entails

$$0 \leqslant B^2 \leqslant AC \tag{345}$$

But $C \equiv \sigma_0^2 \geqslant 0$ so necessarily

$$A \equiv \Big[1 - \Big(\frac{cP}{E}\Big)^2\Big] \geqslant 0$$

Evidently (342) identifies the exceptional condition $A = 0$, which by (345) entails $B = 0$. And this, by (344), entails $\boldsymbol{P}\cdot\boldsymbol{\xi}_0 = \boldsymbol{P}\cdot\langle\boldsymbol{x}\rangle^0_0$. But we have already established that

$$\begin{aligned}
\boldsymbol{P}\cdot\boldsymbol{\xi}_0 &= \boldsymbol{P}\cdot\boldsymbol{\xi}_t - Et\\
&= \int\boldsymbol{x}\cdot\boldsymbol{\mathcal{P}}\,d^3x - Et\\
\boldsymbol{P}\cdot\langle\boldsymbol{x}\rangle^0_0 &= \boldsymbol{P}\cdot\Big\{\langle\boldsymbol{x}\rangle^0_t - \tfrac{1}{M}\boldsymbol{P}t\Big\}\\
&= \Big[\int\boldsymbol{\mathcal{P}}\,d^3x\Big]\cdot\Big[\tfrac{1}{E}\int\boldsymbol{x}\,\mathcal{E}\,d^3x\Big] - \frac{P^2}{E/c^2}\,t
\end{aligned}$$

and the t-terms are rendered equal by the condition $cP/E = 1$ which is now in force. The implication is that

$$B = 0 \quad\Longleftrightarrow \Big[\int\mathcal{E}\,d^3x\Big]\Big[\int\boldsymbol{x}\cdot\boldsymbol{\mathcal{P}}\,d^3x\Big] = \Big[\int\boldsymbol{\mathcal{P}}\,d^3x\Big]\cdot\Big[\int\boldsymbol{x}\,\mathcal{E}\,d^3x\Big]$$

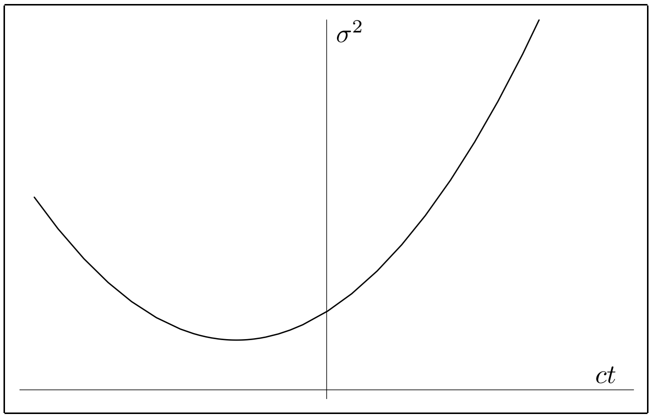


FIGURE 83: *Graph—computed from the right side of* (344)*— of the function* σ_t^2 *that describes (in leading approximation) how the energy in an isolated free field becomes spatially dispersed. This is what would happen to (for example) the Coulomb field of a charge if the charge were suddenly "turned off." It follows immediately from* (344) *that*

$$\tfrac{d}{dt}\sigma_t \rightarrow c \quad \text{as} \quad t\uparrow\infty$$

In the text the fact that the curve cannot cross the time-axis is shown to have important general implications.

which is readily seen to be satisfied if (but *only* if?) it is everywhere and always the case that

$$\boldsymbol{\mathcal{P}}E = \boldsymbol{P}\mathcal{E}$$

This is a very strong condition, for it forces the momentum density $\boldsymbol{\mathcal{P}}$ to be everywhere and always proportional to the constant vector $\hat{\boldsymbol{P}}$:

$$\boldsymbol{\mathcal{P}} = \tfrac{1}{c}\mathcal{E}\hat{\boldsymbol{P}} \tag{346.1}$$

Integration over the isolated free field gives

$$\boldsymbol{P} = \tfrac{1}{c}E\hat{\boldsymbol{P}} \tag{346.2}$$

What can one say about the structure of the electric/magnetic fields which is forced by (what we now recognize to be) the strong condition

$$E = c|\boldsymbol{P}|, \quad \text{equivalently} \quad E = cP \tag{347}$$

On the one hand we have[185]

[185] I hope it will be clear from context when, in the following discussion, E means "total energy" and when it means "magnitude of $\boldsymbol{E}$."

$$\begin{aligned} &E = c|\boldsymbol{P}| \\ &\downarrow \\ \int \frac{E^2+B^2}{2}\,d^3x &= \Big|\int \boldsymbol{E}\times\boldsymbol{B}\,d^3x\Big| \\ &\leqslant \int \big|\boldsymbol{E}\times\boldsymbol{B}\big|\,d^3x \end{aligned} \tag{348.1}$$

with equality if and only if $\boldsymbol{E}\times\boldsymbol{B}$ is *unidirectional*. On the other hand

$$\begin{aligned} \big|\boldsymbol{E}\times\boldsymbol{B}\big|^2 &= E^2B^2 - (\boldsymbol{E}\cdot\boldsymbol{B})^2 \\ &= \Big[\frac{E^2+B^2}{2}\Big]^2 - \Big\{\Big[\frac{E^2-B^2}{2}\Big]^2 + (\boldsymbol{E}\cdot\boldsymbol{B})^2\Big\} \\ &\leqslant \Big[\frac{E^2+B^2}{2}\Big]^2 \quad : \quad \text{equality if and only if } E^2 = B^2 \text{ and } \boldsymbol{E}\cdot\boldsymbol{B} = 0 \end{aligned}$$

so

$$\int \big|\boldsymbol{E}\times\boldsymbol{B}\big|\,d^3x \leqslant \int \frac{E^2+B^2}{2}\,d^3x \tag{348.2}$$

which is (348.1) but with the inequality reversed. If we are to achieve (347) then *both* inequalities must hold: both, in other words, must reduce to equalities. The relation $E = cP$ is seen thus to require that it be everywhere and always the case that

1) $\boldsymbol{E}\times\boldsymbol{B}$ is unidirectional
2) $E^2 = B^2$
3) $\boldsymbol{E}\perp\boldsymbol{B}$

These same conditions will assume major importance when we come to consider plane wave solutions of the free field equations . . . which is curious, since (as was remarked already on page 235) plane waves cannot be "isolated," cannot be considered to comprise "pulses."

The discussion of recent paragraphs illustrates the power of the "momental mode of argument" (and illustrates also the deft genius of Schwinger!), but by no means exhausts the resources of the method: much fruit awaits the picking. More to the immediate point, it shows that basic mechanical properties of electromagnetic fields can be exposed *without direct appeal to Maxwell's equations*. Collectively, those properties encourage us to think of the (free) field as a mechanical object . . . even as a mechanical object which is—to a remarkable degree—"particle-like."

5. Zilch, spin & other exotic constructs. However "particle-like" we may consider the electromagnetic field to be, it does—because a field—possess many more degrees of freedom than a particle (infinitely many!), and can be expected to possess correspondingly many more constants of the motion. That one can actually write some of these down was discovered—by accident, and to everyone's surprise—by D. M. Lipkin in 1964.[186] Lipkin happened somehow to notice that if he defined

$$\left.\begin{aligned} Z^0 &\equiv \boldsymbol{E}\boldsymbol{\cdot}\text{curl}\boldsymbol{E} + \boldsymbol{B}\boldsymbol{\cdot}\text{curl}\boldsymbol{B} \\ \boldsymbol{Z} &\equiv \tfrac{1}{c}\Big[\boldsymbol{E}\times\tfrac{\partial}{\partial t}\boldsymbol{E} + \boldsymbol{B}\times\tfrac{\partial}{\partial t}\boldsymbol{B}\Big] \end{aligned}\right\} \tag{349}$$

then[187] it follows from the free-field Maxwell equations[188] that

$$\partial_0 Z^0 + \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{Z} = 0 \tag{350}$$

This he interpreted to provide local expression of the fact that

$$\text{total "zilch"} \equiv \int Z^0\, d^3x$$

is a constant of the free-field motion. The name he gave his discovery reflects the fact that he had (nor, to this day, does anyone have, so far as I am aware) no sense of what the *physical* significance of "zilch" might be. He drew attention to the fact that field *derivatives*—so conspicuously absent from the stress-energy and angular momentum tensors—enter into the definitions (349).

One is tempted at (350) to write $\partial_\alpha Z^\alpha = 0$, but such an equation would make relativistic good sense only if the Z^α transform as components of a 4-vector . . . which, as it turns out, they do not. One confronts therefore the question: *How to bring Lipkin's discovery into manifest compliance with the principle of relativity*? Persuit of this issue led Lipkin to the identification of nine *additional* *new conservation laws*. More specifically, he was led to write

$$Z^\alpha = V^{00\alpha}$$

where—as T. A. Morgan[189] was quick to discover—the tensor components of $V^{\mu\nu\alpha}$ can be described quite simply as follows:

$$V^{\mu\nu\alpha} \equiv (\partial^\alpha G^\mu{}_\lambda)F^{\lambda\nu} - (\partial^\alpha F^\mu{}_\lambda)G^{\lambda\nu} \tag{351}$$

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[186] "Existence of a new conservation law in electromagnetic theory," J. Math. Phys. **5**, 696 (1964).

[187] PROBLEM 57.

[188] In (65) set $\rho = 0$ and $\boldsymbol{j} = \boldsymbol{0}$.

[189] "Two classes of new conservation laws for the electromagnetic field and other massless fields," J. Math. Phys. **5**, 1659 (1964). See also T. A. Morgan & D. W. Joseph, "Tensor lagrangians and generalized conservation laws for free fields," Nuovo Cimento **39**, 494 (1965) and R. F. O'Connell & D. R. Tompkins, "Generalized solutions for massless free fields and consequent generalized conservation laws," J. Math. Phys. **6**, 1952 (1965). It follows easily from (351) that

$$V^{00\alpha} = -(\boldsymbol{E}\boldsymbol{\cdot}\partial^\alpha\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\cdot}\partial^\alpha\boldsymbol{E})$$

One achieves conformity with (349) by drawing upon the free field equations

This discovery motivated Morgan to write

$$\begin{aligned}V^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} \equiv\ &(\partial^{\alpha_1}\cdots\partial^{\alpha_p}G^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}F^{\lambda\nu})\\ &-(\partial^{\alpha_1}\cdots\partial^{\alpha_p}F^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}G^{\lambda\nu})\end{aligned}$$

$$\begin{aligned}T^{\mu\nu\alpha_1\cdots\alpha_p\beta_1\cdots\beta_q} \equiv \tfrac{1}{2}\Big[\ &(\partial^{\alpha_1}\cdots\partial^{\alpha_p}F^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}F^{\lambda\nu})\\ &+(\partial^{\alpha_1}\cdots\partial^{\alpha_p}G^\mu{}_\lambda)(\partial^{\beta_1}\cdots\partial^{\beta_q}G^{\lambda\nu})\Big]\end{aligned}$$

and to observe that—in consequence of the free field equations and certain fundamental "dualization identities"[190]—each of the above quantities is

1) $\mu\nu$-symmetric: $V^{\mu\nu\cdots} = V^{\nu\mu\cdots}$ and $T^{\mu\nu\cdots} = T^{\nu\mu\cdots}$
2) traceless: $V^\mu{}_\mu{}^{\cdots} = T^\mu{}_\mu{}^{\cdots} = 0$, and
3) locally conserved: $\partial_\mu V^{\mu\nu\cdots} = \partial_\mu T^{\mu\nu\cdots} = 0$.

In the absence of "spectator indices" (*i.e.*, in the case $p = q = 0$) $T^{\mu\nu\cdots}$ reduces to the familiar stress-energy tensor (309), so at least that member of Morgan's <u>infinite population of functionally-independent conservation laws</u> has a strong claim to physical significance. Lipkin's tensor $V^{\mu\nu\alpha}$ has moreover the property (which recommended it to his attention in the first place—namely) that

$$\partial_\alpha V^{\mu\nu\alpha} = 0 \quad : \quad \text{These are } \underline{\text{Lipkin's 10 conservation laws}}$$

. . . but the proof of that fact (see the papers cited above) is intricate, and will be omitted.

The solitary conservation law (350) discovered by Lipkin is seen in retrospect to have been but the tip of an iceberg. Of methodological interest is the observation that it was *relativity* that led from the tip to a perception of the iceberg as a whole. On page 233 we were led from the three components of angular momentum density to the 24 elements of $\mathcal{L}^{\alpha\mu\nu}$. Here the relativistic payoff has been infinitely richer . . . but to what effect? Although the theoretical placement of zilch-like conservation laws has been somewhat clarified,[191] the subject has passed into almost total obscurity: "zilch" is indexed in none of the standard texts, and appears to be on nobody's mind. I know of no argument

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(continued from the preceding page) and upon (compare (5)) the following uncommon but quite elementary identity:

$$\sum_{k=1}^{3}(A_k\boldsymbol{\nabla}B_k - B_k\boldsymbol{\nabla}A_k) = \boldsymbol{A}\times\text{curl}\boldsymbol{B} - \boldsymbol{B}\times\text{curl}\boldsymbol{A} + \boldsymbol{A}\,\text{div}\boldsymbol{B} - \boldsymbol{B}\,\text{div}\boldsymbol{A} - \text{curl}(\boldsymbol{A}\times\boldsymbol{B})$$

Note that the curl$(\boldsymbol{A}\times\boldsymbol{B})$-term makes no contribution to $\boldsymbol{\nabla}\cdot\boldsymbol{Z}$, so can be omitted (Lipkin's option) from the definition of $\boldsymbol{Z}$.

[190] See page 16 in ELEMENTS OF RELATIVITY (1966).

[191] See especially T. W. B. Kibble, "Conservation laws for free fields," J. Math. Phys. **6**, 1022 (1965).

to the effect that zilch is a concept too fundamentally trivial to support useful physics, but the effort to expose that physics appears to lie in the distant future. A place to start might be to describe the zilch-like features of some specific solutions of the free field equations, the objective being to gain a sharper *intuitive* sense of what those infinitely many conservation laws are trying to tell us. "Infinitely many conservation laws" seems a treasure too rich to ignore.

Classical mechanics came into the world as the theory of a particular system—the gravitational two-body system—and it was Newton's descriptive success in that special case that lent credibility to the concepts and methods he had created. But Newton's $\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p}$ was by itself insufficient to support a theory of mechanical-systems-in-general, for it assumed $\boldsymbol{F}$ to be known/given in advance, and had nothing to say about how the forces (most conspicuously: the forces of constraint) internal to multiparticle systems *come* to be known. The *general theory of mechanical systems* had to await the cultivation of ideas that radiate from the work of Lagrange,[192] and only when such a theory was in place could the deepest and most subtle aspects of the original two-body problem be exposed. So it was also in the history of classical field theory: Maxwell gave us the theory of a particular classical field system—a theory which Einstein showed to be "naturally relativistic"—but motivation to create a *general theory of relativistic classical fields* had to await the development of interest a "relativistic theory of gravitation," the theory which by the time it had become ripe enough to fall from the tree had metamorphosed into "general relativity." It emerged that Lagrangian methods provide—ready made—the language of choice for the description of relativistic classical fields, and that the "mechanical properties of fields" are brought into focus (Noether's insight) by <u>conservation laws that reflect symmetries of the dynamical action</u>:[193]

$$S_{\mathcal{R}}[\varphi] \equiv \iiiint_{\mathcal{R}} \mathcal{L}(\varphi, \partial\varphi)\, d^4x$$

Here φ is any *solution* of the field equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{a,\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi_a} = 0$$

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[192] Lagrange's *Mechanique analytique* was published in 1788—101 years after the publication of Newton's *Philosophiae Naturalis Principia Mathematica*. Another near-half-century was to elapse before Hamilton—who took his inspiration directly from what he called Lagrange's "scientific poem"—completed his own contributions to mechanics ("On a general method in dynamics" appeared in 1834, and his "Second essay on a general method in dynamics" in 1835) and it was not until 1918 that Emmy Noether placed the elegant capstone on Lagrangian dynamics.

[193] For more detailed discussion see, for example, CLASSICAL FIELD THEORY (1999), Chapter 1, pages 15–32 or Herbert Goldstein, *Classical Mechanics* (2nd edition 1980), Chapter 12.

$\mathcal{R}$ is any "bubble" in spacetime, and a indexes the individual components of the multi-component field system. When one returns with such general principles to the electrodynamic birthplace of relativistic field theory one acquires deepened insight into the meaning—and a greater respect for the "naturalness" —of constructions that in §§1–3 were introduced in a somewhat improvisatory *ad hoc* manner. Specifically, one finds that (see again (304) and (309)) the $^\nu$-indexed quartet of conservation laws

$$\begin{aligned} \partial_\alpha S^{\alpha\nu} &= 0 \\ S^{\mu\nu} &\equiv F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})g^{\mu\nu} \end{aligned} \tag{352.1}$$

reflects the **translational** symmetry of the electromagnetic free-field action function, and that (see again (330) and (331)) the antisymmetrically $^{\mu\nu}$-indexed sextet of conservation laws

$$\begin{aligned} \partial_\alpha \mathcal{L}^{\alpha\mu\nu} &= 0 \\ \mathcal{L}^{\alpha\mu\nu} &\equiv \tfrac{1}{c}(x^\mu S^{\alpha\nu} - x^\nu S^{\alpha\mu}) \end{aligned} \tag{352.2}$$

reflects the **Lorentz** symmetry of the action. Three of the latter (those that arise from the *rotational* component of the Lorentz group) refer to the conservation of angular momentum $\boldsymbol{L}$, while the other three (those that arise from boosts) refer to the conservation of $\boldsymbol{K}$. We know, however, that Maxwellian electrodynamics is <u>conformally</u> covariant, and that the 4-dimensional conformal group is a 15-parameter group that—in addition to translations, rotations and boosts —contains "dilations" (one parameter) and "Möbius transformations" (four parameters). What are the associated conservation laws? This question was studied by E. Bessel-Hagen (1921), whose work is reviewed in a very accessible paper by B. F. Plybon.[194] It develops that **dilational** symmetry of the action entails

$$\partial_\alpha(S^\alpha{}_\beta x^\beta) = 0 \tag{352.3}$$

while **Möbius** symmetry supplies a $^\mu$-indexed quartet of conservation laws

$$\partial_\alpha(2S^\alpha{}_\beta x^\beta x^\mu - S^{\alpha\mu}\cdot x^\beta x_\beta) = 0 \tag{352.4}$$

Recalling from (310) & (311) that $S^{\mu\nu}$ is <u>symmetric and traceless</u>, we observe (with Plybon) that[195]

- (352.2) follows from (352.1) and the symmetry of $S^{\mu\nu}$
- (352.3) follows from (352.1) and the tracelessness of $S^{\mu\nu}$
- (352.4) follows from (352.1) and the traceless symmetry of $S^{\mu\nu}$

So (352.4) provides no information additional to that conveyed already by the conservation laws (352.1/2/3) and it is therefore pointless to inquire after the

[194] "Observations on the Bessel-Hagen conservation laws for electromagnetic fields," AJP **42**, 998 (1974).
[195] PROBLEM 58.

"independent physical meaning" of the Möbius invariants.[196] The physical meanings of the translational and Lorentz invariants has already been established, while (352.3) supplies the dilational invariant

$$\begin{aligned} D &\equiv \int (S^0{}_\beta x^\beta)\, d^3x \\ &= c\int (\mathcal{E}\,t - \boldsymbol{\mathcal{P}}\cdot\boldsymbol{x})\, d^3x \\ &= c(E\,t - \boldsymbol{P}\cdot\boldsymbol{\xi}_t) \end{aligned}$$

. . . the invariance of which was encountered/exploited already at the bottom of page 237.

This elegant train of thought lends new interest to the zilch-like free-field conservation laws discussed previously, for it is easily demonstrated that those are of a design to which standard "Noetherian analysis" can never lead. This observation led Morgan & Joseph[189] to construct a highly non-standard theory of "tensor Lagrangians"

$$\mathcal{L} \quad\longrightarrow\quad \mathcal{L}_{\text{population of tensor indices}}$$

in which all of the infinitely many "conservation of zilch" statements can be attributed to the translational invariance of the associated tensor Lagrangians. They note, however, that free fields are *unobservable in principle*: that <u>it is by their interactions that systems announce themselves</u> . . . and that it appears to be impossible to build interactions into a tensor Lagrangian theory. It is, in their view, this circumstance that robs "conservation of zilch" of any claim to physical significance, and that explains why only scalar Lagrangians are encountered in theories of the observable real world.

To approach the subject of "spin," as it is (but only rarely!) encountered in classical electrodynamics I must back up a bit. In 1936 A. Proca undertook to apply orthodox Lagrangian methods to the construction of what might be called a "relativistic electrodynamics of massive photons," his hope being that such objects might be identified with Yukawa's conjectured "mesons" (1934: see again page 18). Proca was led[197] to a system of field equations which in

[196] This, however, is not to say that (352.4) is useless. Used in conjunction with (352.1) and the traceless symmetry of $S^{\mu\nu}$ it supplies

$$\partial_\alpha\big[(x^\beta x_\beta)S^{\mu\alpha}\big] - 2S^{\mu\alpha}x_\alpha = 0$$

which in the case $\mu = 0$ was used (at the middle of page 237) to good effect by Schwinger.

[197] Details are developed in CLASSICAL FIELD THEORY (1999), Chapter 2, pages 16–22 and 51–56.

manifestly Lorentz covariant notation read

$$\begin{gathered}G^{\mu\nu} = \partial^\mu U^\nu - \partial^\nu U^\mu \\ \partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} = 0 \\ \partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0 \\ \partial_\nu U^\nu = 0\end{gathered}$$

and in this electrodynamically-inspired notation

$$\|G^{\mu\nu}\| = \begin{pmatrix} 0 & -\mathfrak{E}_1 & -\mathfrak{E}_2 & -\mathfrak{E}_3 \\ \mathfrak{E}_1 & 0 & -\mathfrak{B}_3 & \mathfrak{B}_2 \\ \mathfrak{E}_2 & \mathfrak{B}_3 & 0 & -\mathfrak{B}_1 \\ \mathfrak{E}_3 & -\mathfrak{B}_2 & \mathfrak{B}_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} U^0 \\ U^1 \\ U^2 \\ U^3 \end{pmatrix} = \begin{pmatrix} \phi \\ \boldsymbol{\mathfrak{A}} \end{pmatrix}$$

become

$$\begin{gathered}\boldsymbol{\mathfrak{E}} = -\boldsymbol{\nabla}\phi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{A}} \qquad \text{and} \qquad \boldsymbol{\mathfrak{B}} = \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{A}} \\ \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{E}} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{B}} = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{B}} = 0 \\ \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{E}} = -\varkappa^2\phi \qquad \text{and} \qquad \boldsymbol{\nabla}\times\boldsymbol{\mathfrak{B}} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\mathfrak{E}} = -\varkappa^2\boldsymbol{\mathfrak{A}} \\ \tfrac{1}{c}\tfrac{\partial}{\partial t}\phi + \boldsymbol{\nabla}\cdot\boldsymbol{\mathfrak{A}} = 0\end{gathered}$$

Here

$$\varkappa \equiv mc/\hbar \quad \text{with physical dimension} \quad [\varkappa] = (\text{length})^{-1}$$

is Proca's "mass parameter"—the reciprocal of the λ encountered already on page 18. The formal success of Proca's program resides in the observation that in the limit $\varkappa \downarrow 0$ these equations assume precisely the form of the free-field Maxwell equations (61) in what we will later learn to call the "Lorentz gauge." Noether's argument leads from the translational invariance of Proca's Lagrangian to a stress-energy tensor which is *not symmetric*, but which after "Belinfante symmetrization" becomes[198]

$$\begin{aligned}T^{\mu\nu} &= G^\mu{}_\sigma G^{\sigma\nu} + \mathcal{L}g^{\mu\nu} + \varkappa^2 U^\mu U^\nu \\ &\qquad \mathcal{L} \equiv \tfrac{1}{2}\Big\{G^{\sigma\rho}(U_{\rho,\sigma} - U_{\sigma,\rho}) - \tfrac{1}{2}G^{\rho\sigma}G_{\rho\sigma}\Big\} - \tfrac{1}{2}\varkappa^2 U^\rho U_\rho\end{aligned}$$

which is manifestly symmetric, but traceless only in the limit $\varkappa \downarrow 0$, and which supplies

$$\begin{aligned}\text{energy density} &= \tfrac{1}{2}\Big[\big(G_{01}^2 + G_{02}^2 + G_{03}^2 + G_{12}^2 + G_{23}^2 + G_{31}^2\big) \\ &\qquad\qquad + \varkappa^2\big(U_0^2 + U_1^2 + U_2^2 + U_3^2\big)\Big] \\ &= \tfrac{1}{2}\big\{\mathfrak{E}^2 + \mathfrak{B}^2 + \varkappa^2(\phi^2 + \mathfrak{A}^2)\big\} \geqslant 0 \\ \text{momentum density vector} &= \tfrac{1}{c}\big\{\boldsymbol{\mathfrak{E}}\times\boldsymbol{\mathfrak{B}} + \varkappa^2\phi\boldsymbol{\mathfrak{A}}\big\}\end{aligned}$$

[198] We write $T^{\mu\nu}$ instead of $S^{\mu\nu}$ because S has been preempted by Spin.

These formulæ give back their electromagnetic counterparts in the limit $\varkappa \downarrow 0$ . . . and bring us at last to the main point of this discussion: Noether's argument leads from the Lorentz invariance of Proca's Lagrangian to an angular momentum tensor of the form

$$\begin{aligned}\mathcal{J}^{\mu\alpha\beta} &= \tfrac{1}{c}\big(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}\big)\\ &= \mathcal{L}^{\mu\alpha\beta} + \mathcal{S}^{\mu\alpha\beta}\\ &= \text{orbital component} + \text{intrinsic or ``spin'' component}\end{aligned}$$

with

$$\mathcal{S}^{\mu\alpha\beta} = \tfrac{1}{c}\big(G^{\alpha\mu}U^\beta - G^{\beta\mu}U^\alpha\big)$$

Both $\partial_\mu\mathcal{L}^{\mu\alpha\beta}$ and $\partial_\mu\mathcal{S}^{\mu\alpha\beta}$ fail to vanish, but they do so in such a concerted way that $\partial_\mu\mathcal{J}^{\mu\alpha\beta} = 0$ (which arise from the familiar pair of circumstances: $\partial_\mu T^{\mu\nu} = 0$ and $T^{\mu\nu} = T^{\nu\mu}$). Straightforward extension of (see again page 233) the definition

$$\text{angular momentum density vector} = \begin{pmatrix}\mathcal{L}^{023}\\ \mathcal{L}^{031}\\ \mathcal{L}^{012}\end{pmatrix}$$

supplies

$$\begin{aligned}\text{spin density vector} = \begin{pmatrix}\mathcal{S}^{023}\\ \mathcal{S}^{031}\\ \mathcal{S}^{012}\end{pmatrix} &= \tfrac{1}{c}\begin{pmatrix}G^{20}U^3 - G^{30}U^2\\ G^{30}U^1 - G^{10}U^3\\ G^{10}U^2 - G^{20}U^1\end{pmatrix}\\ &= \tfrac{1}{c}\,\boldsymbol{\mathfrak{E}}\times\boldsymbol{\mathfrak{A}} \qquad (353)\end{aligned}$$

which does check out dimensionally: from

$$[\boldsymbol{\mathfrak{E}}] = \sqrt{\text{energy density}} \quad\text{and}\quad [\boldsymbol{\mathfrak{A}}] = \text{length}\cdot\sqrt{\text{energy density}}$$

we have

$$\begin{aligned}[\text{spin density}] &= \text{time}\cdot\text{energy density}\\ &= \text{action density}\\ &= \text{angular momentum density}\end{aligned}$$

Remarkably, (353) contains no reference to $\varkappa$, therefore no reference to either $\hbar$ or m. We expect it therefore to retain its meaning even in the classical electromagnetic limit . . . or would but for this awkward detail: in Proca theory ($\varkappa \neq 0$) $\partial_\nu U^\nu = 0$ enjoys the status of a field equation, but in the Maxwellian limit ($\varkappa = 0$) it acquires the status of an arbitrarily imposed side condition (the "Lorentz gauge condition," which will acquire major importance later in our work). In electrodynamics we expect therefore to have

$$\text{spin density } \boldsymbol{\mathcal{S}} = \tfrac{1}{c}\,\boldsymbol{E}\times\boldsymbol{A}, \quad \textit{but only in the Lorentz gauge!} \qquad (354)$$

But if (354) requires us to nail down the gauge, it does not require us to nail down the coordinate system, to specify a "reference point": since the expression on the right lacks the "momental structure" of $\boldsymbol{x}\times\boldsymbol{p}$ it is insensitive to where we have elected to place of the origin of the $\boldsymbol{x}$-coordinate system. If $\mathbf{S}$ has anything at all to do with "angular momentum" it must have to do with "*intrinsic* angular momentum" (or "spin").

Equation (354) appears on page 115 of Davison Soper's *Classical Field Theory* (1976) but nowhere else in the pedagogical literature, so far as I have been able to discover. That the construction $\frac{1}{c}\boldsymbol{E}\times\boldsymbol{A}$ does indeed have "something to do with angular momentum" Soper argues as follows: Look to the case

$$\|A^\mu\| = \begin{pmatrix} \varphi \\ \boldsymbol{A} \end{pmatrix} = \begin{pmatrix} 0 \\ -A\sin[k(ct-z)] \\ \pm A\cos[k(ct-z)] \\ 0 \end{pmatrix}$$

Then the Lorentz gauge condition $\partial_\mu A^\mu = 0$ becomes trivial, and

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} = \begin{pmatrix} Ak\cos[k(ct-z)] \\ \pm Ak\sin[k(ct-z)] \\ 0 \end{pmatrix}$$

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \begin{pmatrix} \mp Ak\sin[k(ct-z)] \\ +Ak\cos[k(z-ct)] \\ 0 \end{pmatrix} = \hat{\boldsymbol{z}}\times\boldsymbol{E}$$

describe ↺ / ↻ circularly polarized plane waves of frequency $\omega = kc$, advancing up the z-axis with speed c. We compute

$$\mathbf{S} = \tfrac{1}{c}\boldsymbol{E}\times\boldsymbol{A} = \pm A^2(k/c)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Noting that the energy density is $\mathcal{E} = \frac{1}{2}(E^2 + B^2) = A^2k^2$ we have

$$\mathbf{S} = (\mathcal{E}/\omega)\,\hat{\boldsymbol{z}} \tag{355}$$

To interpret this result, Soper draws upon a "photonic" conception of the electromagnetic field: he imagines it to contain N photons per unit volume, each carrying energy $\hbar\omega$. Then $\mathcal{E} = N\hbar\omega$ gives

$$\mathbf{S} = (N\hbar)\,\hat{\boldsymbol{z}}$$

which Soper interprets to state that

Each photon carries $\hbar$ units of intrinsic angular momentum (356)

In the *quantum* theory of angular momentum one is brought to the conclusion that the

"allowed values" of **orbital** angular momentum are $0, \hbar, 2\hbar, 3\hbar, \ldots$

"allowed values" of **spin** angular momentum are $0, \frac{1}{2}\hbar, \hbar, \frac{3}{2}\hbar, 2\hbar, \ldots$

and is led to assert that "electrons are spin $\frac{1}{2}$ particles," that "fermions carry half-integral spin, bosons carry integral spin," etc. We are (in view of what happens to those statements in the limit $\hbar \downarrow 0$) not surprised to encounter the frequently-repeated to claim—not quite accurate, as it turns out[199]—that "spin is an intrinsically quantum mechanical phenomenon, and at the deepest level a *relativistic* quantum mechanical phenomenon." It becomes in this light interesting to notice that (355) is *classically meaningful as it stands, that the introduction of "photonic language"—though possible—is inessential*. And indeed: the first direct *experimental* support of (355) was reported by authors who, while they allowed themselves to make casual use of "photonic language," employed methods that were in fact entirely classical.[200]

Yet subtleties lurk within the preceding account of the angular momentum of electromagnetic free fields, and literature bearing on the subject remains to this day often confused/misleading. Some authors fall into paradox when they talk about orbital angular momentum but imagine themselves to be talking about spin angular momentum,[201] though by the present account the two could hardly be more different: an unbounded circularly polarized plane wave carries

- infinite spin angular momentum but (by a symmetry argument)
- zero orbital angular momentum.

Richly detailed accounts of orbital angular momentum can be found in §2.7 and Chapter 9 of J. W. Simmons & M. J. Guttmann's *States, Waves and Photons: A Modern Introduction to Light* (1970) and in a recent paper by L. Allen, M. J. Padgett & M. Babiker,[202] but those authors do not share my interest in probing the outer limits of *classical* electrodynamics: they have other fish to fry, and at critical moments reveal themselves to have photons on the brain. Nor are things quite so simple as I have represented them to be: in §3 of the last of the papers mentioned above we encounter the observation that

> "*... there is a considerable literature which warns against such a separation* [as is conveyed by the equation $\boldsymbol{\mathfrak{J}} = \boldsymbol{\mathfrak{L}} + \boldsymbol{\mathfrak{S}}$: they cite sources, and continue ...] *Biedenharn & Louck write 'It is, indeed,*

---

[199] See the material collected in H. C. Corben, *Classical & Quantum Theories of Spinning Particles* (1968).

[200] R. A. Beth, "Mechanical detection and measurement of the angular momentum of light," Phys. Rev. **50**, 115 (1936). Beth used a torsion balance to measure the change in the angular momentum of a circularly polarized light beam on passage through a doubly refracting crystal plate. He worked at Princeton, and was in correspondence with A. H. S. Holbourn (at Cambridge) who obtained similar results at the same time (Nature **137**, 31 (1936)). Beth reports that an equation equivalent to (355) can be found in J. H. Poynting, Proc. Roy. Soc. **A82**, 560 (1909).

[201] See, for example, R. I. Khrapko, "Question #79. Does plane wave not carry a spin?" AJP **69**, 405 (2001).

[202] "The orbital angular momentum of light," Progress in Optics (1999), pages 294–372.

> *not possible to separate the total angular momentum of the photon field into and "orbital" and a "spin" part (this would contradict gauge invariance); the best that can be done is to define the helicity operators ... which is an observable (Beth).'"*[203]

The problems to which these authors allude do *not* arise in the Proca theory. They are parts of an interrelated nest of problems that arise in the Maxwellian limit $\varkappa \downarrow 0$—problems to which we will have occasion to revisit after we have acquired some sharper tools. Appeals to the Proca theory will often prove of assistance in those endeavors.

6. Conclusion. The work of this chapter has shown the electromagnetic field to be richly endowed with "mechanical properties"... to possess, indeed, all the properties that we standardly/intuitively associate with "particles" *except spatial localization*. The results developed are of a practical importance that should by now be obvious. On the philosophical side ... while they do not of themselves "resolve" the question "Is the electromagnetic field 'real'"? they do have clear relevance to any attempt to assess the status of that question. It is my personal opinion that any attempt to dismiss the electromagnetic field as "a computational convenience ... but a physical fiction"

- has overwhelmingly much to answer for
- is therefore quite unlikely to succeed, and
- would almost entail more cost than benefit.

Readers should, however, be aware that some very able physicists—the young Feynman, among others!—have from time to time been motivated to adopt—tentatively, and without compelling success—the opposite view, and that this minor tradition has exposed isolated points of great interest.[204]

Our work has also served—an many points—to illustrate the remarkable *theory-shaping power of special relativity*.

It is for theory-shaping reasons that I have raised (and will raise again) the "reality question." We do not expect the methods of physics ever to part the final veil and reveal "the stark beauty of naked Reality": we are, after all, decendents of Newton, the great natural philosopher who, though he yearned to know what gravity *is*, recognized that he/we must be content to *describe* what gravity *does* ("I do not philosophize ..."). But when we look to the history of physics we find that major developments have often entailed shifts in the points at which we imagine the "reality" in our theories to be invested. So it is for pragmatic reasons that we must pay attention to the "reality question"

[203] Their reference here is to L. C. Biedenharn & J. D. Luock, *Angular Momentum in Quantum Physics: Vol. VIII of the Encyclopaedia of Mathematics & its Applications* (1980).

[204] See the papers reprinted in E. H. Kerner (editor), *The Theory of Action-at-a-Distance in Relativistic Particle Physics* (1972). Also relevant is F. Hoyle & J. V. Narlikar's *Action at a Distance in Physics and Cosmology* (1974).

if it is our ambition to contribute to the next such development. My claims regarding the "reality of the electromagnetic field" draw only weak support from electro/magnetostatics, stronger support from electro*dynamics* . . . but in the end we must admit that in fact we never *observe* $\boldsymbol{E}$-fields or $\boldsymbol{B}$-fields their naked selves: what we observe are (ramifications of) their mechanical properties, the results of their interaction with other mechanical systems (which themselves remain similarly unobservable in isolation!). It might therefore be argued that we should assign tentative "reality" not to $F^{\mu\nu}$ but to objects like $S^{\mu\nu}$. But even then the situation is not entirely clear cut . . . for suppose were were to form

$$
{\color{red}S}^{\mu\nu} \equiv S^{\mu\nu} + \partial_\alpha W^{\alpha\mu\nu}
$$

$$
W^{\alpha\mu\nu} \text{ assumed to be } \begin{cases} \alpha\mu\text{-antisymmetric} \\ \mu\nu\text{-symmetric} \end{cases}
$$

Then

$$
\begin{aligned}
\partial_\mu S^{\mu\nu} = 0 \quad &\iff \quad \partial_\mu {\color{red}S}^{\mu\nu} = 0 \\
S^{\mu\nu} = S^{\nu\mu} \quad &\iff \quad {\color{red}S}^{\mu\nu} = {\color{red}S}^{\nu\mu}
\end{aligned}
$$

and (because $\alpha\mu$-antisymmetry entails $\partial_\alpha W^{\alpha 0\nu} = \sum_k \partial_k W^{k0\nu} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{W}^\nu$)

$$
\begin{aligned}
\int {\color{red}S}^{0\nu}\, d^3x &= \int S^{0\nu}\, d^3x + \int \boldsymbol{\nabla}\cdot\boldsymbol{W}^\nu\, d^3x \\
&= \text{ditto} + \int \boldsymbol{W}^\nu\cdot\boldsymbol{d\sigma} \\
&= \int S^{0\nu}\, d^3x \quad \text{if the surface term is assumed to vanish}
\end{aligned}
$$

show that, while ${\color{red}S}^{\mu\nu}$ assigns

- different energy/momentum densities but
- the same *total* energy/momentum

to the field, it satisfies all formal requirements (symmetry, local conservation) just as well as $S^{\mu\nu}$. We possess therefore as many viable candidate stress-energy tensors as there are ways to assign value to $W^{\alpha\mu\nu}$ *and no principle of choice*. It becomes difficult in such a circumstance to argue that one member of the population has a stronger claim to "reality" than another.[205]

[205] For related discussion see ELECTRODYNAMICAL APPLICATIONS OF THE EXTERIOR CALCULUS (1996) pages 57–61 and "ELECTRODYNAMICS" IN 2-DIMENSIONAL SPACETIME (1997) pages 13–14.

4

POTENTIAL & GAUGE

Introduction. When Newton wrote $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$ he imposed no significant general constraint on the design of the force law $\boldsymbol{F}(\boldsymbol{x}, t)$. God, however, appears to have special affection for *conservative* forces—those (a subset of zero measure within the set of all conceivable possibilities) that conform to the condition

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{0}$$

—those, in other words, that can be considered to derive from a scalar potential:

$$\boldsymbol{F} = -\boldsymbol{\nabla}U \tag{357}$$

Only in such cases is it

- possible to speak of energy conservation
- possible to construct a Lagrangian $L = T - U$
- possible to construct a Hamiltonian $H = T + U$
- possible to quantize.

It is, we remind ourselves, the potential U—not the force $\boldsymbol{F}$—that appears in the Schrödinger equation ... which is rather remarkable, for U has the lesser claim to direct physicality: if U "does the job" (by which I mean: if U reproduces $\boldsymbol{F}$) then so also does

$$U \equiv U + \text{constant} \tag{358}$$

where "constant" is vivid writing that somewhat overstates the case: we require only that $\boldsymbol{\nabla}\!\cdot(\text{constant}) = \boldsymbol{0}$, which disallows $\boldsymbol{x}$-dependence but does not disallow t-dependence.

At (357) a "spook" has intruded into mechanics—a device which we are content to welcome into (and in fact can hardly exclude from) our computational lives ... but which, in view of (358), cannot be allowed to appear nakedly in our final results. The adjustment

$$U \longrightarrow \color{red}{U} = U + \text{constant}$$

provides the simplest instance of what has come in relatively recent times to be called a "gauge transformation."[206] For obvious reasons we require of such physical statements as may contain U that they be *gauge-invariant*. To say the same thing another way: It is permissible to write (say)

$$E = \tfrac{1}{2}m\dot{x}^2 + U(\boldsymbol{x})$$

in the midst of a theoretical argument, but it would be pointless to go to the stockroom in quest of a "U-meter": the best we could do would be to obtain a "potentiometer" ... that has *two* testleads and measures

$$\Delta U = U(\boldsymbol{x}) - U(\boldsymbol{x}_0) \quad : \quad \text{gauge-invariant}$$

Or a "differential potentiometer," that measures $\boldsymbol{\nabla}U$.

Moving deeper into mechanics, we encounter the Lagrangian $L(q, \dot{q}, t)$, which (though seldom described in such terms) must itself be a kind of "potential"—a "spook"—since susceptible to gauge transformations of the form

$$L(q, \dot{q}, t) \longrightarrow \color{red}{L}(q, \dot{q}, t) + \tfrac{d}{dt}(\text{any function of } q \text{ and } t)$$

—the point here being that if L and $\color{red}{L}$ are so related then they give rise to identical equations of motion.

We encountered the scalar potential already when at (17) we had occasion to write

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi \quad : \quad \text{invariant under} \quad \varphi \longrightarrow \color{red}{\varphi} = \varphi + \text{constant} \tag{359.1}$$

and to observe that it is characteristic of the structure of electro*static* fields that

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0} \tag{359.2}$$

In a parallel discussion of *magnetostatic* fields we were led at (92) to the "vector potential;"[207] *i.e.*, to the observation that if we write

[206] The terminology is due, I have read, to Hermann Weyl (the founding father of what became "gauge field theory"), who reportedly had in mind the "gauge" of railway tracks.

[207] The vector potential first appears (∼1835) in work of F. E. Neumann (1798–1895) concerned with the mechanical interaction of current-carrying wires (Ampere's law: see page 58). Maxwell (1831–1879) came independently to the same idea at a much later date, and from a different direction (Faraday's law). Neumann, by the way, was a close associate of Jacobi (1804–1851) from 1833 until the younger man's death, and was the teacher of many of the greatest figures in 19$^{\text{th}}$ Century German physics.

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \quad : \quad \text{invariant under} \quad \boldsymbol{A} \longrightarrow \boldsymbol{A} = \boldsymbol{A} + \boldsymbol{\nabla}\chi \tag{359.3}$$

then

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{359.4}$$

is rendered automatic.

So important is the role played by scalar/vector potentials in all vector field theories—in fluid dynamics, for example, but especially in electrodynamics—that in this chapter I interrupt the flow of the narrative to indicate how those concepts fit within the framework of the manifestly covariant theory of the electromagnetic field. The ideas presented here will be central to all of our subsequent work.

1. How potentials come into play: Helmholtz' decomposition theorem. In three dimensions, a vector field $\boldsymbol{V}(\boldsymbol{x})$ is said to be

- "irrotational" if and only if $\boldsymbol{\nabla}\times\boldsymbol{V} = \boldsymbol{0}$
- "solenoidal" if and only if $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$.

Helmholtz (and later but independently also Maxwell) showed that *every* vector field can be resolved[208]

$$\boldsymbol{V}(\boldsymbol{x}) = \big\{\text{irrotational part } \boldsymbol{I}(\boldsymbol{x})\big\} + \big\{\text{solenoidal part } \boldsymbol{S}(\boldsymbol{x})\big\} \tag{360}$$

Drawing now upon the (unproven) converse of (6) we conclude that $\boldsymbol{I}$ can be considered to arise by

$$\boldsymbol{I} = \boldsymbol{\nabla}\psi$$

from a *scalar potential* ψ, and that $\boldsymbol{S}$ can be considered to arise by

$$\boldsymbol{S} = \boldsymbol{\nabla}\times\boldsymbol{\psi}$$

from a *vector potential* $\boldsymbol{\psi}$. Every vector field $\boldsymbol{V}$ can therefore be displayed

$$\boldsymbol{V} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\times\boldsymbol{\psi} = \text{gradient} + \text{curl} \tag{361}$$

but that display is *non-unique*, since the *potentials are determined only to within gauge transformations*

$$\left.\begin{aligned} \psi &\longrightarrow \psi = \psi + \text{arbitrary constant} \\ \boldsymbol{\psi} &\longrightarrow \boldsymbol{\psi} = \boldsymbol{\psi} + \boldsymbol{\nabla}(\text{arbitrary scalar field}) \end{aligned}\right\} \tag{362}$$

Since susceptible to gauge transformation, the potentials ψ and $\boldsymbol{\psi}$ are released from adherence to such boundary/symmetry/transformation properties as—in specific applications—typically pertain to the "physical" fields $\boldsymbol{V}$.

[208] For proof see R. B. McQuistan, *Scalar & Vector fields: A Physical Interpretation* (1965), page 261.

It is not obvious that the replacement of three objects (the components of the vector $\boldsymbol{V}$) by four (ψ and the components of $\boldsymbol{\psi}$) represents an advance. But in applications it is invariably the case that Helmholtz decomposition (360) serves to *clarify the essential structure* of the theory in question, and is often the case that by exploiting gauge freedom one can *simplify both the formulation of the theory and many of the attendant computations*. The electrodynamical application will serve to illustrate both of those advantages.

Helmholtz decomposition provides the simplest instance of the vastly more general "Hodge decomposition," which (though not usually phrased in such terms) can be considered to pertain to completely antisymmetric tensors of arbitrary rank, inscribed on N-dimensional manifolds of almost arbitrary topology.[209]

2. Application to Maxwellian electrodynamics. Look again to the pair of Maxwell equations that make no reference to source activity:

$$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{B} = 0 \tag{65.2}$$
$$\boldsymbol{\nabla}\!\times\!\boldsymbol{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} = \boldsymbol{0} \tag{65.4}$$

The former asserts that *magnetic fields—not only in the static case, but also dynamically—are solenoidal*, so can be written

$$\boldsymbol{B} = \boldsymbol{\nabla}\!\times\!\boldsymbol{A} \tag{363.1}$$

Returning with this information to (65.4) we obtain $\boldsymbol{\nabla}\!\times\!\big\{\boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}\big\} = \boldsymbol{0}$, according to which $\boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}$ is irrotational, so can be expressed $-\boldsymbol{\nabla}\varphi$, giving

$$\begin{aligned}
\boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} \\
&= \big\{\text{irrotational component arising from charges}\big\} \\
&\qquad\qquad + \big\{\text{component generated by Faraday induction}\big\} \\
&\downarrow \\
&= -\boldsymbol{\nabla}\varphi \quad \text{in the static case}
\end{aligned} \tag{363.2}$$

It was, by the way, to place himself in position to write $\boldsymbol{E}_{\text{Faraday}} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}$ that Maxwell was motivated[207] to re*invent* the vector potential.

The construction (363.1) of $\boldsymbol{B}$ is invariant under $\boldsymbol{A} \longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\chi$. But that adjustment sends

$$\begin{aligned}
\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} \quad \longrightarrow \quad \boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}(\boldsymbol{A} + \boldsymbol{\nabla}\chi) \\
&= -\boldsymbol{\nabla}\big\{\varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi\big\} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}
\end{aligned}$$

[209] See H. Flanders, *Differential Forms, with Applications to the Physical Sciences* (1963), page 138.

and that observation motivates us to write $\varphi \equiv \varphi + \frac{1}{c}\frac{\partial}{\partial t}\chi$. To summarize: the equations (363) are invariant under

$$\left.\begin{aligned} \varphi &\longrightarrow \varphi = \varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi \\ \boldsymbol{A} &\longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\chi \end{aligned}\right\} \tag{364}$$

where χ is an arbitrary scalar field, and where we can look upon the first adjustment as a forced implication of the second.

The source-independent Maxwell equations (65.2) and (65.4) have—by the introduction (363) of the scalar/vector potentials—been rendered *automatic*. We need concern ourselves, therefore, only with the sourcey Maxwell equations

$$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} = \rho \tag{65.1}$$

$$\boldsymbol{\nabla}\!\times\!\boldsymbol{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} = \tfrac{1}{c}\boldsymbol{j} \tag{65.3}$$

which, when expressed in terms of the potentials, become a *pair* of *second order* partial differential equations:

$$\begin{aligned} \boldsymbol{\nabla}\!\cdot\!\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\big\} &= \rho \\ \boldsymbol{\nabla}\!\times\!(\boldsymbol{\nabla}\!\times\!\boldsymbol{A}) - \tfrac{1}{c}\tfrac{\partial}{\partial t}\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}$$

These, after simplification[210] and reorganization, can be rendered

$$\begin{aligned} -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A} - \nabla^2\varphi &= \rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} + \boldsymbol{\nabla}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}$$

or again but more symmetrically (add/subtract a term in the first equation)

$$\left.\begin{aligned} \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} + \quad \boldsymbol{\nabla}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}\right\} \tag{365.1}$$

The field equations (365) are *gauge-invariant*, which is to say: under the substitutional adjustment

$$\begin{aligned} \varphi &\longmapsto \varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi \\ \boldsymbol{A} &\longmapsto \boldsymbol{A} + \boldsymbol{\nabla}\chi \end{aligned}$$

they go over into

$$\left.\begin{aligned} \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} + \quad \boldsymbol{\nabla}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}\right\} \tag{365.2}$$

[210] Recall the identity $\boldsymbol{\nabla}\!\times\!(\boldsymbol{\nabla}\!\times\!\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}) - \nabla^2\boldsymbol{A}$, of which we made use already on page 54.

because all the χ-terms cancel. Gauge freedom can be used to render (365.2) simpler (or, for that matter, more complicated) than (365.1). For example: from $\nabla\cdot\boldsymbol{A} = \nabla\cdot\boldsymbol{A} - \nabla^2\chi$ we learn that if χ is taken to be any solution of

$$\nabla^2\chi = \nabla\cdot\boldsymbol{A}$$

then $\boldsymbol{A}$ satisfies the

$$\textsc{coulomb gauge condition:}\quad \nabla\cdot\boldsymbol{A} = 0$$

and equations (365.2) become

$$\begin{aligned} \nabla^2\varphi &= -\rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} &= \tfrac{1}{c}\boldsymbol{j} - \underbrace{\nabla\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi\big\}}_{\text{formally a kind of ``current''}} \end{aligned}$$

The Coulomb gauge is also known as the "radiation" or "transverse gauge." For discussion see §6.3 in J. D. Jackson's *Classical Electrodynamics* (3^{rd} edition 1999). Of much more general importance is the

$$\textsc{lorentz gauge condition:}\quad \tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \nabla\cdot\boldsymbol{A} = 0 \tag{366}$$

which arises from taking χ to be any solution of

$$\Box\chi = -\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \nabla\cdot\boldsymbol{A}\big\}$$

and which brings (365.2) to the strikingly simple form

$$\left.\begin{aligned} \Box\varphi &= \rho \\ \Box\boldsymbol{A} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}\right\} \tag{367}$$

historical remark: I have been informed by David Griffiths (who learned from J. D. Jackson, while on sabbatical at Berkeley) that (366) first appears in the work (1867)not of H. A. Lorentz (Dutch, 1853–1928) but of L. V. Lorenz (Danish, 1829–1891), so should —in violation of universal practice—be called the "Lorenz gauge condition" (no "t"). For the fascinating historical details see J. D. Jackson & L. B. Okun, "Historical roots of gauge invariance," RMP **73**, 6653 (2001). My own recent effort to discover the facts of the matter took me to the *Dictionary of Scientific Biography* (1973), where I was reminded that the Lorentz article—by Russell McCormmach, an eminent historian of physics who was once my Reed College classmate—provides a splendid short account of the confused state of electrodynamics when Lorentz entered upon the scene. Theories by Weber, Neumann, Riemann, Lorenz and—almost lost in the crowd—Maxwell were then in lively competition. McCormmach makes clear the insightful audacity that Lorentz displayed when he embraced a theory that assigned a central place to a perplexing notion (the field concept) and that declined to address a question that others considered paramount: What *is* charge?

3. Manifestly covariant formulation of the preceding material. The emphasis here must be on the "manifestly." The material developed in §2 is relativistic as it stands (as, indeed, were the Maxwell equations (65) on which it is based) . . . but "covertly" so. It will emerge that our recent work becomes much more transparent when rendered in language that makes the Lorentz covariance manifest. We look first to the notational aspects of the matter, then to its transformational aspects (which will be almost obvious):

Let us—in addition to this familiar variant of (159)

$$\|F_{\mu\nu}\| = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

—agree to write

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \equiv \begin{pmatrix} \varphi \\ \boldsymbol{A} \end{pmatrix}, \quad \text{equivalently} \quad \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \varphi \\ -\boldsymbol{A} \end{pmatrix} \tag{368}$$

where the Lorentz metric $g_{\mu\nu}$ has been used to lower the indices. Then equations (363) become

$$\begin{aligned} B_1 &= F_{32} = -F_{23} = -(\partial_2 A_3 - \partial_3 A_2) \\ B_2 &= F_{13} = -F_{31} = -(\partial_3 A_1 - \partial_1 A_3) \\ B_3 &= F_{21} = -F_{12} = -(\partial_1 A_2 - \partial_2 A_1) \\ E_1 &= F_{01} = -F_{10} = -(\partial_1 A_0 - \partial_0 A_1) \\ E_2 &= F_{02} = -F_{20} = -(\partial_2 A_0 - \partial_0 A_2) \\ E_3 &= F_{03} = -F_{30} = -(\partial_3 A_0 - \partial_0 A_3) \end{aligned}$$

or, more compactly,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{369}$$

The preceding construction is obviously invariant under

$$A_\mu \longrightarrow A_\mu = A_\mu + \partial_\mu \chi \tag{370}$$

which when spelled out in detail becomes precisely (364).

The source-independent pair of Maxwell equations were found at (166) to be expressible

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

which are seen now to follow *automatically* from the construction (369), while the sourcey pair of Maxwell equations—which at (167) we learned to write

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c} j^\nu \quad \text{with} \quad \|j^\nu\| \equiv \begin{pmatrix} c\rho \\ \boldsymbol{j} \end{pmatrix}$$

—become $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{c}j^\nu$ or

$$\square A^\nu - \partial^\nu(\partial_\mu A^\mu) = \tfrac{1}{c}j^\nu \tag{371}$$

The Coulomb gauge condition violates the spirit of relativity (can be adopted by any particular inertial observer, but not simultaneously by all), but that criticism does not pertain to (366), which becomes the

$$\text{LORENTZ GAUGE CONDITION:}\quad \partial_\mu A^\mu = 0 \tag{372}$$

and when in force causes (371) to become

$$\square A^\nu = \tfrac{1}{c}j^\nu \tag{373}$$

which reproduces (367). Imposition of the Lorentz gauge condition does not quite exhaust the available gauge freedom, for

$$\begin{aligned}\partial_\mu A^\mu = 0 \quad\Longrightarrow\quad &\partial_\mu A^\mu = 0\\ &A^\mu = A^\mu + \partial^\mu\chi \quad\text{with } \chi \text{ any solution of}\quad \square\chi = 0\end{aligned}$$

It becomes at this point entirely natural to <u>assume that A_μ transforms as a weightless vector field</u>. It is then automatic—here our "catalog of accidentally tensorial derivative constructions" (pages 120–122) comes again into play—that $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ transforms as a weightless antisymmetric tensor, and that $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$ makes tensorial good sense. On the other hand

- $\partial_\mu F^{\mu\nu} = \frac{1}{c}j^\nu$ is unrestrictedly tensorial if and only if $F^{\mu\nu}$ (whence also j^ν) have *unit* weight
- $\partial_\mu A^\mu = 0$ is unrestrictedly tensorial if and only if A^μ has unit weight

We, however, have interest at the moment in a <u>restricted</u> tensoriality, in *Lorentz covariance* (which means "tensoriality with respect to Lorentz transformations"). Inspection of the arguments used to develop the entries in the "catalog" shows that all weight restrictions arose from the presumption that the elements of the transformation matrix $\mathbb{M} \equiv \|\partial x^m/\partial x^n\|$ change from point to point: $\partial\mathbb{M} \neq \mathbb{O}$. But in that respect the Lorentz transformations—being *linear* transformations—are atypical: one has $\partial\Lambda = \mathbb{O}$, with the consequence that all weight restrictions are lifted. We are brought thus to the conclusion that the numbered equations at the top of the page are *Lorentz covariant as they stand*.

It is now possible—and instructive—to consider afresh this question:

4. So what kind of a thing is Maxwellian electrodynamics? My strategy will be to consider the question not in isolation, but in juxtaposition to a second question: *What kind of a thing is the Proca theory*? ...and it is to the latter question that we look first.

The Proca theory arises fairly naturally when—within the formal context provided by the "classical theory of fields"—one asks for a **relativistic theory of**

a massive vector field. One is led at length to a system of free-field equations that were encountered already on page 246 and are reproduced below:

$$\partial_\mu U^\mu = 0 \tag{374.1}$$
$$\partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0 \tag{374.2}$$

$$G^{\mu\nu} \equiv \partial^\mu U^\nu - \partial^\nu U^\mu \tag{374.3}$$

$$\partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} = 0 \tag{374.4}$$

Here U^μ is the physical field, (374.1) and (374.2) are the field equations, (374.3) introduces a notational device used to simplify the statement of the second field equation—which would otherwise read

$$\Box U^\nu \underbrace{-\partial^\nu(\partial_\mu U^\mu)}_{\text{vanishes by the first field equation}} + \varkappa^2 U^\nu = 0$$

—and (374.4) records a corollary property of the "notational device" $G^{\mu\nu}$. Distinct vector fields—namely those that stand in the relationship

$$\mathcal{U}^\mu = U^\mu + \partial^\mu \chi$$

—give rise to identical $G^{\mu\nu}$-fields, but the field equations are <u>not invariant</u> under $U^\mu \longrightarrow \mathcal{U}^\mu = U^\mu + \partial^\mu\chi$. For if U^μ satisfies

$$\partial_\mu U^\mu = 0$$
$$\partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0$$

then $\mathcal{U}^\mu$ satisfies

$$\partial_\mu(\mathcal{U}^\mu - \partial^\mu\chi) = 0$$
$$\partial_\mu G^{\mu\nu} + \varkappa^2(\mathcal{U}^\mu - \partial^\mu\chi) = 0$$

which become structurally identical to the original equations if and only if

$$\Box\chi = 0 \quad \text{and} \quad \varkappa^2 = 0$$

In the degenerate case $\varkappa^2 = 0$ the Proca free-field equations (374) become structurally identical to the system of equations that was seen above to describe the free electromagnetic field, but in the latter context the "location of the physics" is shifted, and the equations stand suddenly in a different logical relation to one another. One writes

$$\partial_\mu F^{\mu\nu} = 0 \tag{375.1}$$
$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \tag{375.2}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{375.3}$$

$$\partial_\mu A^\mu = 0 \tag{375.4}$$

What was formerly a mere "notational device" $G^{\mu\nu}$ has now become the *physical* field $F^{\mu\nu}$, and what was formerly dismissed as an incidental "corollary property" has at (375.2) been promoted to the status of a field equation. It is to render that field equation "automatic" that we write (375.3), at which point it is the formerly physical vector field that has acquired the status of a "notational device, a crutch"... denied direct physical significance because it is defined *only up to an arbitrary gauge transformation*. Finally, the Lorentz gauge condition (375.4)—which in Proca theory enjoyed the status of a field equation—has in electrodynamics been demoted to the status of an arbitrarily imposed side condition.

The comparative situation (at least so far as concerns *free* Proca/Maxwell fields: no externally impressed sources/currents) can be summarized this way:

> PROCA has given us the manifestly covariant theory of a **physical/observable massive vector field** U^μ.
>
> MAXWELL has given us (what is in effect, or can be rendered as) the manifestly covariant theory of an **unphysical/unobservable massless vector field**—a "gauge field." The observable physics attaches in that theory to the gauge invariant object
>
> $$\text{field tensor } F^{\mu\nu} \equiv \text{curl of the gauge field}$$

The "theory of gauge fields"—quantum mechanical generalizations of A^μ—has, during the second half of the 20^{th} Century, moved to center stage in the theory of elementary particles and their fundamental interactions.[211] Our recent experience indicates that *gauge freedom arises from masslessness*, so we are perhaps not surprised to learn that a major problem in that area has been to figure out a way to endow gauge fields with mass (lots of it! ...as the experimental evidence clearly requires). The "Higgs mechanism" stands as the best available solution of the problem,[212] though it is in some respects unattractive, and has as yet no convincing experimental support.

Further insight into the distinctive structure of the electromagnetic field can be gained by carrying "comparative Proca/Maxwell theory" a bit further:

5. Plane wave solutions of the Proca/Maxwell field equations. Both (374) and (375) are notable for their *linearity*. In both theories a *principle of superposition* is operative, so we expect to be able to write

$$\text{general solution} = \sum\!\!\!\!\!\!\int \text{(simple solutions)}$$

[211] See L. O'Raifeartaigh, *The Dawning of Gauge Theory* (1997) for a splendid account of the major contours of that development.

[212] See the concluding §11.9 in David Griffiths' *Introduction to Elementary Particles* (1987) for a brief account of the essential idea.

The meaning most usefully assigned to "simple solution" is highly context-dependent (selection of a basis *always* is): it serves my present purpose to proceed as Fourier did; *i.e.*, to write

$$U^\mu(x) = \int \mathsf{U}^\mu(k)\cdot e^{ik\,x}\,d^4k \quad \text{with} \quad \|k^\mu\| \equiv \begin{pmatrix} \omega/c \\ \boldsymbol{k} \end{pmatrix},\ \|x^\mu\| \equiv \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix}$$

where $k\,x \equiv k_\alpha x^\alpha \equiv \omega\,t - \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}$ is evidently Lorentz invariant, where the $\mathsf{U}^\mu(k)$ are understood to transform as a k-parameterized population of complex 4-vectors, and where the reality of $U^\mu(x)$ requires $[\mathsf{U}^\mu(k)]^* = \mathsf{U}^\mu(-k)$. At the expense of some notational clutter we could write

$$U^\mu(x) = \int \mathsf{V}^\mu(k)\cos kx\,d^4k + \int \mathsf{W}^\mu(k)\sin kx\,d^4k$$

where V^μ and W^μ are now understood to be *real* 4-vectors. From the field equations $\big\{g^{\alpha\beta}\partial_\alpha\partial_\beta + \varkappa^2\big\}U^\mu = 0$ and $\partial_\mu U^\mu = 0$ we discover that necessarily

$$k^2 \equiv g^{\alpha\beta}k_\alpha k_\beta \equiv k_0^2 - \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{k} = \varkappa^2 \tag{376.1}$$

and

$$k_\mu \mathsf{U}^\mu = 0 \quad \text{equivalently} \quad k_\mu \mathsf{V}^\mu = k_\mu \mathsf{W}^\mu = 0 \tag{376.2}$$

The first condition places the k-vector "on the mass shell" (see again Figure 70), while the second condition requires (the real and imaginary parts of) U^μ to be (in the Lorentzian sense) *normal* to k^μ.[213] The question now arises: How many linearly independent vectors V^μ stand normal to any given timelike vector k^μ? The answer, pretty clearly, is three: the following example illustrates the situation

$$\|k^\mu\| = \begin{pmatrix} \varkappa \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \perp \quad \|\mathsf{V}^\mu\| = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ else } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ else } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and so do all Lorentz transforms of that example.

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[213] The language has become a bit tangled: The mass shell is seen in Figure 70 to live in p-space, while the $\varkappa$-shell lives in k-space. A scale factor distinguishes the one form the other:

$$p = \hbar k \quad \text{and} \quad mc = \hbar\varkappa$$

The "timelike/null (or lightlike)/spacelike" terminology I will carry over from x-space into p-space, though in the latter context it would be more correct to distinguish "energylike" from "momentumlike" 4-vectors. In k-space there is, so far as I am aware, *no* commonly accepted "correct" terminology.

Suppose we write

$$\mathsf{V}^\mu \cos kx = \mathsf{V}^\mu \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t)$$

to describe one of our "simple free Proca fields." A second inertial observer O would write

$$\mathsf{V}^\mu \cos kx = \mathsf{V}^\mu \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t)$$

to describe the same physical situation, but we will persist in language special to our own perception of the situation. Writing

$$\phi(\boldsymbol{x}, t) \equiv \boldsymbol{k}\cdot\boldsymbol{x} - \omega t \equiv \text{phase}$$

or again

$$\boldsymbol{k}\cdot\boldsymbol{x} = \omega t + \text{phase}$$

we see the points of constant phase to lie at time t on a *plane* in 3-dimensional space. From

$$\boldsymbol{\nabla}\phi = \boldsymbol{k} \quad : \quad \text{all } \boldsymbol{x} \text{ and all } t$$

we see that all phase planes stand normal to $\boldsymbol{k}$, which by t-differentiation we have

$$\boldsymbol{k}\cdot\boldsymbol{u} = \omega \quad : \quad \boldsymbol{u} = u\hat{\boldsymbol{k}} \equiv \text{phase velocity}$$

Immediately

$$u = \omega/k = \text{phase speed}$$

From (376.1) we have the "dispersion equation"

$$\omega = c\sqrt{k^2 + \varkappa^2}$$

so

$$u = c\,\frac{\sqrt{k^2+\varkappa^2}}{k} \text{ which } \begin{cases} \text{is } \geqslant c \\ \text{is a } \textit{descending} \text{ function of } k \\ = \infty \text{ at } k = 0 \\ = \ \ c \ \ \text{at } k = \infty \end{cases}$$

On the other hand, we have

$$\begin{aligned} \text{group speed } v &\equiv \frac{d\omega}{dk} \\ &= c\frac{k}{\sqrt{k^2+\varkappa^2}} \text{ which } \begin{cases} \text{is } \leqslant c \\ \text{is an } \textit{ascending} \text{ function of } k \\ = 0 \text{ at } k = 0 \\ = c \text{ at } k = \infty \end{cases} \end{aligned}$$

Different inertial observers will assign different values to u and v, but all will be in agreement that

$$(\text{phase speed}) \cdot (\text{group speed}) = c^2 \quad : \quad \text{all } k$$

The results just developed are standard to all occurrences of the so-called "Klein-Gordon equation" $\Box\psi + \varkappa^2\psi = 0$, which is to say: they are not special to the Proca theory.[214] I turn now to statements that *are* special to the Proca theory. Agree to write $\boldsymbol{e}_\parallel \equiv \hat{\boldsymbol{k}}$, to let $\boldsymbol{e}_1$ be any unit 3-vector normal to $\hat{\boldsymbol{k}}$, and to define $\boldsymbol{e}_2 \equiv \hat{\boldsymbol{k}} \times \boldsymbol{e}_1$, so that $\{\boldsymbol{e}_\parallel, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ comprise a righthanded orthonormal triad in 3-dimensional $\boldsymbol{k}$-space. And recall that $\boldsymbol{k}$ came to us from the 4-vector

$$\|k^\mu\| = \begin{pmatrix} \omega/c \\ \boldsymbol{k} \end{pmatrix} = \begin{pmatrix} \sqrt{k^2 + \varkappa^2} \\ \boldsymbol{k} \end{pmatrix} \quad : \quad \text{gives } k_\mu k^\mu = \varkappa^2$$

Now define the spacelike unit 4-vectors

$$\|\mathsf{V}_1^\mu\| \equiv \begin{pmatrix} 0 \\ \boldsymbol{e}_1 \end{pmatrix} \quad \text{and} \quad \|\mathsf{V}_2^\mu\| \equiv \begin{pmatrix} 0 \\ \boldsymbol{e}_2 \end{pmatrix}$$

Clearly

$$\left.\begin{array}{r} \mathsf{V}_1^\mu \perp \mathsf{V}_2^\mu \\ k^\mu \perp \text{ both } \mathsf{V}_1^\mu \text{ and } \mathsf{V}_2^\mu \end{array}\right\} \text{ in the Lorentzian sense}$$

Finally construct

$$\|\mathsf{V}_\parallel^\mu\| \equiv \gamma \begin{pmatrix} \beta \\ \boldsymbol{e}_\parallel \end{pmatrix}$$

NOTE: β and γ are here to be regarded simply as constants, stripped of all prior relativistic associations.

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[214] And though they pertain the the planewave solutions of certain relativistic free fields, the results just obtained bear a striking resemblence to equations encountered in the theory of relativistic free particles ... for $E = \gamma mc^2$ can be written

$$v = c\,\frac{\sqrt{E^2 - (mc^2)^2}}{E}$$

which describes the speed v of a mass m with energy E. We see that

$$\text{particle speed } v \begin{cases} = 0 \text{ at } E = mc^2 \\ \text{approaches } c \text{ as } E \uparrow \infty \\ \therefore \text{ can never equal or exceed } c \end{cases}$$

while in the limit $m \downarrow 0$ we have

$$\text{speed of a "massless particle" is } \textit{always } v = c$$

We see also that the "massless particle" concept is delicate: it would be senseless to write $p^\mu = 0u^\mu$ or $E = \gamma 0c^2$.

and observe that $\perp$ to both V_1^μ and V_2^μ is (for all β) automatic, while $\perp$ k^μ entails $\beta\sqrt{k^2+\varkappa^2}-k=0$, which requires that we set

$$\beta=\frac{k}{\sqrt{k^2+\varkappa^2}}=\frac{\text{group speed } v}{c}$$

The "spacelike unit vector condition"

$$g_{\mu\nu}\mathsf{V}_\parallel^\mu\mathsf{V}_\parallel^\nu=g_{\mu\nu}\mathsf{V}_1^\mu\mathsf{V}_1^\nu=g_{\mu\nu}\mathsf{V}_2^\mu\mathsf{V}_2^\nu=-1$$

requires finally that we set

$$\gamma=\frac{1}{\sqrt{1-\beta^2}}$$

To summarize: the Proca theory supports <u>plane waves of three types</u>. Specification of the propagation vector $\boldsymbol{k}\neq\boldsymbol{0}$ determines both the direction of propagation $\hat{\boldsymbol{k}}$ and the frequency of oscillation $\omega=c\sqrt{k^2+\varkappa^2}$. The three wave types consist of two linearly independent **transverse waves**

$$U^\mu_{\text{transverse}}(x)=\begin{cases}\mathsf{V}_1^\mu\cos(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega\,t+\delta_1)\\ \mathsf{V}_2^\mu\cos(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega\,t+\delta_2)\end{cases}$$

and a solitary **longitudinal wave**

$$U^\mu_{\text{longitudinal}}(x)=\mathsf{V}_\parallel^\mu\cos(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega\,t+\delta_\parallel)$$

In the degenerate case $\boldsymbol{k}=\boldsymbol{0}$ the "direction of propagation" loses its meaning (there *is* no propagation!), the $\boldsymbol{x}$-dependence drops away, the field oscillates *as a whole* with frequency $\omega_0=c\varkappa$, the "transverse/longitudinal distinction" becomes meaningless, and the orthonormal triad $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\equiv\boldsymbol{e}_\parallel\}$ can be erected arbitrarily. It is as such a "degenerate case" that any Proca field presents itself to any "co-moving observer." The example of page 263 provides an instance of just such a case.

It is in the light of the preceding discussion, and by the formal process $\varkappa^2\downarrow 0$, that we return now to free-field electrodynamics. We have already noted (while discussing the relationship of (375) to (374)) that the transition

$$\varkappa^2 \text{ arbitrarily small} \quad\longrightarrow\quad \varkappa^2=0$$

is formally/qualitatively quite abrupt. The point becomes especially vivid when one looks comparatively to the planewave *solutions* of the Proca/Maxwell field equations. Look first to what happens to the dispersion equation

$$\omega=c\sqrt{k^2+\varkappa^2}\xrightarrow[\varkappa\downarrow 0]{}\omega=ck$$

In Proca theory we found that

$$\text{phase speed}\equiv\omega/k=c\,\frac{\sqrt{k^2+\varkappa^2}}{k}=\frac{\omega}{\sqrt{(\omega/c)^2-\varkappa^2}}$$

is frequency-dependent. Proca fields are "dispersive:" the constituent Fourier components of wavepackets travel at different speeds, and the wavepackets therefore "dissolve." The free electromagnetic field is, on the other hand, *non-dispersive*, since

$$\omega = ck \quad \Longrightarrow \quad \text{phase speed} = \text{group speed} = c \ : \ \text{all } k$$

Look next to what happens to the propagation 4-vector

$$\begin{pmatrix} \sqrt{k^2+\varkappa^2} \\ \boldsymbol{k} \end{pmatrix} \xrightarrow[\varkappa\downarrow 0]{} \begin{pmatrix} k \\ \boldsymbol{k} \end{pmatrix} \quad : \quad \text{clearly a } \textit{null} \text{ vector}$$

By this account, a "co-moving observer"—defined by the condition $\boldsymbol{k} = \mathbf{0}$ —would (because $k^\mu = 0$) see a spatially constant/non-oscillatory potential[215]

$$A^\mu(x) = \mathsf{A}^\mu \quad : \quad \mathsf{A}^\mu \text{ arbitrary}$$
$$\Downarrow$$
$$\text{no electromagnetic } \boldsymbol{E} \text{ or } \boldsymbol{B} \text{ fields at all!}$$

But such use of the "co-moving observer" concept is impossible, for we are informed by Proca theory that such an observer sees the group speed to vanish, while in electrodynamics all inertial observers see the group speed to be c. And it is forbidden to contemplate "inertial observers passing by with the speed of light" because $\mathbb{A}(\boldsymbol{\beta})$ becomes *singular* when $\beta = 1$.

> REMARK: At this point we touch upon a point that engaged the curiosity of the young Einstein, and that contributed later to the invention of special relativity. In his "Autobiographical Notes" (see Paul Schilpp (editor), *Albert Einstein: Philosopher-Scientist* (1951), page 53) he remarks that
>
> > *". . . After ten years of reflection such a principle resulted from a paradox upon which I had already hit at the age of sixteen: if I pursue a light beam with velocity c . . . I should observe such a beam as a spatially oscillatory electromagnetic field at rest. However, there seems to be no such thing, whether on the basis of experience or according to Maxwell's equations . . . It seemed to me intuitively clear that, judged from the standpoint of such an observer, everything would have to happen according to the same laws as for an observer . . . at rest."*
>
> We are in position now to recognize that—beyond his willingness to attach "intuitive clarity" to an impossible fiction —Einstein had (at sixteen!) a somewhat crooked conception of the Maxwellian facts of the matter, but . . .

---

[215] Forgive the too-casual figure of speech: one cannot "see" electromagnetic 4-potentials, except with the mind's eye!

The transverse Proca plane waves described at the bottom of page 265 go over straightforwardly into **transverse electromagnetic plane waves**: we have

$$A^\mu_{\text{transverse}}(x) = \begin{cases} \mathsf{A}^\mu_1 \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t + \delta_1) \\ \mathsf{A}^\mu_2 \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t + \delta_2) \end{cases}$$

where the constant A^μ-vectors differ only notationally from the V^μ-vectors described previously. But Proca's longitudinal plane wave becomes

$$A^\mu_{\text{longitudinal}}(x) = \mathsf{A}^\mu_\parallel \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t + \delta_\parallel)$$

where in light of the ∞ that intrudes into

$$\|\mathsf{V}^\mu_\parallel\| \equiv \gamma \begin{pmatrix} \beta \\ \boldsymbol{e}_\parallel \end{pmatrix} \xrightarrow[\varkappa\downarrow 0]{} \infty \cdot \begin{pmatrix} 1 \\ \boldsymbol{e}_\parallel \end{pmatrix}$$

we have set

$$\begin{pmatrix} 1 \\ \boldsymbol{e}_\parallel \end{pmatrix} \equiv \mathsf{A}^\mu_\parallel$$

We expected to have $\mathsf{A}^\mu_\parallel \perp k^\mu$, but in fact $\mathsf{A}^\mu_\parallel$ is *parallel* to the propagation vector

$$k^\mu = k \cdot \mathsf{A}^\mu_\parallel$$

and $k_\mu \mathsf{A}^\mu_\parallel = 0$ arises from the circumstance that <u>in electrodynamics k^μ is null</u>. Writing

$$A^{\text{transverse}}_\mu(x) = \frac{\text{constant}}{k} \cdot k_\mu e^{i(k_\alpha x^\alpha)}$$

we find

$$\begin{aligned} F^{\text{transverse}}_{\mu\nu} &= \partial_\mu A^{\text{transverse}}_\nu - \partial_\nu A^{\text{transverse}}_\mu \\ &= i\frac{\text{constant}}{k} \cdot (k_\mu k_\nu - k_\nu k_\mu) \\ &= 0 \end{aligned}$$

and conclude that in electrodynamics the potential $A^{\text{transverse}}_\mu$ can be <u>dismissed as an unphysical artifact</u>:

> Massive Proca fields support three polarizational degrees of freedom, but—"because the photon is massless"—the electromagnetic field supports only two, and they are transverse to the direction of propagation.

As things now stand that statement, by the argument from which it sprang, can be claimed to pertain only to the 4-potential, and to hold only in the Lorentz gauge. But later it will be shown to pertain also to the gauge-independent physical fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

One sometimes encounters attempts to attribute the "disappearance of the longitudinal mode" to the proposition that "an observer riding on a photon sees time dilated to a standstill, and the forward space dimension contracted to extinction." I am not entirely sure the idea actually does what it is intended to do, but in any event: such observers *cannot exist*, so can have no role to play in any convincing account of the physical facts. On the other hand, it is (in other contexts) sometimes illuminating to point out that "an observer riding on a very fast massive particle sees time dilated *nearly* to a standstill, and 3-space contracted *nearly* to a wafer."

6. Contact with the methods of Lagrangian field theory.* As Kermit the Frog might say, "It's not easy, bein' massless"... impossible in pre-relativistic physics, and a delicate business in relativistic physics ... whether you are a particle[214] or a field. Looking to Maxwellian electrodynamics as "Proca theory in the massless limit," we have seen (in §4) that electrodynamics—for all its physical importance—lives *right on the outer edge of formal feasibility*, that "turning off the mass"

- strips the vector field A^μ of its former direct physicality
- introduces "gauge freedom" into the theory
- reduces a formerly basic field equation to the status of a mere convention
- shifts the "locus of physicality," from A^μ to $F^{\mu\nu}$.

In all those respects electromagnetic field is fairly typical of massless fields in general, so close study of the way Maxwell's theory is constructed tends to be more broadly informative than one might at first suppose. The sketchy remarks that follow touch on matters that would be fundamental to any such "close study."

A formalism derived straightforwardly from Lagrangian mechanics is today universally acknowledged to provide the language of choice if one's objective is a systematic development of the properties of a field theory.[216] The formalism in outline: Let φ_a signify the fields of interest.[217] The associated field theory acquires its specific structure from the postulated design of a "Lagrange density" —a real number-valued function $\mathcal{L}(\varphi, \partial\varphi)$ of the field and their spatial/temporal derivatives $\partial_\mu \varphi_a$. An extension of Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S \equiv \tfrac{1}{c}\int_{\mathcal{R}} \mathcal{L}\, d^4x$$

leads[193] to an a-indexed system of coupled Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{a,\,\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi_a} = 0$$

---

* This relatively advanced material will *not* be treated in lecture. First-time readers should skip directly to §7.

[216] In some cases of historic importance this was recognized only after the fact: Maxwell, Einstein, Schrödinger, Dirac ... each was led to the field theory that bears his name by methods that made *no* use of the Lagrangian method.

[217] The subscript $_a$ is generic. In specific cases it becomes a set of tensor/spinor indices and other marks used to distinguish one field component from another.

which when spelled out in detail read

$$\frac{\partial^2 \mathcal{L}}{\partial\varphi_{a,\,\mu}\partial\varphi_{b,\,\nu}}\varphi_{b,\,\mu\nu} + \frac{\partial^2 \mathcal{L}}{\partial\varphi_{a,\,\mu}\partial\varphi_{b}}\varphi_{b,\,\mu} - \frac{\partial \mathcal{L}}{\partial\varphi_a} = 0$$

These are, in the general case, non-linear partial differential equations into which, however, the second partials enter linearly, and will be manifestly Lorentz covariant if L is Lorentz invariant. Noether's theorem can be brought now into play to deduce the design of the stress-energy tensor and to develop other mechanical properties of the field system, to identify conservation laws, *etc.*

The Proca theory is an unexceptional relativistic field theory that fits straightforwardly into the Lagrangian rubric. Taking the vector field U_μ to be the field system of interest, one constructs[218]

$$\mathcal{L} = \tfrac{1}{2}g^{\alpha\rho}g^{\beta\sigma}U_{\alpha,\beta}(U_{\rho,\sigma} - U_{\sigma,\rho}) - \tfrac{1}{2}\varkappa^2 g^{\alpha\beta}U_\alpha U_\beta$$

and computes

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial U_{\mu,\,\nu}} - \frac{\partial \mathcal{L}}{\partial U_\mu} = \partial_\nu(U^{\mu,\nu} - U^{\nu,\mu}) + \varkappa^2 U^\mu = 0$$

In short: $\Box U^\mu - \partial^\mu(\partial_\nu U^\nu) + \varkappa^2 U^\mu = 0$, which when hit with ∂_μ supplies

$$\begin{aligned} \varkappa^2(\partial_\mu U^\mu) &= 0 \\ &\Downarrow \\ \partial_\mu U^\mu \ &= 0 \quad \text{if} \quad \varkappa^2 \neq 0 \end{aligned}$$

Returning with this information to the field equation, we obtain (see again (374.1&2))

$$\Box U^\mu + \varkappa^2 U^\mu = 0$$

whereupon we might introduce $G_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu$ as an auxiliary definition. *Alternatively*, we might take $\big\{U_\mu, G_{\mu\nu}\big\}$ to be the field system of interest, and write

$$\mathcal{L} = -\tfrac{1}{4}g^{\alpha\rho}g^{\beta\sigma}G_{\alpha\beta}G_{\rho\sigma} - \tfrac{1}{2}g^{\alpha\rho}g^{\beta\sigma}G_{\alpha\beta}(U_{\rho,\sigma} - U_{\sigma,\rho}) - \tfrac{1}{2}\varkappa^2 g^{\alpha\beta}U_\alpha U_\beta$$

giving

$$\begin{aligned} \partial_\kappa \frac{\partial \mathcal{L}}{\partial G_{\mu\nu,\,\kappa}} - \frac{\partial \mathcal{L}}{\partial G_{\mu\nu}} &= \quad \tfrac{1}{2}G^{\mu\nu} + \tfrac{1}{2}(U^{\mu,\nu} - U^{\nu,\mu}) = 0 \\ \partial_\nu \frac{\partial \mathcal{L}}{\partial U_{\mu,\,\nu}} - \frac{\partial \mathcal{L}}{\partial U_\mu} &= -\tfrac{1}{2}\partial_\nu(G^{\mu\nu} - G^{\nu\mu}) + \varkappa^2 U^\mu = 0 \end{aligned}$$

---

[218] See my CLASSICAL FIELD THEORY (1999), Chapter 2, pages 16–19 for discussion of why this is a relativistically natural thing to do, and for other details.

The former "auxiliary definition" has now acquired the status of a field equation

$$G_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$$

The automatic antisymmetry of $G_{\mu\nu}$ permits the second set of field equations to be written

$$\partial_\nu G^{\nu\mu} + \varkappa^2 U^\mu = 0$$

and from that pair of equations we again recover $\partial_\mu U^\mu = 0$ as a corollary *provided* $\varkappa^2 \neq 0$.

Proca theory supplies us with a way to construct $G_{\mu\nu}$ from U_μ but no way to construct U_μ from $G_{\mu\nu}$. It is therefore not possible to dismiss U_μ from the list of field functions, to consider Lagrangians of the form $\mathcal{L}(G, \partial G)$. Nor are we motivated to do so. But in electrodynamics—where $F_{\mu\nu}$ is physical but the vector field A_μ is *un*physical—that would be our natural instinct. It appears, however, to be impossible to obtain the free-field Maxwell equations

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \\ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} &= 0 \end{aligned}$$

from a Lagrangian of the form $\mathcal{L}(F, \partial F)$: we are *forced to enlist the assistance of the 4-potential*... and then things become easy. If, for example, we borrow from Proca theory the construction[219]

$$\mathcal{L} = -\tfrac{1}{4} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma} - \tfrac{1}{2} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta}(A_{\rho,\sigma} - A_{\sigma,\rho}) + \text{no mass term}$$

then we obtain

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{whence} \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

and

$$\partial_\mu F^{\mu\nu} = 0$$

but because $\varkappa^2 = 0$ we have lost the leverage which would enforce the Lorentz gauge condition $\partial_\mu A^\mu = 0$.

The preceding discussion touches on yet another sense in which Maxwellian electrodyanmics is—for the familiar reason ("masslessness of the photon")—formally exceptional, delicate.

7. Naked potential in the classical/quantum dynamics of particles. Though particles respond to *forces* $\boldsymbol{F} = -\boldsymbol{\nabla} U$, it is the *naked potential* that enters into the design of the Lagrangian $L = T - U$ (which, as was remarked on page 254, is itself a kind of "potential"). We found at (293) that the non-relativistic[220]

[219] For other possibilities see A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (1964), page 102.

[220] Why *non*-relativistic? Because my destination is a result that emerges from non-relativistic quantum mechanics.

motion of a charged particle in an impressed electromagnetic field can be described

$$\tfrac{d}{dt}(m\boldsymbol{v}) = e\big\{\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\big\}$$

which by (363) becomes

$$= e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A}\big\} \tag{377}$$

I begin this discussion with a review of how that equation of motion can be brought within the compass of Lagrangian mechanics. We will not be surprised when we find that φ and $\boldsymbol{A}$ stand nakedly/undifferentiated in our final result.

From $\frac{d}{dt}\boldsymbol{A} = \frac{\partial}{\partial t}\boldsymbol{A} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$ it follows that

$$-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} = -\tfrac{1}{c}\tfrac{d}{dt}\boldsymbol{A} + \tfrac{1}{c}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$$

Moreover

$$\tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A} = \tfrac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - \tfrac{1}{c}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$$

Taken in combination, those two identities supply

$$e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A}\big\} = e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{d}{dt}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A})\big\}$$

But

$$\begin{aligned} e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{d}{dt}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A})\big\}_i &= e\big\{-\varphi_{,i} - \tfrac{1}{c}\tfrac{d}{dt}A_i + \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}_{,i}\big\} \\ &= \left\{\frac{d}{dt}\frac{\partial}{\partial v_i} - \frac{\partial}{\partial x_i}\right\}e(\varphi - \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}) \end{aligned}$$

The implication is that (377) can be written

$$\begin{aligned} &\left\{\frac{d}{dt}\frac{\partial}{\partial v_i} - \frac{\partial}{\partial x_i}\right\}L = 0 \\ &\qquad L \equiv \tfrac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v} - \underbrace{e(\varphi - \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A})} \end{aligned} \tag{378}$$

Classic instance of a “velocity-dependent potential” that gives rise by Lagrange differentiation to a velocity-dependent force: see, for example, Goldstein’s Section I-5.

As anticipated, the potentials stand naked in L.

The “momentum conjugate to $\boldsymbol{x}$” is given by

$$\boldsymbol{p} \equiv \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \tfrac{e}{c}\boldsymbol{A} \tag{379}$$

and must be distinguished from the “mechanical momentum” $m\boldsymbol{v}$. Substitution of $\boldsymbol{v} = \frac{1}{m}\big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\big)$ into $H = \boldsymbol{v}\cdot\boldsymbol{p} - L(\boldsymbol{x},\boldsymbol{v})$ gives the associated Hamiltonian

$$H(\boldsymbol{x},\boldsymbol{p}) = \tfrac{1}{2m}\big(\boldsymbol{p} - \tfrac{e}{c}\boldsymbol{A}\big)\cdot\big(\boldsymbol{p} - \tfrac{e}{c}\boldsymbol{A}\big) + e\varphi \tag{380}$$

Though the motion must necessarily be gauge invariant, the Lagrangian is not: the gauge transformation (364)

$$\begin{aligned}\varphi &\longrightarrow \varphi = \varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi\\ \boldsymbol{A} &\longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\chi\end{aligned}$$

sends

$$\begin{aligned}L \longrightarrow L &= \tfrac{1}{2}m\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{v} - e(\varphi - \tfrac{1}{c}\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{A})\\ &= L - \tfrac{e}{c}\big\{\tfrac{\partial}{\partial t}\chi + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\chi\big\}\\ &= L - \tfrac{d}{dt}\big\{\tfrac{e}{c}\chi\big\}\end{aligned}$$

From the final equation we conclude that, though L and L are distinct, they are (see again page 254) *gauge-equivalent in the sense of Lagrangian mechanics*—in the sense, that is to say, that they give rise to identical Lagrange equations. The *action* associated with any Hamiltonian test-path $\boldsymbol{x}(t)$

$$S[\boldsymbol{x}(t)] \equiv \int_{t_1}^{t_2} L\big(\boldsymbol{x}(t), \boldsymbol{v}(t)\big)\,dt$$

therefore responds to gauge transformation by a rule

$$S \longrightarrow S = S - \tfrac{e}{c}\big\{\chi(\boldsymbol{x}_2) - \chi(\boldsymbol{x}_1)\big\} \tag{381}$$

in which for the first time we see the "naked gauge function" (evaluated here at the specified endpoints of the test-path: $\boldsymbol{x}_1 \equiv \boldsymbol{x}(t_1)$ and $\boldsymbol{x}_2 \equiv \boldsymbol{x}(t_2)$).

Turning now from the classical to the *quantum* mechanics of a charged particle in an impressed field, we are led from (380) to the time-dependent Schrödinger equation

$$\begin{aligned}\mathsf{H}\psi = i\hbar\tfrac{\partial}{\partial t}\psi \quad\text{with}\quad \mathsf{H} &\equiv \tfrac{1}{2m}\big(\tfrac{\hbar}{i}\boldsymbol{\nabla} - \tfrac{e}{c}\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\tfrac{\hbar}{i}\boldsymbol{\nabla} - \tfrac{e}{c}\boldsymbol{A}\big) + e\varphi\\ &= -\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big) + e\varphi\\ g &\equiv e/\hbar c\end{aligned} \tag{382}$$

We expect/require the quantum physics to be gauge-invariant, but observe that H is clearly *not* gauge-invariant. As a first step toward reconciling the latter fact with the former requirement we observe (i) that the Schrödinger equation can be written

$$-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big)\psi = i\hbar(\tfrac{\partial}{\partial t} + igc\varphi)\psi$$

and (ii) that from the "shift rule"

$$e^{-F(u)}\tfrac{\partial}{\partial u}\bullet \equiv \big[\tfrac{\partial}{\partial u} + \tfrac{\partial F}{\partial u}\big]e^{-F(u)}\bullet$$

it follows that if we multiply the left/right sides of the Schrödinger equation by $e^{-ig\chi}$ we obtain an equation that can be written

$$-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}+ig\boldsymbol{\nabla}\chi-ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}+ig\boldsymbol{\nabla}\chi-ig\boldsymbol{A}\big)e^{-ig\chi}\psi = i\hbar\big(\tfrac{\partial}{\partial t}+ig\tfrac{\partial\chi}{\partial t}+igc\varphi\big)e^{-ig\chi}\psi$$

or again

$$-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)e^{-ig\chi}\psi = i\hbar\big(\tfrac{\partial}{\partial t}+igc\varphi\big)e^{-ig\chi}\psi$$

The implication is that if we interpret "gauge transformation" to have this expanded meaning

$$\left.\begin{aligned} \varphi &\longrightarrow \varphi = \varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi \\ \boldsymbol{A} &\longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\chi \\ \psi &\longrightarrow \psi = e^{-ig\chi}\cdot\psi \end{aligned}\right\} \tag{383}$$

then we have achieved a *gauge-covariant quantum theory*

$$\begin{aligned} \Big\{-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)+e\varphi\Big\}\psi &= i\hbar\tfrac{\partial}{\partial t}\psi \\ &\Big\downarrow \text{gauge transformation} \\ \Big\{-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)+e\varphi\Big\}\psi &= i\hbar\tfrac{\partial}{\partial t}\psi \end{aligned}$$

which—more to the point—yields gauge-invariant physical statements, of which the following

$$\begin{aligned} \langle\psi|\mathbf{x}|\psi\rangle &= \langle\psi|\mathbf{x}|\psi\rangle \\ \langle\psi|\mathbf{p}-\tfrac{e}{c}\mathbf{A}|\psi\rangle &= \langle\psi|\mathbf{p}-\tfrac{e}{c}\mathbf{A}|\psi\rangle \end{aligned}$$

are merely illustrative.

To retain the relative simplicity of time-independent quantum mechanics, let us assume for the moment that all potentials and gauge functions depend only upon $\boldsymbol{x}$. We are placed then in position to write

$$\psi(\boldsymbol{x},t) = \int G(\boldsymbol{x},t;\boldsymbol{x}_0,0)\,\psi(\boldsymbol{x}_0,0)\,d^3x_0$$

and thus to describe the *temporal evolution* of the (unobserved) wavefunction. Quantum mechanics provides two alternative descriptions of the "propagator" $G(\boldsymbol{x},t;\boldsymbol{x}_0,t_0)$: the "spectral description"

$$G(\boldsymbol{x},t;\boldsymbol{x}_0,0) = \sum_n e^{-\frac{i}{\hbar}E_nt}\psi_n(\boldsymbol{x})\psi_n^*(\boldsymbol{x}_0)$$

and Feynman's "sum-over-paths description"

$$G(\boldsymbol{x},t;\boldsymbol{x}_0,0) = (\text{normalization factor})\cdot\sum_{\text{paths}}\exp\Big\{\tfrac{i}{\hbar}S[\text{path: }(\boldsymbol{x}_0,0)\rightarrow(\boldsymbol{x},t)]\Big\}$$

Bringing $\psi = e^{ig\chi}\cdot\psi$ to the spectral description we obtain

$$G(\boldsymbol{x},t;\boldsymbol{x}_0,0) = G(\boldsymbol{x},t;\boldsymbol{x}_0,0)\cdot\exp\big\{ig\big[\chi(\boldsymbol{x})-\chi(\boldsymbol{x}_0)\big]\big\}$$

The point of interest is that since (381) can be expressed

$$\tfrac{i}{\hbar}S[\text{path}] = \tfrac{i}{\hbar}S[\text{path}] + ig\big[\chi(\boldsymbol{x})-\chi(\boldsymbol{x}_0)\big]$$

the Feynman method leads immediately to that same conclusion,[221] and does so independently of how we elect to give meaning to the "sum-over-paths" concept.

From $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ it follows that

$$\begin{aligned}\text{magnetic flux } \Phi \text{ through disk bounded by } \mathcal{C} &= \iint_{\text{disk}} \boldsymbol{B}\cdot\boldsymbol{d\sigma}\\ &= \iint_{\text{disk}} (\boldsymbol{\nabla}\times\boldsymbol{A})\cdot\boldsymbol{d\sigma}\\ &= \oint_{\mathcal{C}} \boldsymbol{A}\cdot\boldsymbol{d\ell} \end{aligned}\tag{384}$$

This simple result is of importance for at least *two* reasons:

1. It exposes a *gauge-independent* "naked $\boldsymbol{A}$":

$$\oint_{\mathcal{C}} \boldsymbol{A}\cdot\boldsymbol{d\ell} = \oint_{\mathcal{C}} \boldsymbol{A}\cdot\boldsymbol{d\ell} \quad \text{because} \quad \oint_{\mathcal{C}} \boldsymbol{\nabla}\chi\cdot\boldsymbol{d\ell} = 0 \ (\text{all } \chi)$$

2. It assigns physical importance (as explained below) to certain topological circumstances, and does so for reasons that are of some interest in themselves. The simplest way to expose the points at issue is to consider the "cylindrical" magnetic field shown in Figure 84. The symmetry of the field, and what we know about the geometrical meaning of "curl," suggest that the $\boldsymbol{A}$-field should have (to within gauge) the form indicated in Figure 85:

$$\boldsymbol{A} = A(r)\,\boldsymbol{T}$$

$$\boldsymbol{T} \equiv \text{unit tangent to Amperian circle of radius } r = \begin{pmatrix} -y/r \\ +x/r \\ 0 \end{pmatrix}$$

Working from (384) we therefore have

$$\text{encircled flux} = \begin{cases} \pi\, r^2 B \text{ if } r \leqslant R \\ \pi R^2 B \text{ if } r \geqslant R \end{cases} = 2\pi r\cdot A(r)$$

[221] Or would if we could establish the gauge-independence of the normalization factor. The point becomes trivial if one is willing to borrow from the result of the spectral argument, but (except in the simplest cases) is too intricate to pursue here by methods internal to the Feynman formalism. Evaluation of the normalization factor is in some respects the most delicately problematic aspect of the formalism. Feynman himself was content to assume that

$$\text{normalization factor} = (\boldsymbol{x},\boldsymbol{x}_0)\text{-independent function of } t$$

and to extract its specific design from the requirement that

$$\lim_{t\downarrow 0} G(\boldsymbol{x},t;\boldsymbol{x}_0,0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)$$

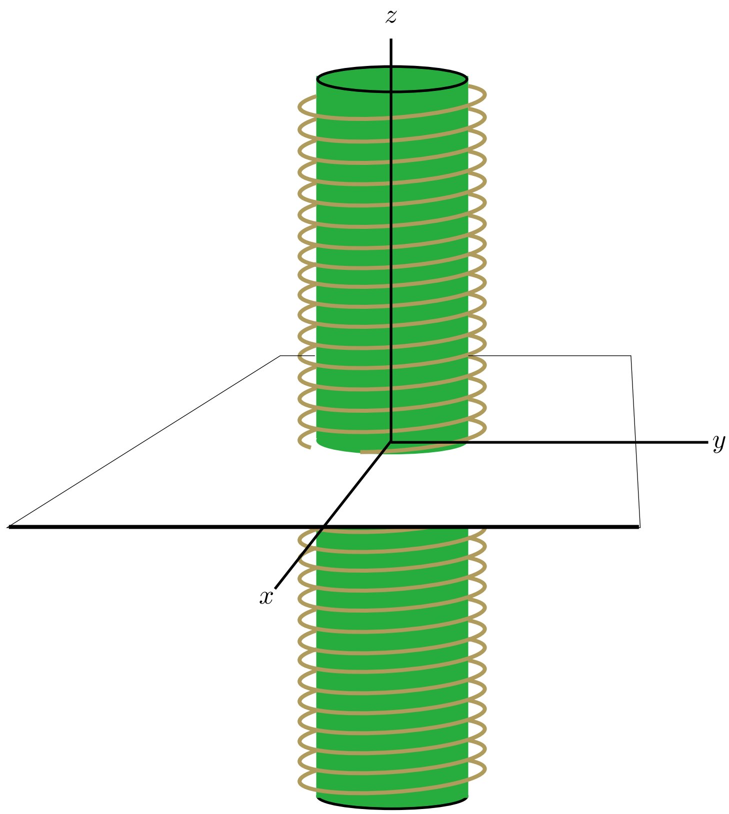

FIGURE 84: *Current flows in an infinitely long straight solenoid, of radius R. The resulting magnetic field is well known to be coaxial and uniform across the interior of the solenoid, but to vanish at all points exterior to the solenoid:*

$$\boldsymbol{B} = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} & \textit{at interior points} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \textit{at exterior points} \end{cases}$$

The "magnetic spaghetti, stretching from one side of Euclidean space to the other," alters the topology of the part of space where $\boldsymbol{B} = \mathbf{0}$, *and this is shown in the text to have some profound physical consequences. Additional spaghetti would make the topological situation even more complicated. The configuration shown has the merit of being simple enough to permit all calculations to be done exactly.*

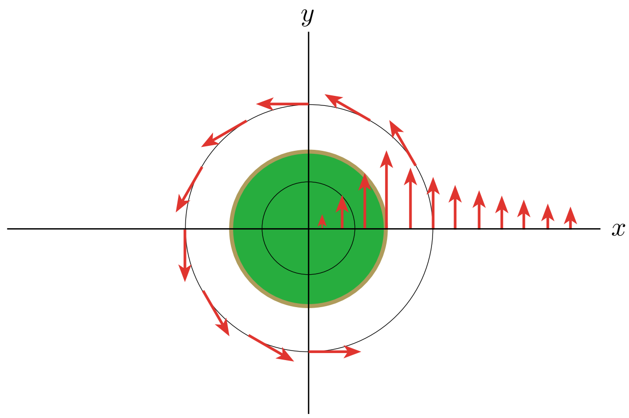

FIGURE 85: *Cross-section of the preceding figure. The black circles (one with radius $r < R$, one with $r > R$) are ↺-oriented "Amperian loops" drawn to capture the symmetry of the system. Red arrows decorate the larger loop, and indicate the anticipated design of the $\boldsymbol{A}$-field. The red arrows that march along the x-axis illustrate how the magnitude of $\boldsymbol{A}$, as computed in the text, depends upon r. The striking fact is that, <u>while $\boldsymbol{B}$ vanishes at exterior points $r > R$, the vector potential $\boldsymbol{A}$ does not</u>.*

A little guesswork has brought us thus to

$$A(r) = \begin{cases} \frac{1}{2}Br & : \quad r \leqslant R \\ \frac{1}{2}BR^2 r^{-1} & : \quad r \geqslant R \end{cases}$$

whence

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \frac{1}{2}B\begin{pmatrix} -y \\ +x \\ 0 \end{pmatrix} & : \quad r \leqslant R \\ \\ \frac{1}{2}BR^2\begin{pmatrix} -y/r^2 \\ +x/r^2 \\ 0 \end{pmatrix} & : \quad r \geqslant R \end{cases} \tag{385}$$

and a quick calculation[222] confirms the accuracy of the guess:

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} & : \quad r \leqslant R \\ \mathbf{0} & : \quad r \geqslant R \end{cases}$$

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[222] PROBLEM 59.

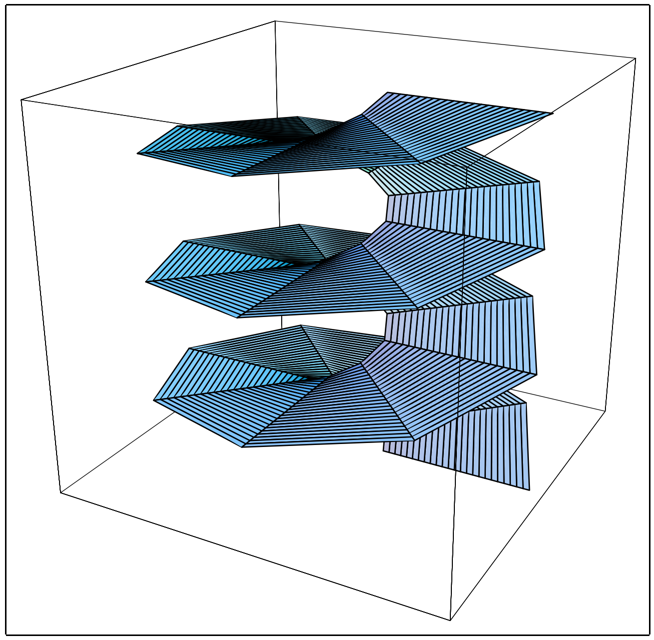

FIGURE 86: *Graph of the multivalued superpotential $\alpha(x,y)$ defined at (387)*

In the exterior region the condition $\boldsymbol{\nabla}\times\boldsymbol{A} = \mathbf{0}$ would be rendered automatic if we wrote

$$\boldsymbol{A} = \boldsymbol{\nabla}\underbrace{\alpha}_{\text{“superpotential”}} \tag{386}$$

The $\boldsymbol{A}$-vectors stand normal to the equi-(super)potential surfaces, so from results in hand we infer that $\alpha(\boldsymbol{x})$ is constant on planes that radiate radially from the z-axis: $\alpha(\boldsymbol{x}) = f(\arctan(y/x))$. On a hunch, we try the simplest instance of such a function

$$\alpha(x, y, z) = \tfrac{1}{2}BR^2 \arctan(y/x) \tag{387}$$

and by quick calculation (ask *Mathematica*) verify that indeed

$$\boldsymbol{\nabla}\alpha = \tfrac{1}{2}BR^2 \begin{pmatrix} -y/r^2 \\ +x/r^2 \\ 0 \end{pmatrix} = \boldsymbol{A}_{\text{exterior}}$$

The superpotential defined at (387) is plotted in Figure 86. It is clearly multivalued, but—a remark of David Griffiths[223] notwithstanding—no physical principle excludes that possibility: we are concerned here not with potentials but with *super*potentials.

[223] *Introduction to Electrodynamics* (1981), page 207, Problem 29.

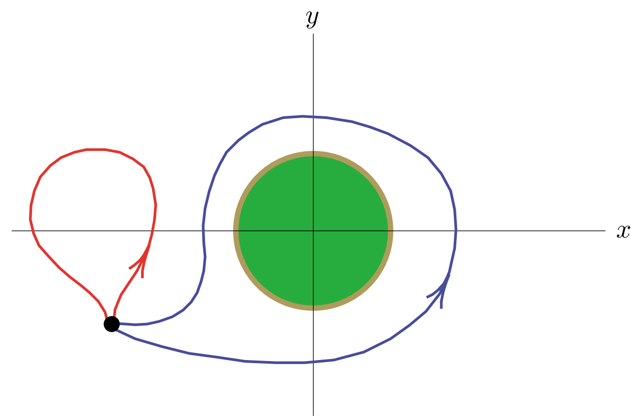

FIGURE 87: *Two (oriented) curves are inscribed on a plane from which a single green hole has been excised. Each curve begins & ends at the point marked $\bullet$. Many curves are equivalent to—the proper phrase is "homotopic to"—the red curve $\mathcal{C}$ in the sense that they could be brought into coincidence with $\mathcal{C}$ by continuous deformation. But the blue curve $\mathcal{C}$ is not among them: the required deformation is impeded by the circumstance that $\mathcal{C}$ winds (once) around the hole. Evidently $\mathcal{C}_1$ and $\mathcal{C}_2$ will be homotopically equivalent*

$$\mathcal{C}_1 \sim \mathcal{C}_2 \text{ iff } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ have the same "winding number"}$$

The idea of resolving the set of all $\bullet$-based curves into homotopic equivalence classes extends straightforwardly to more complex situations (multiple holes in the plane, surfaces of sphere/torus/etc.). Down this road lies "homotopy theory," of which a very good introductory account (written for physicists) can be found in §23.2 of L. S. Schulman's Techniques & Applications of Path Integration (1981).

I allude above to the topological information that can be gained from resolving curves/loops/paths into homotopic equivalence classes. Some physical problems hinge naturally on precisely that mode of classification, and acquire thus a "topological" aspect. One such—but by no means the only such—problem was identified by Bohm & Aharonov in 1959,[224] who contemplate a modification of the "two slit experiment" in which (see Figure 88) a solenoid is tucked behind the slits: particles, in their flight from source to detector, experience no electromagnetic forces, but pass through a region in which $\boldsymbol{A} \neq \mathbf{0}$, and the latter circumstance has (as Bohm & Aharonov were actually not the first to point out) observable consequences. I turn now to a sketch of how the so-called "Bohm-Aharonov effect" comes about:

[224] Y. Aharonov & D. Bohm, "Significance of electromagnetic potentials in the quantum theory," Phys. Rev. **115**, 485 (1959).

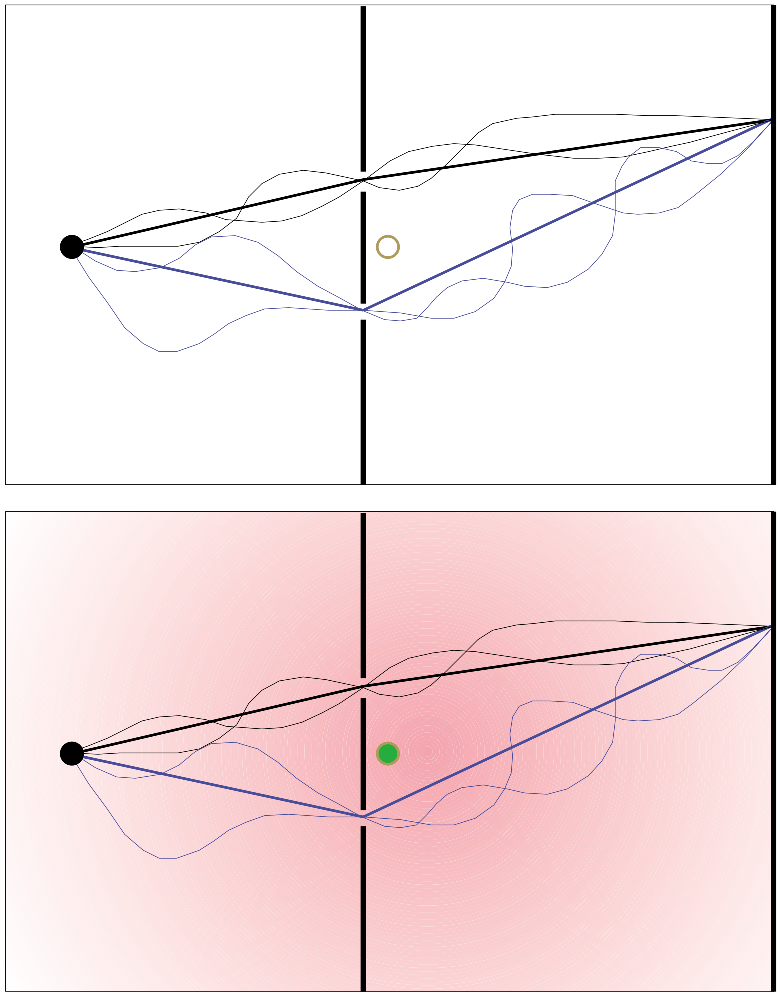

FIGURE 88: *In Bohm/Aharonov's modification (below) of the classic 2-slit experiment (above) a solenoid produces a localized $\boldsymbol{B}$-field. By arrangement, particles—in their flight from source to detector—are excluded from the region where $\boldsymbol{B} \neq \boldsymbol{0}$, but pass through a region now flooded with the associated $\boldsymbol{A}$ field. The latter circumstance was predicted and experimentally found to cause an observable alteration of the interference pattern—the* **Bohm-Aharonov effect**.

In the classic 2-slit set-up (prior to Bohm/Aharonov's modification) a particle proceeds in time t from source *via* slit #1 to detection point $\boldsymbol{x}$ with probability amplitude

$$\psi_1(\boldsymbol{x},t) \sim \sum_{\text{such paths}} e^{\frac{i}{\hbar}S[\text{path via slit \#1}]} \tag{388}$$

where the $\sim$ signals my intention to be casual about normalization factors throughout this discussion. $\psi_2(\boldsymbol{x},t)$ is defined similarly, and the *net* amplitude for arrival at $(\boldsymbol{x},t)$ is given by

$$\psi(\boldsymbol{x},t)=\psi_1(\boldsymbol{x},t)+\psi_2(\boldsymbol{x},t)$$

All three of those functions are solutions of

$$-\tfrac{\hbar^2}{2m}\nabla^2\psi=i\hbar\tfrac{\partial}{\partial t}\psi$$

though ψ_1 and ψ_2 satisfy somewhat different boundary conditions (ψ_1 vanishes at slit #2, ψ_2 vanishes at slit #1). If we write

$$\psi_1(\boldsymbol{x},t)=\sqrt{P_1(\boldsymbol{x},t)}\,e^{i\phi_1(\boldsymbol{x},t)}\quad\text{and}\quad\psi_2(\boldsymbol{x},t)=\sqrt{P_2(\boldsymbol{x},t)}\,e^{i\phi_2(\boldsymbol{x},t)}$$

then the *probability* of detection at $(\boldsymbol{x},t)$ is given by

$$\begin{aligned}P(\boldsymbol{x},t)&=|\psi_1+\psi_2|^2\\&=P_1+P_2+\underbrace{2\sqrt{P_1P_2}\cos\Delta\phi}_{\text{interference term}}\end{aligned}$$

Here $\Delta\phi\equiv\phi_1-\phi_2$ and we dismiss as irrelevant the fact that most detectors are so slow that they report only the value of $P(\boldsymbol{x})\equiv\int_0^\infty P(\boldsymbol{x},t)\,dt$.

Now turn on the current in the solinoid. In place of (388) we have

$$\color{red}{\psi}_1(\boldsymbol{x},t)\sim\sum_{\text{paths}}e^{\frac{i}{\hbar}\left\{S[\text{path via slit \#1}]+\frac{e}{c}\int_{\text{path}}\boldsymbol{A}\cdot d\boldsymbol{x}\right\}}$$

But all paths $\bullet\xrightarrow[\text{via slit \#1}]{}\boldsymbol{x}$ (let such paths be called "paths of type #1") are homotopically equivalent, $\boldsymbol{\nabla}\times\boldsymbol{A}=\boldsymbol{0}$ holds at every point along each, so we have

$$\begin{aligned}\int_{\text{any path of type \#1}}\boldsymbol{A}\cdot d\boldsymbol{x}&=\int_{\text{any other such path}}\boldsymbol{A}\cdot d\boldsymbol{x}\\&=\text{path-independent function of }\boldsymbol{x}\end{aligned}$$

giving

$$\color{red}{\psi}_1(\boldsymbol{x},t)=e^{\frac{i}{\hbar}\frac{e}{c}\int_{\text{typical path of type \#1}}\boldsymbol{A}\cdot d\boldsymbol{x}}\cdot\psi_1(\boldsymbol{x},t)$$

We note in passing that from the operator identity

$$\boldsymbol{\nabla}=e^{-\frac{i}{\hbar}\frac{e}{c}\int_{\#1}\boldsymbol{A}\cdot d\boldsymbol{x}}\big[\boldsymbol{\nabla}-\tfrac{i}{\hbar}\tfrac{e}{c}\boldsymbol{A}\big]e^{\frac{i}{\hbar}\frac{e}{c}\int_{\#1}\boldsymbol{A}\cdot d\boldsymbol{x}}$$

it follows that if ψ_1 satisfies the Schrödinger equation at the top of the page then $\color{red}{\psi}_1$ satisfies

$$-\tfrac{\hbar^2}{2m}\big[\boldsymbol{\nabla}-ig\boldsymbol{A}\big]\cdot\big[\boldsymbol{\nabla}-ig\boldsymbol{A}\big]\color{red}{\psi}_1=i\hbar\tfrac{\partial}{\partial t}\color{red}{\psi}_1$$

—as expected. Identical remarks pertain, of course, to $\color{red}{\psi}_2$.

Which brings us at last to the main point of this discussion. It follows from results now in hand that turning on the solenoidal $\boldsymbol{B}$-field sends

$$P(\boldsymbol{x}) = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}) + 2\sqrt{P_1(\boldsymbol{x})P_2(\boldsymbol{x})}\cos\Big\{\Delta\phi(\boldsymbol{x})\Big\}$$
$$\downarrow$$
$$P(\boldsymbol{x}) = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}) + 2\sqrt{P_1(\boldsymbol{x})P_2(\boldsymbol{x})}\cos\Big\{\Delta\phi(\boldsymbol{x}) + g\Big[\int_{\#1} - \int_{\#2}\Big]\boldsymbol{A}\cdot\boldsymbol{dx}\Big\}$$

But paths $\bullet \xrightarrow[\text{via slit }\#1]{} \boldsymbol{x}$ and $\bullet \xrightarrow[\text{via slit }\#2]{} \boldsymbol{x}$ are homotopically inequivalent: the integrals, instead of cancelling, produce

$$\oint_{\mathcal{C}} \boldsymbol{A}\cdot\boldsymbol{dx} = \text{enveloped magnetic flux } \varPhi$$

because $\mathcal{C}$ encloses the solenoid. So we have

$$P(\boldsymbol{x}) = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}) + 2\sqrt{P_1(\boldsymbol{x})P_2(\boldsymbol{x})}\cos\Big\{\Delta\phi(\boldsymbol{x}) + \tfrac{e}{\hbar c}\varPhi\Big\} \tag{389}$$

which, since $\varPhi$ is $\boldsymbol{x}$-independent, describes an *observably shifted copy* of the original interference pattern $P(\boldsymbol{x})$. Several points now merit comment:

1. The pattern-shift becomes invisible when

$$\varPhi = n\cdot 2\pi\frac{\hbar c}{e} \quad : \quad n = 0, \pm1, \pm2, \ldots \tag{390}$$

This "flux quantization condition" assumes central importance in connection with the physics of superconductors (most notably: that of "superconducting quantum interference devices" or SQUIDs).[225]

2. One sometimes encounters the claim that "The vector potential, though not observable classically, becomes observable in quantum mechanics." The claim is misleading: what becomes quantum mechanically observable is not $\boldsymbol{A}$ itself but the gauge-invariant construct $\oint_{\mathcal{C}}\boldsymbol{A}\cdot\boldsymbol{dx}$, and the element of surprise arises from cases in which $\boldsymbol{B} = \boldsymbol{0}$ everywhere along $\mathcal{C}$. The situation is, however, in some respects quite familiar: at (116) we had

$$\text{Faraday emf} = -\tfrac{1}{c}\tfrac{d}{dt}(\text{enclosed magnetic flux}) = -\tfrac{1}{c}\tfrac{d}{dt}\oint_{\mathcal{C}}\boldsymbol{A}\cdot\boldsymbol{dx}$$

—some engineering applications of which (*e.g.*,, the bevetron) hinge critically on the fact that $\mathcal{C}$ may be remote from the region of changing flux. Here as in the Bohm-Aharanov effect, an element of *non-locality* intrudes.

[225] See F. Schwabl, *Quantum Mechanics* (2^{nd} edition 1995), §§7.5 & 7.6 or Bjørn Felsager, *Geometry, Particles, and Fields* (1998), §2.12. On the cover of my edition of the latter text, by the way, is a version of my Figure 84, promoted by Felsager to the status of an ikon symbolizing the problem area where geometry/topology and the physics of particles/fields intersect.

3. We found at (386) that

$$\boldsymbol{A} = \boldsymbol{\nabla}\alpha \quad \text{in the region exterior to the solenoid}$$

By gauge transformation $\boldsymbol{A} \longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\alpha$ we construct therefore a vector potential $\boldsymbol{A}$ which *vanishes identically in the exterior region* . . . and presents us with a seeming contradiction:

- We know on the one hand that

$$\oint_{\mathcal{C}} \boldsymbol{A}\cdot d\boldsymbol{x} = \oint_{\mathcal{C}} \boldsymbol{A}\cdot d\boldsymbol{x} \quad \text{by gauge-invariance} \tag{391}$$

- but on the other hand it is clear that

$$\oint_{\mathcal{C}} \boldsymbol{0}\cdot d\boldsymbol{x} = 0$$

Why does this not extinguish the Bohm-Aharonov effect?

The "seeming contradiction" is resolved by the observation that (391) holds if (as is standardly the case) the gauge function is single-valued. But the gauge function α that kills the external solenoidal $\boldsymbol{A}$-field is (see again Figure 86) *multi*-valued, and the contours $\mathcal{C}$ of interest wind from one sheet to the next, so

$$\oint_{\mathcal{C}} \boldsymbol{\nabla}\alpha\cdot d\boldsymbol{x} = \alpha(\text{point}) - \alpha(\text{same point on next-lower sheet}) \neq 0$$

Soon after Michael Berry's discovery (1984) of what came to be called "Berry's phase"—soon recognized to be itself a manifestation of a more general phenomenon called "geometrical phase"—it was pointed out by Aharonov himself that the Bohm-Aharonov effect can be portrayed as a special instance of that deeper and ever more pervasive train of physico-geometrical train of thought . . . that, in short, it represents but the tip of an iceberg.[226]

Conclusion. Potentials are usually considered to enter electrodynamics as mere computational crutches, as aids to simplified formulation of the theory. The same—only more so—can be said of the "superpotentials" of which Hertz gave the first systematic account.[227] We have seen, however, by looking upon Maxwell's theory as a limiting case of Proca's theory . . . that the ghostly status of the potential hangs by a precarious thread: that gauge freedom would be lost, that the potential fields would become directly observable/physical participants in the theory "if only the photon were endowed with mass, however slight."

[226] See Y. Aharonov & J. Anandan, "Phase change in cyclic quantum evolution," PR Letters **58**, 1593 (1987) and other classic papers reprinted in A. Shapere & F. Wilczek, *Geometric Phases in Physics* (1989). Also §10.2.4 in David Griffiths' *Introduction to Quantum Mechanics* (1995).

[227] See §13–4 Wolfgang Panofsky & Melba Phillips, *Classical Electricity & Magnetism* (1955).

With the infusion of quantum mechanical ideas the life of A^μ acquires a dramatic new dimension, and the subject acquires a deeply geometrical flavor. Our review of the Bohm-Aharonov effect has served to illustrate the point, and I have alluded to parallel developments in the theory of superconductivity, but historically prior to either of those is a pretty train of thought set into motion by Dirac in 1931. Dirac[228] put

- the classical electrodynamnics of a magnetic monopole and
- the quantum mechanics of an electrically charged particle

in a bag together. . . shook. . . and came away with an explanation for *why electrical charge is quantized*. We are in position to follow the details only the (very instructive) first part of his argument.[229]

We look (with Dirac) to the vector potential

$$\boldsymbol{A} = (g/4\pi)\begin{pmatrix} \frac{y}{r(r-z)} \\ \frac{-x}{r(r-z)} \\ 0 \end{pmatrix} \quad : \quad r^2 \equiv x^2 + y^2 + z^2 \tag{392}$$

and compute[230]

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = (g/4\pi)\,\tfrac{1}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} \text{spherically symmetric radial field of} \\ \text{a magnetic monopole of strengh } g \end{cases}$$

. . . as encountered already on page 227. Notice now that on the z-axis (*i.e.*, at $x = y = 0$)

$$\frac{1}{r(r-z)} = \begin{cases} \infty & : \quad z > 0 \\ 1/2z^2 & : \quad z < 0 \end{cases}$$

The potential (392) is called a "Dirac string potential" because it displays a "string singularity" on the positive z-axis. To clarify the mathematical/physical meaning of the singularity we make use once again of the "regularization trick," first encountered on page 12: we write

$$\boldsymbol{A}_\epsilon = (g/4\pi)\begin{pmatrix} \frac{y}{R(R-z)} \\ \frac{-x}{R(R-z)} \\ 0 \end{pmatrix} \quad : \quad R^2 \equiv r^2 + \epsilon^2$$

(from which we recover (392) in the limit $\epsilon \downarrow 0$) and compute

$$\boldsymbol{B}_\epsilon = \boldsymbol{\nabla}\times\boldsymbol{A}_\epsilon = \boldsymbol{B}_\epsilon^{\text{monopole}} + \boldsymbol{B}_\epsilon^{\text{string}}$$

---

[228] P. A. M. Dirac, Proc. Roy. Soc. London **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948).

[229] For a splendid account of details here omitted see Chapter 9 in Felsager.[225] Also §6.11 in J. D. Jackson, *Classical Electrodynamics* (3rd edition 1999).

[230] PROBLEM 60.

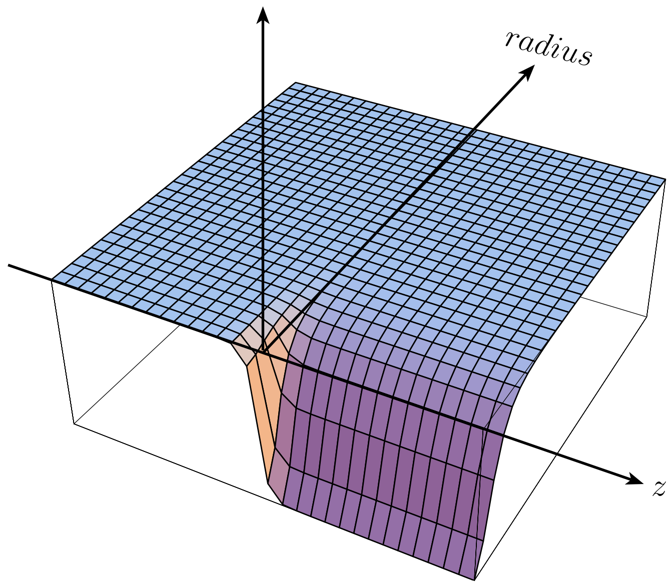

FIGURE 89: *$B_\epsilon^{\text{string}}$ displayed as a function of z and*

radius $s \equiv \sqrt{x^2 + y^2}$

The trough along the positive z-axix gets narrower/deeper as $\epsilon \downarrow 0$. The figure refers to the case $\epsilon = \frac{1}{10}$.

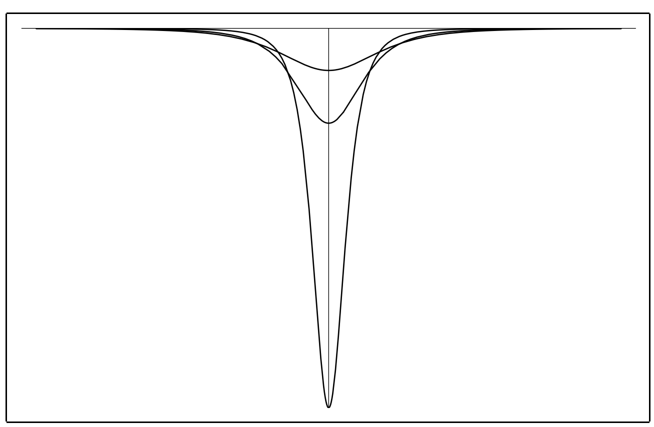

FIGURE 90: *Graphs of the radial dependence of $B_\epsilon^{\text{string}}$ at $z = 1$ in the cases $\epsilon = \frac{3}{10}, \frac{2}{10}, \frac{1}{10}$.*

with

$$\boldsymbol{B}_\epsilon^{\text{monopole}} = (g/4\pi)\,\frac{1}{R^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \boldsymbol{B}_\epsilon^{\text{string}} = (g/4\pi)\begin{pmatrix} 0 \\ 0 \\ B_\epsilon^{\text{string}} \end{pmatrix}$$

where

$$B_\epsilon^{\text{string}} \equiv B^{\text{string}}(z, s; \epsilon) \equiv -\frac{\epsilon^2(2R - z)}{R^3(R - z)^2} \quad : \quad R \equiv \sqrt{s^2 + z^2}$$

Clearly

$$\lim_{\epsilon\downarrow 0} \boldsymbol{B}_\epsilon^{\text{monopole}} = \text{monopole field described earlier}$$

It is from the string term, which runs everywhere parallel to the z-axis, that we have things to learn. Figures 89 & 90 tell the story. *Mathematica* informs us that

$$\begin{aligned}\int_0^\infty B^{\text{string}}(z, s; \epsilon) 2\pi s\, ds &= -\frac{2\pi\epsilon^2}{\sqrt{z^2+\epsilon^2}\,(-z + \sqrt{z^2+\epsilon^2}\,)} \\ &\Big\downarrow \text{limit } \epsilon \downarrow 0 \\ &= \begin{cases} -4\pi & : \quad z > 0 \\ \;\;0 & : \quad z < 0 \end{cases}\end{aligned}$$

We are brought thus to the conclusion that $\boldsymbol{B}_\epsilon^{\text{string}}$ is a field such as would arise from a *solenoid of zero cross-section* wrapped around the positive z-axis and carrying a current given by

$$\boldsymbol{j} = \lim_{\epsilon\downarrow 0} \boldsymbol{j}_\epsilon \quad \text{with} \quad \boldsymbol{j}_\epsilon = c\,\boldsymbol{\nabla}\times\boldsymbol{B}_\epsilon^{\text{string}}$$

We learn, moreover, that (see Figure 91)

total magnetic flux delivered down-string by $\boldsymbol{B}^{\text{string}}$
= total magnetic flux delivered spherically outward by $\boldsymbol{B}^{\text{monopole}}$

so the *net* flux through any closed surface containing a Dirac monopole is zero!

One can show that the "string singularity" encountered at (392) is essential, in the sense that it cannot be gauged away. But pretty clearly (and as one can also show), the string can trace *any* curve "from infinity" to the point where it terminates (called "the monopole").

The second part of Dirac's argument is, as already indicated, quantum mechanical: he looks to the quantum motion of an electrically charged particle in the presence of a monopole and stipulates that <u>the string (irrespective of its shape) shall be quantum mechanically invisible</u>. This requirement, which from one point of view serves to fix the pitch of the multivalued superpotential

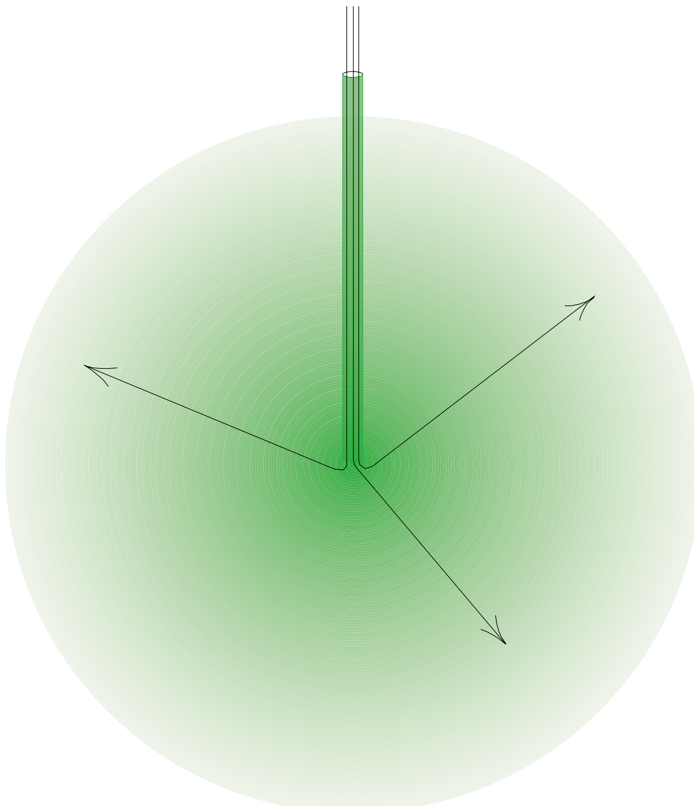

FIGURE 91: *Magnetic field and field lines of the Dirac monopole described in the text. The net magnetic flux through any surface that encloses the monopole is zero. Dirac's idealized "string solenoid" is shown (here as in the text) to be coincident with the positive z-axis, but can in general trace any curve from the location of the monopole "to infinity." My use of the phrase "from the monopole" is perhaps misleading: for Dirac the monopole is the dangling free end of the string solenoid.*

(Figure 86), can be phrased as a requirement that the string give rise to a null Bohm-Aharonov effect (this 25 years before the ostensible *discovery* of the Bohm-Aharonov effect!). One is led thus from (390) to the **Dirac quantization condition**

$$\text{string flux } g = n \cdot 2\pi\frac{\hbar c}{e}$$

This is precisely the condition

$$\text{angular momentum of Tompson's mixed dipole } \frac{eg}{4\pi c} = n \cdot \tfrac{1}{2}\hbar$$

to which we were led on page 232 by quite another (and less compelling) line of argument. The strongest conclusion that can be drawn from either argument is that the product $e\cdot g$ is quantized:

$$eg = n\cdot 2\pi\hbar c$$

A fundamentally new idea would be required to account theoretically for this observed fact of Nature:

e—and therefore also g—are *individually* quantized

Dirac's argument does not quite do the job; to pretend otherwise (a common practice) is to engage in some wishful thinking . . . and to decline an invitation to invention.

The Bohm-Aharonov effect and its siblings—seen now to include flux and charge quantization—are topological children of a liaison between A^μ and quantum mechanics. Gauge field theory is, if anything, even more deeply geometrical. Drawing covertly upon ideas (covariant differentiation, curvature) borrowed from differential geometry, it portrays electrodynamics as "the price one pays" in order to promote the global phase invariance

$$\psi \longrightarrow \psi = \quad e^{i\,g\chi}\cdot\psi \quad : \quad \chi \text{ any real constant}$$

standard to quantum theory . . . to an invariance with respect to *local* phase transformations

$$\psi \longrightarrow \psi = e^{i\,g\chi(x)}\cdot\psi \quad : \quad \chi(x) \text{ any real-valued function of } x$$

This is accomplished by in effect pursuing in reverse the argument which on pages 273–274 was used to establish the electrodynamical gauge-invariance of quantum mechanics: we adjust the meaning (of momentum; *i.e.*, of) the differentiation operator

$$\partial_\mu \longrightarrow \mathcal{D}_\mu \equiv \partial_\mu - igA_\mu$$

and achieve the desired local phase (or gauge) invariance by stipulating that the "compensating field" A_μ will participate in the transformation by the rule (383). Finally (by a mechanism natural to Lagrangian field theory) we launch the compensating field into motion and find that it satisfies precisely the equation

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = \tfrac{1}{c}j^\nu$$

that at (371) was found to comprise "Maxwell's theory in a nutshell." The theory leads, moreover, to an explicit description of the current 4-vector j_μ. Directly observable "physicality" is assigned—from a formal point of view almost as an afterthought!—to the gauge-invariant construction

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad : \quad \text{analog of geometrical "curvature"}$$

No mere crutch, A_μ has by this point become arguably the *principal object* in Maxwellian electrodynamics—the wellspring from which the theory flows. And in quantum electrodynamics (QED) it is, moreover, A_μ—not $F_{\mu\nu}$ but A_μ—that is "quantized."[231]

For several decades the program just described was dismissed as a formal curiosity, an exercise that led to nothing not already known. But in the 1950's it was discovered (by Yank & Mills, Shaw, Umazawa[211]) that it admits readily and elegantly of profound generalization, that it can be used to construct Maxwell-like theories of the non-electromagnetic interactions among elementary particles—"non-Abelian gauge theories" that appear to be in generally excellent agreement with the observational facts.[232] Physics provides no more persuasive evidence that Truth and stunning Beauty come often to the same thing.

It may be fair, as I did at the outset, to refer to potentials (and, more generally, to gauge fields) as "spooks," as sirens who discretely hide their nakedness, but such language leaves half the story untold: they are spooks who spring from the deepest darkest places, who come to us murmuring of the most obscure symmetries of Nature . . . and who appear to be in formal control of Reality.

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[231] See, for example, J. M. Jauch & F. Rohrlich, *The Theory of Photons & Electrons* (1955), §2–4.

[232] For an elementary introduction to this inexhaustibly rich subject see, for example, the final Chapter 11 in David Griffiths' *Introduction to Elementary Particles* (1987).

5

LIGHT IN VACUUM

Theory of optical polarization

Introduction. In regions empty of matter—empty more particularly of *charged* matter—the electromagnetic field is described by equations that we have learned to write in various ways:

$$\left.\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= 0 \\ \boldsymbol{\nabla}\times\boldsymbol{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} &= \mathbf{0} \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{\nabla}\times\boldsymbol{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} &= \mathbf{0} \end{aligned}\right\} \tag{65}$$

$$\left.\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \\ \partial_\alpha \epsilon^{\alpha\rho\sigma\nu} F_{\rho\sigma} &= 0 \end{aligned}\right\} \tag{168}$$

$$\left.\begin{array}{c} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0 \quad : \quad \text{arbitrary gauge} \\ \downarrow \\ \Box A^\nu = 0 \quad : \quad \text{Lorentz gauge} \end{array}\right\} \tag{371}$$

And we have learned that, whichever language we adopt, multiple instances of the wave equation hover close by. It was Maxwell himself who first noticed that equations (65) can be "decoupled by differentiation" to yield six copies of the wave equation:

$$\Box \boldsymbol{E} = \Box \boldsymbol{B} = \mathbf{0}$$

The manifestly covariant version of Maxwell's argument is less familiar: to

$$\partial^a \epsilon_{ars\nu} \cdot \partial_\alpha \epsilon^{\alpha\rho\sigma\nu} F_{\rho\sigma} = 0$$

bring the identity[233]

$$\begin{aligned}
\epsilon_{ars\nu}\epsilon^{\alpha\rho\sigma\nu} = \tfrac{1}{g}\delta^{\alpha\rho\sigma}{}_{ars} &\equiv \tfrac{1}{g}\begin{vmatrix} \delta^\alpha{}_a & \delta^\alpha{}_r & \delta^\alpha{}_s \\ \delta^\rho{}_a & \delta^\rho{}_r & \delta^\rho{}_s \\ \delta^\sigma{}_a & \delta^\sigma{}_r & \delta^\sigma{}_s \end{vmatrix} \\
&= \tfrac{1}{g}\big\{ \;\; \delta^\alpha{}_a(\delta^\rho{}_r\delta^\sigma{}_s - \delta^\sigma{}_r\delta^\rho{}_s) \\
&\qquad + \delta^\alpha{}_r(\delta^\rho{}_s\delta^\sigma{}_a - \delta^\sigma{}_s\delta^\rho{}_a) \\
&\qquad + \delta^\alpha{}_s(\delta^\rho{}_a\delta^\sigma{}_r - \delta^\sigma{}_a\delta^\rho{}_r)\big\}
\end{aligned}$$

and obtain

$$\Box(F_{rs} - F_{sr}) + \partial^a\big\{\partial_r(F_{sa} - F_{as}) + \partial_s(F_{ar} - F_{ra})\big\} = 0$$

whence (by the antisymmetry of $F_{\mu\nu}$)

$$\begin{aligned}
\Box F_{\mu\nu} &= \tfrac{1}{c}(\partial_\mu j_\nu - \partial_\nu j_\mu) \\
&\downarrow \\
&= 0 \quad \text{in charge-free space: } j_\mu = 0
\end{aligned}$$

Finally, at (371) we obtained four copies of the wave equation by covariant specialization of the gauge.

We will be concerned in these pages with certain particular solutions of the preceding free-field equations that bear on the classical physics of light. Two points should be born in mind:

- All of the equations ennumerated above are satisfied by the Coulomb field of an isolated charge *except at the location of the charge itself*. They are satisfied by the Lorentz transforms of such a field (field of a charge drifting by), by the field of a static population of such charges, by the magnetic field of a current-carrying wire *except at the location of the wire itself*,

[233] For discussion of the "generalized Kronecker deltas" see pages 7–8 in "Electrodynamical applications of the exterior calculus" (1996). The notational resources of the exterior calculus render the following argument—though it looks here a little contrived—entirely and transparently natural. Incidentally, g has recently signified magnetic charge, and before that was the name of a coupling constant: $g \equiv e/\hbar c$. In the following lines g is restored to its original meaning: $g \equiv \det \|g_{\mu\nu}\|$.

by the fields produced by drifting populations of such wires. In none of those situations are the fields detectable by the apparatus of optics (photometers, etc.); none of them present the diffraction/interference phenomena characteristic of wave physics; to each of them the language of optics would appear alien (except quantum mechanically, where one attributes electrostatic interaction to an "exchange of photons"). What we at present lack is a sharp criterion for distinguishing "light-like" from "other" solutions of the free-field equations.

- We will be studying the physics of light-in-the-absence-of-matter, of light *in vacuuo*. But such light is invisible, an inferential abstraction! For it is only by its interaction with matter (production by radiative processes, transmission through media, manipulation by lenses/mirrors/filters and other such devices,detection by eyes/photometers) that we "see" light, that we become aware of its existence as a fact of Nature—reportedly the *first* fact.[234] But before we can construct a theory of the light-matter interaction we must possess a theory of (the electromagnetic properties of) matter . . . and toward that objective—since matter and most production/absorption processes are profoundly quantum mechanical—classical physics can carry us only a short part of the way (yet far enough to account phenomenologically for most of classical optics).

Nevertheless . . . the ideas to which we will be led are absolutely fundamental to the physics of light, whatever the depth of the physical detail and conceptual sophistication with which we elect to pursue that subject.

The physics of light is in several important (but too seldom remarked) respects "exceptional, surprising." In order to highlight the points at issue, which remain invisible until placed in broader context, I will (as I have several times already) draw occasionally on Proca's theory of "massive light."

1. Fourier decomposition of the wave field. On pages 291 & 292 we encountered several instances of the wave equation

$$\Box\varphi = 0 \qquad i.e., \qquad \big\{\tfrac{1}{C^2}\partial_t^2 - \nabla^2\big\}\varphi(t,\boldsymbol{x}) = 0$$

It is mathematically natural—alien to the spirit of relativity, but an option available to every *particular* inertial observer—to "split off the time variable,"

[234] "In the beginning God created the heavens and the earth. The earth was without form, and void, and darkness was on the face of the deep. Then God said, 'Let there be light'; and there was light. And God saw the light, that it was good; and God divided the light from the darkness. . . " (Genesis I: 1–4). For an absorbing account of the philosophical contemplation of relationships among God, Good and Light that, after more than two millennia, had led by the 16$^{\text{th}}$ Century to the conception of physical space—the non-obvious one we now take for granted—that "made physics possible" see Max Jammer's slim masterpiece *Concepts of Space: The History of Theories of Space in Physics* (1954), with forward by Albert Einstein.

writing $\varphi(t,\boldsymbol{x}) = f(t)\cdot\phi(\boldsymbol{x})$. Then

$$\tfrac{1}{c^2}\ddot{f} = -k^2 f \quad\text{and}\quad (\nabla^2 + k^2)\phi = 0$$

where k^2 is a positive separation constant, with the physical dimension of $(\text{length})^{-2}$. We are led thus to solutions of the *monochromatically oscillatory* form

$$\varphi_\omega(t,\boldsymbol{x}) = e^{i\omega t}\cdot\phi_\omega(\boldsymbol{x}) \quad\text{with}\quad \omega \equiv kc$$

where ω can assume any (positive or negative) real value.[235]

In Cartesian coordinates the

$$\textsc{helmholtz equation}: \qquad (\nabla^2 + k^2)\phi = 0$$

reads

$$\Big\{(\tfrac{\partial}{\partial x})^2 + (\tfrac{\partial}{\partial y})^2 + (\tfrac{\partial}{\partial z})^2 + k^2\Big\}\phi(x,y,z) = 0$$

The separation of variables technique can be carried to completion, and yields solutions of the form

$$\phi(x,y,z) = (\text{constant})\cdot e^{i\,k_1 x}\cdot e^{i\,k_2 y}\cdot e^{i\,k_3 z}$$

with $k_1^2 + k_2^2 + k_3^2 = k^2$.[236] But it has been known since 1934 that separation can be carried to completion in a total of *eleven* coordinate systems; namely,

1. Cartesian (or rectangular) coordinates
2. Circular-cylinder (or polar) coordinates
3. Elliptic-cylinder coordinates
4. Parabolic-cylinder coordinates
5. Spherical coordinates
6. Prolate spheroidal coordinates
7. Oblate spheroidal coordinates
8. Parabolic coordinates
9. Conical coordinates
10. Ellipsoidal coordinates
11. Paraboloidal coordinates

---

[235] We make casual use here and henceforth of the familiar "complex variable trick," with the understanding that one has direct *physical* interest only in the real/imaginary parts of φ_ω.

[236] Separation of three variables brings only two separation constants into play. Why, therefore, do we appear in the present instance to encounter three? By notational illusion. Look upon (say) k_2 and k_3 as separation constants, and regard $k_1 \equiv \sqrt{k^2 - k_2^2 - k_3^2}$ as an enforced definition.

so the question arises: Why are all but the first largely absent from literature pertaining to the physics of light? Why do theorists in this area so readily capitulate to "Cartesian tyranny." For *several* reasons:

- In non-Cartesian coordinates the description of ∇^2 becomes complicated, so separation of the Helmholtz equation leads to a system of three typically fairly complicated ordinary differential equations, the solutions of which are typically "higher functions" (Bessel functions, Legendre functions, Mathieu functions, etc.).[237] For example (looking only to the simplest case): in circular-cylinder coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

  the Helmholtz equation becomes

$$\Big\{\big(\tfrac{\partial}{\partial r}\big)^2 + \tfrac{1}{r}\tfrac{\partial}{\partial r} + \tfrac{1}{r^2}\big(\tfrac{\partial}{\partial \theta}\big)^2 + \big(\tfrac{\partial}{\partial z}\big)^2 + k^2\Big\}\phi = 0$$

  We write $\phi = R(r)\cdot\Theta(\theta)\cdot Z(z)$ and obtain

$$\left.\begin{aligned} \tfrac{d^2R}{dr^2} + \tfrac{1}{r}\tfrac{dR}{dr} - \big(\tfrac{\alpha}{r^2}+\beta\big)R &= 0 \\ \tfrac{d^2\Theta}{d\theta^2} + \alpha\Theta &= 0 \\ \tfrac{d^2Z}{dz^2} + (k^2+\beta)Z &= 0 \end{aligned}\right\} \quad \alpha \text{ and } \beta \text{ are separation constants}$$

  The second equation gives

$$\Theta(\theta) = a_2\sin\sqrt{\alpha}\,\theta + b_2\cos\sqrt{\alpha}\,\theta$$

  which by a single-valuedness requirement enforces

$$\sqrt{\alpha} = n \quad : \quad 0, \pm1, \pm2, \ldots$$

  The third equation (no single-valuedness requirement is here in force, since z is not a periodic variable) gives

$$Z(z) = a_3\sin\sqrt{k^2+\beta}\,z + b_3\cos\sqrt{k^2+\beta}\,z$$

  For the first equation *Mathematica* supplies

$$R(r) = a_1\texttt{BesselI[}n, r\sqrt{\beta}\texttt{]} + b_1\texttt{BesselI[}-n, r\sqrt{\beta}\texttt{]}$$

---

[237] Details are spelled out in various mathematical handbooks, of which my favorite in this connection is P. Moon & D. E. Spencer, *Field Theory Handbook* (1961).

- All the coordinate systems listed—with the sole exception of the Cartesian coordinate system(s)—possess *singularities* (recall the behavior of the circular-cylinder and spherical coordinate systems on the z-axis).
- Description of the translations/rotations/Lorentz transformations of physical interest is *awkward except in Cartesian coordinates*. Notice in particular that

$$\begin{aligned}\phi &= (\text{constant})\cdot e^{i\omega t}\cdot e^{i\,k_1x}\cdot e^{i\,k_2y}\cdot e^{i\,k_3z}\\ &= (\text{constant})\cdot e^{i(k_0x^0+k_1x^1+k_2x^2+k_3x^3)} \quad\text{with}\quad k_0\equiv\omega/c\\ &= (\text{constant})\cdot e^{ikx}\end{aligned}$$

where $kx\equiv k_\alpha x^\alpha$ becomes Lorentz invariant if we stipulate that

$$k\equiv\begin{pmatrix}k^0\equiv\omega/c\\ k^1\\ k^2\\ k^3\end{pmatrix}\equiv\begin{pmatrix}k^0\\ \boldsymbol{k}\end{pmatrix}\quad\text{transforms as a covariant 4-vector}$$

Notice also that

$$\begin{aligned}\Box e^{ikx} &= i^2 g^{\alpha\beta}k_\alpha k_\beta\, e^{ikx}\\ &= 0 \quad\text{if and only if } k \text{ is null: } k_\alpha k^\alpha = 0\end{aligned}$$

It is impossible to argue so neatly in non-Cartesian coordinates.
- In Cartesian coordinates—uniquely—we *gain direct access to the powerful techniques of Fourier transform theory*... for by superposition of the plane waves just described we obtain

$$\begin{aligned}\phi(x) &= \tfrac{1}{(2\pi)^2}\iiiint a(k)\delta(k_\alpha k^\alpha - 0)e^{ikx}\,dk^0dk^1dk^2dk^3\\ &= \text{Fourier transform of } a(k)\delta(k_\alpha k^\alpha - 0)\end{aligned}$$

- Last but most important: When we write (say) $\Box F^{\mu\nu}=0$ we have interest not in independent ${}^{\mu\nu}$-indexed solutions of the wave equation, but in solutions so interrelated that they satisfy the ${}^{\nu}$-indexed side-conditions $\partial_\mu F^{\mu\nu}=0$ and $\partial_\alpha\epsilon^{\alpha\rho\sigma\nu}F_{\rho\sigma}=0$. Similarly, when we write $\Box A^\mu=0$ we have interest not in independent ${}^{\mu}$-indexed solutions of the wave equation, but in solutions so interrelated that they satisfy the side-condition $\partial_\mu A^\mu=0$. *Implications of the side conditions are far easier to work out in Cartesian coordinates than in any other coordinate system.*

So we yield uncomplainingly to "Cartesian tyranny," and expect soon to see concrete evidence of the advantages of doing so.

One further point merits preparatory comment. If solutions

$$\phi_n(x) = a_n e^{ik_nx}$$

of the wave equation are required to satisfy linear side conditions

$$\sum_n \phi_n(x) = 0$$

then pretty clearly it is essential that $k_1 = k_2 = \ldots$; *i.e.*, that they *buzz in synchrony*.

Look now to these plane wave solutions

$$\begin{aligned}
E_1(x) &= E_1 \cdot e^{ik_1x}\\
E_2(x) &= E_2 \cdot e^{ik_2x}\\
E_3(x) &= E_3 \cdot e^{ik_3x}\\
B_1(x) &= B_1 \cdot e^{ik_4x}\\
B_2(x) &= B_2 \cdot e^{ik_5x}\\
B_3(x) &= B_3 \cdot e^{ik_6x}\\
&\qquad\ \ \uparrow_{\text{constants}}
\end{aligned}$$

upon Maxwell's equations (65) impose what amount to a set of eight linear side conditions, which there is no hope of satisfying unless the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ "buzz in synchrony":

$$k_{1\alpha} = k_{2\alpha} = k_{3\alpha} = k_{4\alpha} = k_{5\alpha} = k_{6\alpha}$$

So we adopt this sharpened hypothesis:

$$\left.\begin{aligned}
\boldsymbol{E}(x) &= \boldsymbol{E} \cdot e^{ikx} = \boldsymbol{E} \cdot \exp\left\{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})\right\}\\
\boldsymbol{B}(x) &= \boldsymbol{B} \cdot e^{ikx} = \boldsymbol{B} \cdot \exp\left\{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})\right\}
\end{aligned}\right\} \tag{393}$$

Maxwell's equations (65) now become a set of conditions

$$\begin{aligned}
\boldsymbol{k}\cdot\boldsymbol{E} &= 0\\
\boldsymbol{k}\times\boldsymbol{B} + \tfrac{\omega}{c}\boldsymbol{E} &= \mathbf{0}\\[4pt]
\boldsymbol{k}\cdot\boldsymbol{B} &= 0\\
\boldsymbol{k}\times\boldsymbol{E} - \tfrac{\omega}{c}\boldsymbol{B} &= \mathbf{0}
\end{aligned}$$

that serve to constrain the relationships among $\boldsymbol{E}$, $\boldsymbol{B}$ and the propagation vector $\boldsymbol{k}$. The 1$^{\text{st}}$ and 3$^{\text{rd}}$ conditions tell us that

$$\boxed{\boldsymbol{E}\text{ and }\boldsymbol{B}\text{ lie necessarily in the plane normal to }\boldsymbol{k}}$$

Crossing $\boldsymbol{k}$ into the 2$^{\text{nd}}$ equation gives

$$\tfrac{\omega}{c}\boldsymbol{k}\times\boldsymbol{E} + \underbrace{\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{B})}_{=\,(\boldsymbol{k}\cdot\boldsymbol{B})\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{B} \,=\, \mathbf{0} - \left(\frac{\omega}{c}\right)^2\boldsymbol{B}} = \mathbf{0}$$

which is redundant with the 4$^{\text{th}}$ equation. Dotting $\boldsymbol{E}$ into the 4$^{\text{th}}$ equation we discover that

$$\boxed{\boldsymbol{E}\text{ and }\boldsymbol{B}\text{ are normal to each other}}$$

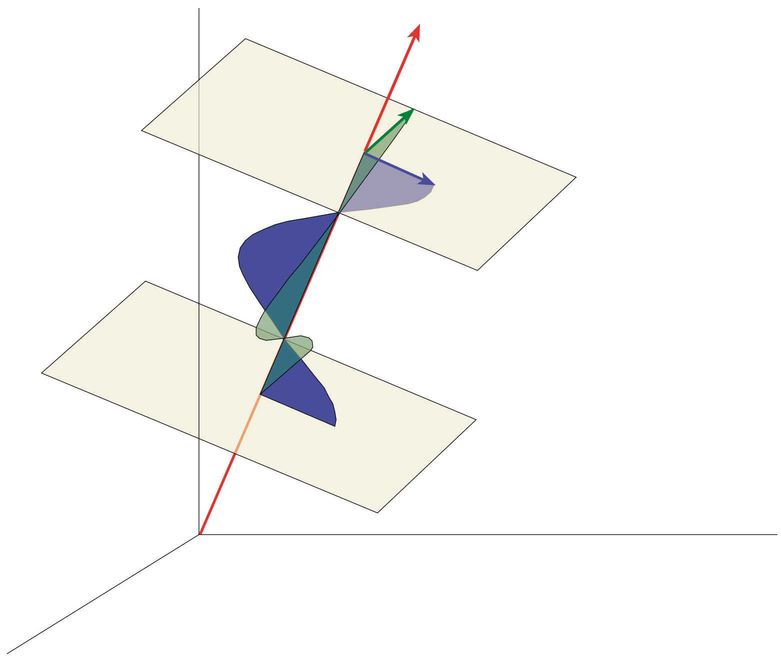

FIGURE 92: *Snapshot of a monochromatic electromagnetic plane wave. Normal to all planes-of-constant-phase (two are shown) is the "propagation or wave vector"* $\boldsymbol{k}$. *The blue sinusoid represents the* $\boldsymbol{E}$*-vector. Normal to it (and of the same amplitude and phase) is the green* $\boldsymbol{B}$*-vector. In animation the electric/magnetic waves would be seen to slide rigidly along* $\boldsymbol{k}$ *with phase speed* c.

Finally, dot the 4$^{\text{th}}$ equation into itself to obtain

$$\begin{aligned}\left(\tfrac{\omega}{c}\right)^2\boldsymbol{B}\cdot\boldsymbol{B} &= \underbrace{(\boldsymbol{k}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E})}\\ &= (\boldsymbol{k}\cdot\boldsymbol{k})(\boldsymbol{E}\cdot\boldsymbol{E}) - (\boldsymbol{k}\cdot\boldsymbol{E})^2 = \left(\tfrac{\omega}{c}\right)^2\boldsymbol{E}\cdot\boldsymbol{E} - 0\end{aligned}$$

$$\boxed{\boldsymbol{E}\text{ and }\boldsymbol{B}\text{ are of equal magnitude}}$$

It now follows that if $\boldsymbol{k}$ and $\boldsymbol{E}$ are given/known, then $\boldsymbol{B}$ can be computed from

$$\boldsymbol{B} = \hat{\boldsymbol{k}}\times\boldsymbol{E} \tag{394}$$

We saw already on page 264 that

$$\text{phase} = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$$

is constant on planes $\perp\ \boldsymbol{k}$ that slide along with

$$\text{phase speed } \omega/k = c$$

so are led to the image of an *electromagnetic plane wave* shown in Figure 92.

The vector $\boldsymbol{E}$ can be inscribed in two linearly independent ways on the phase plane. With that fact in mind ...

- go to some arbitrary "inspection point,"
- face into the onrushing plane wave,
- inscribe an arbitrarily unit vector $\boldsymbol{e}_1$ on the phase plane,
- construct $\boldsymbol{e}_2 \equiv \hat{\boldsymbol{k}} \times \boldsymbol{e}_1$, a unit vector $\perp \boldsymbol{e}_1$.

The "flying $\boldsymbol{E}$-vector" can by these conventions be described

$$\boldsymbol{E}(t) = E_1\boldsymbol{e}_1 + E_2\boldsymbol{e}_2 = \begin{pmatrix} E_1(t) \\ E_2(t) \end{pmatrix} \tag{395.1}$$

with

$$\left.\begin{aligned} E_1(t) &= \mathcal{E}_1\cos(\omega t + \delta_1) \\ E_2(t) &= \mathcal{E}_2\cos(\omega t + \delta_2) \end{aligned}\right\} \tag{395.2}$$

Equations (395) will provide the point of departure for the main work of this chapter.

Suppose we had elected to work in the language of potential theory; *i.e.*, from[238]

$$A^\mu(x) = \underbrace{A^\mu}_{\text{constant 4-vector}} \cdot\, e^{ik\,x}$$

where

$$\Box A^\mu(x) = 0 \quad \text{requires } k^\mu \text{ to be null: } k_\mu k^\mu = 0$$
$$\text{The Lorentz gauge condition } \partial_\mu A^\mu = 0 \text{ requires } k_\mu A^\mu = 0$$

Borrowing notation from pages 296 and 259

$$\|k^\mu\| = \begin{pmatrix} k^0 \\ \boldsymbol{k} \end{pmatrix} \quad \text{with} \quad k^0 \equiv \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}} = \omega/c$$
$$\|A^\mu\| = \begin{pmatrix} \varphi \\ \boldsymbol{A} \end{pmatrix}$$

we find that

$$k_\mu A^\mu = 0 \quad \Longleftrightarrow \quad \varphi = \hat{\boldsymbol{k}}\cdot\boldsymbol{A}$$

so our potential plane wave can be described

$$A_\mu(x) = A_\mu \cdot e^{ik\,x} \quad \text{with} \quad \|A_\mu\| = \begin{pmatrix} \hat{\boldsymbol{k}}\cdot\boldsymbol{A} \\ -\boldsymbol{A} \end{pmatrix}$$

[238] See again page 268. We employ the "complex variable trick" to simplify the writing: extract the real part to obtain the physics.

This we use to obtain

$$\begin{aligned}
E_1 = F_{01} &= \partial_0 A_1 - \partial_1 A_0 = i(k_0 A_1 - k_1 A_0)\cdot e^{ik\,x}\\
E_2 = F_{02} &= \partial_0 A_2 - \partial_2 A_0 = i(k_0 A_2 - k_2 A_0)\cdot e^{ik\,x}\\
E_3 = F_{03} &= \partial_0 A_3 - \partial_3 A_0 = i(k_0 A_3 - k_3 A_0)\cdot e^{ik\,x}\\
B_1 = F_{32} &= \partial_3 A_2 - \partial_2 A_3 = i(k_3 A_2 - k_2 A_3)\cdot e^{ik\,x}\\
B_2 = F_{13} &= \partial_1 A_3 - \partial_3 A_1 = i(k_1 A_3 - k_3 A_1)\cdot e^{ik\,x}\\
B_3 = F_{21} &= \partial_2 A_1 - \partial_1 A_2 = i(k_2 A_1 - k_1 A_2)\cdot e^{ik\,x}
\end{aligned}$$

whence

$$\begin{aligned}
\boldsymbol{E} &= -(\omega/c)\big[\boldsymbol{A} - (\hat{\boldsymbol{k}}\cdot\boldsymbol{A})\hat{\boldsymbol{k}}\big]\cdot ie^{ik\,x}\\
&= -(\omega/c)\boldsymbol{A}_\perp\cdot ie^{ik\,x} &(396.1)\\
\boldsymbol{B} &= -(\omega/c)\big[\hat{\boldsymbol{k}}\times\boldsymbol{A}\big]\cdot ie^{ik\,x}\\
&= \hat{\boldsymbol{k}}\times\boldsymbol{E} &(396.2)
\end{aligned}$$

Notice that

- there are two linearly independent ways to inscribe $\boldsymbol{A}_\perp$ on the plane normal to $\boldsymbol{k}$
- $\boldsymbol{A}_\parallel$ makes no contribution to $\boldsymbol{E}$ or $\boldsymbol{B}$, no contribution therefore to the physics . . . so can be discarded, the reason being that
- $\boldsymbol{A}_\parallel$ can be very simply *gauged away*: take $\chi = e^{ik\,x}$ and notice that

  $$\partial^\mu\chi = ik^\mu\chi \text{ is parallel to } k^\mu$$

  Moreover
  $$\partial_\mu(\partial^\mu\chi) = -(k_\mu k^\mu)\chi = 0 \text{ because } k^\mu \text{ is null}$$

  so such a gauge transformation respects the Lorentz gauge condition.

The argument just completed has led us back again—but rather more swiftly/luminously—to precisely the physical results obtained earlier by other means.

It is instructive to consider how electromagnetic plane wave physics would be altered "if the photon had mass." According to Proca,[239] we would have interest then the plane wave solutions

$$A_\mu(x) = A_\mu\cdot e^{ik\,x}$$

of

$$(\Box + \varkappa^2)A_\mu = 0 \quad\text{and}\quad \partial_\mu A^\mu = 0$$

[239] We borrow here from §5 in Chapter 4, but use A^μ rather than U^μ to denote the "massive vector Proca field."

The first condition supplies $k_\mu k^\mu = \varkappa^2$ or $k^0 = \sqrt{\boldsymbol{k}\cdot\boldsymbol{k} + \varkappa^2}$, while the second condition supplies $A^0 = (\boldsymbol{k}\cdot\boldsymbol{A})/k^0$. The argument that led to (396) now leads to

$$\begin{aligned}
\boldsymbol{E} &= -\big[k_0\boldsymbol{A} - A_0\boldsymbol{k}\big]\cdot ie^{ik\,x}\\
&= -k_0\left[\boldsymbol{A} - \frac{(\boldsymbol{k}\cdot\boldsymbol{A})\boldsymbol{k}}{k_0^2}\right]\cdot ie^{ik\,x}\\
&= -k_0\big[\boldsymbol{A} - \wp^2(\hat{\boldsymbol{k}}\cdot\boldsymbol{A})\hat{\boldsymbol{k}}\big]\cdot ie^{ik\,x} \qquad (397.1)\\
\boldsymbol{B} &= -k\big[\hat{\boldsymbol{k}}\times\boldsymbol{A}\big]\cdot ie^{ik\,x}\\
&= \wp\cdot(\hat{\boldsymbol{k}}\times\boldsymbol{E}) \qquad (397.2)
\end{aligned}$$

where what I call the "Proca factor"

$$\begin{aligned}
\wp \equiv k/k_0 &= \frac{\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}}{\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}+\varkappa^2}} \quad \text{with}\begin{cases} k \;\equiv \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}\\ k_0 = \omega/c\end{cases}\\
&\downarrow\\
&= 1 \quad \text{in the Maxwellian limit } \varkappa^2 \downarrow 0
\end{aligned}$$

We distinguish two cases:

$\boxed{\textsc{Case}\ \boldsymbol{A}\perp\boldsymbol{k}}$ This can happen in two ways. Because $\hat{\boldsymbol{k}}\cdot\boldsymbol{A} = 0$ we have

$$\begin{aligned}
\boldsymbol{E} &= -(\omega/c)\boldsymbol{A}_\perp\cdot ie^{ik\,x}\\
\boldsymbol{B} &= \wp\cdot(\hat{\boldsymbol{k}}\times\boldsymbol{E})
\end{aligned}$$

which differs from (396) only in the presence of the $\wp$-factor, which diminishes the strength of the $\boldsymbol{B}$-field.

$\boxed{\textsc{Case}\ \boldsymbol{A}\parallel\boldsymbol{k}}$ Writing $\boldsymbol{A} = A_\parallel\hat{\boldsymbol{k}}$ we have

$$\begin{aligned}
\boldsymbol{E} &= -(\omega/c)(1-\wp^2)A_\parallel\hat{\boldsymbol{k}}\cdot ie^{ik\,x}\\
\boldsymbol{B} &= \wp\cdot(\hat{\boldsymbol{k}}\times\boldsymbol{E})\\
&= \boldsymbol{0} \quad \text{because } \boldsymbol{E}\parallel\hat{\boldsymbol{k}}
\end{aligned}$$

The *electric field has acquired an oscillatory longitudinal component which possesses no magnetic counterpart*, and both longitudinal fields vanish in the Maxwellian limit.

2. Stokes parameters. The "flying $\boldsymbol{E}$-vector" of (395) traces/retraces the simplest of Lissajous figures—an ellipse—on the (E_1, E_2)-plane. The flight of $\boldsymbol{E}(t)$ is, at optical frequencies ($\omega \sim 10^{15}$ Hz), much too brisk to be observed, but the *figure* of the ellipse (size, shape, orientation) and the ↺ / ↻ sense in which it is pursued *are* observable—detectable by the "slow" devices of classical optics (eyes, photometers, filters of various types). They give rise to the phenomenology of **optical polarization**, the theory of which will concern us in this and the next few sections.

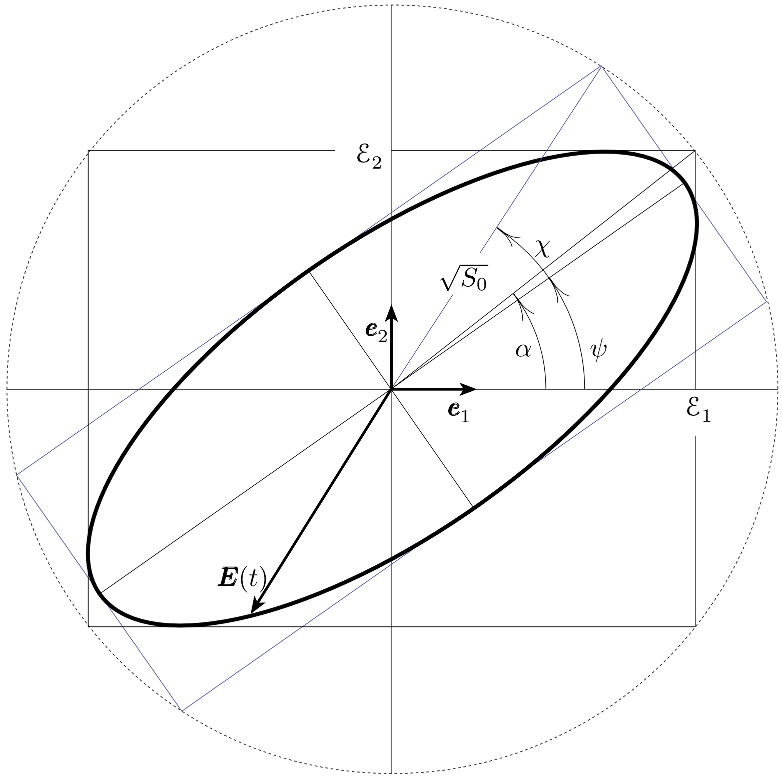

FIGURE 93: *Ellipse traced by the* $\boldsymbol{E}$*-vector of an electromagnetic plane wave, with* $\boldsymbol{k}$ *up out of the page. It is a remarkable property of ellipses that all circumscribing rectangles (two are shown) have the same diagonal measure, which can be taken to set the* size *of the ellipse. The angle* ψ *describes the* orientation *of the principal rectangle, which is of long dimension* $2a$*, short dimension* $2b$*. The* shape *of the ellipse is usually described in terms of the*

$$\text{ellipticity} \equiv \sqrt{1 - (b/a)^2}$$

but—as Stokes appreciated—is equally well described by

$$\chi \equiv \arctan(b/a)$$

Helicity *information is absent from* (398)*, but from* (395.2) *we discover—look to* $\frac{d}{dt}\boldsymbol{E}(t)$ *at conveniently chosen points, or argue that if* $E_2(t)$ *leads* $E_1(t)$ *(i.e., if* $\delta_2 > \delta_1$*) the circulation is clockwise, and in the contrary case counterclockwise—that the circulation is* $\circlearrowright$ *or* $\circlearrowleft$ *according as* $0 < \delta \equiv \delta_2 - \delta_1 < \pi$ *or* $-\pi < \delta < 0$.

Eliminating t between equations (395.2) we obtain[240]

$$\mathcal{E}_2^2 \cdot E_1^2 - 2\mathcal{E}_1\mathcal{E}_2\cos\delta \cdot E_1E_2 + \mathcal{E}_1^2 \cdot E_2^2 = \mathcal{E}_1^2\mathcal{E}_2^2\sin^2\delta \tag{398}$$
$$\delta \equiv \delta_2 - \delta_1 \equiv \text{phase difference}$$

Equations (395.2) provide a parametric description, and (398) an implicit description . . . of the ellipse[241] shown in Figure 93. Some elementary analytical geometry—the details are fun but uninformative, and (since they have nothing specifically to do with electrodynamics) will be omitted—leads to the following conclusions:

$$S_0 = \mathcal{E}_1^2 + \mathcal{E}_2^2$$
$$\sin 2\chi = \sin 2\alpha \cdot \sin\delta = \frac{2\mathcal{E}_1\mathcal{E}_2\sin\delta}{\mathcal{E}_1^2 + \mathcal{E}_2^2} \equiv \frac{S_3}{S_0}$$
$$\tan 2\psi = \tan 2\alpha \cdot \cos\delta = \frac{2\mathcal{E}_1\mathcal{E}_2\cos\delta}{\mathcal{E}_1^2 - \mathcal{E}_2^2} \equiv \frac{S_2}{S_1}$$

where
$$S_1 \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2$$

Notice that helicity—which was observed above to be controlled by the sign of δ—could as well be said (since $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-negative) to be controlled by the sign of χ, and that (as is clear from the figure) χ ranges on the restricted interval $\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$. Recasting and extending the results summarized above, we have

$$\left.\begin{aligned} S_0 &= \mathcal{E}_1^2 + \mathcal{E}_2^2 \\ S_1 &= \mathcal{E}_1^2 - \mathcal{E}_2^2 &&= S_0\cos 2\chi\cos 2\psi \\ S_2 &= 2\mathcal{E}_1\mathcal{E}_2\cos\delta &&= S_0\cos 2\chi\sin 2\psi \\ S_3 &= 2\mathcal{E}_1\mathcal{E}_2\sin\delta &&= S_0\sin 2\chi \end{aligned}\right\} \tag{399}$$

These equations define the so-called **Stokes parameters**, which were introduced by G. G. Stokes in 1852 to facilitate the discussion of some experimental results. There is reason to think that Stokes himself was unaware of the extraordinary power of his creation . . . which took nearly a century, and the work of many hands, to be revealed. Today his lovely idea is recognized to be central to every classical/statistical/quantum account of the phenomenology of optical polarization.

It is evident that
$$S_1^2 + S_2^2 + S_3^2 = S_0^2 \tag{400}$$

and that *rotational sense (helicity) can be read from the sign of* S_3.

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[240] PROBLEM 61.
[241] PROBLEM 62.

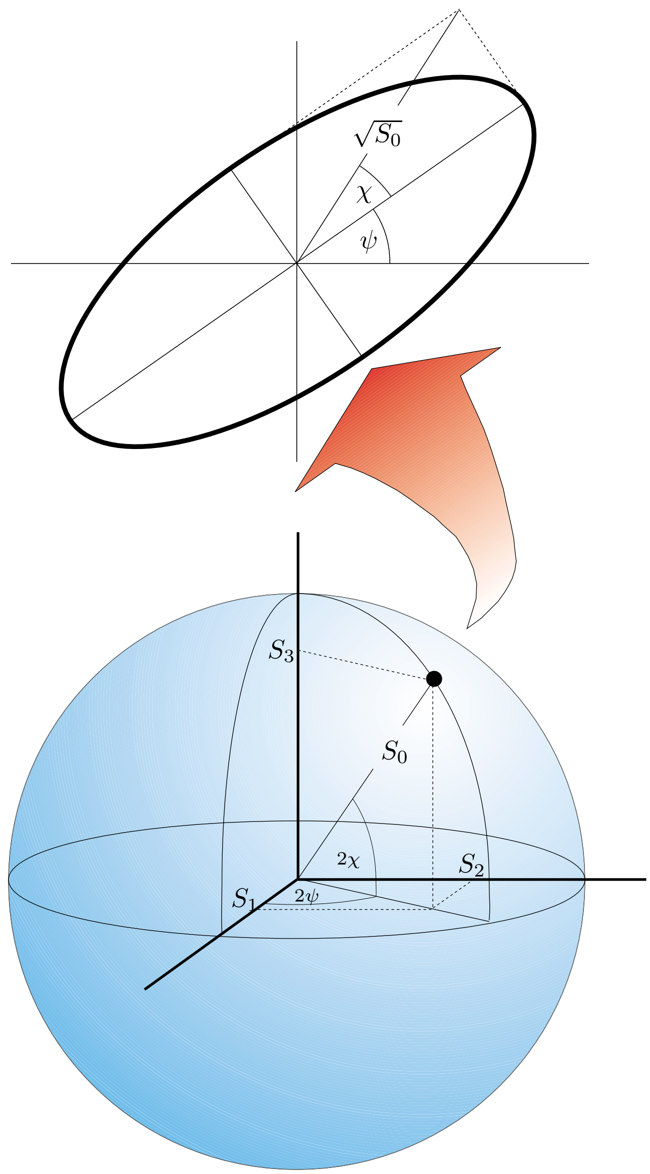
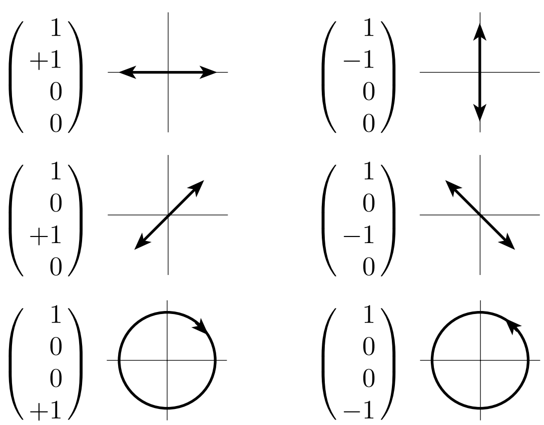


FIGURE 94: *Equations* (399) *serve to associate points on the Stokes sphere of radius* S_0 *with centered ellipses of fixed size and all possible figures & orientations. Points in the northern hemisphere* $(S_3 > 0)$ *are assigned* ↻ *helicity, points in the southern hemisphere are assigned* ↺ *helicity. In the case* $S_0 = 1$ *the Stokes sphere becomes the Poincaré sphere.*

Henri Poincaré (1892) observed that, in view of the structure second stack of equalities in (399), it is natural to *place the polarizational states of electromagnetic plane waves in one-one association with the points* $\boldsymbol{S}$ *that comprise the surface of a sphere of radius* S_0 *in 3-dimensional "Stokes space,"* as indicated in Figure 94. It becomes obvious from the figure that specification of $\{S_0, S_1, S_2, S_3\}$ is equivalent to specification of the intuitively more immediate parameters $\{S_0, \psi, \chi\}$. We need $\boldsymbol{k}$ to describe the *direction of propagation* and *frequency/wavelength* of the monochromatic plane wave, but if we have only "slow detectors" to work with then $\{S_0, S_1, S_2, S_3\}$ summarize <u>all that we can experimentally verify</u> concerning the polarizational state of the wave.[242]

Reading from Figure 94, we find the polarizational states which correspond to (for example) the axial positions on the Poincaré sphere to be those illustrated below:



It becomes in this light natural to say (with Stokes) of a pair of plane waves that they are *"oppositely polarized" if and only if their Stokes*

$$\boldsymbol{S} \equiv \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \quad \textit{and} \quad \boldsymbol{S} \equiv \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

vectors point in diametrically opposite directions:

$$\boldsymbol{S} = -\lambda^2 \boldsymbol{S}$$

of which

$$S_0 = +\lambda^2 S_0$$

[242] It is because they *relate so directly to the observational realities* that Stokes parameters become central to the quantum theory of photon spin. See §2–8 in J. M. Jauch & F. Rohrlich, *The Theory of Photons & Electrons* (1955) where, by the way, I was first introduced to this pretty subject.

is—by (400)—a corollary. From (399) we see that

$$\boldsymbol{S} \longrightarrow \boldsymbol{S} = -\lambda^2 \boldsymbol{S}$$

can, in more physical terms, be described

$$\begin{aligned} \mathcal{E}_1 &\longrightarrow \mathcal{E}_1 = +\lambda\mathcal{E}_2 \\ \mathcal{E}_2 &\longrightarrow \mathcal{E}_2 = -\lambda\mathcal{E}_1 \\ \delta &\longrightarrow \delta \;=\; \delta \end{aligned}$$

so the "oppositely polarized" associates of

$$\boldsymbol{E}(t) = \boldsymbol{e}_1\mathcal{E}_1 \cos\omega t + \boldsymbol{e}_2\mathcal{E}_2 \cos(\omega t + \delta)$$

have the form

$$\boldsymbol{E}(t) = \boldsymbol{e}_1\lambda\mathcal{E}_2 \cos(\omega t + \alpha) - \boldsymbol{e}_2\lambda\mathcal{E}_1 \cos(\omega t + \delta + \alpha)$$

where λ and α are arbitrary. As is intuitively evident, as Fresnel (∼1816) demonstrated experimentally,[243] and as we will soon be in position to prove, *oppositely polarized plane waves to not interfere*.

I propose now to make more secure the recent claim[242] that Stokes parameters pertain *directly* to the *observational* properties of plane waves. *Energy flux* is described (see again page 216) by the

$$\text{Poynting vector } \boldsymbol{S}(t) = c(\boldsymbol{E}\times\boldsymbol{B})$$

For a plane wave

$$\boldsymbol{B} = \hat{\boldsymbol{k}}\times\boldsymbol{E}$$

so

$$= cE^2(t)\hat{\boldsymbol{k}}$$

The *magnitude* of the Poynting vector is given therefore by

$$S(t) = cE^2(t) = c\big\{\mathcal{E}_1^2 \cos^2\omega t + \mathcal{E}_2^2 \cos^2(\omega t + \delta)\big\}$$

and the *intensity* of the wave ($S(t)$ averaged over a period τ) by

$$I \equiv \tfrac{1}{\tau}\int_0^\tau S(t)\,dt = \tfrac{1}{2}c\big\{\mathcal{E}_1^2 + \mathcal{E}_2^2\big\}$$

So

$$S_0 \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \tfrac{2}{c}I \tag{401}$$

<u>can be measured directly by a "*J*-meter,"</u> *i.e.*, by a photometer that has been re-scaled so that it displays

$$J \equiv \tfrac{2}{c}\cdot(\text{intensity})$$

[243] Augustin Jean Fresnel (1788–1827) was an engineer who took up optics while a political exile with time on his hands. It was his study of polarization that led him to propose that light was to be understood in terms of *transverse* waves, not the longitudinal waves postulated by Huygens, Young and others. Practical problems of lighthouse design led him to the invention of the Fresnel lens and to fundamental contributions to theoretical optics (diffraction).

If an arbitrarily polarized wave

$$\boldsymbol{E}_{\text{in}}(t) \ = \boldsymbol{e}_1\mathcal{E}_1\cos\omega t + \boldsymbol{e}_2\mathcal{E}_2\cos(\omega t+\delta)$$

is incident upon a $\longleftrightarrow$ linear polarizer then the exit beam can be described

$$\boldsymbol{E}_{\text{out}}(t) = \boldsymbol{e}_1\mathcal{E}_1\cos\omega t$$

so—arguing from (399)—we have

$$\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{out}} = \begin{pmatrix} \mathcal{E}_1^2\\ \mathcal{E}_1^2\\ 0\\ 0\end{pmatrix} = \begin{pmatrix} \frac12(S_0+S_1)\\ \frac12(S_0+S_1)\\ 0\\ 0\end{pmatrix} = \begin{pmatrix} \frac12 & \frac12 & 0 & 0\\ \frac12 & \frac12 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{in}}$$

We are led thus to these descriptions of *the action of some typical polarizers*:

$$\longleftrightarrow \text{ polarizer}: \quad \begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac12 & \frac12 & 0 & 0\\ \frac12 & \frac12 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{in}} \tag{402.1}$$

$$\updownarrow \text{ polarizer}: \quad \begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac12 & -\frac12 & 0 & 0\\ -\frac12 & \frac12 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{in}} \tag{402.2}$$

$$\swarrow\!\!\nearrow \text{ polarizer}: \quad \begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac12 & 0 & \frac12 & 0\\ 0 & 0 & 0 & 0\\ \frac12 & 0 & \frac12 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{in}} \tag{402.3}$$

$$\nwarrow\!\!\searrow \text{ polarizer}: \quad \begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac12 & 0 & -\frac12 & 0\\ 0 & 0 & 0 & 0\\ -\frac12 & 0 & \frac12 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{in}} \tag{402.4}$$

$$\circlearrowright \text{ polarizer}: \quad \begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac12 & 0 & 0 & \frac12\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \frac12 & 0 & 0 & \frac12\end{pmatrix}\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{in}} \tag{402.5}$$

$$\circlearrowleft \text{ polarizer}: \quad \begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac12 & 0 & 0 & -\frac12\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\frac12 & 0 & 0 & \frac12\end{pmatrix}\begin{pmatrix} S_0\\ S_1\\ S_2\\ S_3\end{pmatrix}_{\text{in}} \tag{402.6}$$

Arguing again from (399), we find that the action

$$\begin{aligned} \boldsymbol{E}_{\text{in}}(t) \ &= \boldsymbol{e}_1\mathcal{E}_1\cos\omega t + \boldsymbol{e}_2\mathcal{E}_2\cos(\omega t+\delta)\\ &\downarrow\\ \boldsymbol{E}_{\text{out}}(t) &= \boldsymbol{e}_1 e^{-\alpha}\mathcal{E}_1\cos\omega t + \boldsymbol{e}_2 e^{-\alpha}\mathcal{E}_2\cos(\omega t+\delta) \end{aligned}$$

of a neutral filter can be described

$$\text{neutral filter}: \quad \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = e^{-2\alpha} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \tag{402.7}$$

Suppose, now, that we present a plane wave serially to

0) a neutral filter F_0 with $e^{-2\alpha} = \frac{1}{2}$,
1) a $\longleftrightarrow$ polarizer F_1,
2) a $\swarrow\!\!\!\nearrow$ polarizer F_2,
3) a $\circlearrowright$ polarizer F_3

and in each case use a J-meter to measure the intensity of the output, obtaining

$$\big[S_0\big]_{\text{out}} = \begin{cases} J_0 = \frac{1}{2}S_0 & \text{when } \mathsf{F}_0 \text{ used} \\ J_1 = \frac{1}{2}(S_0 + S_1) & \text{when } \mathsf{F}_1 \text{ used} \\ J_2 = \frac{1}{2}(S_0 + S_2) & \text{when } \mathsf{F}_2 \text{ used} \\ J_3 = \frac{1}{2}(S_0 + S_3) & \text{when } \mathsf{F}_3 \text{ used} \end{cases}$$

Algebraically deconvolving the output data, we obtain

$$\left.\begin{aligned} S_0 &= 2J_0 \\ S_1 &= 2J_1 - 2J_0 \\ S_2 &= 2J_2 - 2J_0 \\ S_3 &= 2J_3 - 2J_0 \end{aligned}\right\} \tag{403}$$

Alternative sets of filters would serve as well, but would require some algebraic adjustment at (403). The implication is that

> With four suitably selected filters and a photometer one can *measure* Stokes' parameters, and thus fully characterize the intensity/polarization/helicity of a (coherent monochromatic) plane wave.

3. Mueller calculus. A light beam—modeled, for the moment, as a plane wave—with attributes

$$\big\{\boldsymbol{k}, S_0, S_1, S_2, S_3\big\}_{\text{in}}$$

is presented to a passive device, from which a beam with attributes

$$\big\{\boldsymbol{k}, S_0, S_1, S_2, S_3\big\}_{\text{out}}$$

emerges. A description of how the output variables depend upon the input variables would comprise a *characterization of the device*. In view of the fact that

- mirrors/lenses typically *change the direction* of the beam, and scatterers typically spray a beam in multiple directions
- some crystals *change the frequency* of a monochromatic beam
- some materials/devices *alter the coherence properties* of an incident beam, others alter the *degree of polarization* (of which more later)

we recognize that some physical restriction is involved when agree to limit our concern to devices that conform to the following scheme:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \longrightarrow \boxed{\text{device}} \longrightarrow \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}}$$

Since (400) pertains *generally* to monochromatic plane waves, we see that for *every* such device

$$\big[S_0^2 - S_1^2 - S_2^2 - S_3^2\big]_{\text{out}} = \big[S_0^2 - S_1^2 - S_2^2 - S_3^2\big]_{\text{in}} = 0 \tag{403.1}$$

while for every *passive* device (since passive devices are—unlike lasers—not connected to an external energy source, and therefore may absorb energy from, but cannot inject energy into... the transmitted light beam) energy conservation requires

$$0 \leqslant \big[S_0\big]_{\text{out}} \leqslant \big[S_0\big]_{\text{in}} \tag{403.2}$$

A <u>general theory of passive devices</u> would result from an effort to describe the functional relationships

$$S_{\mu\,\text{out}} = \mathcal{D}_\mu(S_{0\,\text{in}}, S_{1\,\text{in}}, S_{2\,\text{in}}, S_{3\,\text{in}}) \quad : \quad \mu = 0, 1, 2, 3$$

permitted by (403). Remarkably, such an effort, if based upon (403.1) alone, would lead back again to the *conformal group*, which was encountered earlier in quite another connection.[244] When (403.2) is brought into play certain group elements are excluded: one is left with what might be called the "device semigroup."[245]

A far simpler theory—which is, however, adequate to most practical needs —is obtained if one imposes the additional assumption that the parameters $S_{\mu\,\text{out}}$ are *linear* functions of $S_{\mu\,\text{in}}$:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}} \longrightarrow \boxed{\text{linear passive device}} \longrightarrow \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{out}} = \mathbb{M}\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{\text{in}}$$

One is led then to the linear fragment of the conformal group; *i.e.*, to the condition (compare (185.2) on pages 129 & 164)

$$\mathbb{M}^{\mathsf{T}} g\, \mathbb{M} = m^2 g \tag{404.1}$$

subject to the proviso that one must exclude cases that place one in violation of (403.2). Evidently $\det \mathbb{M} = m^4$, so in non-singular cases one can state that $\mathbb{M}/m$ is Lorentzian:

$$\mathbb{M} \text{ (if non-singular) possesses the structure } \mathbb{M} = m \cdot \mathbb{\Lambda} \tag{404.2}$$

---

[244] See again Chapter 2, §6. For a brief sketch of the resulting theory of optical devices see pages 353–354 in CLASSICAL ELECTRODYNAMICS (1980).

[245] A *semigroup* is a "group without inversion."

> REMARK: One must carefully resist any temptation to conclude from the design of (404) that the Stokes parameters S_μ transform as the components of a 4-vector. Their Lorentz transformation properties are inherited—*via* the definitions (399)—from those of the electromagnetic field, and are in fact quite intricate. The subject is treated on pages 436 *et seq* in my ELECTRODYNAMICS (1972).

The idea of using 4×4 matrices to describe the action of linear passive optical devices was first developed in a report by Hans Mueller . . . which, however, he never published. Such matrices are called "Mueller matrices," and their use (discussed below) is the subject matter of the "Mueller calculus."

The 4×4 matrices encountered in (402.1–6) are readily shown to satisfy

$$\mathbb{M}^{\mathsf{T}}\mathcal{g}\,\mathbb{M} = \mathbb{O}, \text{ which is (404.1) with } m = 0 \tag{405}$$

and to be always in compliance with (403.2).[246] So each is a Mueller matrix. Each is found, moreover, to possess[247] the "projection property"[248]

$$\mathbb{M}^2 = \mathbb{M} \tag{406}$$

Calculation shows, moreover, that in each case

$$\det(\mathbb{M} - \lambda\mathbb{I}) = \lambda^3(\lambda - 1) \tag{407}$$

so

$\mathbb{M}\,S_{\text{in}} = 0$	has three linearly independent solutions; the device extinguishes such beams
$\mathbb{M}\,S_{\text{in}} = S_{\text{in}}$	has but one; the device is transparent to such beams (scalar multiples of one another)

EXAMPLE: Noting that $3^2 + 4^2 + 12^2 = 13^2$ let us, by contrivance, take

$$S_{\text{in}} = \begin{pmatrix} 13 \\ 3 \\ 4 \\ 12 \end{pmatrix}$$

and let us take $\mathbb{M}$ to be the Mueller matrix of (402.1) that describes the action of a $\longleftrightarrow$ polarizer. Then (by quick calculation)

---

[246] PROBLEM 63.

[247] PROBLEM 64.

[248] From (406) it follows, by the way, that $(\det\mathbb{M})^2 = \det\mathbb{M}$ whence

$$\det\mathbb{M} = \begin{cases} 1 & \text{if } \mathbb{M} \text{ is the trivial projector } \mathbb{I} \\ 0 & \text{otherwise} \end{cases}$$

The zero on the right side of (405) can be therefore be looked upon as a forced consequence of projectivity.

$$S_{\text{out}} = \mathbb{M}S_{\text{in}} = \begin{pmatrix} 8 \\ 8 \\ 0 \\ 0 \end{pmatrix}, \text{ projected component of } \begin{pmatrix} 13 \\ 3 \\ 4 \\ 12 \end{pmatrix}$$

The exit beam is 100% $\longleftrightarrow$ polarized, but dimmer:

$$S_{0\text{out}} = 8 \;<\; S_{0\text{in}} = 13$$

A second pass through the device (second such projection) has no effect (that being the upshot of $\mathbb{M}^2 = \mathbb{M}$):

$$\mathbb{M}\begin{pmatrix} 8 \\ 8 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 0 \\ 0 \end{pmatrix}$$

To describe the action of an *arbitrary polarizer*: let $\boldsymbol{\sigma}$ be an arbitrary unit 3-vector and construct

$$\mathbb{M}(\boldsymbol{\sigma}) \equiv \tfrac{1}{2}\cdot\begin{pmatrix} 1 & \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma_1\sigma_1 & \sigma_1\sigma_2 & \sigma_1\sigma_3 \\ \sigma_2 & \sigma_2\sigma_1 & \sigma_2\sigma_2 & \sigma_2\sigma_3 \\ \sigma_3 & \sigma_3\sigma_1 & \sigma_3\sigma_2 & \sigma_3\sigma_3 \end{pmatrix} \tag{408.1}$$

One can show[249] that $\mathbb{M}(\boldsymbol{\sigma})$ satisfies (405/6/7) and that

$$\mathbb{M}(\boldsymbol{\sigma})\begin{pmatrix} 1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \tag{408.2}$$

Moreover

$$\mathbb{M}(-\boldsymbol{\sigma})\mathbb{M}(+\boldsymbol{\sigma}) = \mathbb{O} \quad : \quad \text{all } \boldsymbol{\sigma} \tag{409}$$

which supplies neat support for Stokes' claim (page 305) that diametrically opposite points on the Stokes sphere refer to "opposite polarizations," and conforms precisely to the pattern evident when one compares (402.2) with (402.1), (402.4) with (402.3), (402.6) with (402.5). In the case

$$\boldsymbol{\sigma} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Equation (409) might be notated

$$\mathbb{M}(\updownarrow)\mathbb{M}(\leftrightarrow) = \mathbb{O}$$

---

[249] PROBLEM 65.

and interpreted to express the familiar fact that *no light passes through crossed polarizers*.

Suppose, however, we were to interpose (between $\mathbb{M}(\updownarrow)$ and $\mathbb{M}(\leftrightarrow)$) a third device: let it be (say) the linear polarizer represented (see again Figure 94) by

$$\mathbb{M}(\psi) \equiv \mathbb{M}(\boldsymbol{\sigma}) \quad \text{with} \quad \boldsymbol{\sigma} = \begin{pmatrix} \cos 2\psi \\ \sin 2\psi \\ 0 \end{pmatrix}$$

With the assistance of *Mathematica* we compute

$$\mathbb{M}(\updownarrow)\mathbb{M}(\psi)\mathbb{M}(\leftrightarrow) = \begin{pmatrix} \frac{1}{8}\sin^2 2\psi & \frac{1}{8}\sin^2 2\psi & 0 & 0 \\ -\frac{1}{8}\sin^2 2\psi & -\frac{1}{8}\sin^2 2\psi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq \mathbb{O}$$

which illustrates the basis of an experimental technique standard to microscopy and engineering: one places a microscope slide or the stressed Lucite model of a machine part between crossed polarizers, and examines the transmitted image.

The preceding calculation also illustrates the central idea of the "Mueller calculus": To determine the net effect of cascaded optical devices one simply *multiplies the corresponding Mueller matrices*.

"Optical devices" exist in considerable variety. At (402.7) we encountered the Mueller matrices

$$\mathbb{M} = e^{-2\alpha} \cdot \mathbb{I} \tag{410}$$

that describe the action of "neutral filters." Such a device is transparent at $\alpha = 0$, and becomes progressively more absorptive (optically dense) as α increases.

Mueller matrices of major practical importance arise if at (404.2) we set $m = 1$ and assume $\mathbb{M} = \mathbb{\Lambda}$ to have (see again (208) on page 155) the rotational design

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathbb{R} & \\ 0 & & & \end{pmatrix} \tag{411}$$

$$\mathbb{R} \equiv \exp\left\{ 2\theta \begin{pmatrix} 0 & -\sigma_3 & \sigma_2 \\ \sigma_3 & 0 & -\sigma_1 \\ -\sigma_2 & \sigma_1 & 0 \end{pmatrix} \right\} \ : \ \text{a rotation matrix}$$

Such an $\mathbb{M}$ leaves S_0 invariant (no absorption) but causes

$$\boldsymbol{S} \equiv \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

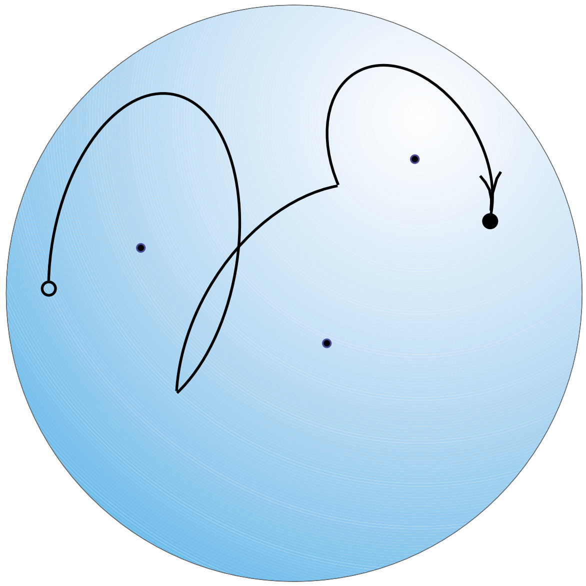


FIGURE 95: *The input beam in Stokes state* ○ *is passes through three successive devices of type* (411) *to produce an output beam in Stokes state* ●. *Dots mark the centers of rotation (ends of the* $\boldsymbol{\sigma}$ *vectors). Because rotations possess the group property, the net effect of the three rotational beam transformations could have been achieved by a* single *such transformation.*

to experience righthanded (↻) rotation through the angle 2θ about the axis defined by the unit vector $\boldsymbol{\sigma}$. In the special case

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(411) gives

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta & 0 \\ 0 & \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the action of which (see again Figure 94) is to rotate the plane of polarization:

$$\psi \to \psi + \theta$$

Such devices exploit the *optical activity* phenomenon, and are called “rotators.” The case

$$\boldsymbol{\sigma} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

gives

$$\mathbb{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 2\theta & -\sin 2\theta \\ 0 & 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}$$

which achieves

$$\delta \to \delta + 2\theta$$

Such devices are called "compensators" or "phase shifters." It is clear that Mueller matrices of type (411) are non-singular: $\mathbb{M}^{-1}$ is again a Mueller matrix, which means that the action of such a device could be undone by a suitably chosen second such device. Projection, on the other hand, is a *non*-invertible operation: the action of a polarizer, when undone by subsequent polarizers, always entails attenuation of the beam. To illustrate the point, we return to the example of page 312 and by computation find that

$$\mathbb{M}(\leftrightarrow)\mathbb{M}(\psi)\mathbb{M}(\leftrightarrow) = \tfrac{1}{2}\cos^4\psi \cdot \mathbb{M}(\leftrightarrow)$$

Looking back again to (404.2), it becomes natural in view of the foregoing to assign $\mathbb{\Lambda}$ the "boost" design of (209), writing

$$\mathbb{M} = m \cdot \begin{pmatrix} \gamma & \beta_1\gamma & \beta_2\gamma & \beta_3\gamma \\ \beta_1\gamma & 1+(\gamma-1)\beta_1\beta_1/\beta^2 & (\gamma-1)\beta_1\beta_2/\beta^2 & (\gamma-1)\beta_1\beta_3/\beta^2 \\ \beta_2\gamma & (\gamma-1)\beta_2\beta_1/\beta^2 & 1+(\gamma-1)\beta_2\beta_2/\beta^2 & (\gamma-1)\beta_2\beta_3/\beta^2 \\ \beta_3\gamma & (\gamma-1)\beta_3\beta_1/\beta^2 & (\gamma-1)\beta_3\beta_2/\beta^2 & 1+(\gamma-1)\beta_3\beta_3/\beta^2 \end{pmatrix}$$

where the β's are "device parameters" that have now *nothing to do with velocity*. Immediately

$$\begin{aligned} S_{0\text{out}} &= m\gamma(S_{0\text{in}} + \boldsymbol{\beta}\cdot\boldsymbol{S}_\text{in}) \\ &\qquad\qquad \boldsymbol{S}_\text{in} = S_{0\text{in}}\hat{\boldsymbol{S}}_\text{in} \text{ by (400)} \\ &= m\gamma(1 + \boldsymbol{\beta}\cdot\hat{\boldsymbol{S}}_\text{in})S_{0\text{in}} \\ &= m\gamma(1 + \beta\cos\omega)S_{0\text{in}} \quad : \quad \omega \text{ is the angle between } \boldsymbol{\beta} \text{ and } \boldsymbol{S}_\text{in} \end{aligned}$$

so to achieve universal compliance with the passivity condition (403.2) we must have

$$0 < m \leqslant \sqrt{\tfrac{1-\beta}{1+\beta}} \leqslant 1$$

where it is understood that $0 \leqslant \beta < 1$. It is not at all difficult to show of such Mueller matrices that though $\mathbb{M}^{-1}$ exists—and is, in fact, easy to describe

$$[m\mathbb{\Lambda}(\boldsymbol{\beta})]^{-1} = m^{-1}\mathbb{\Lambda}(-\boldsymbol{\beta})$$

—it stands in violation of the passivity condition, so cannot be realized by a passive device. On pages 361/2 of some notes already cited[244] I explore some of the finer details of this subject, and argue that *it should be possible* *to mimic 4-dimensional relativity (composition of non-colinear boosts, Thomas precession, etc.) by experiments performed on a linear optical bench*!

In some respects more elegantly efficient—but in other physical respects more limited—than the Mueller calculus is the "Jones calculus," devised by R. Clark Jones one summer in the early 1940's while he was employed in the laboratory of Edwin Land as a Harvard undergraduate. In Jones' formalism Stokes' parameters are folded into the design of a complex 2-vector, and devices are represented by complex 2×2 matrices. The formalism is developed in elaborate detail in my "Ellipsometry" (1999) and in the literature cited there, but it would take us too far afield to attempt to treat the subject here.

4. Partially polarized plane waves. The "plane waves" considered thus far are highly idealized abstractions: they

- are of infinite temporal duration
- are of infinite spatial extent . . . and therefore
- carry infinite energy and momentum, and moreover
- are spatially/temporally perfectly coherent.

But so also—and in much the same way—is the Euclidean plane an idealized abstraction. Euclidean geometry becomes relevant to physical geometry only in contexts (very numerous indeed!) in which it is sensible to conflate the local geometry of the curved surface with the local geometry of the tangent plane. So it is in classical electrodynamics: ideas borrowed from the idealized physics of plane waves become relevant to the physics of realistic radiation fields only as local approximants,[250] and can be expected to lose their utility "in the large," as also in the vicinity of charges, caustics, "kinks" in the field.

But radiation fields the gross properties of which display any degree of spatial/temporal variability cannot be precisely monochromatic. We expect natural fields to acquire also some degree of spatial/temporal incoherence from the radiation production mechanism, whatever it might be. We are led thus to the concept of a **quasi-monochromatic plane wave**—led, that is, to the replacement

$$\boldsymbol{E}(t) = \big\{\boldsymbol{e}_1\mathcal{E}_1e^{i\delta_1} + \boldsymbol{e}_2\mathcal{E}_2e^{i\delta_2}\big\}e^{i\omega t} \tag{395}$$

$$\downarrow$$

$$\boldsymbol{E}(t) = \big\{\boldsymbol{e}_1\mathcal{E}_1(t)e^{i\delta_1(t)} + \boldsymbol{e}_2\mathcal{E}_2(t)e^{i\delta_2(t)}\big\}e^{i\omega t} \tag{412}$$

where ω sets the nominal frequency and $\mathcal{E}_1(t)$, $\mathcal{E}_2(t)$, $\delta_1(t)$ and $\delta_2(t)$ are assumed to change

- slowly with respect to $e^{i\omega t}$ but (in typical cases)
- rapidly with respect to the response time of our photometers.

[250] Beware! Plane waves are, in one critical respect, *not* representative of the typical local facts. I refer to the circumstance that, while $\boldsymbol{E}\perp\boldsymbol{B}$ is characteristic of plane waves, it is <u>not a property of fields in general</u> (superimposed plane waves). See below, page 332.

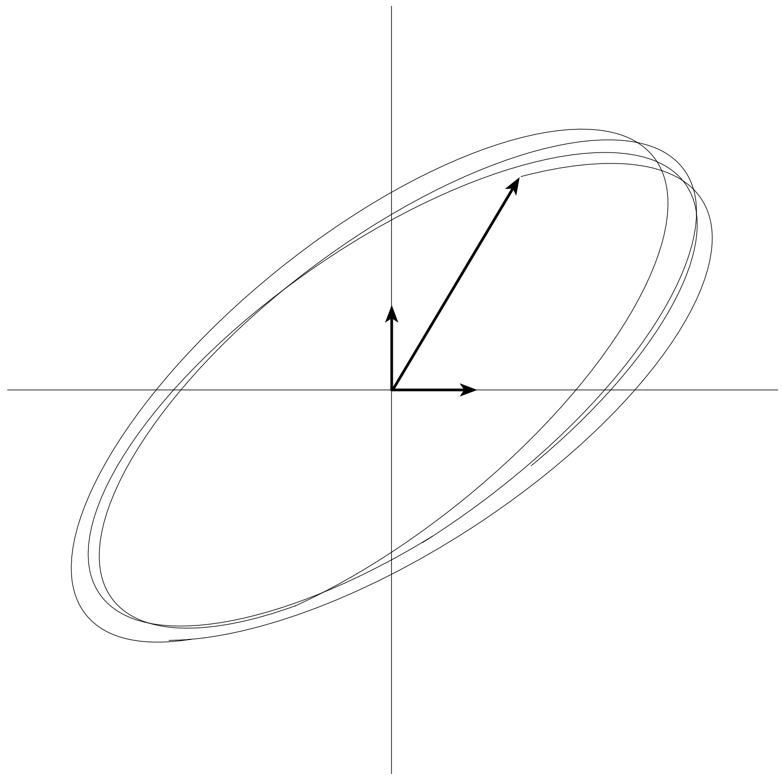


FIGURE 96: *Imperfectly elliptical flight (compare Figure 93) of the* $\boldsymbol{E}$*-vector when the plane wave is only* quasi-*monochromatic*.

Notice that we make no attempt to tinker with the *spatial* properties of the wave (our photometer looks, after all, to only a *local sample* of the physical wave), and that the procedure we have adopted is frankly "phenomenological" in the sense that we do not ask how $\mathcal{E}_1(t)$, $\mathcal{E}_2(t)$, $\delta_1(t)$ and $\delta_2(t)$ might be constrained by Maxwell's equations.

From (412) we conclude that, as illustrated above, $\boldsymbol{E}(t)$ traces an ellipse only in the shortrun—an ellipse with "instantaneous" Stokes parameters given (see again (399)) by

$$\left.\begin{aligned} S_0(t) &= \mathcal{E}_1^2(t) + \mathcal{E}_2^2(t) \\ S_1(t) &= \mathcal{E}_1^2(t) - \mathcal{E}_2^2(t) \\ S_2(t) &= 2\mathcal{E}_1(t)\mathcal{E}_2(t)\cos\delta(t) \\ S_3(t) &= 2\mathcal{E}_1(t)\mathcal{E}_2(t)\sin\delta(t) \end{aligned}\right\} \tag{413}$$

$$\delta(t) \equiv \delta_2(t) - \delta_1(t)$$

The ellipse jiggles about, constantly changing is figure/orientation, in a manner determined by the (let us say steady) *statistical properties of the wave*. The functions $\mathcal{E}_1(t), \mathcal{E}_2(t)$ and $\delta(t)$—whence also $S_0(t)$, $S_1(t)$, $S_2(t)$ and $S_3(t)$—have, in other words, assumed the character of random variables. Our filters and (slow) J-meters, used as described on page 308, supply information not about the functions $S_\mu(t)$ but about their *mean values*:

$$\mathcal{S}_\mu \equiv \langle S_\mu(t)\rangle \equiv \tfrac{1}{T}\int_0^T S_\mu(t)\,dt \quad : \quad \begin{cases} T \text{ might refer to the response} \\ \text{time of the instrument} \end{cases}$$

Proceeding in this light from (403) and (413) we have

$$\left.\begin{aligned} \mathcal{S}_0 &= 2J_0 &&= \langle\mathcal{E}_1^2\rangle + \langle\mathcal{E}_2^2\rangle \\ \mathcal{S}_1 &= 2J_1 - 2J_0 &&= \langle\mathcal{E}_1^2\rangle - \langle\mathcal{E}_2^2\rangle \\ \mathcal{S}_2 &= 2J_2 - 2J_0 &&= 2\langle\mathcal{E}_1\mathcal{E}_2\cos\delta\rangle \\ \mathcal{S}_3 &= 2J_3 - 2J_0 &&= 2\langle\mathcal{E}_1\mathcal{E}_2\sin\delta\rangle \end{aligned}\right\} \tag{414}$$

Evidence that Stokes' parameters are, if not by initial intent, nevertheless wonderfully well-adapted to discussion of the dominant statistical properties of physical lightbeams emerges from the following little argument: working from (414) we have

$$\mathcal{S}_0^2 = \langle\mathcal{E}_1^2\rangle^2 + 2\langle\mathcal{E}_1^2\rangle\langle\mathcal{E}_2^2\rangle + \langle\mathcal{E}_2^2\rangle^2 \tag{415.1}$$

$$\begin{aligned} \mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2 &= \langle\mathcal{E}_1^2\rangle^2 - 2\langle\mathcal{E}_1^2\rangle\langle\mathcal{E}_2^2\rangle + \langle\mathcal{E}_2^2\rangle^2 + \langle 2\mathcal{E}_1\mathcal{E}_2\cos\delta\rangle^2 + \langle 2\mathcal{E}_1\mathcal{E}_2\sin\delta\rangle^2 \\ &= \mathcal{S}_0^2 + 4\big\{\langle\mathcal{E}_1\mathcal{E}_2\cos\delta\rangle^2 + \langle\mathcal{E}_1\mathcal{E}_2\sin\delta\rangle^2 - \langle\mathcal{E}_1^2\rangle\langle\mathcal{E}_2^2\rangle\big\} \end{aligned} \tag{415.2}$$

But if x and y are any random variables (however distributed) then from $\langle(\lambda x + y)^2\rangle = \lambda^2\langle x\rangle^2 + 2\lambda\langle xy\rangle + \langle y\rangle^2 \geqslant 0$ (all λ) it follows that in all cases $\langle xy\rangle^2 \leqslant \langle x^2\rangle\langle y^2\rangle$, so we have

$$\begin{aligned} \langle\mathcal{E}_1\mathcal{E}_2\cos\delta\rangle^2 &\leqslant \langle\mathcal{E}_1^2\rangle\langle\mathcal{E}_2^2\cos^2\delta\rangle \\ \langle\mathcal{E}_1\mathcal{E}_2\sin\delta\rangle^2 &\leqslant \langle\mathcal{E}_1^2\rangle\langle\mathcal{E}_2^2\sin^2\delta\rangle \end{aligned}$$

giving

$$\mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2 \leq \mathcal{S}_0^2 + 4\underbrace{\big\{\langle\mathcal{E}_1^2\rangle\langle\mathcal{E}_2(\cos^2\delta + \sin^2\delta)^2\rangle - \langle\mathcal{E}_1^2\rangle\langle\mathcal{E}_2^2\rangle\big\}}_{0}$$

We are led thus to the important inequality

$$\mathcal{S}_0^2 - \mathcal{S}_1^2 - \mathcal{S}_2^2 - \mathcal{S}_3^2 \geqslant 0 \tag{416}$$

with—according to (400)—equality if (but not only if!) the beam is literally monochromatic. Looking back again to Figure 94, we see that (416) serves to place the vector

$$\boldsymbol{\mathcal{S}} \equiv \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix}$$

inside the Stokes sphere of radius $\mathcal{S}_0$, and that $\boldsymbol{\mathcal{S}}$ reaches all the way to the surface of the Stokes sphere if and only if the beam is, in a fairly evident sense, *statistically equivalent to a monochromatic beam*.

If $\mathcal{E}_1$, $\mathcal{E}_2$ and δ are *statistically independent* random variables then we can in place of (414) write

$$\begin{aligned}
\mathcal{S}_0 &= \langle\mathcal{E}_1^2\rangle + \langle\mathcal{E}_2^2\rangle \\
\mathcal{S}_1 &= \langle\mathcal{E}_1^2\rangle - \langle\mathcal{E}_2^2\rangle \\
\mathcal{S}_2 &= 2\langle\mathcal{E}_1\rangle\langle\mathcal{E}_2\rangle\langle\cos\delta\rangle \\
\mathcal{S}_3 &= 2\langle\mathcal{E}_1\rangle\langle\mathcal{E}_2\rangle\langle\sin\delta\rangle
\end{aligned}$$

If, moreover, all δ-values are equally likely, then $\langle\cos\delta\rangle = \langle\sin\delta\rangle = 0$, and we have $\mathcal{S}_2 = \mathcal{S}_3 = 0$. If, moreover, $\langle\mathcal{E}_1\rangle = \langle\mathcal{E}_2\rangle$ then $\mathcal{S}_1 = 0$. The resulting beam

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{is said to be } \textit{unpolarized} : \boldsymbol{\mathcal{S}} = \mathbf{0}$$

It becomes on this basis natural to introduce the

$$\text{“degree of polarization”}\ P \equiv \frac{\sqrt{\mathcal{S}_1^2 + \mathcal{S}_2^2 + \mathcal{S}_3^2}}{\mathcal{S}_0} \quad : \quad 0 \leqslant P \leqslant 1 \tag{417}$$

and to write

$$\begin{aligned}
\begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix} &= \begin{pmatrix} P\mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix} + \begin{pmatrix} (1-P)\mathcal{S}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \text{100\% polarized component} \quad + \quad \text{unpolarized component}
\end{aligned}$$

When an unpolarized beam is presented to (for example) the linear polarizer of (402.1) one obtains

$$\begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix}_{\text{in}} \xrightarrow[\text{linear polarizer at } 0^\circ]{} \begin{pmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{pmatrix}_{\text{out}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{S}_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\text{in}} = \begin{pmatrix} \frac{1}{2}\mathcal{S}_0 \\ \frac{1}{2}\mathcal{S}_0 \\ 0 \\ 0 \end{pmatrix}$$

Here

$$\begin{aligned} P_{\text{in}} &= 0 &&: \quad \text{the entry beam is unpolarized, but} \\ P_{\text{out}} &= 1 &&: \quad \text{the exit beam is 100\% polarized} \end{aligned}$$

And when the exit beam is presented to a second linear polarizer, described by the $\mathbb{M}(\psi)$ of page 312, one obtains[251] the "Law of Malus":

$$\frac{\text{output intensity}}{\text{input intensity}} = \tfrac{1}{4}(1 + \cos 2\psi) = \tfrac{1}{2}\cos^2\psi$$

A quasi-monochromatic beam is said to be

$$\left.\begin{array}{r} \text{unpolarized} \\ \text{partially polarized} \\ \text{completely polarized} \end{array}\right\} \text{ according as } \left\{\begin{array}{l} 0 = P \\ 0 < P < 1 \\ \quad\;\; P = 1 \end{array}\right.$$

An unpolarized beam necessarily *is* polarized in the shortrun, but in the longer term the $\boldsymbol{E}$-vector traces an orientation-free scribble. Partial polarization results when the scribble is *somewhat* oriented (fuzzy): this requires that $\mathcal{E}_1(t)$, $\mathcal{E}_2(t)$, $\delta_1(t)$ and $\delta_2(t)$ more somewhat in concert; *i.e.*, that they be *statistically correlated*. It is important to note that the numbers $\mathcal{S}_\mu$ provide a very incomplete description of the beam statistics, and that even complete knowledge of the statistical properties of the beam would leave the actual t-dependence of $\boldsymbol{E}$ indeterminate. *Many* beams are—even in the case of complete polarization—consistent with any prescribed/measured set of $\mathcal{S}_\mu$-values.

We are by those remarks into position to appreciate the import of Stokes'

> **Principle of Optical Equivalence**: Lightbeams with identical Stokes parameters are "equivalent" in the sense that they interact identically with devices which detect or alter the intensity and/or polarizational state of the incident beam.

and the depth of his insight into the physics of light. But one does not say of objects that they are, in designated respects, "equivalent" unless there exist other respects—whether overt or covert—in which they are at the same time inequivalent; implicit in the formulation of Stokes' principle is an assertion that physical light beams possess properties beyond those to which the Stokes parameters allude, properties to which photometer-like devices are insensitive. There are many ways to render a page gray with featureless squiggles, many ways to assemble an unpolarized light beam. What such beams, such statistical assemblages share is, according to (414), not "identity" but only the property that a certain quartet of numbers arising from their low-order moments and correlation coefficients are equi-valued.

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[251] PROBLEM 66. Étiènne Louis Malus (1775–1812) was a French engineer/ physicist.

We have, in effect, been alerted by Stokes to the existence of a "statistical optics"—to the possibility that instruments (more subtle in their action than photometers) might be devised which are sensitive to higher moments of an incident optical beam. And we have been alerted to the possible existence and potential usefulness of an *ascending hierarchy* of "higher order analogs" of the parameters that bear Stokes' name, formal devices that serve to capture successively more refined statistical properties of optical beams. Examination of the literature[252] shows all those expectations to be borne out by fairly recent developments. It becomes interesting in the light of these remarks to recall the title of the paper in which the Stokes parameters were first described: "On the composition and resolution of streams of polarized light from different sources" (Trans. Camb. Phil. Soc. **9**, 399 (1852)). Stokes brought the theory of physical light beams to a state somewhat analogous to that encountered in thermodynamics, where a few operationally defined variables mask a rich time-dependent microphysics, yet serve to support a formalism which is—surprisingly—closed/self-consistent/complete . . . and which accounts accurately for the phenomenological facts.

Already on page 318 we began to accumulate evidence that the Mueller calculus is as "robust" as the Stokes formalism upon which it is based. To our former population of Mueller matrices $\mathbb{M}$ it might now seem appropriate to add (for example)

$$\mathbb{M} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-u} & 0 & 0 \\ 0 & 0 & e^{-u} & 0 \\ 0 & 0 & 0 & e^{-u} \end{pmatrix} \quad : \quad u \geqslant 0 \tag{418}$$

which evidently describes the action of an *isotropic depolarizer*, where the adjective refers to isotropy not in physical space but in Stokes space. The interesting point—which stands as an open invitation to formal/physical invention—is that the $\mathbb{M}$ described above *does not satisfy the fundamental Mueller condition* (404.1). Relatedly: I am informed by Morgan Mitchell, my optical colleague, that while active "polarization scramblers" do exist, a "*passive* depolarization device" would be a "tall order." [253,254]

5. Optical beams. Listed at the beginning are several respects in which "plane waves are highly idealized abstractions." With the introduction of the notion of "quasi-monochromaticity" we were able to introduce an element of realism into the discussion, but

- infinite temporal duration
- infinite spatial extent
- infinite energy/momentum

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[252] See, for example, E. L. O'Neill, *Introduction to Statistical Optics* (1963); J. W. Simmons & M. J. Guttmann, *States, Waves and Photons: A Modern Introduction to Light* (1970); C. Brosseau,*Fundamentals of Polarized Light: A Statistical Optics Approach* (1998).

[253] PROBLEM 67.

[254] PROBLEM 68.

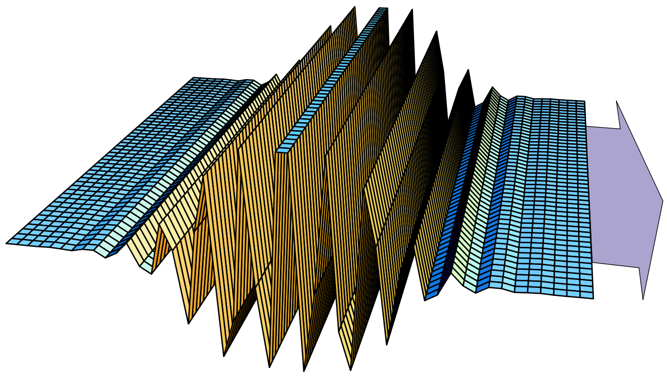

FIGURE 97: *Representation of the function $\varphi(t,x,0,z)$ described at (419) below. The Gaussian wavepacket glides rigidly, as indicated by the arrow. It is temporally confined, but spatially unconfined.*

are unphysical abstractions that survived untouched in the ensuing discussion of beam statistics and imperfect polarization. Temporal confinement is fairly easy to achieve, as the following remark makes clear:

Write $e^{i(kct+0x+0y-kz)}$ to describe a plane wave running up the z-axis. Write

$$\begin{aligned}\varphi(t,x,y,z) &= \int_{-\infty}^{+\infty} f(k)e^{ik(ct-z)}\,dk\\ &= \int_{-\infty}^{+\infty} g(\omega)e^{i\omega(t-z/c)}\,d\omega\end{aligned}$$

to describe a weighted superposition of such waves. Take $g(\omega)$ to have, in particular, the form of a normalized Gaussian centered at Ω:

$$\begin{aligned}g(\omega) &\equiv \tfrac{1}{\sqrt{2\pi}}Te^{-\frac{1}{2}T^2(\omega-\Omega)^2} \quad : \; T>0 \text{ has the physical dimension of TIME}\\ &\downarrow\\ &= \delta(\omega-\Omega) \quad \text{as } T\uparrow\infty\end{aligned}$$

Then

$$\begin{aligned}\varphi(t,x,y,z) &= \int_{-\infty}^{+\infty} \tfrac{1}{\sqrt{2\pi}}Te^{-\frac{1}{2}T^2(\omega-\Omega)^2}e^{i\omega(t-z/c)}\,d\omega\\ &= e^{-\frac{1}{2}T^{-2}(t-z/c)^2}\cdot e^{i\Omega(t-z/c)} \qquad (419)\\ &\downarrow\\ &= e^{i\Omega(t-z/c)} \quad \text{as } T\uparrow\infty\end{aligned}$$

The physical (*i.e.*, the real) part of the expression on the right side of (419) is plotted in Figure 97.

I have occasionally allowed myself to speak informally of "beams" when the objects to which I referred were actually plane waves. We confront now the mathematical force of the distinction. While the waves sampled by astronomers are good approximations to plane waves, when we go into the laboratory to perform optical experiments we deal most commonly with laterally confined light *beams*.[255] The mathematical description of lateral confinement poses a number of delicate problems entirely absent from the theory of temporal confinement. The subject acquired new urgency from the invention of the laser, and it is from a classic contribution to that literature[256] that I have adapted the following remarks:

Setting aside, for the moment, the fact that electromagnetic radiation is properly described by a *transverse vector* field, we look for laterally confined monochromatic solutions $\varphi(t,x,y,z) = e^{i\omega t}\cdot\phi(x,y,z)$ of the *scalar* wave equation $\Box\varphi = 0$. Which is to say (see again page 294): we look for laterally confined solutions of the Helmholtz equation

$$\Big\{(\tfrac{\partial}{\partial x})^2 + (\tfrac{\partial}{\partial y})^2 + (\tfrac{\partial}{\partial z})^2 + k^2\Big\}\phi(x,y,z) = 0$$

We have interest in laterally confined waves propagating in the z-direction, so look for solutions of the form

$$\phi(x,y,z) = e^{-ikz}\cdot\psi(x,y,z) \quad : \quad k = \omega/c$$

Which is to say: we look for laterally confined solutions of

$$\Big\{(\tfrac{\partial}{\partial x})^2 + (\tfrac{\partial}{\partial y})^2 + (\tfrac{\partial}{\partial z})^2\Big\}\psi(x,y,z) = 2ik\tfrac{\partial}{\partial z}\psi(x,y,z)$$

We agree to work in the **approximation** that $\psi(x,y,z)$ changes so gradually in the z-direction that the red $(\frac{\partial}{\partial z})^2$-term can be dropped. We arrive then at an equation

$$\tfrac{1}{2k}\Big\{(\tfrac{\partial}{\partial x})^2 + (\tfrac{\partial}{\partial y})^2\Big\}\psi(x,y,z) = \quad i\tfrac{\partial}{\partial z}\psi(x,y,z) \tag{420}$$

which is structurally reminiscent of the <u>Schrödinger equation for a particle free to move in two dimensions</u>:

$$\tfrac{\hbar}{2m}\Big\{(\tfrac{\partial}{\partial x})^2 + (\tfrac{\partial}{\partial y})^2\Big\}\psi(x,y,t) = -i\tfrac{\partial}{\partial t}\psi(x,y,t)$$

Both equations have unlimitedly many solutions, depending

- in quantum mechanics upon the form assigned to $\psi(x,y,t)$ at an initial time t_0, commonly taken to be $t_0 = 0$

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[255] We do not speak of "star beams," and it is only for local meteorological reasons that we speak sometimes of "sun beams."

[256] H. Kogelnik & T. Li, "Laser beams and resonators," Applied Optics **5**, 1550 (1966). See also §4.5 in O. Svelto, *Principles of Lasers* (3rd edition 1989).

- in beam theory upon the form assigned to $\psi(x,y,z)$ at some prescribed axial point z_0; we will find it convenient to take $z_0 = 0$.

To illustrate the point, the authors of quantum texts[257] often take $\psi(x,y,t_0)$ to be Gaussian

$$\psi(x,y,0) = Ae^{-a(x^2+y^2)}$$

and by one or another of the available computational techniques obtain

$$\psi(x,y,0) \xrightarrow[t]{} \psi(x,y,t) = A\tfrac{1}{1+i(t/T)}\exp\Big\{-\tfrac{a(x^2+y^2)}{1+i(t/T)}\Big\} \quad : \quad T \equiv m/2a\hbar$$

which they use to demonstrate the characteristic *temporal diffusion of initially localized quantum states*. Exactly the same mathematics lies at the base of the "theory of Gaussiam beams." Suppose it to be the case that

$$\psi(x,y,z) = Be^{-a(x^2+y^2)} \quad \text{at } z = 0$$

The exact solution of (420) is given then at other axial points z by[258]

$$\begin{aligned}\psi(x,y,z) &= B\tfrac{1}{1-iz/Z}e^{-a(x^2+y^2)/(1-iz/Z)} \quad : \quad Z \equiv k/2a\\ &= B\tfrac{1}{1+(z/Z)^2}[1+i(z/Z)]\exp\Big\{-ar^2\tfrac{1}{1+(z/Z)^2}[1+i(z/Z)]\Big\}\end{aligned}$$

$$[1+i(z/Z)] = \sqrt{1+(z/Z)^2}\,e^{i\Phi} \text{ with } \Phi \equiv \arctan(z/Z)$$

$$= \tfrac{B}{\sqrt{1+(z/Z)^2}}\exp\Big\{-ar^2\tfrac{1}{1+(z/Z)^2}\Big\}\exp\Big\{i\big[\Phi(z) - ar^2\tfrac{z/Z}{1+(z/Z)^2}\big]\Big\}$$

$$r^2 \equiv x^2 + y^2$$

We are brought thus to a beam of the design

$$\varphi(t,x,y,z) \sim \tfrac{1}{\rho(z)}\exp\Big\{-\big[\tfrac{r}{\rho(z)}\big]^2\Big\}\cdot e^{i\big[\omega t - kz + \Phi(z) - (r/\rho)^2(z/Z)\big]} \tag{421}$$

where

$$\rho(z) \equiv \sqrt{\frac{1+(z/Z)^2}{a}} \quad \text{describes the "spot radius" at } z$$

Evidently

$$\rho_{\text{min}} \equiv \rho_0 = \rho(0) = \sqrt{1/a} \quad : \quad \text{called the "beam waist"}$$

and at this point the a-notation—a relic of Griffiths' discussion of another subject—has outworn its usefulness: we agree henceforth to write $1/\rho_0^2$ in place of a. In this new notation we have

$$\rho(z) = \rho_0\sqrt{1+(z/Z)^2} \qquad i.e., \qquad (\rho/\rho_0)^2 - (z/Z)^2 = 1 \tag{422}$$

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[257] See, for example, David Griffiths, *Introduction to Quantum Mechanics* (1995), page 50: Problem 2.22.

[258] PROBLEM 69.

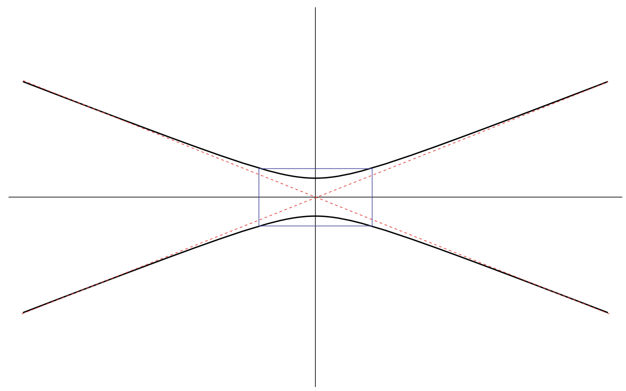


FIGURE 98: *Graph of the function* $\rho(z) = \rho_0\sqrt{1 + (z/Z)^2}$ *that lends the Gaussian beam its hyperbolic profile. The asymptotes are shown in red. The blue box is of length* L. *Its ends are positioned at* $z = \pm Z$, *where the spot radius has grown from* ρ_0 *to* $\sqrt{2}\rho_0$. *The figure was drawn with* $Z/\rho_0 = 10$, *and is in that respect misleading: in realistic cases* $Z/\rho_0 \sim 10^4$ *and the angle between the asymptotes (beam divergence) is* much(!) *reduced*.

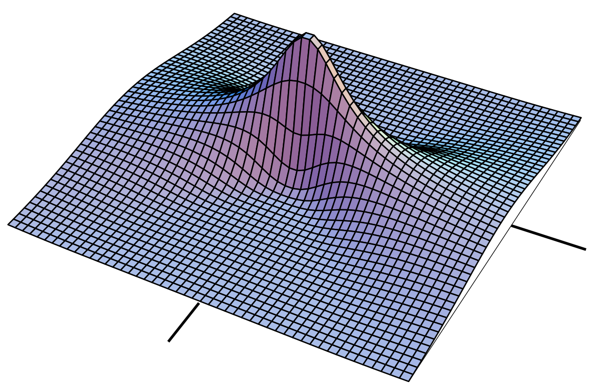


FIGURE 99: *Graph of the factor that, according to* (421), *controls the amplitude of a Gaussian beam. The values assigned to* ρ_0 *and* Z *are the same as those in the preceding figure, and are unrealistic in the sense already explained. The running-wave modulation would be much too finely detailed to be displayed at the same scale.*

which shows that the growth of the spot radius is hyperbolic (see Figure 98 & Figure 99), with asymptotes

$$\rho_{\text{asymptotic}} = \pm(\rho_0/Z)z$$

the slopes of which are typically very shallow: from the definition

$$Z \equiv \tfrac{1}{2}L \equiv k\rho_0^2/2 = \pi\rho_0^2/\lambda$$

we have

$$\left.\begin{aligned} \text{beam waist} &= 0.3989\sqrt{L\lambda} \\ \text{beam divergence} &= 0.7978\sqrt{\lambda/L} \end{aligned}\right\} \tag{423}$$

where the numerics arise from $\sqrt{1/2\pi}$ and $\sqrt{2/\pi}$ respectively. In a typical case $L \sim 1$ meter and $\lambda \sim 7.0\times10^{-7}$ meter, giving

$$\begin{aligned} \text{beam waist} &= 0.33 \text{ mm} \\ \text{beam divergence} &= 6.67\times10^{-4} \text{ (dimensionless)} \end{aligned}$$

Such a beam must travel about 15 meters for the spot radius to grow to 1 cm.

Looking back again to (421), we set $r = 0$ and find that the

$$\text{axial phase at } z = \omega t - kz + \arctan(z/Z)$$

Arguing from $\frac{d}{dt}(\text{axial phase at } z) = 0$ we compute

$$\begin{aligned} \text{phase velocity at } z &= \left[k - \tfrac{Z}{Z^2+z^2}\right]^{-1}\cdot\omega \\ &= \left[1 - \tfrac{Z\lambda}{2\pi(Z^2+z^2)}\right]^{-1}\cdot c \quad \text{by } Z/k = Z\lambda/2\pi = \tfrac{1}{2}\rho_0^2 \\ &= \left\{1 + \tfrac{\lambda}{2\pi Z} + \left(\tfrac{\lambda}{2\pi Z}\right)^2 + \cdots\right\}\cdot c \gtrsim c \quad \text{at } z = 0 \\ &\downarrow \\ &= c \quad \text{as } z \to \infty \end{aligned}$$

—the interesting point being that as z becomes large the axial phase velocity approaches c *from above*. Looking next to the *geometry of the near-axial equiphase surfaces*, we study

$$kz - \arctan(z/Z) + \frac{r^2}{\rho_0^2[1+(z/Z)^2]}(z/Z) = kz_0 - \arctan(z_0/Z) \tag{424}$$

where z_0 marks the point at which the surface in question intersects the z-axis. Taking both r^2 and $z_0 - z$ to be small and asking *Mathematica* to develop the arctan as a power series in $(z - z_0)$, we obtain

$$\frac{r^2}{\rho_0^2[1+(z_0/Z)^2]}(z_0/Z) = \left\{k - \frac{1}{Z[1+(z_0/Z)^2]}\right\}(z_0 - z) + \cdots \tag{425}$$

Define R in such a way that

$$\frac{k}{2R} \equiv \frac{1}{\rho_0^2[1+(z_0/Z)^2]}(z_0/Z)$$

which is to say: let $R \equiv z[1+(Z/z)^2]$, so that the expression on the left side of

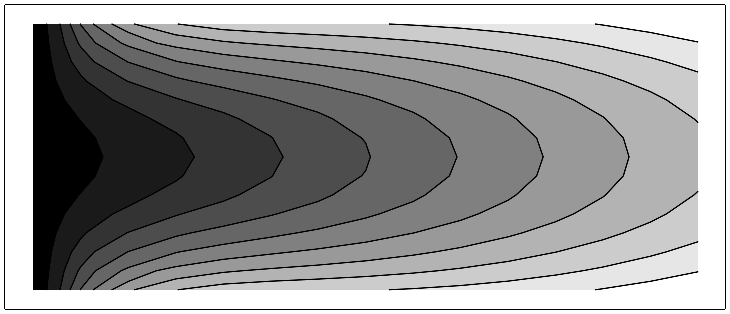


FIGURE 100: *Equiphase contours, taken from the expression on the left side of* (424).

(425) can be written $kr^2/2R$. Next, notice that

$$\frac{1}{Z[1+(z_0/Z)^2]} < \frac{1}{Z} = \frac{2}{L} \ll \pi\frac{2}{\lambda} = k$$

so the second term in braces can be abandoned, giving (see Figure 100)

$$z_0 - z = (1/2R)(x^2+y^2) \ : \ \begin{cases}\text{parabola-of-revolution, opening}\\ \text{to the left, with apex at } z_0\end{cases} \tag{425}$$

That

$$R = \text{radius of curvature at the apex}$$

follows from the observations (i) that

$$[z-(z_0-R)]^2 + x^2 + y^2 = R^2 \tag{426}$$

describes a sphere of radius R that is centered on the z-axis and intersects that axis at $z = z_0$ and $z = z_0 - 2R$, and (ii) that expansion of (426) gives back (425) if a small $(z_0 - z)^2$-term is abandoned. This information might be used to design the concave mirrors placed at the ends of a "Gaussian laser."

The Gaussian beam discussed above can be used as the "seed" from which to grow an infinite population of "Gaussian beams of higher order." These (at least those of lower order) are of physical importance when taken individually, and collectively enable one (by weighted superposition) to fabricate beams of unlimited variety. The generative idea is quite elementary

> If φ is a solution of $\Box\varphi = 0$ and if $\mathcal{D}$ is a differential operator that *commutes* with $\Box$
>
> $$\mathcal{D}\Box = \Box\mathcal{D}$$
>
> then so also is $\mathcal{D}\varphi$ a solution.

but must be adapted to the approximation scheme that was seen on page 322 to lie at the base of Gaussian beam theory: we write

$$\varphi(t,x,y,z) = e^{i(\omega t - kx)} \cdot \psi(x,y,z)$$

and require that ψ be an exact solution of the "Schrödinger equation"[259]

$$\Big\{(\tfrac{\partial}{\partial x})^2+(\tfrac{\partial}{\partial y})^2\Big\}\psi(x,y,z)=i(4Z/\rho_0^2)\tfrac{\partial}{\partial z}\psi(x,y,z) \tag{427}$$

Taking from page 323 the demonstrably exact solution

$$\psi_{00}=\rho_0\,\frac{1}{\rho_0[1-iz/Z]}\,\exp\Big\{-\frac{x^2+y^2}{\rho_0^2[1-iz/Z]}\Big\}$$

—which by $\rho_0[1-iz/Z]=\rho_0\sqrt{1+(z/Z)^2}\,e^{-i\arctan(z/Z)}\equiv\rho(z)e^{-i\Phi(z)}$ can also be written

$$\begin{aligned}&=\rho_0\,\frac{1}{\rho(z)}e^{i\Phi(z)}\cdot\exp\Big\{-\Big[\frac{x}{\sigma(z)}\Big]^2-\Big[\frac{y}{\sigma(z)}\Big]^2\Big\}\\ &\qquad\qquad\qquad\qquad\sigma(z)\equiv\sqrt{\rho_0\rho(z)}\,e^{-i\frac{1}{2}\Phi}\\ &=\rho_0\,\tfrac{1}{\rho}e^{i\Phi}\cdot e^{-\xi^2-\eta^2}\quad:\quad\xi\equiv x/\sigma\text{ and }\eta\equiv y/\sigma\end{aligned}$$

—as our "seed," we harvest this fairly natural fruit:

$$\begin{aligned}\psi_{mn}&\equiv\big(-\rho_0\tfrac{\partial}{\partial x}\big)^m\big(-\rho_0\tfrac{\partial}{\partial y}\big)^n\,\psi_{00}\\ &=\rho_0\tfrac{1}{\rho}e^{i\Phi}\big(-\rho_0\tfrac{1}{\sigma}\tfrac{\partial}{\partial\xi}\big)^m\big(-\rho_0\tfrac{1}{\sigma}\tfrac{\partial}{\partial\eta}\big)^n e^{-\xi^2-\eta^2}\\ &=\rho_0\tfrac{1}{\rho}e^{i\Phi}\big(\rho_0\tfrac{1}{\sigma}\big)^{m+n}\big(-\tfrac{\partial}{\partial\xi}\big)^m\big(-\tfrac{\partial}{\partial\eta}\big)^n e^{-\xi^2-\eta^2}\\ &=\big(\rho_0\tfrac{1}{\rho}\big)^{1+\frac{1}{2}(m+n)}\,e^{i[1+\frac{1}{2}(m+n)]\Phi}H_m(\xi)H_n(\eta)\cdot e^{-\xi^2-\eta^2}\end{aligned} \tag{428}$$

In the final line we have recalled[260] Rodrigues' construction

$$H_m(\xi)=e^{\xi^2}\big(-\tfrac{\partial}{\partial\xi}\big)^m e^{-\xi^2}$$

of the Hermite polynomials:

$$\begin{aligned}H_0(\xi)&=1\\ H_1(\xi)&=2\xi\\ H_2(\xi)&=4\xi^2-2\\ H_3(\xi)&=8\xi^3-12\xi\\ H_4(\xi)&=16\xi^4-48\xi^2+12\\ &\vdots\end{aligned}$$

That the functions ψ_{mn} constructed in this way do in fact exactly satisfy (427) can be demonstrated (for small m, n) by *Mathematica*-assisted calculation, but that they *must* do so follows transparently from the observation that

$$\tfrac{\partial}{\partial x}\text{ and }\tfrac{\partial}{\partial y}\text{ commute with }\Big\{(\tfrac{\partial}{\partial x})^2+(\tfrac{\partial}{\partial y})^2\Big\}-i(4Z/\rho_0^2)\tfrac{\partial}{\partial z}$$

---

[259] This is just (420) with $k\mapsto 2aZ=2Z/\rho_0^2$.

[260] See, for example, Chapter 24 in J. Spanier & K. B. Oldham, *An Atlas of Functions* (1987).

The Gaussian factor

$$e^{-\xi^2-\eta^2} = \exp\left\{-\frac{x^2+y^2}{\rho^2(z)}[1+i(z/Z)]\right\}$$

is a shared feature of *all* the ψ_{mn}-functions, which give rise therefore to identical populations of equiphase surfaces (Figure 100). Using *Mathematica*'s

`HermiteH[n,x]`

command to evaluate the complex prefactors

$$g_{mn}(x,y) \equiv \big(\rho_0\tfrac{1}{\rho}\big)^{1+\frac{1}{2}(m+n)}\,e^{i[1+\frac{1}{2}(m+n)]\Phi}H_m\big(\tfrac{x}{\sqrt{\rho_0\rho}}e^{i\frac{1}{2}\Phi}\big)H_n\big(\tfrac{y}{\sqrt{\rho_0\rho}}e^{i\frac{1}{2}\Phi}\big)$$

in some low-order cases, we find

$$\begin{aligned}
g_{00} &= (\rho_0/\rho)e^{i\Phi}\\
g_{10} &= (\rho_0/\rho)2(x/\rho)e^{2i\Phi}\\
&\ \ \vdots\\
g_{20} &= (\rho_0/\rho)4(x/\rho)^2e^{3i\Phi} - 2(\rho_0/\rho)^2e^{2i\Phi}\\
g_{11} &= (\rho_0/\rho)\big\{2(x/\rho)\big\}\big\{2(y/\rho)\big\}e^{3i\Phi}\\
&\ \ \vdots\\
g_{30} &= (\rho_0/\rho)8(x/\rho)^3e^{4i\Phi} - 12(\rho_0/\rho)^2(x/\rho)e^{3i\Phi}\\
g_{21} &= (\rho_0/\rho)\big\{4(x/\rho)^2 2(y/\rho)e^{4i\Phi} - 2(\rho_0/\rho)^2e^{3i\Phi}\big\}\big\{2(y/\rho)\big\}\\
&\ \ \vdots
\end{aligned}$$

The red terms depart from the result asserted by Kogelnik & Li and quoted by Svelto[256]:

$$g_{mn} = (\rho_0/\rho)H_m(x/\rho)H_n(y/\rho)\,e^{i[1+m+n]\Phi}$$

Their results[261] and mine are, however, in precise agreement at $z = 0$, where $\rho = \rho_0$ and $\Phi = 0$ give

$$\psi_{mn}(x,y,0) = H_m(x/\rho_0)H_n(y/\rho_0)\exp\left\{-\frac{x^2+y^2}{\rho_0^2}\right\}$$

This striking result acquires special interest from the orthogonality relation

$$\int_{-\infty}^{+\infty} H_\mu(u)H_\nu(u)e^{-u^2}\,du = \sqrt{\pi}\mu!2^\mu\delta_{\mu\nu}$$

---

[261] ...which are not incorrect (as I for awhile supposed) but refer to a *distinct population* of beam modes: the point is developed in §§3 & 4 of a companion essay "Toward an exact theory of lightbeams" (2002).

For if we introduce the "normalized Gaussian beam functions"

$$\Psi_{mn}(x,y,z) \equiv \tfrac{1}{\rho_0\sqrt{m!2^m n!2^n\pi}}\psi_{mn}(x,y,z)$$

then we have

$$\int_{-\infty}^{+\infty}\!\!\int_{-\infty}^{+\infty} \Psi_{\mu\nu}(x,y,0)\Psi_{mn}(x,y,0)\exp\left\{\frac{x^2+y^2}{\rho_0^2}\right\}dxdy = \delta_{\mu m}\delta_{\nu n}$$

which we can use to evaluate the coefficients c_{mn} that enter into the description

$$\psi(x,y,0) = \sum_{m,n} c_{mn}\Psi_{mn}(x,y,0)$$

of *beam structure at the waist*. We then write

$$\begin{aligned}\varphi(t,x,y,z) &= e^{i(\omega t-kz)}\cdot\sum_{m,n} c_{mn}\Psi_{mn}(x,y,z) \\ &= \sum_{\text{modes}} \text{Gaussian beams of various “modes” (identified by } m,n)\end{aligned} \tag{429}$$

to describe the generalized Gaussian beam possessing that prescribed structure at the waist.

Physically more realistic beam models would be obtained if we

- used the mechanism described on page 321 to *turn the beam on/off* (this would entail loss of strict monochromaticity)
- constructed *statistical linear combinations* of such beams.

But the beams thus constructed could not possibly describe laser beams: they are *scalar* beams ("acoustic" beams), whereas physical laser beams must be endowed with the transverse vectorial properties known to be characteristic of all electromagnetic radiation. This is a circumstance we were content to set aside on page 322, but would like now to find some way to accommodate. I invite you to turn on *Mathematica* and follow along...

Exponential solutions of the "Schrödinger equation" (427) can be described

$$\exp\left\{i\left[-px-qy+\frac{p^2+q^2}{4Z/\rho_0^2}z\right]\right\} \quad : \quad \text{all real } p,q$$

and minimal tinkering leads to the discovery that

$$\begin{aligned}\iint_{-\infty}^{+\infty} \tfrac{\rho_0^2}{4\pi}e^{-\frac{1}{4}\rho_0^2(p^2+q^2)}\cdot\exp\left\{i\left[-px-qy+\frac{p^2+q^2}{4Z/\rho_0^2}z\right]\right\}dpdq \\ = \frac{1}{[1-iz/Z]}\exp\left\{-\frac{x^2+y^2}{\rho_0^2[1-iz/Z]}\right\} \\ = \psi_{00}(x,y,z) \text{ of page 327}\end{aligned} \tag{430.1}$$

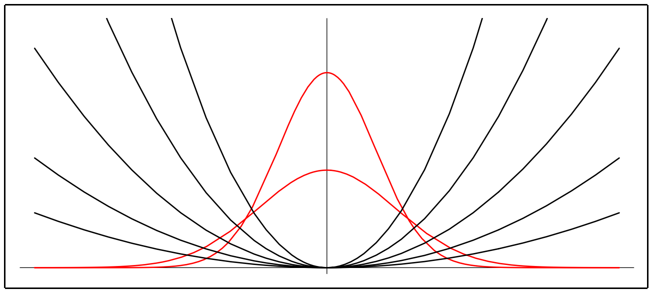


FIGURE 101: *As* ρ_0 *increases the* *Gaussian* $g = \frac{\rho_0^2}{4\pi}e^{-\frac{1}{4}\rho_0^2(p^2+q^2)}$ *becomes narrower, while at higher frequencies the parabolic term* $f = \frac{1}{4\pi}(p^2+q^2)\lambda$ *becomes shallower. At sufficiently high frequencies the Gaussian discriminates against the* (p,q)*-values where* f *departs significantly from zero, and it is this circumstance that justifies the approximation upon which Gaussian beam theory is based.*

But (see again the bottom of page 323) $4Z/\rho_0^2 = 2k$ so we have

$$\begin{aligned}\varphi_{00}(t,x,y,z) &= e^{i(\omega t - kz)}\cdot\psi_{00}(x,y,z) \quad \text{with} \quad \omega = kc \\ &= \iint_{-\infty}^{+\infty} \tfrac{\rho_0^2}{4\pi}e^{-\frac{1}{4}\rho_0^2(p^2+q^2)}\cdot\exp\Big\{i\Big[\omega t - px - qy - \Big(k - \frac{p^2+q^2}{2k}\Big)z\Big]\Big\}\,dpdq\end{aligned} \tag{430.2}$$

From

$$(\omega/c = k)^2 - p^2 - q^2 - \Big(k - \frac{p^2+q^2}{2k}\Big)^2 = -\Big(\frac{p^2+q^2}{2k}\Big)^2$$

we see that the wave vector

$$\begin{pmatrix}k^0\\k^1\\k^2\\k^3\end{pmatrix} = \begin{pmatrix}\omega/c\\p\\q\\k-[(p^2+q^2)/2k]\end{pmatrix}$$

is <u>not null (as the wave equation $\square\varphi_{00} = 0$ requires) but spacelike</u>: we encounter here the **force of the approximation** made on page 322. Notice in this connection that (because $k = 2\pi/\lambda$)

$$\frac{p^2+q^2}{2k} = \frac{p^2+q^2}{4\pi}\cdot\lambda \quad : \quad \text{vanishes at high frequencies}$$

so for given ρ_0 the approximation becomes better and better as $\lambda\uparrow\infty$, while for given λ the approximation becomes progressively better as the Gaussian $e^{-\rho_0^2(p^2+q^2)}$ becomes narrower; *i.e.*, as ρ_0 becomes larger (see the figure).

We want now to extract from (430) the description of a Gaussian *light* beam. To that end we must replace the scalar plane waves encountered at (430) with *electromagnetic* plane waves, and that effort presents certain problems. I will carry this discussion only far enough to expose the problems and some

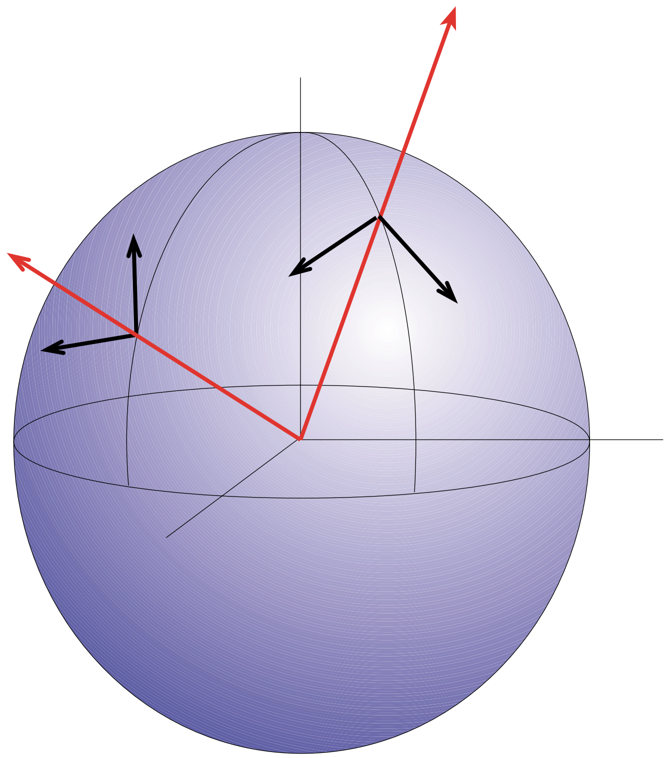


FIGURE 102: *To each vector* $\boldsymbol{k}(p,q)$ *we associate a pair of unit vectors* $\boldsymbol{e}(p,q)$ *and* $\boldsymbol{f}(p,q)$ *in such a way that* $\{\hat{\boldsymbol{k}},\boldsymbol{e},\boldsymbol{f}\equiv\hat{\boldsymbol{k}}\times\boldsymbol{e}\}$ *comprise an orthonormal triad. Let* $\boldsymbol{k}_1$ *and* $\boldsymbol{k}_2$ *be two such wave vectors. If* $\boldsymbol{k}_1$ *and* $\boldsymbol{k}_2$ *are not parallel then specification of* $\boldsymbol{e}_1$ *exerts no geometrically compelling constraint on the selection of* $\boldsymbol{e}_2$.

points of principle: to carry it farther would to risk becoming lost in bewildering detail.

Let (430.2) be notated

$$\varphi_{00}(t,\boldsymbol{x}) = \iint_{-\infty}^{+\infty} \tfrac{\rho_0^2}{4\pi} e^{-\frac{1}{4}\rho_0^2(p^2+q^2)}\cdot\exp\Big\{i\Big[\omega t - \boldsymbol{k}(p,q)\cdot\boldsymbol{x}\Big]\Big\}\,dpdq$$

with

$$\boldsymbol{k}(p,q)\equiv\begin{pmatrix} p\\ q\\ k-[(p^2+q^2)/2k]\end{pmatrix} = k\sqrt{1+\big(\tfrac{p^2+q^2}{2k^2}\big)^2}\cdot\hat{\boldsymbol{k}}(p,q)$$

To every such $\boldsymbol{k}(p,q)$ assign unit vectors $\boldsymbol{e}(p,q)$ and $\boldsymbol{f}(p,q)$ in such a way that $\{\hat{\boldsymbol{k}}(p,q),\boldsymbol{e}(p,q),\boldsymbol{f}(p,q)\}$ is an orthonormal triad. The first point of interest is that *this can be accomplished in infinitely many ways*: the triads erected at the points of (p, q)-space are independent creations (see Figure 102). Given such an assignment, form

$$\boldsymbol{E}(t,\boldsymbol{x}) = \iint_{-\infty}^{+\infty} \tfrac{\rho_0^2}{4\pi} e^{-\frac{1}{4}\rho_0^2(p^2+q^2)} \cdot \boldsymbol{\mathcal{E}}(p,q) \exp\Big\{ i\Big[\omega t - \boldsymbol{k}(p,q)\cdot\boldsymbol{x}\Big]\Big\}\, dpdq \tag{431}$$

with

$$\boldsymbol{\mathcal{E}}(p,q) \equiv \mathcal{E}_1(p,q)\boldsymbol{e}(p,q) + \mathcal{E}_2(p,q)e^{i\delta(p,q)}\boldsymbol{f}(p,q)$$

where further arbitrariness enters into the design of the functions $\mathcal{E}_1(p,q)$, $\mathcal{E}_2(p,q)$ and $\delta(p,q)$. The constructions

$$\begin{aligned}
\boldsymbol{E}_{p,q}(t,\boldsymbol{x}) &= \big\{\mathcal{E}_1(p,q)\boldsymbol{e}(p,q) + \mathcal{E}_2(p,q)e^{i\delta(p,q)}\boldsymbol{f}(p,q)\big\} \exp\Big\{ i\Big[\omega t - \boldsymbol{k}(p,q)\cdot\boldsymbol{x}\Big]\Big\}\\
\boldsymbol{B}_{p,q}(t,\boldsymbol{x}) &= \boldsymbol{k}(p,q)\times\boldsymbol{E}_{p,q}(t,\boldsymbol{x})\\
&= \big\{\mathcal{E}_1(p,q)\boldsymbol{f}(p,q) - \mathcal{E}_2(p,q)e^{i\delta(p,q)}\boldsymbol{e}(p,q)\big\} \exp\Big\{ i\Big[\omega t - \boldsymbol{k}(p,q)\cdot\boldsymbol{x}\Big]\Big\}
\end{aligned}$$

serve—in the approximation $(p^2+q^2)/2k \approx 0$—to associate a monochromatic polarized electromagnetic plane wave (propagating in the direction $\hat{\boldsymbol{k}}(p,q)$) with each point of (p,q)-space, and (431) describes a Gauss-weighted superposition of such plane waves. The "bewildering detail" to which I have referred arises (even in the simplest of the cases I have studied) when one undertakes to do the integration.

"Electromagnetic Gaussian beams" exist, by this account, in infinite variety. Evidently one must look to the finer particulars of laser design to discover how the physical device "selects among options," how to construct acceptable models of the laser beams encountered in laboratories.

The fields

$$\begin{aligned}
\boldsymbol{E}(t,\boldsymbol{x}) &= \iint_{-\infty}^{+\infty} \tfrac{\rho_0^2}{4\pi} e^{-\frac{1}{4}\rho_0^2(p^2+q^2)} \cdot \boldsymbol{E}_{p,q}(t,\boldsymbol{x})\, dpdq\\
\boldsymbol{B}(t,\boldsymbol{x}) &= \iint_{-\infty}^{+\infty} \tfrac{\rho_0^2}{4\pi} e^{-\frac{1}{4}\rho_0^2(p^2+q^2)} \cdot \boldsymbol{B}_{p,q}(t,\boldsymbol{x})\, dpdq
\end{aligned}$$

possess a property worthy of notice which I will expose by considering the superposition of only *two* electromagnetic plane waves. Let

$$\begin{aligned}
\boldsymbol{E}_1(t,\boldsymbol{x}) &= \boldsymbol{\mathcal{E}}_1 \exp\Big\{ i(\omega t - \boldsymbol{k}_1\cdot\boldsymbol{x})\Big\} \qquad : \quad \boldsymbol{\mathcal{E}}_1 \perp \boldsymbol{k}_1\\
\boldsymbol{B}_1(t,\boldsymbol{x}) &= \hat{\boldsymbol{k}}_1\times\boldsymbol{\mathcal{E}}_1 \exp\Big\{ i(\omega t - \boldsymbol{k}_1\cdot\boldsymbol{x})\Big\}
\end{aligned}$$

describe one monochromatic plane wave, and

$$\begin{aligned}
\boldsymbol{E}_2(t,\boldsymbol{x}) &= \boldsymbol{\mathcal{E}}_2 \exp\Big\{ i(\omega t - \boldsymbol{k}_2\cdot\boldsymbol{x} + \delta)\Big\} \quad : \quad \boldsymbol{\mathcal{E}}_2 \perp \boldsymbol{k}_2\\
\boldsymbol{B}_2(t,\boldsymbol{x}) &= \hat{\boldsymbol{k}}_2\times\boldsymbol{\mathcal{E}}_2 \exp\Big\{ i(\omega t - \boldsymbol{k}_2\cdot\boldsymbol{x} + \delta)\Big\}
\end{aligned}$$

describe another. Let $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ and $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$. Then

$$\boldsymbol{E}\cdot\boldsymbol{B} = \underbrace{\big\{\boldsymbol{\mathcal{E}}_1\cdot(\hat{\boldsymbol{k}}_2\times\boldsymbol{\mathcal{E}}_2) + \boldsymbol{\mathcal{E}}_2\cdot(\hat{\boldsymbol{k}}_1\times\boldsymbol{\mathcal{E}}_1)\big\}}_{\substack{= (\hat{\boldsymbol{k}}_1 - \hat{\boldsymbol{k}}_2)\cdot(\boldsymbol{\mathcal{E}}_1\times\boldsymbol{\mathcal{E}}_2) \\ \neq \boldsymbol{0} \text{ except under obvious special conditions}}} \exp\Big\{i(2\omega t - [\boldsymbol{k}_1 + \boldsymbol{k}_2]\cdot\boldsymbol{x} + \delta)\Big\}$$

shows that, in general, *superimposed plane waves do not share the* $\boldsymbol{E}\perp\boldsymbol{B}$ *condition characteristic of individual plane waves*. In particular: $\boldsymbol{E}\perp\boldsymbol{B}$ will not be found in the superpositions that produce "beams."

Let us look to a concrete example. Working from

$$\hat{\boldsymbol{k}}(p,q) \equiv k^{-1}\begin{pmatrix} p \\ q \\ k - [(p^2+q^2)/2k] \end{pmatrix} \quad \text{in the approximation } \big(\tfrac{p^2+q^2}{2k^2}\big)^2 \approx 0$$

we complete the dimensionless orthonormal triad by writing

$$\begin{aligned} \boldsymbol{e}(p,q) &\equiv \tfrac{1}{\sqrt{p^2+q^2}}\begin{pmatrix} +q \\ -p \\ 0 \end{pmatrix} \\ \boldsymbol{f}(p,q) &\equiv \tfrac{1}{2k^2\sqrt{p^2+q^2}}\begin{pmatrix} p[2k^2-(p^2+q^2)] \\ q[2k^2-(p^2+q^2)] \\ -2k(p^2+q^2) \end{pmatrix} = \hat{\boldsymbol{k}}(p,q)\times\boldsymbol{e}(p,q) \end{aligned}$$

and to achieve tractable integrals set

$$\begin{aligned} \mathcal{E}_1(p,q) &= \mathcal{E}_1\ell\cdot\sqrt{p^2+q^2} \\ \mathcal{E}_2(p,q) &= \mathcal{E}_2\ell\cdot\sqrt{p^2+q^2} \\ \delta(p,q) &= \text{constant} \end{aligned}$$

Here ℓ (introduced to cancel the physical dimension of $\sqrt{p^2+q^2}$) is a constant of arbitrary value and the dimensionality of length, so $\mathcal{E}_1\ell$ and $\mathcal{E}_2\ell$ have the dimensionality of electric potential. Working from (431) with $k = 2Z/\rho_0^2$ and

$$\begin{aligned} \boldsymbol{\mathcal{E}}(p,q) &= \mathcal{E}_1\ell\begin{pmatrix} +q \\ -p \\ 0 \end{pmatrix} + \mathcal{E}_2\ell e^{i\delta}\begin{pmatrix} p[1-(p^2+q^2)/2k^2] \\ q[1-(p^2+q^2)/2k^2] \\ -(p^2+q^2)/k \end{pmatrix} \\ \boldsymbol{\mathcal{B}}(p,q) &= \mathcal{E}_1\ell\begin{pmatrix} p[1-(p^2+q^2)/2k^2] \\ q[1-(p^2+q^2)/2k^2] \\ -(p^2+q^2)/k \end{pmatrix} - \mathcal{E}_2\ell e^{i\delta}\begin{pmatrix} +q \\ -p \\ 0 \end{pmatrix} \end{aligned}$$

we entrust the $\iint$'s to *Mathematica*, who supplies

$$\left.\begin{aligned} \boldsymbol{E}(t,\boldsymbol{x}) &= \mathcal{E}_1\ell\,\boldsymbol{e}(t,\boldsymbol{x}) + \mathcal{E}_2\ell e^{i\delta}\,\boldsymbol{f}(t,\boldsymbol{x}) \\ \boldsymbol{B}(t,\boldsymbol{x}) &= \mathcal{E}_1\ell\,\boldsymbol{f}(t,\boldsymbol{x}) - \mathcal{E}_2\ell e^{i\delta}\,\boldsymbol{e}(t,\boldsymbol{x}) \end{aligned}\right\} \tag{432}$$

with

$$\left.\begin{aligned} \boldsymbol{e}(t,\boldsymbol{x}) &= e^{G}\cdot\begin{pmatrix} -\mathcal{A}y \\ +\mathcal{A}x \\ 0 \end{pmatrix} \\ \boldsymbol{f}(t,\boldsymbol{x}) &= e^{G}\cdot\begin{pmatrix} -\mathcal{B}x \\ -\mathcal{B}y \\ \mathcal{C} \end{pmatrix} \end{aligned}\right\} \tag{433}$$

where

$$e^{G} \equiv \exp\Big\{-\tfrac{x^2+y^2}{\rho^2}[1+i(z/Z)]+i[\omega t-kz]\Big\}$$

is familiar already (see again page 323) from the *scalar* theory of Gaussian beams, and where

$$\begin{aligned} \mathcal{A} &= \tfrac{2iZ^2}{\rho_0^2(Z-iz)^2} \\ &\equiv Ae^{i\alpha} \quad \text{with} \quad A=\sqrt{\tfrac{\{0\}^2+\{2Z^2\}^2}{\rho_0^4(Z^2+z^2)^2}} \\ \mathcal{B} &= \tfrac{-2Z\rho_0^2(z+iZ)+iZ^2[r^2-2(z+iZ)^2]}{\rho_0^2(z+iZ)^4} \\ &\equiv Be^{i\beta} \quad \text{with} \quad B=\sqrt{\tfrac{\{\text{stuff}\}^2+\{\text{more stuff}\}^2}{\rho_0^4(Z^2+z^2)^4}} \\ \mathcal{C} &= \tfrac{2Z^2r^2-2\rho_0^2Z(Z-iz)}{\rho_0^2(Z-iz)^3} \\ &\equiv Ce^{i\gamma} \quad \text{with} \quad C=\sqrt{\tfrac{\{\text{stuff}\}^2+\{\text{more stuff}\}^2}{\rho_0^4(Z^2+z^2)^3}} \end{aligned}$$

I have indicated how the invariable reality of A, B and C comes about, but have omitted details too complicated to be informative, and have also omitted (as irrelevant to the purposes at hand) explicit description of the phase factors α, β and γ (which could be expressed as the arctangents of the obvious ratios). Notice that the functions described above depend upon x and y only through $r^2 \equiv x^2+y^2$; they are, in short, *axially symmetric*. Notice also that $[A]=[B]=(\text{length})^{-2}$ while $[C]=(\text{length})^{-1}$.

Returning now with (433) to (432), we have $\boldsymbol{E}(t,\boldsymbol{x})=\boldsymbol{E}_1(t,\boldsymbol{x})+\boldsymbol{E}_2(t,\boldsymbol{x})$ with

$$\begin{aligned} \boldsymbol{E}_1 &= \mathcal{E}_1\ell e^{-(r/\rho)^2}\cdot e^{i\{(\omega t-kz)-(r/\rho)^2(z/Z)\}}\left\{ Ae^{i\alpha}\begin{pmatrix} -y \\ +x \\ 0 \end{pmatrix} \right\} \\ \boldsymbol{E}_2 &= \mathcal{E}_2 e^{i\delta}\ell e^{-(r/\rho)^2}\cdot e^{i\{(\omega t-kz)-(r/\rho)^2(z/Z)\}}\left\{ Be^{i\beta}\begin{pmatrix} -x \\ -y \\ 0 \end{pmatrix}+Ce^{i\gamma}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

The associated magnetic fields are

$$\boldsymbol{B}_1 = \quad \mathcal{E}_1\ell e^{-(r/\rho)^2}\cdot e^{i\{(\omega t-kz)-(r/\rho)^2(z/Z)\}}\left\{\quad Be^{i\beta}\begin{pmatrix}-x\\-y\\0\end{pmatrix}+Ce^{i\gamma}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$\boldsymbol{B}_2 = \mathcal{E}_2 e^{i\delta}\ell e^{-(r/\rho)^2}\cdot e^{i\{(\omega t-kz)-(r/\rho)^2(z/Z)\}}\left\{-Ae^{i\alpha}\begin{pmatrix}-y\\+x\\0\end{pmatrix}\right\}$$

It is understood that to extract the *physical* fields, and <u>before</u> we assemble such quadratic constructions as (field)$\cdot$(field) and (field)$\times$(field), we must make the replacements

$$e^{i(\text{stuff})}\longmapsto\cos(\text{stuff})$$

That done, we obtain finally

$$\begin{aligned}
\boldsymbol{E}_1 &= \mathcal{E}_1\ell e^{-(r/\rho)^2}\left\{\quad A\cos(\vartheta+\alpha)\begin{pmatrix}-y\\+x\\0\end{pmatrix}\right\}\\
\boldsymbol{E}_2 &= \mathcal{E}_2\ell e^{-(r/\rho)^2}\left\{\quad B\cos(\vartheta+\beta+\delta)\begin{pmatrix}-x\\-y\\0\end{pmatrix}+C\cos(\vartheta+\gamma+\delta)\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\\
\boldsymbol{B}_1 &= \mathcal{E}_1\ell e^{-(r/\rho)^2}\left\{\quad B\cos(\vartheta+\beta)\begin{pmatrix}-x\\-y\\0\end{pmatrix}\qquad+C\cos(\vartheta+\gamma)\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\\
\boldsymbol{B}_2 &= \mathcal{E}_2\ell e^{-(r/\rho)^2}\left\{-A\cos(\vartheta+\alpha+\delta)\begin{pmatrix}-y\\+x\\0\end{pmatrix}\right\}
\end{aligned}$$

with $\vartheta\equiv\omega t-kz-(r/\rho)^2(z/Z)$. It is immediately evident that at every spacetime point

$$\begin{aligned}\boldsymbol{E}_1\perp\boldsymbol{E}_2\,,\quad \boldsymbol{B}_1\perp\boldsymbol{B}_2\\ \boldsymbol{E}_1\perp\boldsymbol{B}_1\,,\quad \boldsymbol{E}_2\perp\boldsymbol{B}_2\end{aligned}$$

but from

$$\begin{aligned}
\boldsymbol{E}\cdot\boldsymbol{B} &= (\boldsymbol{E}_1+\boldsymbol{E}_2)\cdot(\boldsymbol{B}_1+\boldsymbol{B}_2)\\
&= \mathcal{E}_1\mathcal{E}_2\ell^2e^{-2(r/\rho)^2}\Big\{r^2\big[B^2\cos(\vartheta+\beta)\cos(\vartheta+\beta+\delta)\\
&\qquad\qquad -A^2\cos(\vartheta+\alpha)\cos(\vartheta+\alpha+\delta)\big]\\
&\qquad\qquad +C^2\cos(\vartheta+\gamma)\cos(\vartheta+\gamma+\delta)\Big\}\\
&\neq 0 \text{ except under non-obvious special conditions: note, however, that}\\
&\downarrow\\
&= 0\quad\text{as } r\to\infty \text{ because the fields die at points far from the beam axis}
\end{aligned}$$

we see that—consistently with the remark developed on page 332—the net fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are typically *not* perpendicular: at axial points ($r=0$) they are, in fact, *parallel*! The *energy flux* and *momentum density* at the spacetime point are proportional to

$$\boldsymbol{E}\times\boldsymbol{B} = (\boldsymbol{E}_1+\boldsymbol{E}_2)\times(\boldsymbol{B}_1+\boldsymbol{B}_2) \equiv e^{-2(r/\rho)^2}\cdot\boldsymbol{F}$$

where according to *Mathematica*

$$\begin{aligned}
F_1 = {}& xAC\big[\mathcal{E}_1^2\cos(\vartheta+\alpha)\cos(\vartheta+\gamma)+\mathcal{E}_2^2\cos(\vartheta+\alpha+\delta)\cos(\vartheta+\gamma+\delta)\big]\\
&-yBC\mathcal{E}_1\mathcal{E}_2\big[\cos(\vartheta+\beta+\delta)\cos(\vartheta+\gamma)-\cos(\vartheta+\beta)\cos(\vartheta+\gamma+\delta)\big]\\
F_2 = {}& yAC\big[\mathcal{E}_1^2\cos(\vartheta+\alpha)\cos(\vartheta+\gamma)+\mathcal{E}_2^2\cos(\vartheta+\alpha+\delta)\cos(\vartheta+\gamma+\delta)\big]\\
&+xBC\mathcal{E}_1\mathcal{E}_2\big[\cos(\vartheta+\beta+\delta)\cos(\vartheta+\gamma)-\cos(\vartheta+\beta)\cos(\vartheta+\gamma+\delta)\big]\\
F_3 = {}& r^2AB\big[\mathcal{E}_1^2\cos(\vartheta+\alpha)\cos(\vartheta+\beta)+\mathcal{E}_2^2\cos(\vartheta+\alpha+\delta)\cos(\vartheta+\beta+\delta)\big]
\end{aligned}$$

This is of the design

$$\begin{aligned}
\boldsymbol{F} &= a\begin{pmatrix}x\\y\\0\end{pmatrix}+b\begin{pmatrix}-y\\+x\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}\\
&= \boldsymbol{F}_{\text{radial}}+\boldsymbol{F}_{\text{tangential}}+\boldsymbol{F}_{\text{axial}}
\end{aligned}$$

where the vectors $\boldsymbol{F}_{\text{radial}}$ stand normal to the z-axis (beam-axis) and are of constant magnitude on circles concentric about that axis, the vectors $\boldsymbol{F}_{\text{tangential}}$ are (also constant on but) tangent to such circles and have ↺ or ↻ handedness according as $b \gtrless 0$, and the vectors $\boldsymbol{F}_{\text{axial}}$ (also constant on such circles) run parallel to the z-axis. The "constants" a, b and c are in fact horribly complicated functions of the variables $\{t,z,r\}$ and of the parameters $\{\omega,\rho_0,Z,\mathcal{E}_1,\mathcal{E}_2,\delta\}$.

We are in position now to state that the momentary *momentum density of the beam field* at any designated point $\boldsymbol{x}$ can be described (see again page 216)

$$\boldsymbol{\mathcal{P}} = \tfrac{1}{c}e^{-2(r/\rho)^2}\boldsymbol{F}$$

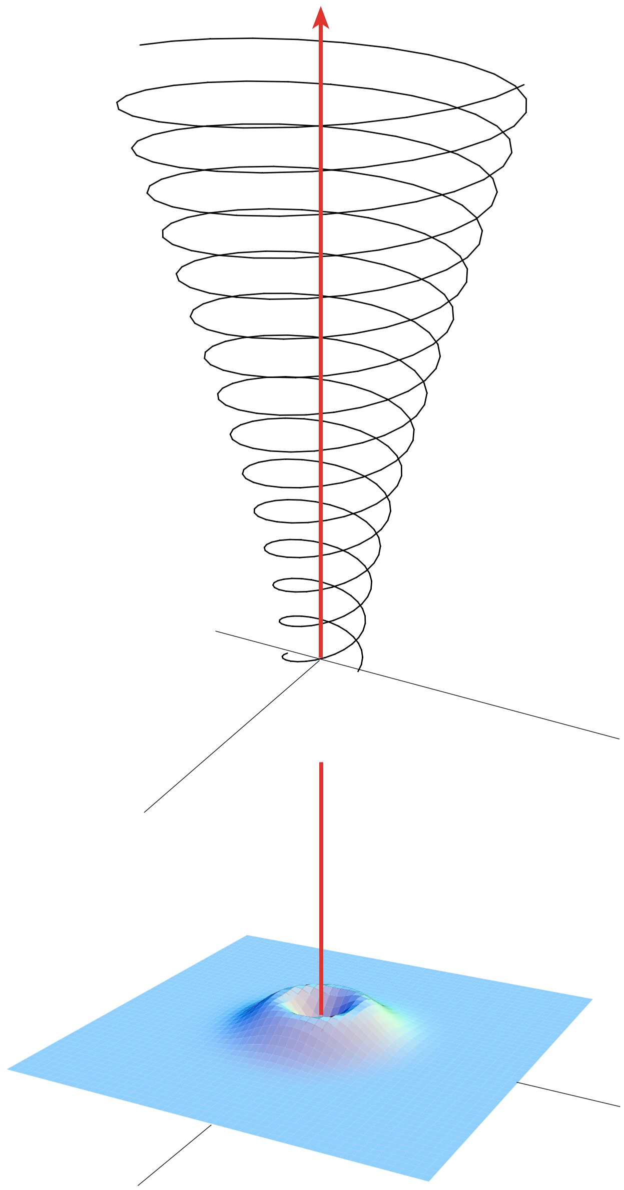
We observe that

- $\boldsymbol{\mathcal{P}}$ vanishes far from the beam axis because of Gaussian attenuation
- $\boldsymbol{\mathcal{P}}$ vanishes *on* the beam axis by the design of a, b and c
- <u>field momentum traces a divergent spiral</u> in the near neighborhood of the beam axis unless $b=0$.

The *angular momentum density* of the beam field is given by[262]

$$\begin{aligned}
\boldsymbol{\mathcal{L}} = \boldsymbol{x}\times\boldsymbol{\mathcal{P}} &= \tfrac{1}{c}e^{-2(r/\rho)^2}\left\{-bz\begin{pmatrix}x\\y\\0\end{pmatrix}+(az-c)\begin{pmatrix}-y\\+x\\0\end{pmatrix}+br^2\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\\
&= \boldsymbol{\mathcal{L}}_{\text{radial}}+\boldsymbol{\mathcal{L}}_{\text{tangential}}+\boldsymbol{\mathcal{L}}_{\text{axial}}
\end{aligned}$$

[262] Don't be confused by the fact that c is used here to mean two entirely different things.



FIGURES 103 & 104: *The upper figure portrays the* <u>*spiroform deployment of the momentum*</u> *in the electromagnetic field of the Gaussian beam described in the text. Displayed below is the resulting angular momentum density (presented as a function of x and y at the beam waist: $z = 0$). The figures show that/why it makes sense to say that "the angular momentum lives at the fringes of the beam."*

The first two components (by an elementary symmetry argument) can make no net contribution to the total angular momentum of the beam, which is given therefore by

$$\boldsymbol{L} = L\begin{pmatrix}0\\0\\1\end{pmatrix} \quad \text{with} \quad L = \iiint \tfrac{1}{c}e^{-2(r/\rho)^2}br^2\,dxdydz$$

We notice that L vanishes if $b = 0$, and that this happens when $\delta = 0$, for in the latter circumstance the equations at near the top of page 336 assume the much-simplified form

$$\begin{aligned}
F_1 &= \ xAC\big[\mathcal{E}_1^2 + \mathcal{E}_2^2\big]\cos(\vartheta+\alpha)\cos(\vartheta+\gamma) + \text{no } y\text{-term}\\
F_2 &= \ yAC\big[\mathcal{E}_1^2 + \mathcal{E}_2^2\big]\cos(\vartheta+\alpha)\cos(\vartheta+\gamma) + \text{no } x\text{-term}\\
F_3 &= r^2AB\big[\mathcal{E}_1^2 + \mathcal{E}_2^2\big]\cos(\vartheta+\alpha)\cos(\vartheta+\beta)
\end{aligned}$$

One occasionally encounters the claim that "the angular momentum transported by a laser beam lives at the fringes of the beam," but in support of that claim authors who possess only a *scalar* theory of beams must argue rather vaguely that

i) beam angular momentum must arise from momentum circulation

ii) there can be no circulation *at the axis* of an axially-symmetric beam

iii) all $\boldsymbol{\mathcal{P}}$-circulation must therefore occur between the axis and the remote regions where the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields have fallen off to zero—in short: "at the fringes" of the beam.

My effort has been to carry a *vector* theory of beams far enough to illuminate the details of the matter. Having achieved that objective, I must be content now to abandon my little "electromagnetic theory of beams"... but feel an obligation to list some of the respects in which the theory remains incomplete:

- It should be feasible (by the method sketched on page 321) to turn such beams on and off; *i.e.*, to construct laterally confined quasi-monochromatic Gaussian wavepackets—"classical photons," if you will.
- It should be feasible, moreover, to construct trains of such wavepackets, and to describe the coherence/polarization properties of such trains.
- One would like to be in position to describe the energy, momentum and angular momentum transported by such a "classical photon," and to identify conditions under which they stand in the quantum relationships

$$E = cP = \omega L$$

- To that end one would need to clarify certain salient properties of and interrelationships among the complicated functions a, b and c.
- Identical values of $\mathcal{E}_1$, $\mathcal{E}_2$ and δ were assigned to each of the plane waves from which our Gaussian beams were assembled. Do the Stokes parameters implicit (by (399)) in $\{\mathcal{E}_1, \mathcal{E}_2, \delta\}$ speak usefully about the polarization properties of the *assembled* beam?

It should be borne always in mind that the theory sketched above proceeds from an **inoffensive approximation** (page 322) and—within the bounds of that approximation—from a **convenient specification** (page 333) of the manner in which orthonormal vectors will be attached to $\hat{\boldsymbol{k}}$-vectors (and weighted). The theory is rich enough to support easily the notion of "higher beam modes," but this question remains open: *Is the theory—as I suspect—rich enough to account for the observed properties of the optical beams encountered in laboratories*?

6

SOLUTION OF FIELD EQUATIONS

Construction & application of the electromagnetic propagators

Introduction. Working in the Lorentz gauge, our problem—acquired at (373)—is to describe the *solution* of

$$\square A^\nu = \tfrac{1}{c} j^\nu \tag{434}$$

which results when

i) the source term $j^\nu(x)$ and

ii) initial & boundary conditions

are prescribed. We will have then only to construct $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ to obtain descriptions of the physical fields $\boldsymbol{E}(x)$ and $\boldsymbol{B}(x)$ that arise under the conditions specified. Our problem is made tractable by two circumstances:

- Equations (434) are *uncoupled* (though the j^μ are constrained by charge conservation to satisfy $\partial_\mu j^\mu = 0$ and the A^μ to satisfy the Lorentz gauge condition $\partial_\mu A^\mu = 0$). This means that it is suffient to study the generic equation

$$\square \phi(x) = \rho(x) \tag{435}$$

- Equations (434) are *linear*. This means that we can employ *Green's technique*; *i.e.*, that (recall the discussion on pages 16–17) we can undertake to solve (435) by weighted superposition of the solutions of

$$\square \phi(x) = \delta(x) \tag{436.1}$$

$$\square \phi(x) = 0 \tag{436.2}$$

We anticipate on these grounds that the solution of (435) can be developed

$$\begin{aligned}\phi(x) &= \phi_0(x) + \int G(x-y)\rho(y)\,d^4y \qquad (437)\\ &= \left\{\begin{matrix}\text{solution of the homogeneous equation (436.2)}\\ \text{into which we have folded the } \textit{initial value data}\end{matrix}\right\}\\ &\qquad + \big\{\text{particular solution of (435)}\big\}\end{aligned}$$

and that the physical solutions of (434) admit of similar description:

$$A^\mu(x) = A_0^\mu(x) + \underbrace{\tfrac{1}{c}\int D_{\mathrm{R}}(x-y)\,j^\mu(y)\,d^4y} \qquad (438.1)$$

Here $A_0^\mu(x)$ denotes the <u>field which has evolved from any initially present ambient field</u>, and

$$\equiv A_{\mathrm{R}}^\mu(x) \qquad (438.2)$$

denotes the <u>field generated by past source activity</u> (the subscript $_\mathrm{R}$ stands for "retarded").

We look first to the detailed substance of the preceding rough remarks, and in subsequent sections to a graded sequence of illustrative applications.

1. Green's function techniques in classical electrodynamics: construction of the propagators. I start with remarks that—though they may seem at first to be in mathematical left field—will place us in position to say powerful things about the source-independent term $A_0^\mu(x)$.

If in Gauss' theorem

$$\iiint_{\mathcal{R}} \boldsymbol{\nabla}\cdot\boldsymbol{A}\,d^3x = \iint_{\partial\mathcal{R}} \boldsymbol{A}\cdot\boldsymbol{d\sigma}$$

we set $\boldsymbol{A} = \varphi\boldsymbol{\nabla}\psi$ we obtain

$$\iiint_{\mathcal{R}} \big\{\varphi\nabla^2\psi + \boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\psi\big\}\,d^3x = \iint_{\partial\mathcal{R}} \varphi\boldsymbol{\nabla}\psi\cdot\boldsymbol{d\sigma}$$

from which (interchange φ and ψ, subtract) follows **Green's theorem**

$$\iiint_{\mathcal{R}} \big\{\varphi\nabla^2\psi - \psi\nabla^2\varphi\big\}\,d^3x = \iint_{\partial\mathcal{R}} \big\{\varphi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\varphi\big\}\cdot\boldsymbol{d\sigma}$$

Green's theorem lies at the heart of many notable existence and uniqueness theorems. And it is quite robust: it extends to spaces of any dimension, and of non-Euclidean metric structure. In 4-dimensional spacetime it reads

$$\iiiint_{\mathcal{R}} \big\{\varphi\Box\psi - \psi\Box\varphi\big\}\,d^4x = \iiint_{\partial\mathcal{R}} \big\{\varphi\,\partial^\alpha\psi - \psi\,\partial^\alpha\varphi\big\}\,d\sigma_\alpha \qquad (439)$$

To prepare for the application specifically at hand we

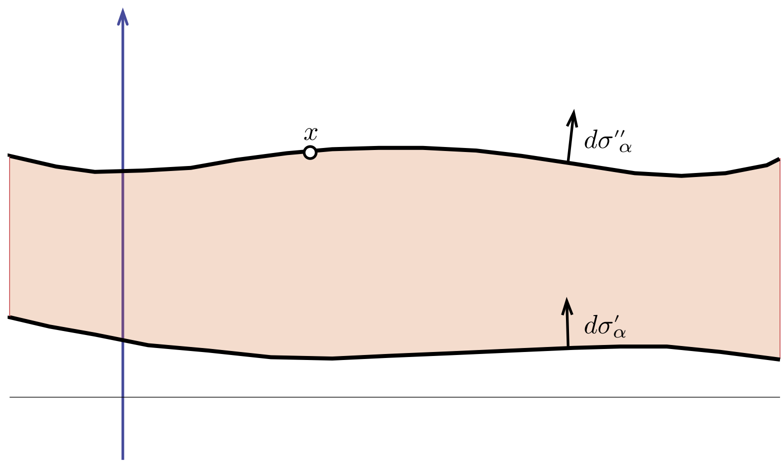


FIGURE 105: *Spacetime sandwich, bounded by surfaces the normals to which are everywhere* <u>*timelike and future directed*</u> *(the former by construction, the latter by convention). The upper and lower spacelike surfaces (or timeslices) σ' and σ'' jointly comprise the boundary $\partial\mathcal{R}$ of the region $\mathcal{R}$ which we bring in the text to a distinctive application of Green's theorem.*

1) assume both φ and ψ to satisfy (436.2): $\Box\varphi = \Box\psi = 0$

2) assume $\mathcal{R}$ to be the disk-like region bounded by the everywhere-spacelike surfaces σ' and σ'', where σ'' contains x—the field point of interest. It is out intention to spread **Cauchy data** (*i.e.*; initial data sufficient to identify/determine a solution) on σ'... like so much peanut butter & jelly.

3) assume the surface differentials $d\sigma'_\alpha$ and $d\sigma''_\alpha$ to be (not "outer-directed" but) <u>future-directed</u> (see the figure).

Green's equation (439), on the strength of those assumptions, becomes

$$0 = \int_{\sigma''} \{\varphi\partial^\alpha\psi - \psi\partial^\alpha\varphi\}\, d\sigma''_\alpha - \int_{\sigma'} \{\varphi\partial^\alpha\psi - \psi\partial^\alpha\varphi\}\, d\sigma'_\alpha$$

or

$$\underbrace{\int_{\sigma''}\Big\{\varphi(x'')\,\partial^\alpha\psi(x'') - \psi(x'')\,\partial^\alpha\varphi(x'')\Big\}\, d\sigma''_\alpha}$$
$$= \int_{\sigma'}\Big\{\varphi(x')\,\partial^\alpha\psi(x') - \psi(x')\,\partial^\alpha\varphi(x')\Big\}\, d\sigma'_\alpha$$

Now $\qquad = \varphi(x) \qquad (440)$

<u>if an appropriately specialized meaning is assigned to ψ</u>. If we agree to write

$\psi(x'') \equiv D_0(x''-x)$ and to interpret x as a "continuously adjustable parameter" then we achieve (440) by stipulating that

$$\left.\begin{aligned} \Box D_0(x''-x) &= 0 \\ \int_{\sigma''} f(x'')\,\partial^\alpha D_0(x''-x)\,d\sigma''_\alpha &= f(x) \quad : \quad \begin{cases} \text{all } f\text{, and} \\ \text{all timeslices } \sigma'' \text{ through } x \end{cases} \\ D_0(x''-x) &= 0 \qquad : \quad x''-x \text{ spacelike} \\ D_0(0) &= 0 \end{aligned}\right\} \tag{441}$$

It is by no means obvious that such a $D_0(\bullet)$ exists, but if it did (and it does!...as will soon be established *by construction*) we would have

$$\phi_0(x) = \int_{\sigma'} \Big\{ \underbrace{\phi_0(x')}_{\text{Cauchy data}}\,\partial^\alpha D_0(x'-x) - D_0(x'-x)\,\underbrace{\partial^\alpha\phi_0(x')}_{\text{more Cauchy data}} \Big\}\,d\sigma'_\alpha \tag{442}$$

which describes $\phi(x)$ in terms of the prescribed initial data; *i.e.*, in terms of the stipulated values assumed by ϕ and $\partial\phi$ on the spacelike surface σ'. The construction of $D_0(\bullet)$ follows (as it happens) directly from that of $D_{\text{R}}(\bullet)$, so it is to the latter—simpler—problem that I now turn:

Let $\tilde\phi(k)$ and $\tilde\rho(k)$ be the Fourier transforms of $\phi(x)$ and $\rho(x)$:

$$\begin{aligned} \phi(x) &= \big(\tfrac{1}{\sqrt{2\pi}}\big)^4 \iiiint \tilde\phi(k) e^{i(k^0x^0-\,\boldsymbol{k\cdot x})}\,dk^0dk^1dk^2dk^3 \\ &\equiv \tfrac{1}{(2\pi)^2}\int \tilde\phi(k)e^{ikx}\,d^4k && (443.1) \\ &\qquad \tilde\phi(k) = \tfrac{1}{(2\pi)^2}\int \phi(x)e^{-ikx}\,d^4x && (443.2) \\ \rho(x) &= \tfrac{1}{(2\pi)^2}\int \tilde\rho(k)e^{ikx}\,d^4k && (443.3) \\ &\qquad \tilde\rho(k) = \tfrac{1}{(2\pi)^2}\int \rho(x)\,e^{-ikx}\,d^4x && (443.4) \end{aligned}$$

The Fourier transform of $\Box\phi(x) = \rho(x)$ is *algegbraic*

$$-k^2\tilde\phi(k) = \tilde\rho(k)$$

and admits of immediate solution:[263]

$$\tilde\phi(k) \overset{\Downarrow}{=} -\tfrac{1}{k^2}\,\tfrac{1}{(2\pi)^2}\int \rho(x)\,e^{-ikx}\,d^4x$$

[263] This development is typical of the effective application of integral transform techniques to the solution of differential equations. And it illustrates why the inhomogeneous equation $\Box\phi(x) = \rho(x)$ is so much easier to discuss than its homogeneous counterpart.

Returning with this information to (443.1), we reverse the order of integration to obtain

$$\phi(x) = \int \Big\{ -\tfrac{1}{(2\pi)^4}\int k^{-2} e^{ik(x\,-\,x)}\, d^4k \Big\} \rho(x)\, d^4x \tag{444}$$

Comparison with (437) gives

$$G(x - x) = -\tfrac{1}{(2\pi)^4}\int k^{-2} e^{ik(x\,-\,x)}\, d^4k$$
$$k^2 = k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}$$

But the integrand is <u>singular on the null-cone in k-space</u>, so the integral is *meaningless until <u>assigned</u> a meaning*. To that end, we write

$$= -\tfrac{1}{(2\pi)^3}\iiint e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}\,-\,\boldsymbol{x})} \Big\{ \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}} e^{ik_0(x^0 -\, x^0)}\, dk_0 \Big\}\, d^3k \tag{445}$$

which serves to localize the pathology at a pair of *points*: $k_0 = \pm\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$. Next we resort to some standard trickery: we complexify k_0, reinterpret $\int_{-\infty}^{+\infty}$ as a contour integral $\oint$, and circumvent the simple poles at $k_0 = \pm\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$ by contour deformation. Equation (445) is replaced thus by the meaningful but contour-dependent equation

$$G_{\text{C}}(x - x) = -\tfrac{i}{(2\pi)^3}\iiint e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}\,-\,\boldsymbol{x})} \tag{446}$$
$$\cdot \Big\{ \tfrac{1}{2\pi i}\oint_{\text{C}} \frac{1}{k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}} e^{ik_0(x^0 -\, x^0)}\, dk_0 \Big\}\, d^3k$$

where (by the "method of partial fractions")

$$\frac{1}{k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k}} = \frac{1}{2k}\Big[\frac{1}{k_0 - k} - \frac{1}{k_0 + k}\Big]$$

with $k \equiv \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$.

We have physical interest not in all possible G_{C}-functions (all possible contours C, of which there are only a handful of truly distinct options: see RELATIVISTIC CLASSICAL FIELDS (1973) page 167) but only in that particular G_{C}—denoted $D_{\text{R}}(x - x)$—which conforms to our conception of "retarded causal action." It is, therefore, for *physical* reasons (see below) that we take C to have the form illustrated in Figure 106. Writing $k_0 = r + is$, we have

$$e^{ik_0(x^0 -\, x^0)} = e^{-s(x^0 -\, x^0)} \cdot e^{ir(x^0 -\, x^0)}$$

and it becomes clear that to achieve a finite result we must have $s \to \pm\infty$ according as $x^0 \gtrless x^0$; *i.e.*, that we must *close the contour on the upper or lower half-plane according as the source point x lies in the past or the future of the field point x*. The detours around the poles (see the figure) are now dictated by the physical requirement that <u>present field physics shall be insensitive to future source activity</u>. It now follows by the residue theorem that

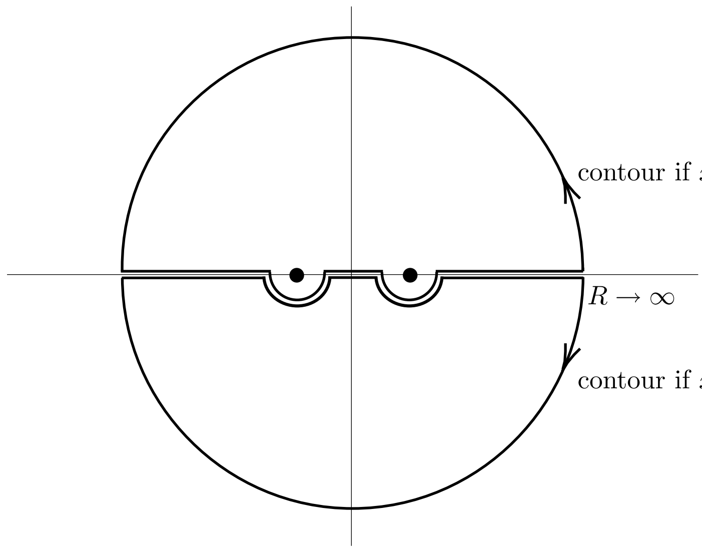


FIGURE 106: *Causal contour, inscribed on the complex k_0-plane: close on the upper half-plane if the field point x lies in the future of the source-point x ($x^0 > x^0$), and on the lower half-plane in the contrary case. The upper contour encloses the poles at $k_0 = \pm\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}$; the lower contour excludes them, so gives $\oint_C = 0$.*

$$\Big\{\text{etc.}\Big\} = \begin{cases} \dfrac{1}{2k}\Big[e^{ik(x^0 - x^0)} - e^{-ik(x^0 - x^0)}\Big] = i\,\dfrac{\sin k(x^0 - x^0)}{k} & \text{if} \quad x^0 > x^0 \\ \\ 0 & \text{if} \quad x^0 < x^0 \end{cases}$$

so

$$D_{\mathrm{R}}(x - x) = \begin{cases} \dfrac{1}{(2\pi)^3}\iiint \dfrac{\sin k(x^0 - x^0)}{k} e^{-i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{x})}\, d^3k \\ \\ 0 \end{cases} \tag{447}$$

To facilitate evaluation of the $\iiint$ we introduce spherical coordinates into $\boldsymbol{k}$-space (3-axis parallel to $\boldsymbol{x} - \boldsymbol{x}$) and (in the case $x^0 > x^0$) obtain

$$= \frac{1}{(2\pi)^3}\int_0^\infty\!\!\int_0^\pi\!\!\int_0^{2\pi} \frac{\sin k\xi^0}{k} e^{-ik\xi\cos\phi} k^2 \sin\phi \, d\theta d\phi dk$$

where $\xi^0 \equiv x^0 - x^0$ and $\xi \equiv \sqrt{(\boldsymbol{x} - \boldsymbol{x})\cdot(\boldsymbol{x} - \boldsymbol{x})} \geqslant 0$. Immediately

$$= \frac{1}{(2\pi)^2}\int_0^\infty \underbrace{2\sin\xi^0 k\cdot\frac{\sin\xi k}{\xi}}_{=\frac{1}{\xi}\Big[\cos k(\xi^0-\xi)-\cos k(\xi^0+\xi)\Big]}dk$$

$$= \frac{1}{(2\pi)^2}\frac{1}{\xi}\lim_{k\to\infty}\Big[\frac{\sin k(\xi^0-\xi)}{(\xi^0-\xi)}-\frac{\sin k(\xi^0+\xi)}{(\xi^0+\xi)}\Big]$$

But $\delta(x)=\frac{1}{\pi}\lim_{k\to\infty}\frac{\sin kx}{x}$ provides a standard *parameterized representation of the Dirac δ-function*,[264] so

$$= \frac{1}{(2\pi)^2}\frac{\pi}{\xi}\Big[\delta(\xi^0-\xi)-\delta(\xi^0+\xi)\Big] \tag{448}$$

The 2nd δ-function is moot when $\xi^0>0$ (*i.e.*, when x^0 and x^0 stand in causal sequence: $x^0>x^0$), while according to (447) *both* terms are extinguished when $x^0<x^0$. We come thus to the conclusion that

$$D_\mathrm{R}(x-x)=\begin{cases}\dfrac{1}{4\pi\xi}\delta(\xi^0-\xi) & : \quad \xi^0>0\\ \\ 0 & : \quad \xi^0<0\end{cases} \tag{449.1}$$

Were we to deform the contour C so as instead to favor *advanced action* (fields responsive to *future* source activity!) we would, by the same analysis, be led to

$$D_\mathrm{A}(x-x)=\begin{cases}0 & : \quad \xi^0>0\\ \\ \dfrac{1}{4\pi\xi}\delta(\xi^0+\xi) & : \quad \xi^0<0\end{cases} \tag{449.2}$$

The *retarded* and *advanced propagators* (or Green's functions) $D_\mathrm{R}(\bullet)$ and $D_\mathrm{A}(\bullet)$ are, in an obvious sense, "natural companions." The former, according to (448), <u>vanishes except on the lightcone that extends backwards from the fieldpoint x</u>, while $D_\mathrm{A}(\bullet)$ vanishes except on the <u>forward</u> lightcone: see Figure 107.

What about the function $D_0(x-x)$? It has, as I will show, been sitting quitely on the right side of (448):

$$\begin{aligned} D_0(x-x) &= \frac{1}{4\pi\xi}\Big[\delta(\xi^0-\xi)-\delta(\xi^0+\xi)\Big] \quad : \quad \text{all } \xi^0 \\ &= D_\mathrm{R}(x-x)-D_\mathrm{A}(x-x) \end{aligned} \tag{450}$$

Note first that $D_0(x-x)$—thus described, and thought of as a function of x—clearly vanishes except on the lightcone that extends backward and forward

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[264] To see how the representation does its job, use *Mathematica* to `Plot` the function $\frac{\sin kx}{\pi x}$ for several values of k, and also to evaluate $\int_{-\infty}^{+\infty}\frac{\sin kx}{\pi x}\,dx$.

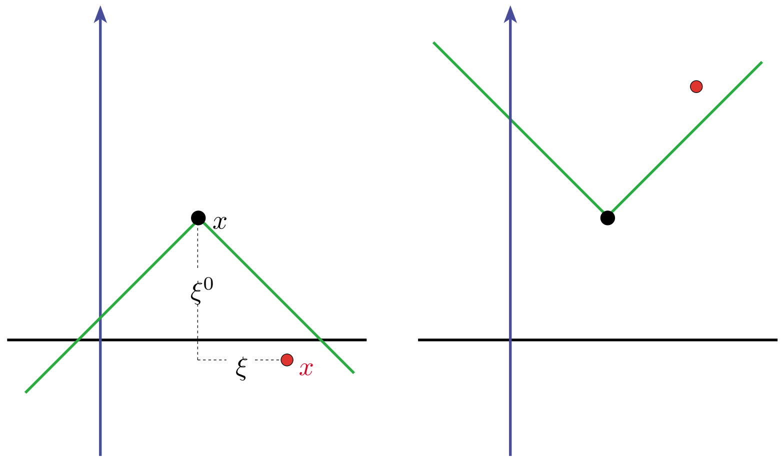

FIGURE 107: *The retarded propagator* $D_{\mathrm{R}}(\bullet)$ *harvests source data written onto the lightcone (shown at left) that extends backward from the fieldpoint* $\bullet$. *The advanced propagator* $D_{\mathrm{A}}(\bullet)$ *looks similarly to the forward lightcone. Source data at the* $\bullet$ *shown at left is actually invisible to the fieldpoint* $\bullet$, *since it lies* <u>*interior to rather than on*</u> *the backward cone (but it would become visible* if the photon had mass). *Ditto at right.*

from x, so the 3rd of the conditions (441) is clearly satisfied. Writing

$$D_0(x - x) \equiv \mathcal{D}(\xi^0, \xi)$$

we observe that $\mathcal{D}(\xi^0, \xi)$ is, by (450), an *odd* function of ξ^0, so

$$\mathcal{D}(0, \xi) = 0 \quad : \quad \text{all } \xi$$

which serves to establish the 4th of the conditions (441). That $\Box D_0(x - x) = 0$ (the 1st of those conditions) follows from the remarks (i) that the functions $G_{\mathrm{C}}(x - x)$ described at (446) satisfy $\Box G_{\mathrm{C}} = 0$ for *every* contour C, and (ii) that $G_{\mathrm{C}} \to D_0$ if we take C to be (topologically equivalent to) the <u>bounded</u> contour shown in Figure 108. Finally, we observe (see again (447))that

$$\begin{aligned}\frac{\partial}{\partial x^0} D_0(x - x) &= \frac{1}{(2\pi)^3} \iiint \cos k(x^0 - x^0) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x})}\, d^3k \\ &\downarrow \\ &= \frac{1}{(2\pi)^3} \iiint e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x})}\, d^3k \quad \text{when} \quad x^0 = x^0\end{aligned}$$

But $$= \delta(\boldsymbol{x} - \boldsymbol{x})$$

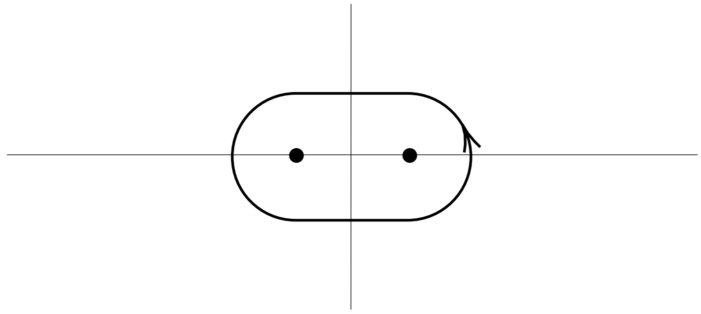

FIGURE 108: *The* <u>*bounded*</u> *contour that, when introduced into* (446), *yields the function* D_0. *The contours shown in Figure 106 have the property that they are "this or that, depending on the sign of the time," and it is because they "flip" that they give rise to a solution of the* inhomogeneous *wave equation. The contour shown above entails no such flip, so gives rise to a solution of the* homogeneous *wave equation. The point is developed in the text, and—in much great detail—in a reference cited.*

by the *Fourier integral theorem*,[265] and this expresses the upshot of the 2^{nd} of the conditions (441). Further analysis would show that the $D_0(x-x)$ described above is the *unique* realization of the conditions (441).

Returning with (450) to (447) we obtain

$$\begin{aligned} D_{\mathrm{R}}(x-x) &= \quad\theta(\ \ x^0 - x^0)\cdot D_0(x-x)\\ D_{\mathrm{A}}(x-x) &= -\theta(-x^0 + x^0)\cdot D_0(x-x) \end{aligned}$$

where $\theta(x)$ is the *Heaviside step function*:

$$\theta(x) = \int_{-\infty}^{x} \delta(\xi)\,d\xi = \begin{cases} 0 & x < 0\\ \frac{1}{2} & x = 0\\ 1 & x > 0 \end{cases}$$

It's occurance in this context can be traced to the sign-of-the-times-dependent "contour flipping" that enters into the definitions of $D_{\mathrm{R}}(x-x)$ and $D_{\mathrm{A}}(x-x)$

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[265] The Fourier integral theorem asserts that

$$\phi(x) = \tfrac{1}{\sqrt{2\pi}}\int e^{ikx}\Big\{\tfrac{1}{\sqrt{2\pi}}\int e^{-ikx}\phi(x)\,dx\Big\}dk$$

for "all" $\phi(x)$. Reversing the order of integration, we obtain the identity used in the text

$$\delta(x-x) = \tfrac{1}{2\pi}\int e^{-ik(x-x)}\,dk$$

which can be considered to lie at the heart of Fourier's theorem and of Fourier analysis.

(see again Figure 106) but is absent from the definition of $D_0(x-x)$.[266]

From the fact that $D_0(\bullet)$ is attached to *both sectors* of the lightcone we conclude (see again (442)) that if we know the values assumed by the free ambient field ϕ_0 and its derivatives $\partial\phi_0$ on *some* spacelike surface σ' then we know the values assumed *everywhere* by ϕ_0: the free field equations allow us both to *predict* and to *retrodict*. But the field equations do not, in general, allow us to predict *source* motion, which is typically of semi-extrinsic origin (we haven't yet decided whether to flip the light switch or not!) ...and it is for this reason that we have—"by hand," not from mathematical (or deep physical?) necessity—inserted $D_\text{R}(\bullet)$ rather than $D_\text{A}(\bullet)$ into (438.1).

The preceding analysis has been somewhat "heavy." But it has yielded results—see again (438), (442), (449) & (450)—of remarkable simplicity and high plausibility. It has employed analytical methods which have in fact long been standard to several branches of "linearity-dominated" physics and engineering (though their importation into classical/quantum electrodynamics is—oddly—of relatively recent date: it was accomplished in the late 1940's and early 1950's by Julian Schwinger) ...and which are, beneath the surface clutter, really rather pretty (Richard Crandall's "favorite stuff"). I turn now to discussion of some of the specific electrodynamical implications of the material now in hand.

2. Application: the Liénard-Wiechert potential. Let the values—values consistent with the Lorentz gauge condition—assumed by the 4-potential A^μ and its first derivatives $\partial^\alpha A^\mu$ on some everywhere-spacelike surface σ be given/prescribed. Then (see again (442): also Figure 109)

$$A^\mu(x) = \int_\sigma \Big\{ A^\mu(x)\partial^\alpha D_0(x-x) - D_0(x-x)\partial^\alpha A^\mu(x)\Big\}\, d\sigma_\alpha \tag{451}$$

describes the "evolved values" that—in forced consequence of the equations of free-field motion—are assumed by our "ambient field" at points x which lie off the "data surface" σ. Any *particular* inertial observer would in most cases find it most natural to take σ to be a time-slice, and in place of (451) to write

$$= \iiint \Big\{ A^\mu(x)\frac{\partial}{\partial x^0} D_0(x-x) - D_0(x-x)\frac{\partial}{\partial x^0} A^\mu(x)\Big\} dx^1 dx^2 dx^3$$

While every particular observer has that option (Figure 110), it must be borne in mind that *the time-slice concept is not boost invariant*: the point was illustrated in Figure 58, and is familiar as the "breakdown of non-local simultaneity." The preceding equation states explicitly how the value of $A^\mu(x)$ depends upon the initial value and initial time derivative of the field, and establishes the sense in which "launching a free electromagnetic field" is like throwing a ball.[267]

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[266] This topic is developed in unusual detail in §§3 & 4 of my "Simplified production of Dirac δ-function identities," (1997).
[267] PROBLEM 70.

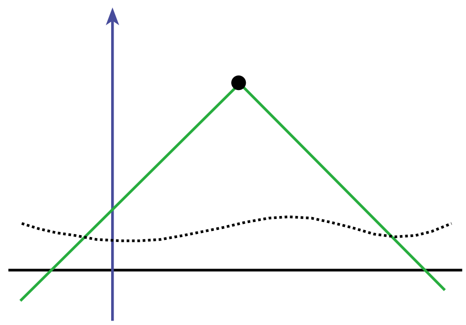

FIGURE 109: *Cauchy data is written onto the dotted surface* σ. *The function* $D_0(x - x)$ *vanishes* *except on the lightcone*: *it serves in* (451) *to describe how data at the intersection of* σ *with the lightcone is conflated to produce the value assumed by* A^μ *at the fieldpoint* $\bullet$. *As the temporal coordinate of* $\bullet$ *increases the intersection becomes progressively more remote, until finally it enters a region where (in typical cases) the initial data was null ... which is to say: the ambient field at any given spatial location can be expected ultimately to die away. The die-off is reenforced by the* $(4\pi\xi)^{-1}$ *which was seen at (450) to enter into the design of* D_0.

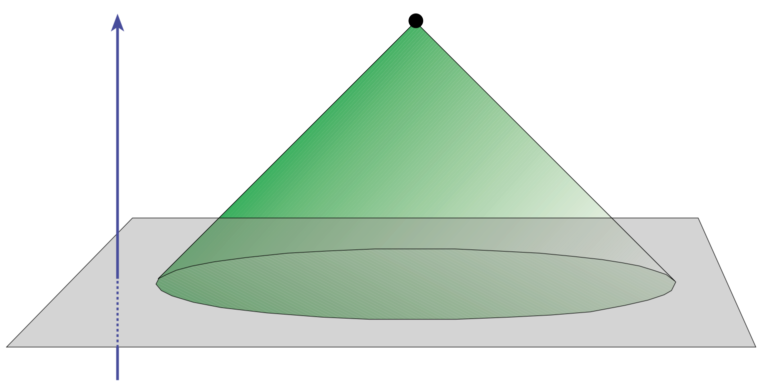

FIGURE 110: *An inertial observer has exercised his non-covariant option to deposit his Cauchy data on a time-slice. Only data at the spherical* *intersect* *of the lightcone and the time-slice contribute to the value assumed at* $\bullet$ *by* A^μ, *though "if the photon had mass" then data interior to the sphere would also contribute.*[2]

We turn our attention now to the component of the A^μ-field that arises from source activity, which according to (438/449) can be described

$$A^\mu(x) = \tfrac{1}{c}\int D_{\mathrm{R}}(x - x)\, j^\mu(x)\, d^4x \tag{452.1}$$

$$D_{\mathrm{R}}(x - x) = \begin{cases} \dfrac{1}{4\pi R}\delta(cT - R) & : \quad T > 0 \\ 0 & : \quad T < 0 \end{cases} \tag{452.2}$$

with $cT \equiv x^0 - x^0$ and $R \equiv |\boldsymbol{x} - \boldsymbol{x}|$. We can therefore state that the value assumed by A^μ at the field point x arises (by superposition) entirely from the source activity sampled by the lightcone which extends backward from x. In an effort to expose more clearly the meaning of this result we consider $j^\mu(x)$ to arise from a solitary point charge e in arbitrarily prescribed motion: we assume, in other words, that $j^\mu(x)$ can be described (see again (323))

$$j^\mu(x) = ec\int_{-\infty}^{+\infty} u^\mu(\tau)\delta(x - x(\tau))\, d\tau$$

Immediately

$$\begin{aligned} A^\mu_{\mathrm{R}}(x) &= e\int D_{\mathrm{R}}(x - x) \int_{-\infty}^{+\infty} u^\mu(\tau)\delta(x - x(\tau))\, d\tau d^4x \\ &= e\int_{-\infty}^{+\infty}\int u^\mu(\tau)D_{\mathrm{R}}(x - x)\delta(x - x(\tau))\, d^4x d\tau \\ &= e\int_{-\infty}^{+\infty} u^\mu(\tau)D_{\mathrm{R}}(x - x(\tau))\, d\tau \\ &= \frac{e}{4\pi}\int_{-\infty}^{+\infty} u^\mu(t)\frac{1}{R(\tau)}\delta(G(\tau))\, d\tau \\ &\qquad\qquad G(\tau) \equiv cT(\tau) - R(\tau) \end{aligned}$$

An elementary change-of-variables argument[268] leads to the important general conclusion that

$$\delta(g(x)) = \sum_\alpha \frac{1}{|g'(x_\alpha)|}\delta(x - x_\alpha) \tag{453}$$

where $g'(x_\alpha) \equiv \frac{d}{dx}g(x)$ and where (see Figure 111) the x_α locate the zeros of $g(x)$. It follows by way of application to the problem at hand that

$$= \frac{e}{4\pi}\int_{-\infty}^{+\infty} u^\mu(t)\frac{1}{R(\tau)}\frac{1}{|G'(\tau_0)|}\delta(\tau - \tau_0)\, d\tau \tag{454}$$

where τ_0 is the proper time at which $x(\tau)$ punctures the backward lightcone, and where $G' \equiv \frac{d}{d\tau}G$. If t_0, $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$ refer to the source-particle at the instant of puncture, then we have (borrowing a trick from page 192)

[268] See ELECTRODYNAMICS (1972), page 304. An alternative argument—that makes transparent the origin of the perplexing absolute value bars—can be found in the little paper cited just above.[264]

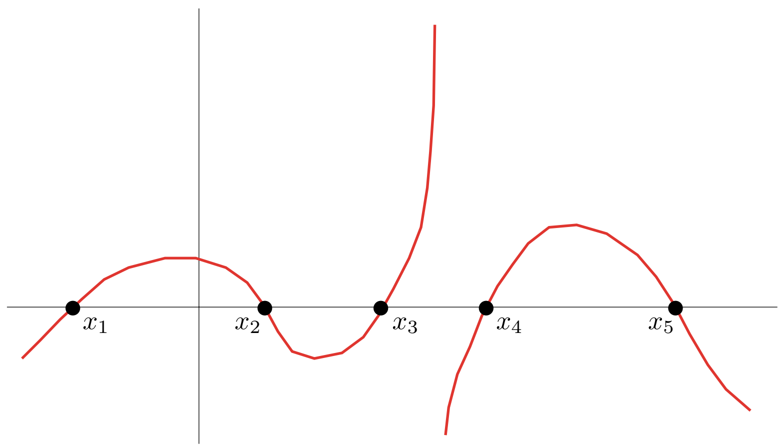

FIGURE 111: *Zeros x_α of a function $g(x)$. The numbers x_α and $g(x_\alpha)$ enter into the formulation of the important identity* (453).

$$\begin{aligned} G'(\tau_0) &= \gamma_0 \tfrac{d}{dt_0}\Big\{c(t-t_0) - \sqrt{\boldsymbol{R}\cdot\boldsymbol{R}}\Big\} \quad \text{with} \quad \boldsymbol{R} \equiv \boldsymbol{x} - \boldsymbol{x}_0 \\ &= \gamma_0\big(-c + \hat{\boldsymbol{R}}\cdot\boldsymbol{v}_0\big) \\ &= -c\gamma_0(1-\beta_\parallel)_0 \end{aligned} \tag{455}$$

$$\beta_\parallel \equiv \tfrac{1}{c}\hat{\boldsymbol{R}}\cdot\boldsymbol{v} \equiv \begin{cases} \text{magnitude of the component} \\ \text{of } \boldsymbol{\beta} \text{ that is parallel to } \boldsymbol{R} \end{cases}$$

Returning with (455) to (454) we obtain[269]

$$A^\mu_{\mathrm{R}}(x) = \tfrac{e}{4\pi}\Big[\frac{1}{c\gamma(1-\beta_\parallel)R}u^\mu\Big]_0 \tag{456.1}$$

$\uparrow$ signifies "evaluation at the puncture point"

which—recall $A = \begin{pmatrix} \varphi \\ \boldsymbol{A} \end{pmatrix}$ and $u = \gamma\begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$—can also be formulated

$$\left.\begin{aligned} \varphi_{\mathrm{R}}(x) &= \tfrac{e}{4\pi}\Big[\frac{1}{(1-\beta_\parallel)R}\Big]_0 \\ \boldsymbol{A}_{\mathrm{R}}(x) &= \tfrac{e}{4\pi}\Big[\frac{1}{(1-\beta_\parallel)R}\boldsymbol{\beta}\Big]_0 \end{aligned}\right\} \tag{456.2}$$

Equations (456)—which are, in view of the complexity of the argument from which they derive, remarkably simple, and which describe the potential

[269] Beware! The R on the left is intended to signify "retarded," while on the right R means "length of $\boldsymbol{R}$."

fields generated by the retarded action of a moving point charge—were first obtained by A. Liénard (1898) and E. Wiechart (1900), and describe what are universally known as the **Liénard-Wiechart potentials**. The "retarded potential" idea was apparently original to B. Riemann (1859), and the essence of (452) can reportedly be found in work (1867) of Ludwig Lorenz (who, as previously remarked, is to be distinguished from H. A. Lorentz). The work of Riemann and of Lorenz was known to Maxwell, but one gets the impression (see *Treatise on Electricity & Magnetism*, §§805 and 861–end) that Maxwell was not much impressed. Which—though historically explicable—is too bad, for equations (456) are, as will emerge, *fundamental to the theory of radiative processes*.

The "advanced analogs" of (456) can be obtained by reversing the signs of all $\beta_{\|}$-terms and evaluating $\big[$etc.$\big]$ at the future puncture point.

The Liénard-Wiechart potential (456.2) gives back the familiar Coulomb potential

$$\begin{aligned}\varphi(x) &= \frac{e}{4\pi R}\\ \boldsymbol{A}(x) &= \mathbf{0}\end{aligned}$$

when the source is at rest (see the figure), and the "retarded evaluation" idea

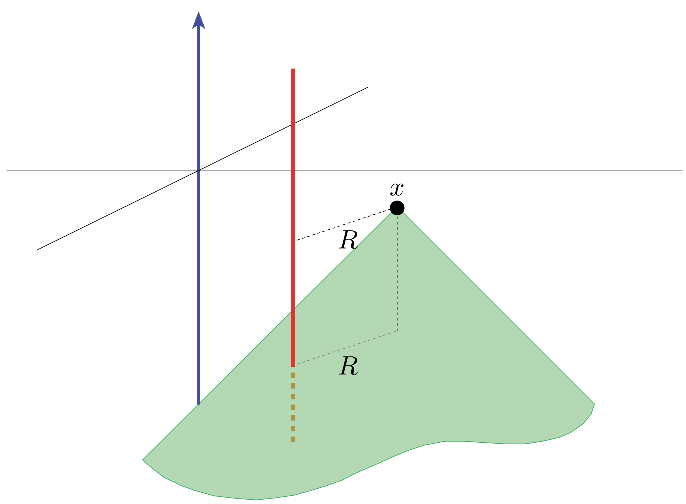

FIGURE 112: *Show in red is the worldline of a charged particle at rest (with respect to the inertial observer who drew the diagram). The distance from the field point x to the puncture point on the backward lightcone was seen to be R . . . and so—as yet unbeknownst to the field point—it has remained.*

conforms nicely to our physical intuition. It is, therefore, the $\gamma(1-\beta_{\|})$-term in (456.1) and the $(1-\beta_{\|})$-term in (456.2) that demand "explanation" if we are to say that we "understand" (456). Now . . . if θ is the angle subtended by $\boldsymbol{\beta}$

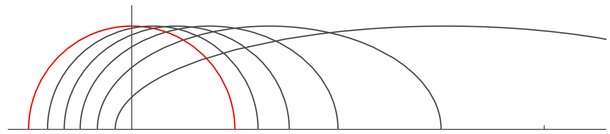

FIGURE 112: *Polar plots showing the* θ*-dependence of the Doppler factor* $\sqrt{1-\beta^2}/(1-\beta\cos\theta)$*, with* $\beta = 0, 0.2, 0.4, 0.6, 0.8, 0.95$.

and $\boldsymbol{R}$ we have

$$\frac{1}{\gamma(1-\beta_\parallel)} = \frac{1}{\gamma(1-\beta\cos\theta)} = \begin{cases} \sqrt{\dfrac{1+\beta}{1-\beta}} > 1 \text{ at } \theta = 0 \\ \qquad\qquad = 1 \text{ at } \theta = \arccos\left[\dfrac{1-\sqrt{1-\beta^2}}{\beta}\right] \\ \qquad \dfrac{1}{\gamma} < 1 \text{ at } \theta = 90^\circ \\ \sqrt{\dfrac{1-\beta}{1+\beta}} < 1 \text{ at } \theta = 180^\circ \end{cases}$$

—results of which the preceding figure provides vivid graphic interpretations. The expressions $[(1+\beta)/(1-\beta)]^{\pm 1}$ are familiar (recall again PROBLEM 43) as the eigenvalues of $\mathbb{\Lambda}(\beta)$: they are found, morover, to be fundamental to the description of the *relativistic Doppler effect*,[270] so

$$\frac{1}{\gamma(1-\beta_\parallel)} \equiv \text{Doppler factor}$$

becomes[271] a natural terminology. Looking back again to (456.1), we see that the Doppler factor

- serves to *enhance* the value of A^μ_{R} if the source point is seen by the field point to be *approaching* at the moment of puncture:

$$0 \leqslant \theta_0 \leqslant \cos^{-1}\left[\tfrac{1-\sqrt{1-\beta^2}}{\beta}\right]_0$$

- serves in the contrary case to *diminish* the value of A^μ_{R}

...which is what one would expect if (see Figure 114) the lightcone possessed some small but finite "thickness," for in the former case the field point would then get a relatively "longer look" at the source point, and in the latter case a "briefer look." Note that it is not the Doppler factor itself but the

$$\text{truncated Doppler factor} \equiv \frac{1}{(1-\beta_\parallel)}$$

that stands in (456.2).

[270] See ELECTRODYNAMICS (1972) page 239.
[271] Compare A. Sommerfeld, *Electrodynamics* (1952), page 250.

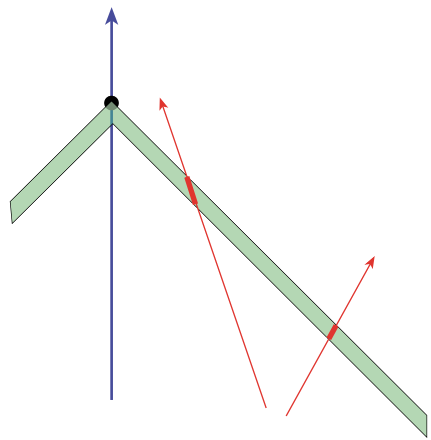

FIGURE 114: *If the lightcone had "thickness" then the presence of the Doppler factor in (456) could be understood qualitatively to result from the relatively "longer look" that the field point gets at approaching charges, the relatively "briefer look" at receding charges.*

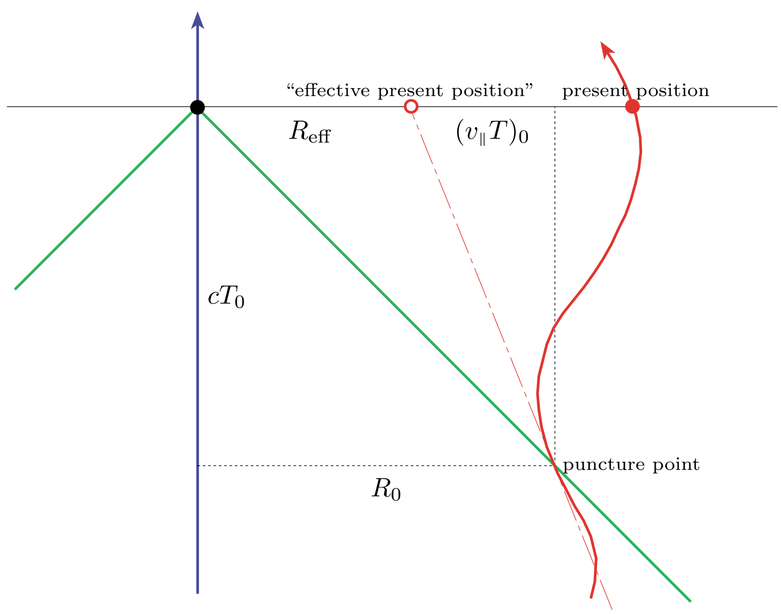

FIGURE 115: *Construction used to define the "effective present distance" from source to field point:*

$$R_{\text{eff}} = (R - v_{\parallel}T)_0 = (R - \beta_{\parallel}cT)_0 = (1 - \beta_{\parallel})_0 R_0$$

Some textbook writers make much of the curious fact that it is possible (see Figure 115) by linear extrapolation from the puncture point data to arrive at an "physical interpretation" of the expression $[(1-\beta_\|)R]_0$

$$[(1-\beta_\|)R]_0 = R_{\text{eff}} \equiv \begin{cases} \textit{present} \text{ distance from field point to charge if} \\ \text{the charge had moved uniformly/rectilinearly} \\ \text{since the moment of puncture} \end{cases}$$

and in this notation to cast (456.2) in the form

$$\begin{aligned} \varphi_{\text{R}}(x) &= \frac{e}{4\pi R_{\text{eff}}} \\ \boldsymbol{A}_{\text{R}}(x) &= \frac{e}{4\pi R_{\text{eff}}}\boldsymbol{\beta}_0 \end{aligned}$$

My own view is that the whole business, though memorably picturesque, should be dismissed as a *mere* curiosity . . . on grounds that it is too alien to the spirit of relativity—and to the letter of the principle of manifest Lorentz covariance—to be of "deep" significance. More worthy of attention, as will soon be demonstrated, is the fact that equations (456) admit[272] of the following *manifestly covariant* formulation

$$A^\mu_{\text{R}}(x) = \frac{e}{4\pi}\Big[\frac{u^\mu}{R_\alpha u^\alpha}\Big]_0 \tag{457}$$

where $R^\mu \equiv x^\mu - x^\mu(\tau)$

3. Field of a point source in arbitrary motion. What we want now to do is to evaluate

$$F^{\mu\nu}_{\text{R}}(x) = \partial^\mu A^\nu_{\text{R}}(x) - \partial^\nu A^\mu_{\text{R}}(x)$$

where $A^\mu_{\text{R}}(x)$ is given most conveniently by (457)

So the physics of what follows is conceptually straightforward. The point is worth keeping in mind, for the computational details are—like the final result—quite intricate.

Turning now, therefore, to the evaluation of

$$\partial^\mu A^\nu_{\text{R}}(x) = g^{\mu m}\frac{\partial}{\partial x^m}\Big\{\frac{e}{4\pi}\Big[\frac{u^\nu}{R_\alpha u^\alpha}\Big]_0\Big\}$$

. . . it is critically important to notice that (see the following figure) *variation of the field point x induces a variation of the proper time of puncture*; *i.e.*, that τ_0 is x-dependent: $\tau_0 = \tau_0(x)$. Formally,

$$\frac{\partial}{\partial x^m} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m} + \frac{\partial\,\tau}{\partial x^m}\frac{\partial}{\partial\,\tau}$$

where $\boldsymbol{\partial}_m$ senses *explicit* x-dependence and $(\partial_m\tau)\frac{\partial}{\partial\tau}$ senses *covert* x-dependence.

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[272] PROBLEM 71.

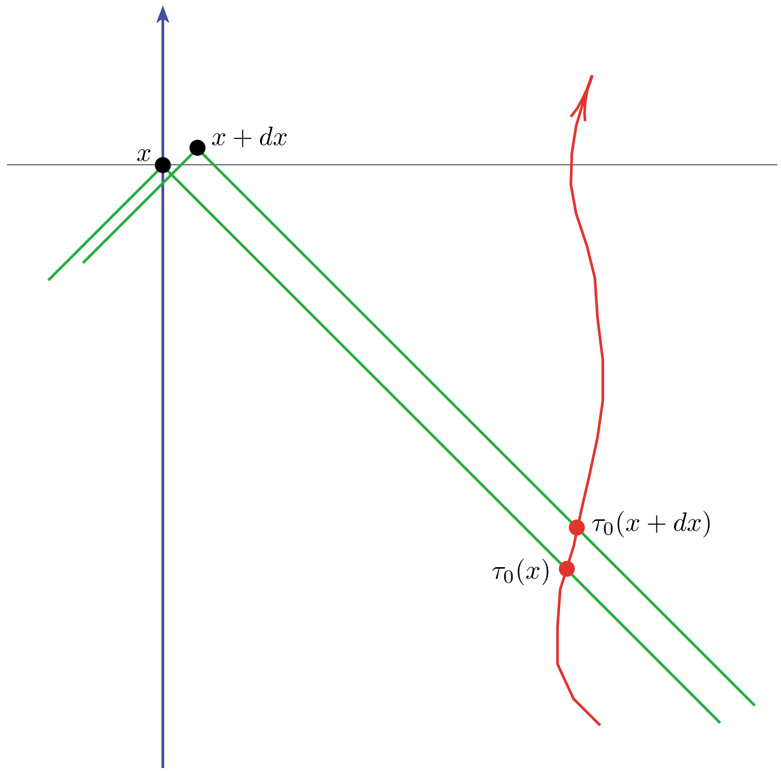

FIGURE 116: *Variation of the field point x typically entails variation also of the puncture point, and it is this circumstance that makes evaluation of the electromagnetic field components so intricate.*

Proceeding thus from

$$\partial^\mu A^\nu_{\mathrm{R}}(x) = \frac{e}{4\pi} g^{\mu m}\Big[\Big\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x^m} + \frac{\partial\,\tau}{\partial x^m}\frac{\partial}{\partial\,\tau}\Big\}\frac{u^\nu(\tau)}{R_\alpha(x,\tau)u^\alpha(\tau)}\Big]_0\Big\}$$

$$R_\alpha(x,\tau) \equiv x_\alpha - x_\alpha(\tau)$$

we are led by straightforward calculation to the following result:

$$\begin{aligned} &= \frac{e}{4\pi}\Big[\frac{1}{(R_\alpha u^\alpha)^2}\big(c^2 g^{\mu m}\frac{\partial\,\tau}{\partial x^m} - u^\mu\big)u^\nu\Big]_0 \\ &\quad + \frac{e}{4\pi}\Big[\frac{1}{(R_\alpha u^\alpha)} g^{\mu m}\frac{\partial\,\tau}{\partial x^m}\Big\{a^\nu - \frac{(R_\alpha a^\alpha)}{(R_\beta u^\beta)}u^\nu\Big\}\Big]_0 \end{aligned} \tag{458}$$

Here use has been made of $u^\alpha u_\alpha = c^2$ and also of

$$a^\mu \equiv \frac{du^\mu(\tau)}{d\tau} = \text{4-acceleration of the source particle}$$

Notational adjustments make this result easier to write, if not immediately easier to comprehend. Let r be the Lorentz-invariant length defined[270]

$$r \equiv \tfrac{1}{c}R_\alpha u^\alpha = \gamma(1-\beta_\parallel)R$$

and let w^μ be the dimensionless 4-vector defined

$$w^\mu \equiv c\partial^\mu\tau - \tfrac{1}{c}u^\mu$$

Then

$$\partial^\mu\tau = \frac{cw^\mu + u^\mu}{c^2}$$

Easily $\partial_\mu(R_\alpha R^\alpha) = 2\big\{R_\mu - (\partial_\mu\tau)(R_\alpha u^\alpha)\big\}$. From this and the fact that R^μ is (by definition of "puncture point") invariably null at the puncture point

$$\big[R_\alpha R^\alpha\big]_0 = 0, \quad \text{therefore} \quad \big[\partial_\mu(R_\alpha R^\alpha)\big]_0 = 0$$

it follows that

$$\big[\partial^\mu\tau\big]_0 = \big[R^\mu/(R_\alpha u^\alpha)\big]_0 = \tfrac{1}{c}\big[R^\mu/r\big]_0$$

from which

$$\begin{aligned}\big[u_\alpha w^\alpha\big]_0 &= 0\\ \big[w_\alpha w^\alpha\big]_0 &= -1\end{aligned}$$

$$\big[a_\alpha\partial^\alpha\tau\big]_0 = \left[\frac{R_\alpha a^\alpha}{R_\beta u^\beta}\right]_0 = \tfrac{1}{c}\big[a_\alpha w^\alpha\big]_0$$

follow as fairly immediate corollaries.[273] When we return with this information to (458) we obtain

$$\partial^\mu A^\nu_{\text{R}}(x) = \frac{e}{4\pi}\left[\frac{1}{r^2}w^\mu b^\nu\right]_0 + \frac{e}{4\pi c^2}\left[\frac{1}{r}(w^\mu + b^\mu)\big(a^\nu - (aw)b^\nu\big)\right]_0$$

$$b \equiv \tfrac{1}{c}u = \gamma\begin{pmatrix}1\\ \boldsymbol{\beta}\end{pmatrix}$$

---

[273] For detailed proof see CLASSICAL RADIATION (1974), pages 523/4. But beware! I have now altered slightly the definitions of r and w^μ.

Consequently

$$
\begin{aligned}
F_{\mathrm{R}}^{\mu\nu}(x) &\equiv \text{electromagnetic field at } x \text{ due to past source activity}\\
&= \frac{e}{4\pi}\Big[\frac{1}{r^2}(w^\mu b^\nu - w^\nu b^\mu)\Big]_0 \qquad (459)\\
&\quad + \frac{e}{4\pi c^2}\Big[\frac{1}{r}\big\{(b^\mu a^\nu - b^\nu a^\mu) + (w^\mu a^\nu - w^\nu a^\mu) - (aw)(w^\mu b^\nu - w^\nu b^\mu)\big\}\Big]_0\\
&= \boxed{\begin{array}{l}\text{acceleration-independent term} \sim 1/r^2,\\ \text{dominant near the worldline of the source}\end{array}}\\
&\qquad\qquad + \boxed{\begin{array}{l}\text{acceleration-dependent term} \sim 1/r,\\ \text{dominant far from the worldline of the source}\end{array}}\\
&= \boxed{\text{“velocity field”}} + \boxed{\text{“acceleration field”}}\\
&= \boxed{\text{“near field”}} + \boxed{\text{“far field”}}\\
&= \boxed{\text{generalized } \textbf{Coulomb} \text{ field}} + \boxed{\textbf{radiation} \text{ field}}
\end{aligned}
$$

This result is complicated (the physics is complicated!), but not “impossibly” complicated. By working in a variety of notations, from a variety of viewpoints, and in contact with a variety of special applications it is possible to obtain—ultimately—a fairly sharp feeling for the extraordinarily rich physical content of (459). As preparatory first steps toward that objective . . .

We note that, using results developed on the preceding page,

$$w^\mu = c\partial^\mu\tau - b^\mu$$

becomes

$$= \big[R^\mu/r - b^\mu\big]_0$$

which when spelled out in detail reads

$$\begin{pmatrix} w^0 \\ \boldsymbol{w}\end{pmatrix} = \frac{1}{\gamma(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})R}\begin{pmatrix} R \\ \boldsymbol{R}\end{pmatrix} - \gamma\begin{pmatrix} 1 \\ \boldsymbol{\beta}\end{pmatrix}$$

with $\hat{\boldsymbol{R}} \equiv \boldsymbol{R}/R$. A little manipulation (use $\gamma^{-2} = 1 - \boldsymbol{\beta}\cdot\boldsymbol{\beta}$) brings this result to the form

$$= \frac{\gamma}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\begin{pmatrix} (\hat{\boldsymbol{R}}-\boldsymbol{\beta})\cdot\boldsymbol{\beta} \\ \hat{\boldsymbol{R}} - \boldsymbol{\beta} + \underbrace{(\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})\boldsymbol{\beta} - (\boldsymbol{\beta}\cdot\boldsymbol{\beta})\hat{\boldsymbol{R}}}_{=\,\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\hat{\boldsymbol{R}})}\end{pmatrix} \tag{460.1}$$

It follows similarly from (270) that

$$a = \begin{pmatrix} \dot{u}^0 \\ \dot{\boldsymbol{u}}\end{pmatrix} = \gamma^4\begin{pmatrix} \boldsymbol{a}\cdot\boldsymbol{\beta} \\ \boldsymbol{a} + \boldsymbol{\beta}\times(\boldsymbol{\beta}\times\boldsymbol{a})\end{pmatrix} \tag{460.2}$$

where $\boldsymbol{a} \equiv d\boldsymbol{v}/dt$.

To extract $\boldsymbol{E}(x)$ from (460) we have only (see again page 108) to set $\nu = 0$ and to let μ range on $\{1, 2, 3\}$:

$$\begin{aligned}\boldsymbol{E}(x) &= \frac{e}{4\pi}\Big[\frac{1}{r^2}(\boldsymbol{w}b^0 - w^0\boldsymbol{b})\Big]_0\\ &\quad+\frac{e}{4\pi c^2}\Big[\frac{1}{r}\big\{(\boldsymbol{b}\dot{u}^0 - b^0\dot{\boldsymbol{u}}) + (\boldsymbol{w}\dot{u}^0 - w^0\dot{\boldsymbol{u}}) - (aw)(\boldsymbol{w}b^0 - w^0\boldsymbol{b})\big\}\Big]_0\end{aligned}$$

It follows readily from (460) that

$$\begin{aligned}\boldsymbol{w}b^0 - w^0\boldsymbol{b} &= \frac{1}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}(\hat{\boldsymbol{R}} - \boldsymbol{\beta})\\ \boldsymbol{b}\,\dot{u}^0 - b^0\,\dot{\boldsymbol{u}} &= -\gamma^3\boldsymbol{a}\\ \boldsymbol{w}\dot{u}^0 - w^0\dot{\boldsymbol{u}} &= -\gamma^3\frac{1}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\boldsymbol{\beta}\times\big(\boldsymbol{a}\times(\hat{\boldsymbol{R}} - \boldsymbol{\beta})\big)\\ (aw) &= -\gamma^3\frac{1}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\Big\{(1-\beta^2)\,(\hat{\boldsymbol{R}}\cdot\boldsymbol{a}) - (1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\boldsymbol{a})\Big\}\end{aligned}$$

so after some unilluminating manipulation we obtain

$$\begin{aligned}\boldsymbol{E}(x) &= \frac{e}{4\pi}\Big[\frac{1}{r^2}\frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})}(\hat{\boldsymbol{R}} - \boldsymbol{\beta})\Big]_0\\ &\quad+\frac{e}{4\pi c^2}\Big[\frac{1}{r}\frac{\gamma}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^2}\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}} - \boldsymbol{\beta})\times\boldsymbol{a}\big)\Big]_0\end{aligned}\tag{461.1}$$

A similar[274] computation addressed to the evaluation of $\boldsymbol{B}(x)$ leads to a a result which can be expressed very simply/economically:

$$\boldsymbol{B}(x) = \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}(x)\Big]_0 \tag{461.2}$$

It should be noted that equations (459) and (461) describe precisely the same physics: they differ only notationally. And both are exact (no approximations). I remarked earlier, in connection with equations (456), that "the 'retarded evaluation' idea $[\quad]_0$ conforms nicely to our physical intuition," but must now admit that (461) contains many non-intuitive details: in this sense it is evidently *easier to think reliably about potentials (which are "spooks") than about fields (which are "real")*!

Notice also that if we insert the expressions that appear on the right sides of equations (461) into Lorentz' $\boldsymbol{F} = q(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})$ then we obtain, in effect, a description of the retarded position/velocity/acceleration-dependent action on one charge upon another—a description <u>free from any direct allusion to the field concept</u>! It was with the *complexity* of this and similar results in mind that I suggested (page 250) that life without fields "would . . . entail more cost than benefit."

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[274] . . . and similarly tedious: generally speaking, one can expect tediousness to increase in proportion to how radically one departs—as here—from adherence to the principle of manifest covariance.

We have encountered evidence (pages 240, 297) of what might be called a "tendency toward $\boldsymbol{B}\perp\boldsymbol{E}$," but have been at pains to stress (page 332) that $\boldsymbol{B}\perp\boldsymbol{E}$ remains, nevertheless, an exceptional state of affairs. It is, in view of the latter fact, a little surprising to discover that $\boldsymbol{B}\perp\boldsymbol{E}$ does pertain—everywhere and exactly—to the field produced by a single point source in arbitrary motion. The key word here is "single," as I shall now demonstrate: write

- $\boldsymbol{E}$ and $\boldsymbol{B}=\hat{\boldsymbol{R}}\times\boldsymbol{E}$ to describe (at x) the field generated by e;
- $\boldsymbol{E}$ and $\boldsymbol{B}=\hat{\boldsymbol{R}}\times\boldsymbol{B}$ to describe the field generated by e.

Clearly $\boldsymbol{B}\perp\boldsymbol{E}$ and $\boldsymbol{B}\perp\boldsymbol{E}$. The question before us: "Is $(\boldsymbol{B}+\boldsymbol{B})\perp(\boldsymbol{E}+\boldsymbol{E})$?" . . . can be formulated "Does $(\hat{\boldsymbol{R}}\times\boldsymbol{E}+\hat{\boldsymbol{R}}\times\boldsymbol{E})\cdot(\boldsymbol{E}+\boldsymbol{E})=0$?" and after a few elementary simplifications becomes "Does $(\boldsymbol{E}\times\boldsymbol{E})\cdot(\hat{\boldsymbol{R}}-\hat{\boldsymbol{R}})=0$?" Pretty clearly, (461.1) carries no such implication unless restrictive conditions are imposed upon $\boldsymbol{\beta}$, $\boldsymbol{a}$, $\boldsymbol{\beta}$ and $\boldsymbol{a}$.[275, 276]

My plan now is to describe a (remarkably simple) physical interpretation of the acceleration-independent leading term in (461). This effort will motivate the introduction of certain diagramatic devices that serve to clarify the meaning also of the 2nd term. With our physical intuition thus sharpened, we will move in the next chapter to a discussion of the "radiative process."

4. Generalized Coulomb fields. The leading term in (459/461) provides an *exact* description of $\boldsymbol{E}(x)$ and $\boldsymbol{B}(x)$ if the source—as seen from x—is unaccelerated at the moment of puncture (*i.e.*, if $\boldsymbol{a}_0=\boldsymbol{0}$), and it becomes universally exact (*i.e.*, exact for *all* fieldpoints x) for free sources (*i.e.*, for sources with rectilinear worldlines). Evidently

$$\boldsymbol{E}=\frac{e}{4\pi}\Big[\frac{1}{r^2}\frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})}(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\Big]_0 \tag{462.1}$$

$$r\equiv\gamma(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})R\quad:\quad\text{see page 359}$$

$$\boldsymbol{B}=\Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}(x)\Big]_0 \tag{462.2}$$

—which become "Coulombic" for sources seen to be at rest ($\boldsymbol{\beta}=\boldsymbol{0}$)—describe the *Lorentz transform of the electrostatic field generated by an unaccelerated*[277] *point charge*. They describe, in other words, our perception of the Coulomb field of a passing charge. Explicit proof—and interpretive commentary—is provided below.

We are, let us suppose, certifiably inertial. So also is O, whom we see to be drifting by with speed $\boldsymbol{\beta}$ (and whose habit it is to use red ink when writing

[275] PROBLEM 72.
[276] PROBLEM 73.
[277] "Unaccelerated" is, we now see, redundant—implied already by the word "electro*static*." Readers may find it amusing/useful at this point to review the ideas developed in §2 of Chapter 1.

down his physical equations). It happens (let us assume) that O's frame is related irrotationally to ours; *i.e.*, by a pure boost $\mathbb{A}(\boldsymbol{\beta})$. Then (see again §5 in Chapter 2) the coordinates which he/we assign to a spacetime point stand in the relation

$$\left.\begin{aligned} t &= \gamma t + (\gamma/c^2)\,\boldsymbol{v}\cdot\boldsymbol{x} \\ \boldsymbol{x} &= \boldsymbol{x} + \big\{\gamma t + (\gamma-1)\,(\boldsymbol{v}\cdot\boldsymbol{x})/v^2\big\}\boldsymbol{v} \end{aligned}\right\} \tag{210.1}$$

which can be notated

$$\left.\begin{aligned} \begin{pmatrix} t \\ x_\parallel \end{pmatrix} &= \gamma\begin{pmatrix} 1 & v/c^2 \\ v & 1 \end{pmatrix}\begin{pmatrix} t \\ x_\parallel \end{pmatrix} \\ \boldsymbol{x}_\perp &= \boldsymbol{x}_\perp \end{aligned}\right\} \tag{210.2}$$

while the electric/magnetic fields which he/we assign to any given spacetime point stand in the relation

$$\left.\begin{aligned} \boldsymbol{E} &= (\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\parallel + \gamma(\boldsymbol{E}-\boldsymbol{\beta}\times\boldsymbol{B})_\perp \\ \boldsymbol{B} &= (\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\parallel + \gamma(\boldsymbol{B}+\boldsymbol{\beta}\times\boldsymbol{E})_\perp \end{aligned}\right\} \tag{263}$$

Let us suppose now that <u>O sees a charge e to be sitting at his origin</u>, and no magnetic field: $\boldsymbol{E} = \frac{e}{4\pi}R^{-2}\,\hat{\boldsymbol{x}}$ and $\boldsymbol{B} = \boldsymbol{0}$. The latter condition brings major simplifications to (263): we have

$$\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp \quad\text{with}\quad \begin{cases} \boldsymbol{E}_\parallel = \boldsymbol{E}_\parallel \\ \boldsymbol{E}_\perp = \gamma\boldsymbol{E}_\perp \end{cases}$$

$$\boldsymbol{B} = \boldsymbol{B}_\parallel + \boldsymbol{B}_\perp \quad\text{with}\quad \begin{cases} \boldsymbol{B}_\parallel = \boldsymbol{0} \\ \boldsymbol{B}_\perp = \gamma(\boldsymbol{\beta}\times\boldsymbol{E})_\perp = \boldsymbol{\beta}\times\boldsymbol{E} \end{cases}$$

which we see to be time-dependent (because we see the charge to be in motion). We use the notations introduced in Figure 117 to work out the detailed meaning of the preceding statements:

O sees a radial electric field:

$$\frac{E_\parallel}{E_\perp} = \frac{R_\parallel}{R_\perp}$$

But

$$\frac{R_\parallel}{R_\perp} = \frac{\gamma R_\parallel}{R_\perp} \quad : \quad \text{the } \|\text{-side of our space triangle is Lorentz contracted}$$

$$\frac{E_\parallel}{E_\perp} = \frac{E_\parallel}{\gamma^{-1}E_\perp} \quad : \quad \text{the } \perp\text{-component of our } \boldsymbol{E}\text{-field is Lorentz dilated}$$

so

$$\frac{E_\parallel}{E_\perp} = \frac{R_\parallel}{R_\perp} \quad : \quad \text{we } \textit{also} \text{ see a radial electric field}$$

But while O sees a spherical "pincushion," we (as will soon emerge) see a

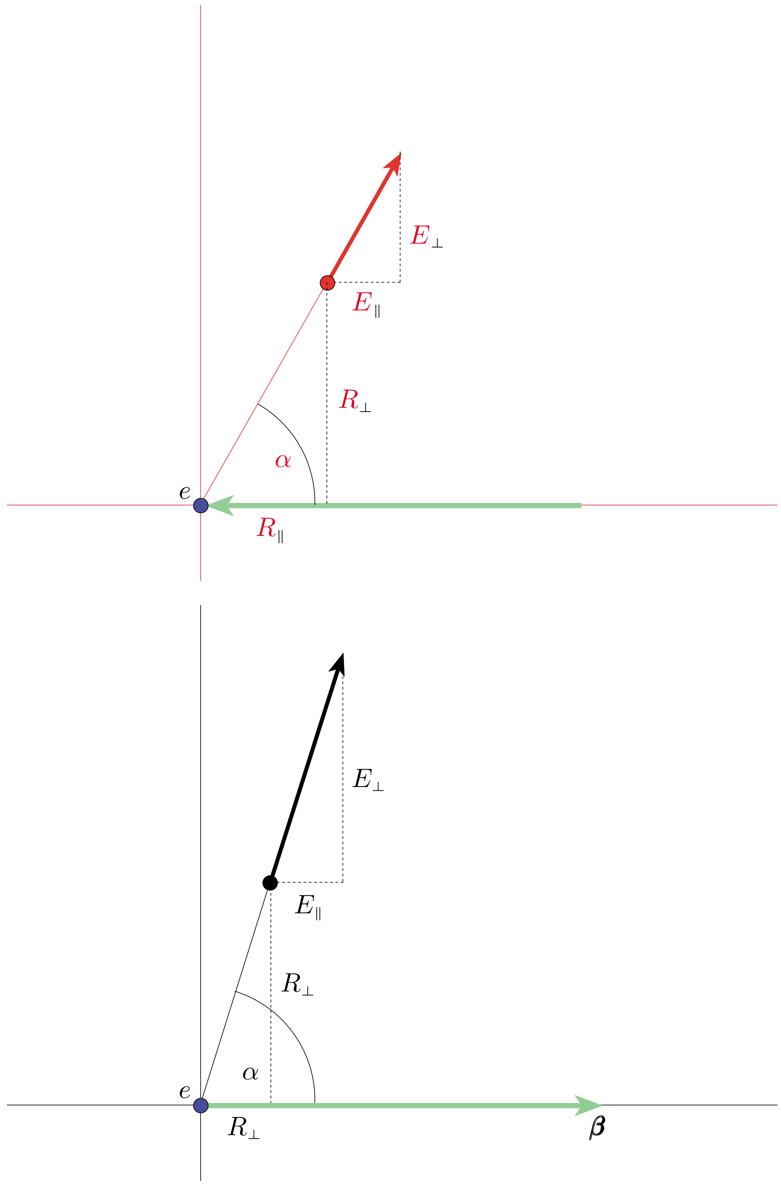

FIGURE 117: *Figures drawn on the space-plane that contains the charge •, the field-point in question, and the $\boldsymbol{\beta}$-vector with which the observer sees the other to be passing by. The upper figure defines the notation used by O to describe the Coulomb field of the charge sitting at his origin. The lower figure defines the notation we (in the text) use to describe our perception of that field.*

flattened pincushion. More precisely: O sees the field intensity to be given by

$$E = \frac{e}{4\pi R^2}\;, \text{ independently of } \alpha$$

It follows, on the other hand, from the figure that

$$E = \sqrt{(E\cos\alpha)^2 + (\tfrac{1}{\gamma}E\sin\alpha)^2} = E\sqrt{\cos^2\alpha + \tfrac{1}{\gamma^2}\sin^2\alpha}$$

so

$$E = \frac{e}{4\pi R^2}\frac{1}{\sqrt{\cos^2\alpha + \frac{1}{\gamma^2}\sin^2\alpha}}$$

Similarly,

$$R = \sqrt{(\gamma R\cos\alpha)^2 + (R\sin\alpha)^2} = \gamma R\sqrt{\cos^2\alpha + \tfrac{1}{\gamma^2}\sin^2\alpha}$$

so

$$\begin{aligned} E &= \frac{e}{4\pi R^2}\frac{1}{\gamma^2\big(\cos^2\alpha + \frac{1}{\gamma^2}\sin^2\alpha\big)^{\frac{3}{2}}} \\ &= \frac{e}{4\pi R^2}\frac{1-\beta^2}{\big(1-\beta^2\sin^2\alpha\big)^{\frac{3}{2}}} \end{aligned} \tag{463.1}$$

which is to be inserted into

$$\boldsymbol{E} = E\hat{\boldsymbol{R}} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{\beta}\times\boldsymbol{E} \tag{463.2}$$

—the upshot of which is illustrated in Figures 118 & 119.

The results developed above make intuitive good sense, but do not much resemble (462). The discrepency is illusory, and arises from the circumstance that (462) is formulated in terms of the *retarded* position $\boldsymbol{R}_0$, while (463) involves the *present* position $\boldsymbol{R}$. Working from Figures 120 & 121 we have

$$\boldsymbol{R} = \boldsymbol{R}_0 - R_0\boldsymbol{\beta}$$

which is readily seen[278] to entail

$$R = R_0\sqrt{1 - 2\hat{\boldsymbol{R}}_0\cdot\boldsymbol{\beta} + \beta^2} = R_0\sqrt{1 - 2\beta\cos\theta + \beta^2}$$

Also[278]

$$\sin^2\alpha = (R_0/R)^2\sin^2\theta = \frac{1-\cos^2\theta}{1-2\beta\cos\theta+\beta^2}$$

and with this information—together with the observation that

$$\boldsymbol{\beta}\times\boldsymbol{E} = \frac{\boldsymbol{R}_0 - \boldsymbol{R}}{R_0}\times E\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}_0\times\boldsymbol{E}$$

—it is an easy matter to recover (462) from (463).[278]

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[278] PROBLEM 74.

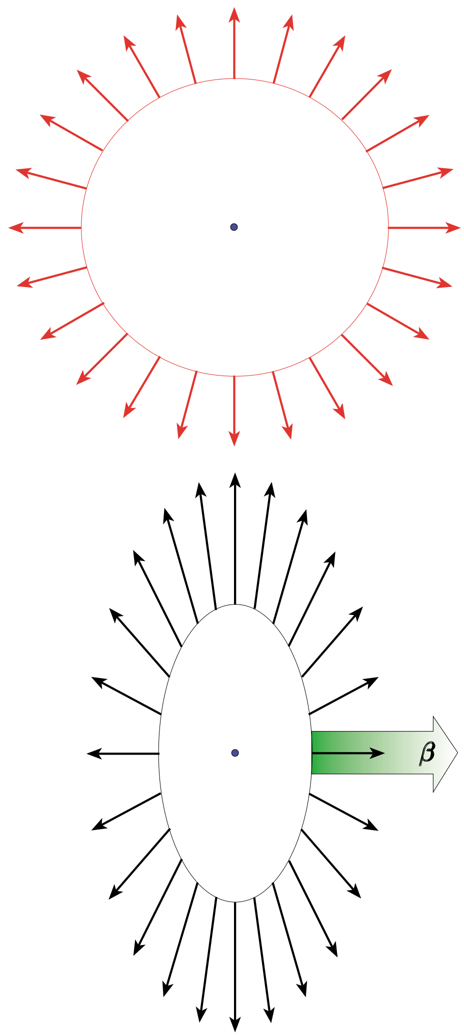

FIGURE 118: *Above: cross section of the "spherical pincushion" that O uses to represent the Coulomb field of a charge • which he sees to be at rest. We see the charge to be in uniform rectilinear motion. The "flattened pincushion" in the lower figure (axially symmetric about the $\boldsymbol{\beta}$-vector) describes our perception of that same electric field. Additionally, we see a solinoidal magnetic field given by*

$$\boldsymbol{B} = \boldsymbol{\beta} \times \boldsymbol{E}$$

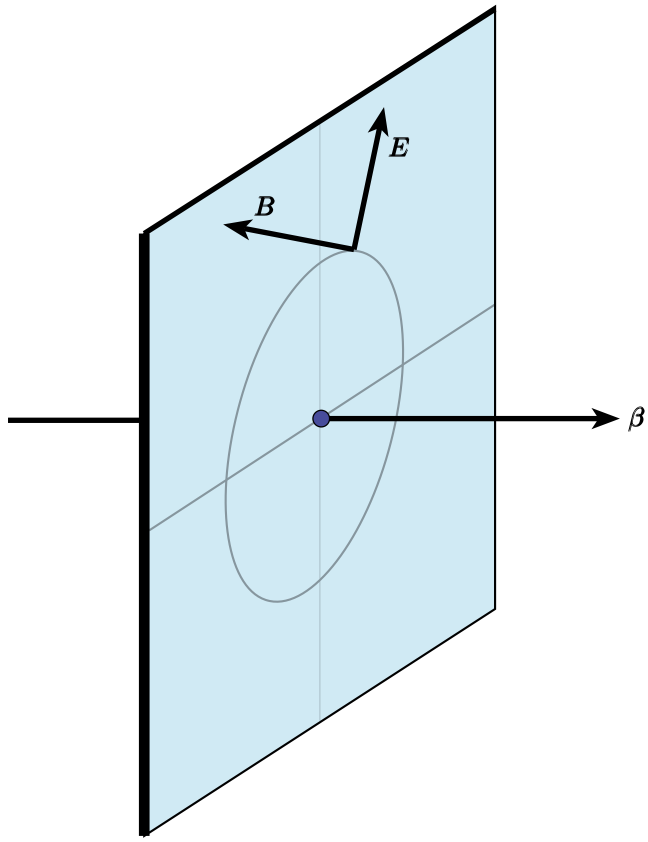

FIGURE 119: *Ultrarelativistic version of the preceding figure, showing also the solenoidal magnetic field. The "pincushion" has become a "pancake:" the field of the rapidly-moving charge is seen to be very nearly confined to a plane, outside of which it nearly vanishes, but within which it has become very strong.*

A **curious cautionary remark** is now in order. We have several times spoken casually/informally of the Coulomb fields "seen" by O and by us. Of course, one does not literally "see" a Coulomb field as one might see/photograph a passing *object* (a *literal* pincushion). The photographic appearance of an object (assume infinitely fast film and shutter) depends actually upon whether it is continuously/intermittently illuminate/self-luminous: the remarks which follow are (for simplicity) specific to continuously self-luminous objects. An object traces a "worldtube" in spacetime. The worldtubes of objects in motion (relative to us) are Lorentz-contracted in the $\boldsymbol{\beta}$-direction. What we see/ photograph is the *intersection of the Lorentz-contracted worldtube with the lightcone that extends into the past from the eye/camera*. The point—once stated—is obvious, but its surprising consequences passed unnoticed until 1959,

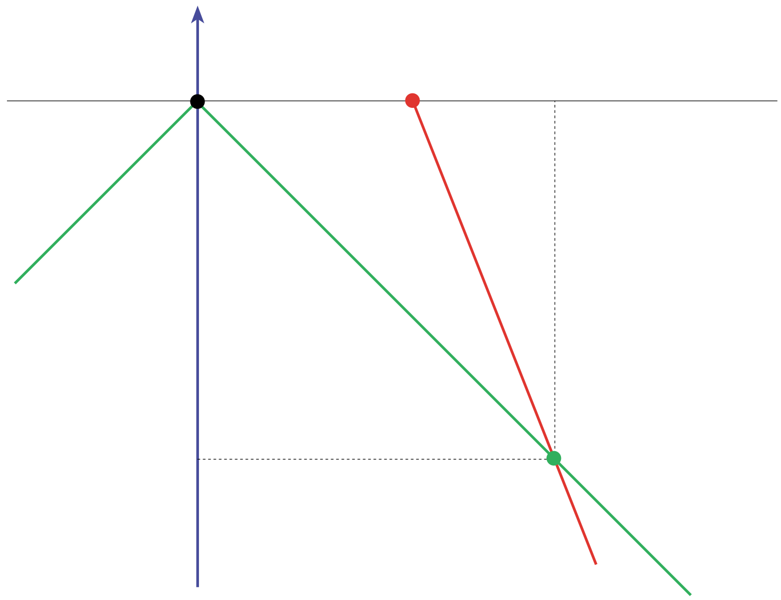

FIGURE 120: *Variant of Figure 115 in which the motion of the charge is not just "pretend unaccelerated" but really unaccelerated. In this spacetime diagram the chosen field point is marked ●, the puncture point visible from ● is marked ●, while ● marks the present position of the charge.*

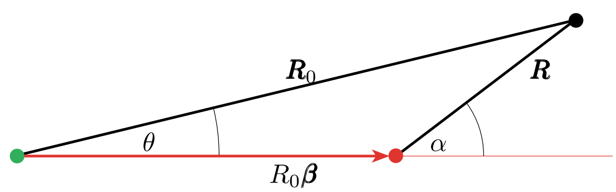

FIGURE 121: *Representation of the spatial relationship among the points ●, ● and ●, which lie necessarily in a plane. A signal proceeds ● → ● with speed c in time $T_0 = R_0/c$, during which time the charge has advanced a distance $vT_0 = \beta R_0$ in the direction $\hat{\boldsymbol{\beta}}$. This little argument accounts for the lable that has been assigned to the red base of the triangle (i.e., to the charge displacement vector).*

when they occurred independently to J. Terrell and R. Penrose. For discussion, computer-generated figures and detailed references see (for example) G. D. Scott & H. J. van Driel, "The geometrical appearance of large objects moving at relativistic speeds," AJP **33**, 534 (1965); N. C. McGill, "The apparent shape of rapidly moving objects in special relativity," Contemp. Phys. **9**, 33 (1968); Ya. A. Smorodinskiĭ & V. A. Ugarov, "Two paradoxes of the special theory of relativity," Sov. Phys. Uspekhi **15**, 340 (1972). I am sure a search would turn up also many more recent sources.

It is important to appreciate that our principal results—equations (462) and (463)—might alternatively have been derived by a *potential-theoretic line of argument*, as sketched below: O, who sees the charge e to be at rest, draws upon (363) to write

$$\begin{aligned}\boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\\ \boldsymbol{B} &= \boldsymbol{\nabla}\times\boldsymbol{A}\end{aligned}$$

where

$$A = \begin{pmatrix}\varphi\\ \boldsymbol{A}\end{pmatrix} \equiv \begin{pmatrix}e/4\pi R\\ \boldsymbol{0}\end{pmatrix}$$

entails

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi = (e/4\pi R^2)\hat{\boldsymbol{R}} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{0}$$

O sees $\boldsymbol{E}$ to be normal to the equipotentials (surfaces of constant φ), which are themselves spherical (see again the upper part of Figure 118). On the other hand we—who see the charge to be in uniform motion—write

$$A = \mathbb{A}(-\boldsymbol{\beta})A = \gamma\phi\begin{pmatrix}1\\ \boldsymbol{\beta}\end{pmatrix}$$

with

$$\phi(x) = \varphi(\boldsymbol{x}(\boldsymbol{x},t)) = \frac{e}{4\pi\sqrt{\gamma^2(\boldsymbol{x}_\parallel - \boldsymbol{v}t)\cdot(\boldsymbol{x}_\parallel - \boldsymbol{v}t) + \boldsymbol{x}_\perp\cdot\boldsymbol{x}_\perp}}$$

and (drawing similarly upon (363)) obtain

$$\begin{aligned}\boldsymbol{E} &= -\big\{\boldsymbol{\nabla} + \boldsymbol{\beta}\tfrac{1}{c}\tfrac{\partial}{\partial t}\big\}\varphi \quad \text{with} \quad \varphi \equiv \gamma\phi\\ \boldsymbol{B} &= -\big\{\boldsymbol{\beta}\times\boldsymbol{\nabla}\big\}\varphi\end{aligned}$$

from which (462/463) can (with labor) be recovered. Note that *we* consider the equipotentials to be *ellipsoidal* (see again the lower part of Figure 118), and that the $\boldsymbol{\beta}\frac{1}{c}\frac{\partial}{\partial t}\varphi$-term causes the $\boldsymbol{E}$-field to be <u>no longer normal</u> to the equipotentials.

Useful geometrical insight into analytical results such as those developed above (and in the next chapter) can be obtained if one looks to the structure of the so-called "equiphase surfaces" which (see Figure 122) are inscribed on timeslices by lightcones projected forward from source points. The points which collectively comprise an equiphase surface "share a puncture point," but in the general case (*i.e.*, except when the source is seen to be momentarily at rest) share little else. To the experienced eye they do, however, indicate at least the qualitative essentials of field structure . . . as will emerge.

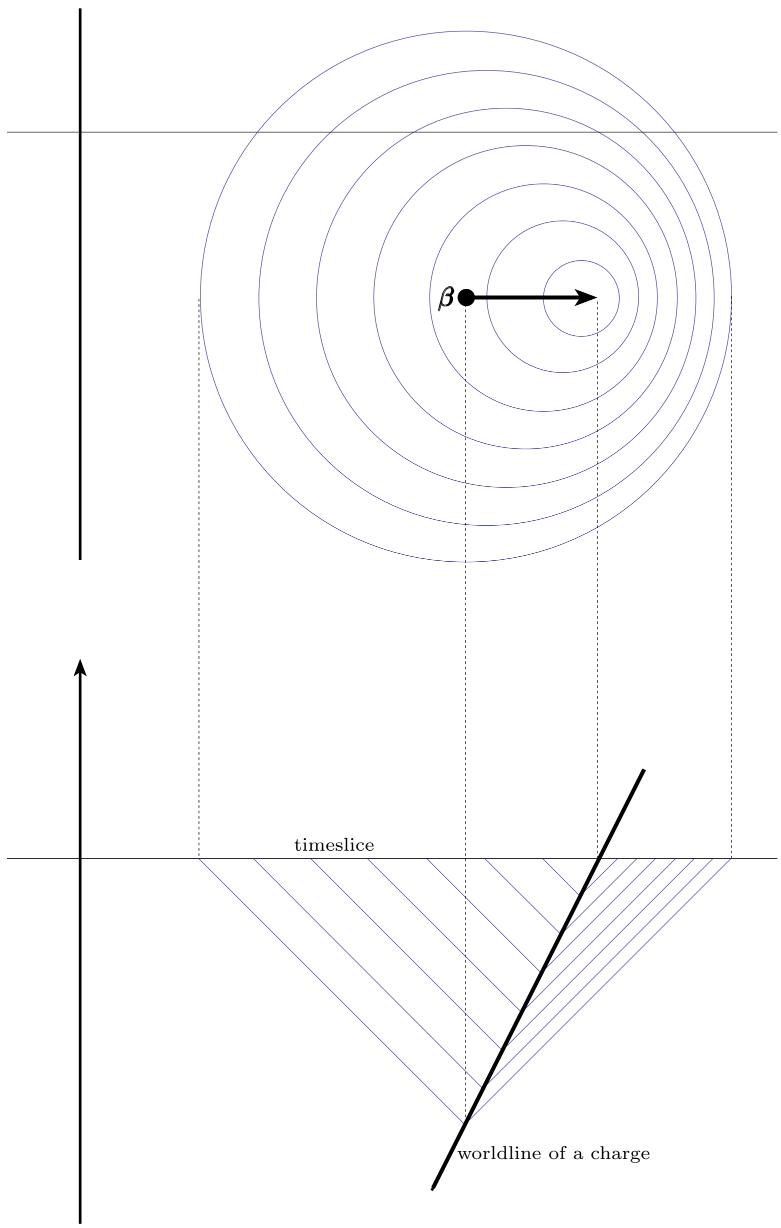

FIGURE 122: *Above: "equiphase surfaces" inscribed on a timeslice by (in this instance) a solitary charge in uniform motion (lower spacetime diagram). More complicated variants of the figure will be encountered in the next chapter.*

7

RADIATIVE PROCESSES

Introduction. It was established in §4 of the preceding chapter that the leading term on the right side of (459/461)—the acceleration-independent term that falls off as $1/r^2$—admits straightforwardly of interpretation as the Coulomb field of the source, as seen from the field point, where the phrase "as seen from" alludes to

- a "retardation effect:" the field point senses not the "present location" of the source (a notion that relativity declares to be meaningless) but the location of the puncture point—the point at which the worldline of the source punctured the lightcone that extends backward from the field point (a notion that *does* make relativistic good sense);
- the fact that if the field point sees the source to be *moving* at the moment of puncture then it sees not the familiar "Coulomb field of a charge at rest" but a *Lorentz transform* of that field.

We turn now to discussion of the structure and physical ramifications of the remaining term on the right side of (459/461)—the acceleration-dependent term that falls off as $1/r^1$. This is physics for which elementary experience provides no sharp intuitive preparation, but which lies at the base of much that is most characteristic of classical electrodynamics. The details are occasionally a bit intricate, and their theoretical/phenomenological/technological consequences remarkably diverse . . . which is why I give the subject a chapter of its own.

1. Radiation fields. Dropping the Coulombic component from the field (459) of a moving charge we obtain the radiation field

$$F^{\mu\nu} = \frac{e}{4\pi c^2}\Big[\frac{1}{r}\big\{(b^\mu a^\nu - b^\nu a^\mu) + (w^\mu a^\nu - w^\nu a^\mu) - (aw)(w^\mu b^\nu - w^\nu b^\mu)\big\}\Big]_0$$

But (see again page 359)

$$w_0^\mu = \Big[\frac{R^\mu}{r} - b^\mu\Big]_0$$
$$b^\mu \equiv \tfrac{1}{c}u^\mu$$
$$r \equiv \tfrac{1}{c}R_\alpha u^\alpha = (Rb) = \gamma(1-\beta_\parallel)R$$

so after a short calculation we find

$$\begin{aligned} F^{\mu\nu} &= \frac{e}{4\pi}\Big[\frac{1}{(Ru)^2}\Big\{(R^\mu a^\nu - R^\nu a^\mu) - \frac{(Ra)}{(Ru)}(R^\mu u^\nu - R^\nu u^\mu)\Big\}\Big]_0 \\ &= \frac{e}{4\pi}\Big[\frac{1}{(Ru)^2}(R^\mu a_\perp^\nu - R^\nu a_\perp^\mu)\Big]_0 \end{aligned} \tag{464.1}$$

where
$$a_\perp^\mu \equiv a^\mu - \frac{(Ra)}{(Ru)}u^\mu \tag{464.2}$$

is (in the Lorentzian sense) $\perp$ to R^μ: $(Ra_\perp) = 0$. Note the manifest covariance of this rather neat result.

3-vector notation—though contrary to the spirit of the principle of manifest covariance, and though always uglier—is sometimes more useful. Looking back again, therefore, to (461), we observe that[279]

$$\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\boldsymbol{a}\big) = -(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})\underbrace{\Big\{\boldsymbol{a} - \frac{\hat{\boldsymbol{R}}\cdot\boldsymbol{a}}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\Big)\Big\}}_{\perp\,\boldsymbol{R}}$$

and that on this basis the radiative part of (461) can be written[280]

$$\boldsymbol{E} = -\frac{e}{4\pi c^2}\Big[\frac{1}{R(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^2}\Big\{\boldsymbol{a} - \frac{\hat{\boldsymbol{R}}\cdot\boldsymbol{a}}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\Big)\Big\}\Big]_0 \tag{465.1}$$

$$\boldsymbol{B} = \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}\Big]_0 \tag{465.2}$$

Equations (464) & (465) provide notationally distinct but physically equivalent descriptions of the radiation field generated by an accelerated point charge.

It is instantaneously possible to have $\boldsymbol{v} = \boldsymbol{0}$ but $\boldsymbol{a} \neq \boldsymbol{0}$; *i.e.*, for a point momentarily at rest to be accelerating. In such a circumstance (465.1) becomes

$$\begin{aligned} \boldsymbol{E} &= -\frac{e}{4\pi c^2}\Big[\frac{1}{R}\boldsymbol{a}_\perp\Big]_0 \\ &\qquad \boldsymbol{a}_\perp = \boldsymbol{a} - (\hat{\boldsymbol{R}}\cdot\boldsymbol{a})\hat{\boldsymbol{R}} = -\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a}) \\ &= \frac{e}{4\pi c^2}\Big[\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})\Big]_0 \end{aligned} \tag{466}$$

with consequences which are illustrated in Figures 123 & 124.

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[279] PROBLEM 75.

[280] We make use here of $r \equiv \gamma(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})R$: see again page 359.

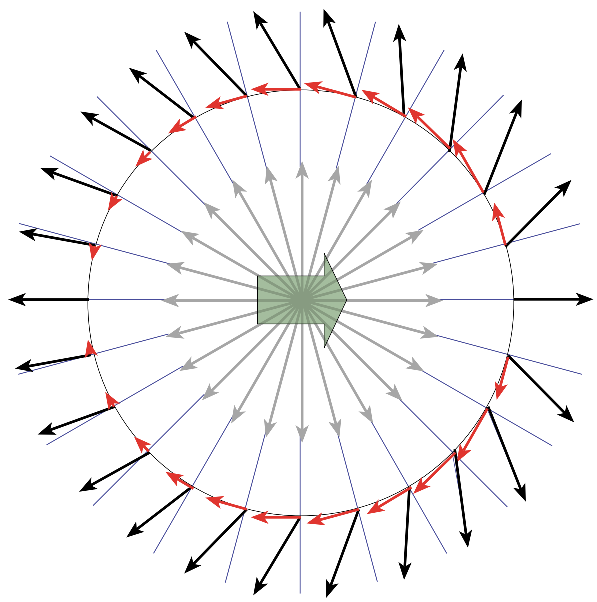


FIGURE 123: *Electric field at points that look back to the same puncture point, where they see the charge to be momentarily at rest but accelerating (in the direction indicated by the green arrow). The red* $\boldsymbol{E}$*-vectors arise from the radiative term (466). Addition of the Coulombic component produces the black* $\boldsymbol{E}$*-vectors. The grey arrows are unit vectors* $\hat{\boldsymbol{R}}$*. The figure is deceptive in one respect: every* $\boldsymbol{E}$*-vector on the left should, according to (466), have the same length as its counterpart on the right.*

The intricate details of (461) are well-adapted to computer-graphic analysis. In this connection every student of electrodynamics should study the classic little paper by R. Y. Tsien,[281] from which I have taken Figures 125–128. Tsien assumes the source orbit to lie in every case in a plane, and it is in that plane that he displays the "electric lines of force." From his figures one can read off the *direction* of the retarded $\boldsymbol{E}$-field, but information pertaining directly to the *magnitude* of the $\boldsymbol{E}$-field (and all information pertaining to the $\boldsymbol{B}$-field) has been discarded. Nor does Tsien attempt to distinguish the radiative from the Coulombic component of $\boldsymbol{E}$.

[281] "Pictures of Dynamic Electric Fields," AJP **40**, 46 (1972). Computers and software have come a very long way in thirty years: the time is ripe for someone to write (say) a *Mathematica* program that would permit students to do interactively/experimentally what Tsien labored so hard to do with relatively primitive resources. Tsien, by the way, is today a well-known biophysicist, who in 1972 was still an undergraduate at Harvard, a student of E. M. Purcell, whose influential *Electricity & Magnetism* (Berkeley Physics Course, Volume II) was then recent.

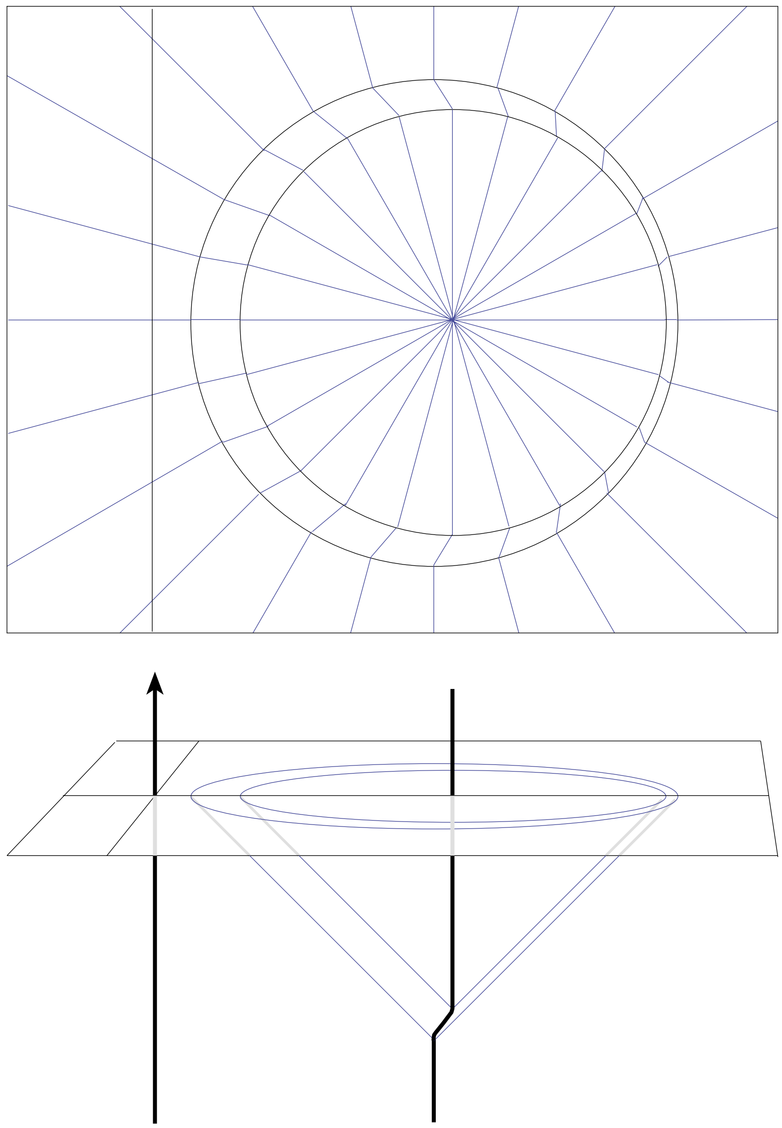


Figure 124: *Shown below: the worldline of a charged particle—initially at rest—that begins abruptly to accelerate to the right, then promptly decelerates, returning again to rest. Shown above is the resulting* $\boldsymbol{E}$*-field. The remote radial section is concentric about the original position, the inner radial section is concentric about the altered position. The acceleration-dependent interpolating field has the form shown in Figure 123. Indeed: it was from this figure—not (466)—that I took the details of Figure 123. The next figure speaks more precisely to the same physics.*

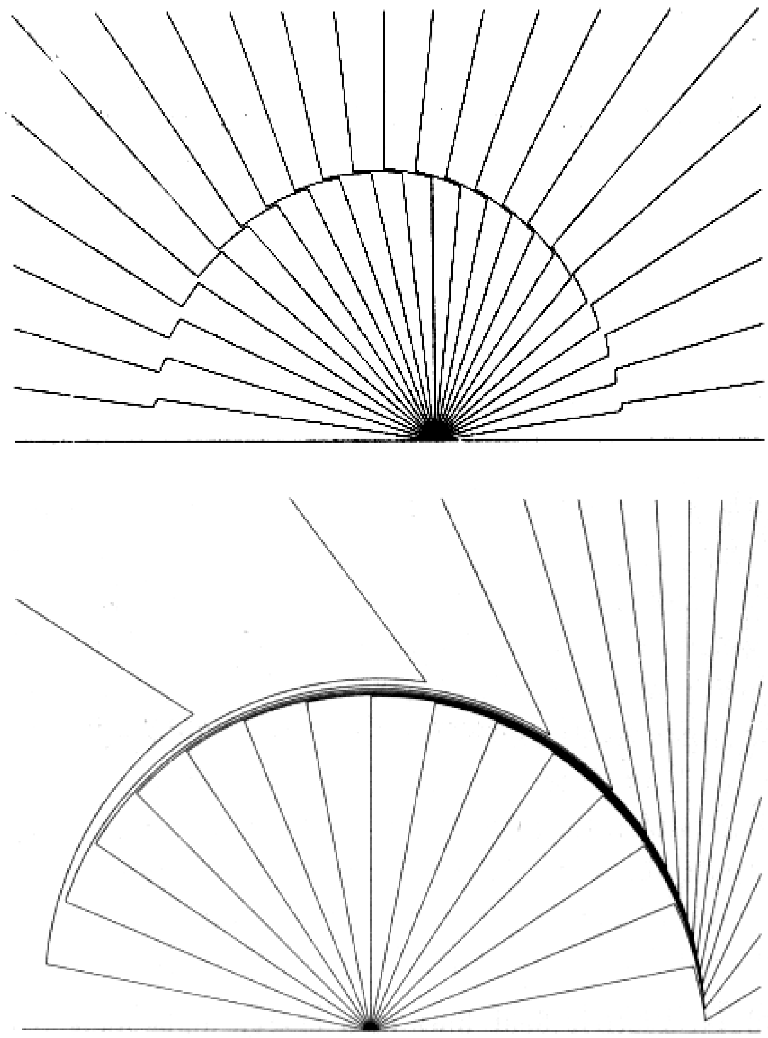


FIGURE 125: *Snapshots of electric field lines derived from the* $\boldsymbol{E}$*-field generated by a charge which abruptly decelerates while moving in the* $\rightarrow$ *direction. The initial velocity was* $\beta = 0.20$ *in the upper figure,* $\beta = 0.95$ *in the lower figure. I am indebted to Fred Lifton for the digitization of Tsien's figures, and regret that the available technology so seriously degraded the quality of Tsien's wonderfully sharp images. See the originals in Tsien's paper*[264]*... or better: run Tsien's algorithm on* Mathematica *to produce* <u>*animated*</u> *versions of the figures.*

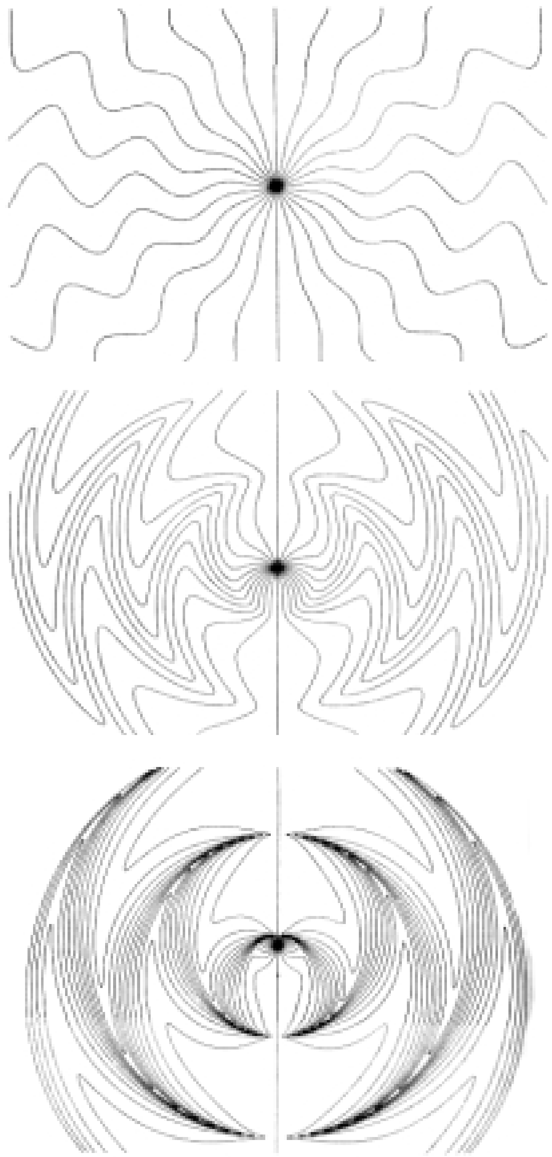


FIGURE 126: *Snapshots of the electric field lines generated by a charge undergoing simple harmonic motion in the* $\updownarrow$ *direction. In the upper figure* $\beta_{\rm max} = 0.10$*, in the middle figure* $\beta_{\rm max} = 0.50$*, in the lower figure* $\beta_{\rm max} = 0.90$.

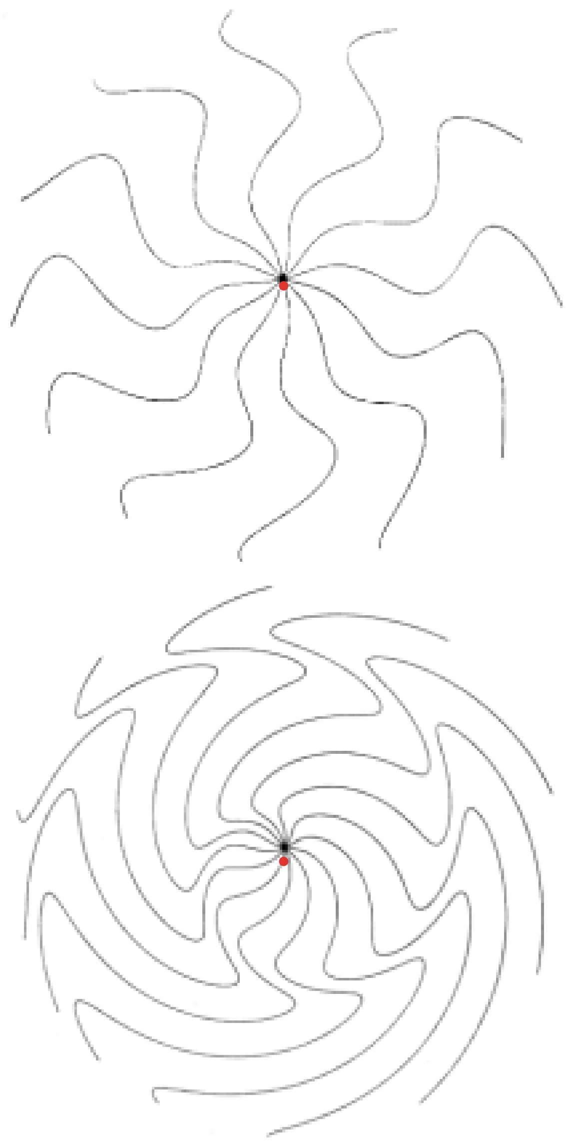


FIGURE 127: *Snapshots (not to the same scale) of the electric field lines generated by a charge undergoing uniform circular motion* ↻ *about the point marked* •. *In the upper figure* $\beta = 0.20$*, in the lower figure* $\beta = 0.50$*. In the upper figure the field is—pretty evidently—dominated by the Coulombic component of* (459/461).

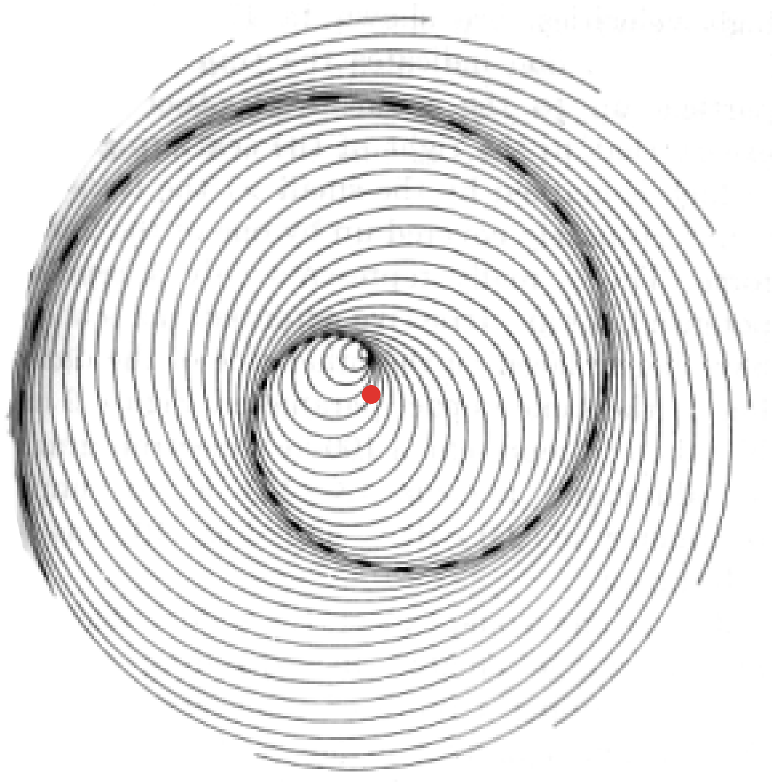


FIGURE 128: *Enlargement of the same physics as that illustrated in Figure 127, except that now* $\beta = 0.95$. *The figure can be animated by placing it on a phonograph turntable: since phonographs turn* ↻ *the spiral will appear to expand. Beyond a certain radius the field lines will appear to move faster than the speed of light. That violates no physical principle, since the field lines themselves are diagramatic fictions: marked features of the field (for example: the kinks) are seen* not *to move faster than light. At such high speeds the field is dominated by the radiative part of* (459/461). *This is "synchrotron radiation," and (as Tsien remarks) the kinks account for the rich harmonic content of relativistic synchrotron radiation.*

2. Energetics of fields produced by a single source. To discuss this topic all we have in principle to do is to introduce (459/461)—which describe the *field generated by a point charge in arbitrary motion*—into (309, page 215)—which describes the *stress/energy/momentum associated with an arbitrarily prescribed electromagnetic field*. The program is clear-cut, but the details can easily become overwhelming . . . and we are forced to look only at the <u>physically most characteristic/revealing features of the physically most important special cases</u>.

The experience thus gained will, however, make it relatively easy to think *qualitatively* about more realistic/complex problems.

We will need to know (see again page 216) that

$$\begin{aligned} \mathcal{E} &= \tfrac{1}{2}(E^2 + B^2) && \text{describes } \textbf{energy density} \\ \boldsymbol{S} &= c(\boldsymbol{E}\times\boldsymbol{B}) && \text{describes } \textbf{energy flux} \\ \boldsymbol{\mathcal{P}} &= \tfrac{1}{c}(\boldsymbol{E}\times\boldsymbol{B}) && \text{describes } \textbf{momentum density} \end{aligned}$$

but will have no direct need of the other nine components $\mathbb{T}$ of the stress-energy tensor $S^{\mu\nu}$. Mechanical properties of the fields generated by *accelerated sources* lie at the focal point of our interest, but to place that physics in context we look first to a couple of simpler special cases:

FIELD ENERGY/MOMENTUM OF A CHARGE AT REST In the rest frame of an unaccelerated charge e we have

$$\boldsymbol{E} = \frac{e}{4\pi}\frac{1}{R^2}\hat{\boldsymbol{R}} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{0}$$

giving

$$\mathcal{E} = \tfrac{1}{2}\Big(\frac{e}{4\pi}\Big)^2\frac{1}{R^4} \qquad \text{and} \qquad \boldsymbol{S} = \boldsymbol{\mathcal{P}} = \boldsymbol{0}$$

If (as in PROBLEM 10) we center a (mental) sphere of radius a on the charge we find the <u>field energy exterior to the sphere</u> to be given by

$$W(a) = \int_a^\infty \mathcal{E}(R)4\pi R^2\,dR = \frac{e^2}{8\pi a} \tag{467}$$

. . . which—“self-energy problem”—becomes infinite as $a \downarrow 0$, and which when we set

$$= mc^2$$

gives rise to the “classical radius” $a = e^2/8\pi mc^2$ of the massive point charge e.

FIELD ENERGY/MOMENTUM OF A CHARGE IN UNIFORM MOTION Drawing now upon (463) we have

$$\boldsymbol{E} = \frac{e}{4\pi\gamma^2}\frac{1}{\big(1-\beta^2\sin^2\alpha\big)^{\frac{3}{2}}}\frac{1}{R^2}\hat{\boldsymbol{R}} \qquad \text{and} \qquad \boldsymbol{B} = \boldsymbol{\beta}\times\boldsymbol{E}$$

But $B^2 = (\boldsymbol{\beta}\times\boldsymbol{E})\cdot(\boldsymbol{\beta}\times\boldsymbol{E}) = (\boldsymbol{\beta}\cdot\boldsymbol{\beta})(\boldsymbol{E}\cdot\boldsymbol{E}) - (\boldsymbol{\beta}\cdot\boldsymbol{E})^2 = \beta^2E^2\sin^2\alpha$, so

$$\mathcal{E} = \tfrac{1}{2}\Big(\frac{e}{4\pi\gamma^2}\Big)^2\frac{1+\beta^2\sin^2\alpha}{\big(1-\beta^2\sin^2\alpha\big)^3}\frac{1}{R^4}$$

The momentum density $\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ is oriented as shown in the first of the following figures. From

$$\mathcal{P}^2 = \tfrac{1}{c^2}\big\{(\boldsymbol{E}\cdot\boldsymbol{E})(\boldsymbol{B}\cdot\boldsymbol{B}) - (\boldsymbol{E}\cdot\boldsymbol{B})^2\big\} = \tfrac{1}{c^2}E^2B^2 = \tfrac{1}{c^2}\beta^2E^4\sin^2\alpha$$

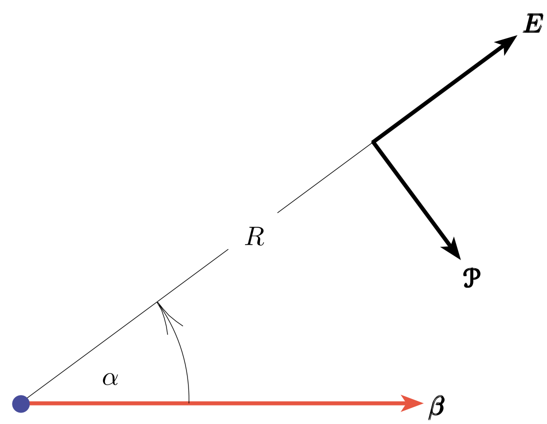


FIGURE 127: *The solinoidal $\boldsymbol{B}$ field is up out of page at the point shown, so $\boldsymbol{\mathcal{P}} = \frac{1}{c}(\boldsymbol{E}\times\boldsymbol{B})$ lies again on the page. Only $\mathcal{P}_{\parallel}$—the component parallel to $\boldsymbol{\beta}$—survives integration over all of space..*

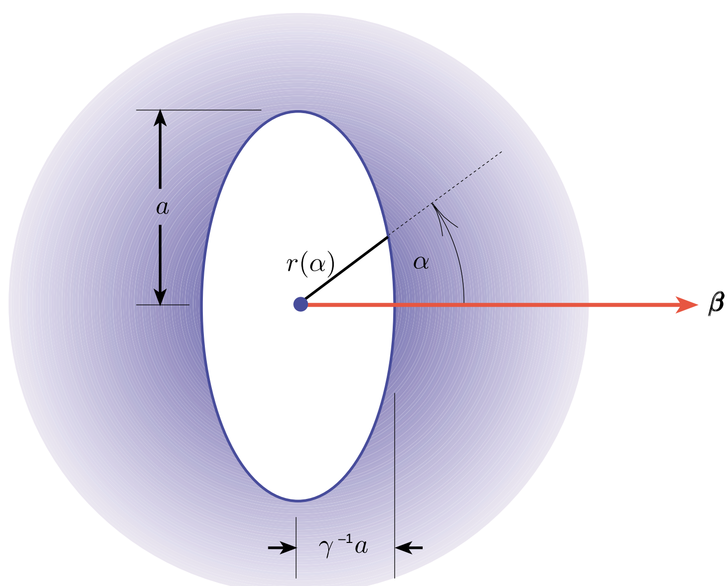


FIGURE 130: *Lorentz contracted geometry of what in the rest frame of the charge was the familiar "sphere of radius a," exterior to which we compute the total energy and total momentum. The figure is rotationally symmetric about the $\boldsymbol{\beta}$-axis.*

we find that the magnitude of $\mathcal{P}$ is given by

$$\mathcal{P} = \tfrac{1}{c}\beta E^2 \sin\alpha = \tfrac{1}{c}\beta\Big(\frac{e}{4\pi\gamma^2}\Big)^2 \frac{1}{\big(1-\beta^2\sin^2\alpha\big)^3}\frac{1}{R^4}$$

Turning now to the evaluation of the integrated *field energy and field momentum exterior to the spherical region considered previously*—a region which appears now to be Lorentz contracted (see the second of the figures on the preceding page)—we have

$$W = \int_0^\pi \int_{r(\alpha)}^\infty \mathcal{E}\cdot 2\pi R^2 \sin\alpha \, dR d\alpha \tag{468.1}$$

and $\boldsymbol{P} = P\hat{\boldsymbol{\beta}}$ with

$$P = \int_0^\pi \int_{r(\alpha)}^\infty \mathcal{P}\sin\alpha\cdot 2\pi R^2 \sin\alpha \, dR d\alpha \tag{468.2}$$

where $r(\alpha)$, as defined by the figure, is given[282] by

$$r(\alpha) = \frac{a}{\gamma}\frac{1}{\sqrt{1-\beta^2\sin^2\alpha}}$$

The R-integrals are trivial: we are left with

$$\begin{aligned} W &= \quad \pi\Big(\frac{e}{4\pi\gamma^2}\Big)^2\frac{\gamma}{a}\int_0^\pi \frac{1}{(1-\beta^2\sin^2\alpha)^{\frac{5}{2}}}\big\{\sin\alpha + \beta^2\sin^3\alpha\big\}\,d\alpha \\ P &= \frac{\beta}{c}2\pi\Big(\frac{e}{4\pi\gamma^2}\Big)^2\frac{\gamma}{a}\int_0^\pi \frac{1}{(1-\beta^2\sin^2\alpha)^{\frac{5}{2}}}\sin^3\alpha\,d\alpha \end{aligned}$$

Entrusting the surviving integrals to *Mathematica*, we are led to results that can be written[283]

$$W = \big(1-\tfrac{1}{4\gamma^2}\big)\cdot\gamma M c^2 \tag{469.1}$$

$$\boldsymbol{P} = \gamma M\boldsymbol{v} \tag{469.2}$$

with

$$M \equiv \frac{4}{3}\frac{e^2}{8\pi a c^2} = \tfrac{4}{3}m \tag{470}$$

---

[282] The argument runs as follows: we have

$$\frac{x^2}{(a/\gamma)^2} + \frac{y^2}{a^2} = 1 \quad \text{whence} \quad \gamma^2(r\cos\alpha)^2 + (r\sin\alpha)^2 = a^2$$

Divide by γ^2 and obtain

$$r^2(1-\sin^2\alpha) + (1-\beta^2)r^2\sin^2\alpha = (a/\gamma)^2$$

Simplify, solve for r.

[283] PROBLEM 76.

The curious velocity-dependent factor

$$\left(1-\tfrac{1}{4\gamma^2}\right)=\begin{cases}\tfrac{3}{4} & : \quad \beta=0\\[2ex] 1 & : \quad \beta=1\end{cases}$$

Were that factor absent (which is to say: in the approximation that $\left(1-\frac{1}{4\gamma^2}\right)\sim 1$) we would have

$$P^0\equiv \tfrac{1}{c}W=\tfrac{4}{3}m\cdot\gamma c \qquad\text{and}\qquad \boldsymbol{P}=\tfrac{4}{3}m\cdot\gamma\boldsymbol{v}$$

which (see again (276) page 193) we recognize to be the relativistic relationship between the energy and momentum of a free particle with mass $\frac{4}{3}m$. This fact inspired an ill-fated attempt by M. Abraham, H. Poincaré, H. A. Lorentz and others ($\sim$ 1900, immediately *prior* to the invention of relativity) to develop an "**electromagnetic theory of mass**,"[284] distant echos of which can be detected in modern theories of elementary particles. We note in passing that

- (469.1) gives back (467) in the limit $v\downarrow 0$: the $\frac{3}{4}$ neatly cancels the curious $\frac{4}{3}$, which would not happen if (on some pretext) we yielded to the temptation to drop the otherwise unattractive $\left(1-\frac{1}{4\gamma^2}\right)$-factor.
- Equations (469) and (467) are <u>not boost-equivalent</u>:

$$\begin{pmatrix}W/c\\ \boldsymbol{P}\end{pmatrix}\neq\mathbb{A}(\boldsymbol{v})\begin{pmatrix}mc\equiv e^2/8\pi ac\\ \mathbf{0}\end{pmatrix}$$

  The reason is that $P^0\equiv W/c$ and $\boldsymbol{P}$ arise by integration from a *subset* $S^{\mu 0}$ of the sixteen components of the $S^{\mu\nu}$ tensor, and the four elements of the subset are not transformationally disjoint from the other twelve components.

- It becomes rather natural to ask: Could a more satisfactory result be achieved if we assumed that Maxwell's equations must be modified in the close proximity of charges? That relativity breaks down at small distances?

3. Energy radiated by an accelerated charge momentarily at rest. It is in the interest mainly of analytical simplicity that we now assume $\boldsymbol{v}=\mathbf{0}$, a condition that (when $\boldsymbol{a}\neq\mathbf{0}$) can hold only instantaneously. But the calculation is less artificial than might at first appear: it leads to <u>results that are *nearly exact in the non-relativistic regime* $v\ll c$</u>.

[284] For a good general review—with bibliography—see R. L. Dendy, "A history of the Abraham–Lorentz electromagnetic theory of mass" (Reed College, 1964). See also Chapter 2 in F. Rohrlich, *Classical Charged Particles* (1965) and R. P. Feynman's *Lectures on Physics* (1964), Volume II, Chapter 28.

Borrowing now from (461) we have (set $\boldsymbol{\beta} = \mathbf{0}$)

$$\begin{aligned} \boldsymbol{E} &= \frac{e}{4\pi}\Big[\frac{1}{R^2}\hat{\boldsymbol{R}}\Big]_0 + \frac{e}{4\pi c^2}\Big[\frac{1}{R}\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})\Big]_0 \equiv \boldsymbol{E}^{\mathsf{C}} + \boldsymbol{E}^{\mathsf{R}} \\ \boldsymbol{B} &= \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}\Big]_0 \equiv \boldsymbol{B}^{\mathsf{C}} + \boldsymbol{B}^{\mathsf{R}} \end{aligned}$$

where the superscript $^{\mathsf{C}}$ identifies the "Coulombic component," and $^{\mathsf{R}}$ the "radiative component." We want to study energy loss (radiation from the vicinity of the charge) so we look not to $\mathcal{E}$ or $\boldsymbol{\mathcal{P}}$ but the energy *flux* vector

$$\begin{aligned} \boldsymbol{S} &= c(\boldsymbol{E}\times\boldsymbol{B}) \\ &= \boldsymbol{S}^{\mathsf{CC}} + \boldsymbol{S}^{\mathsf{CR}} + \boldsymbol{S}^{\mathsf{RC}} + \boldsymbol{S}^{\mathsf{RR}} \quad \text{where} \quad \begin{cases} \boldsymbol{S}^{\mathsf{CC}} \equiv c(\boldsymbol{E}^{\mathsf{C}}\times\boldsymbol{B}^{\mathsf{C}}) \sim 1/R^4 \\ \boldsymbol{S}^{\mathsf{CR}} \equiv c(\boldsymbol{E}^{\mathsf{C}}\times\boldsymbol{B}^{\mathsf{R}}) \sim 1/R^3 \\ \boldsymbol{S}^{\mathsf{RC}} \equiv c(\boldsymbol{E}^{\mathsf{R}}\times\boldsymbol{B}^{\mathsf{C}}) \sim 1/R^3 \\ \boldsymbol{S}^{\mathsf{RR}} \equiv c(\boldsymbol{E}^{\mathsf{R}}\times\boldsymbol{B}^{\mathsf{R}}) \sim 1/R^2 \end{cases} \end{aligned}$$

$\boldsymbol{S}^{\mathsf{CC}}$, $\boldsymbol{S}^{\mathsf{CR}}$ and $\boldsymbol{S}^{\mathsf{RC}}$ may be of importance—even dominant importance—in the "near zone," but they *fall off faster than geometrically*: only $\boldsymbol{S}^{\mathsf{RR}}$ can pertain to the "transport of energy to infinity"—the process of present concern. We look therefore to

$$\boldsymbol{S}^{\mathsf{RR}} = c(\boldsymbol{E}^{\mathsf{R}}\times\boldsymbol{B}^{\mathsf{R}}) \tag{471}$$

with

$$\begin{aligned} \boldsymbol{B}^{\mathsf{R}} &= \Big[\hat{\boldsymbol{R}}\times\boldsymbol{E}^{\mathsf{R}}\Big]_0 \\ \boldsymbol{E}^{\mathsf{R}} &= \frac{e}{4\pi c^2}\Big[\frac{1}{R}\hat{\boldsymbol{R}}\times(\hat{\boldsymbol{R}}\times\boldsymbol{a})\Big]_0 \end{aligned}$$

Clearly $\hat{\boldsymbol{R}}\cdot\boldsymbol{E} = 0$ so $\boldsymbol{E}\times(\hat{\boldsymbol{R}}\times\boldsymbol{E}) = (\boldsymbol{E}\cdot\boldsymbol{E})\hat{\boldsymbol{R}} - (\hat{\boldsymbol{R}}\cdot\boldsymbol{E})\boldsymbol{E}$ gives[285]

$$\begin{aligned} \boldsymbol{S} &= S\hat{\boldsymbol{R}} \\ S &= c(\boldsymbol{E}\cdot\boldsymbol{E}) = \frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\Big(\frac{a}{R}\Big)^2\sin^2\vartheta \end{aligned} \tag{472}$$

where $\vartheta \equiv$ (angle between $\hat{\boldsymbol{R}}$ and $\boldsymbol{a}$). The temporal rate at which field energy is seen ultimately to stream through the remote surface differential $\boldsymbol{d\sigma}$ is given by $dP = \boldsymbol{S}\cdot\boldsymbol{d\sigma}$. But $d\Omega \equiv R^{-2}\hat{\boldsymbol{R}}\cdot\boldsymbol{d\sigma}$ is just the *solid angle* subtended (at e) by $\boldsymbol{d\sigma}$. We conclude that the power radiated into the solid angle $d\Omega$ is given by

$$dP = \underbrace{\Big\{\frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)a^2\sin^2\vartheta\Big\}}_{\text{so-called "sine squared distribution"}}d\Omega \tag{473}$$

The "sine squared distribution" will be shown to be characteristic of *dipole radiation*, and has the form illustrated in the first of the following figures.

[285] PROBLEM 77. Here and henceforth I drop the superscripts $^{\mathsf{R}}$.

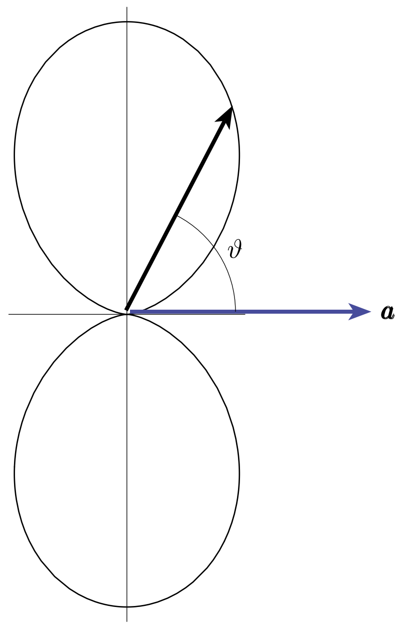


FIGURE 131: *The "sine squared distribution" arises when* $\boldsymbol{v} \sim \mathbf{0}$ *but* $\boldsymbol{a} \neq \mathbf{0}$. *The distribution is axially symmetric about the* $\boldsymbol{a}$*-vector, and describes the relative amounts of energy dispatched in various* ϑ*-directions. The radiation is predominantly* $\perp$ *to* $\boldsymbol{a}$.

Integrating over the "sphere at infinity" we find the instantaneous *total radiated power* to be given by[286]

$$P = \frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)a^2 \cdot 2\pi \int_0^\pi \sin^2\vartheta \, d\vartheta = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{a^2}{c^3} \tag{474}$$

This is the famous **Larmor formula**, first derived by Joseph Larmor in 1897. The following figure schematizes the physical assumptions which underlie (474). We note that while energy may also be dispatched into the solid angle $d\Omega$ by the $\boldsymbol{S}^{\mathsf{CC}}$, $\boldsymbol{S}^{\mathsf{CR}}$ and $\boldsymbol{S}^{\mathsf{RC}}$ it is attenuated too rapidly to contribute to the net "energy flux across the sphere at infinity."

From the c^{-3}-dependence of P_{Larmor} we conclude that *it is not easy to radiate*. Finally, I would emphasize once again that we can expect Larmor's formula to pertain in good approximation *whatever* the non-relativistic (!) motion of the source.

4. Energy radiated by a charge in arbitrary motion. When one turns to the general case the basic strategy (study $\boldsymbol{S}^{\mathsf{RR}}$ in the far zone) is unchanged, but the details[287] become a good deal more complicated. In the interests of brevity

[286] PROBLEM 78.
[287] See CLASSICAL RADIATION (1974), pages 558–571.

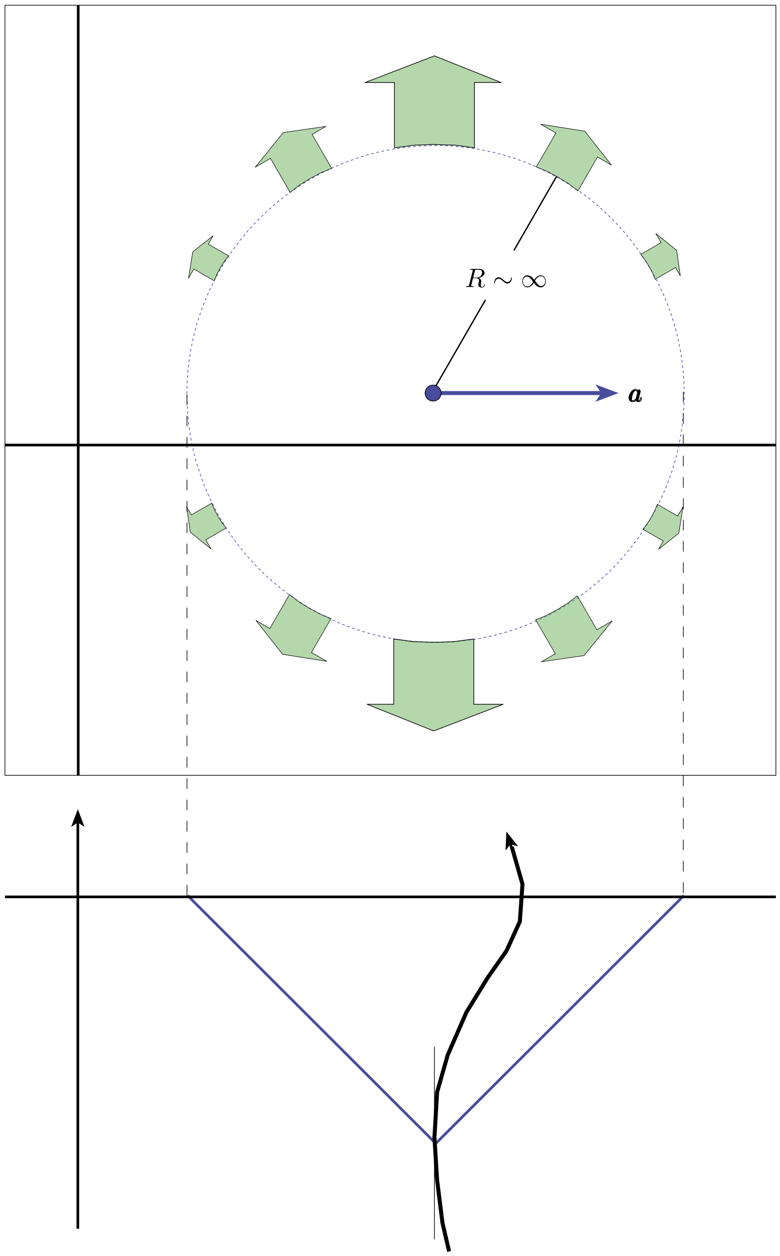


Figure 132: *Above: representation of the sine-squared radiation pattern produced by a charge seen (below) at the moment of puncture to have* $\boldsymbol{v} \sim \mathbf{0}$ *but* $\boldsymbol{a} \neq \mathbf{0}$.

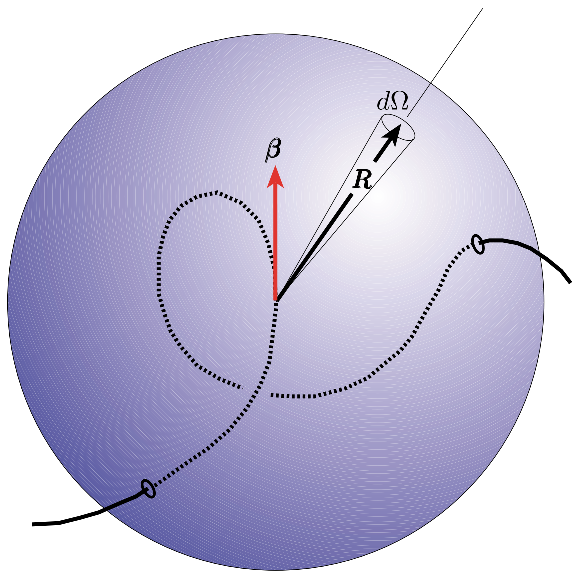


FIGURE 133: *A charged particle e pursues an arbitrary path in physical 3-space. We are concerned with the energy radiated into the solid angle $d\Omega$ identified by the direction vector $\boldsymbol{R}$. The vector $\boldsymbol{\beta}$ refers to the particle's velocity at the radiative moment, and—adhering to the convention introduced in Figures 127 & 128—we write*

$$\alpha \equiv \textit{angle between } \boldsymbol{R} \textit{ and } \boldsymbol{\beta}$$

No attempt has been made here to represent the instantaneous acceleration vector $\boldsymbol{a}$.

and clarity I must therefore be content to report and discuss here only the *results* of the detailed argument. It turns out that (see the preceding figure) an accelerated charge e radiates energy into the solid angle $d\Omega$ (direction $\hat{\boldsymbol{R}}$) at—in τ-time—a temporal rate given by

$$dP = \frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^5}\cdot\frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\big|\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\boldsymbol{a}\big)\big|^2 d\Omega \tag{475}$$

. . . which gives back (473) when $\boldsymbol{\beta}=\boldsymbol{0}$.

The "Dopplerean prefactor"

$$D(\alpha)\equiv\frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^5}=\frac{1}{(1-\beta\cos\alpha)^5}$$

is plotted in Figure 134. Evidently

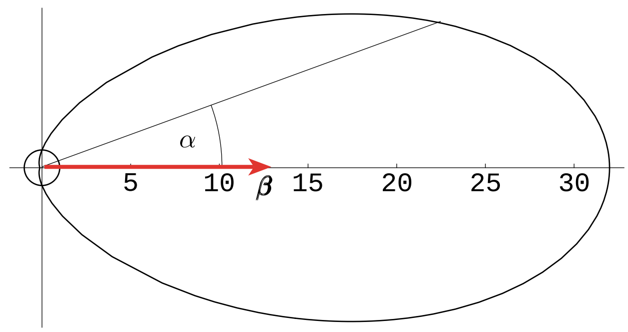


FIGURE 134: *Graph of the Dopplerean factor $D(\alpha)$, the cross-section of a figure of revolution about the $\boldsymbol{\beta}$-axis. Also shown, for purposes of comparison, is the unit circle. The figure refers to the specific case $\beta = 0.5$.*

$$D(\alpha)_{\max} = D(0) = \frac{1}{(1-\beta)^5} \longrightarrow \infty \quad \text{as} \quad \beta \uparrow 1$$
$$D(\alpha)_{\min} = D(\pi) = \frac{1}{(1+\beta)^5} \longrightarrow \tfrac{1}{32} \quad \text{as} \quad \beta \uparrow 1$$

and

$$D(\tfrac{\pi}{2}) = 1 \quad : \quad \text{all } \beta$$

We conclude that the ($\boldsymbol{a}$-independent) Doppler factor serves to favor the forward hemisphere:

Fast charges tend to throw their radiation forward.

Looking back again to (475), we see that the $D(\alpha)$-factor competes with (or modulates) a factor of the form $\big|\hat{\boldsymbol{R}}\times\big((\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\boldsymbol{a}\big)\big|^2$. A simple argument shows that the latter factor <u>vanishes if and only if $(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\parallel\boldsymbol{a}$</u>. This entails that $\hat{\boldsymbol{R}}$ lie in the $(\boldsymbol{\beta},\boldsymbol{a})$-plane, and that within that plane it have one or the other of the values $\hat{\boldsymbol{R}}_1$ and $\hat{\boldsymbol{R}}_2$ described in Figure 135. $\hat{\boldsymbol{R}}_1$ and $\hat{\boldsymbol{R}}_2$ describe the so-called "nodal directions" which are <u>instantaneously radiation-free</u>. Reading from the figure, we see that

- in the **non-relativistic limit** $\hat{\boldsymbol{R}}_1$ and $\hat{\boldsymbol{R}}_2$ lie fore and aft of the $\boldsymbol{a}$-vector, independently (in lowest order) of the magnitude/direction of $\boldsymbol{\beta}$: this is a property of the "sine squared distribution" evident already in Figure 131.
- in the **ultra-relativistic limit** $\hat{\boldsymbol{R}}_1 \to \boldsymbol{\beta}$ while $\hat{\boldsymbol{R}}_2$ gives rise to a "dangling note," the location of which depends conjointly upon $\boldsymbol{\beta}$ and $\boldsymbol{a}$.

From preceding remarks we conclude that the distribution function that describes the rate at which a charge "sprays energy on the sphere at ∞" is (in the general case) quite complicated. Integration over the sphere can, however,

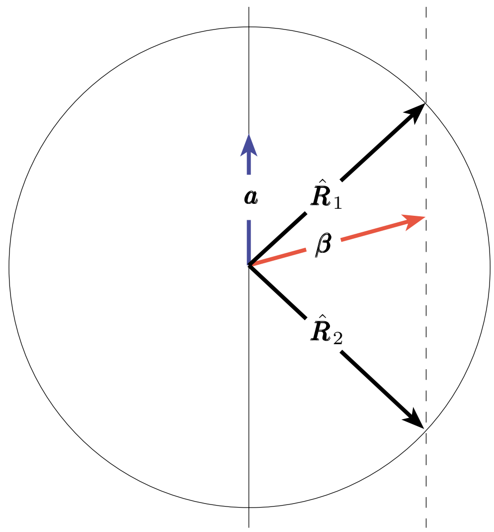


FIGURE 135: *Geometrical construction of the vectors* $\hat{\boldsymbol{R}}_1$ *and* $\hat{\boldsymbol{R}}_2$ *that locate the nodes of the radiative distribution in the general case.*

be carried out in closed form . . . and gives rise (compare (474)) to the following description of the **total power instantaneously radiated by an arbitrarily moving source**:

$$
\begin{aligned}
P &= -\tfrac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{(aa)}{c^3} && (476)\\
&= \tfrac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\cdot\Big\{\gamma^4(\boldsymbol{a}\cdot\boldsymbol{a})+\gamma^6(\boldsymbol{a}\cdot\boldsymbol{\beta})^2\Big\}\\
&= \text{———————}\cdot\ \gamma^6\Big\{(\boldsymbol{a}\cdot\boldsymbol{a})-(\boldsymbol{a}\times\boldsymbol{\beta})\cdot(\boldsymbol{a}\times\boldsymbol{\beta})\Big\}
\end{aligned}
$$

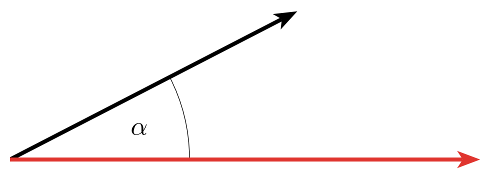
Equation (476) is *manifestly Lorentz covariant*, shows explicitly the sense in which Larmor's formula (474) is a "non-relativistic approximation," and has been extracted here from the relativistic bowels of electrodynamics . . . but was first obtained by A. Liénard in 1898, only one year after the publication of Larmor's result, and *seven years prior to the invention of special relativity*!

More detailed commentary concerning the physical implications of (473–476) is most usefully presented in terms of **special cases & applications** . . . as below:

$$\boxed{\text{CASE } \boldsymbol{a} \parallel \boldsymbol{\beta}}$$

This is the "most favorable case" in the sense that it is parallelism ($\boldsymbol{a}\times\boldsymbol{\beta}=\mathbf{0}$) that (see the last of the equations just above) maximizes P. The distribution itself can in this case be described

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{\sin^2\alpha}{(1-\beta\cos\alpha)^5}\frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\,\boldsymbol{a}\cdot\boldsymbol{a} \qquad (477)\\ &= D(\alpha)\cdot\Big[\text{sine squared distribution}\Big]\end{aligned}$$



The distribution is symmetric about the ($\boldsymbol{a}\|\boldsymbol{\beta}$)-axis (the nodes lie fore and aft), and has the cross section illustrated below:

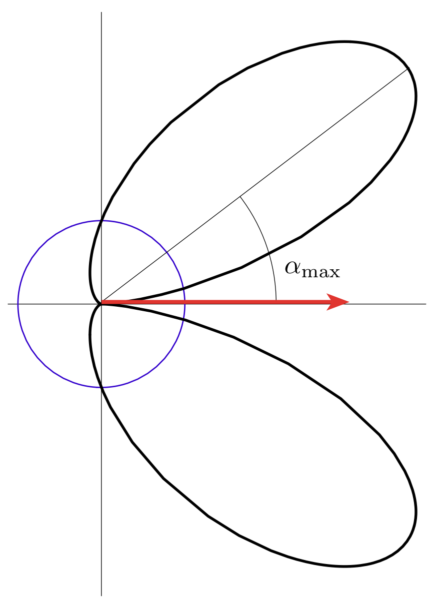


FIGURE 136: *Radiation pattern in the case* $\boldsymbol{a}\|\boldsymbol{\beta}$, *to be read as the cross section of a figure of revolution. The figure as drawn refers to the specific case* $\beta = 0.5$. *The circle has radius* $\frac{1}{4\pi c^3}(\frac{e^2}{4\pi})a^2$, *and sets the scale. The ears of the sine squared distribution (Figure 131) have been thrown forward (independently of whether* $\boldsymbol{a}$ *is parallel or antiparallel to* $\boldsymbol{\beta}$*).*

The ears of the sine squared distribution (Figure 131) have been thrown forward (independently of whether $\boldsymbol{a}$ is parallel or antiparallel to $\boldsymbol{\beta}$) by action of the

Doppler factor $D(\alpha)$. *How much* they are thrown forward is measured by

$$\begin{aligned}\alpha_{\max} &= \cos^{-1}\left\{\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right\} = \tfrac{\pi}{2} - \tfrac{5}{2}\beta + \tfrac{325}{48}\beta^3 - \cdots \\ &= \cos^{-1}\left\{\frac{4\sqrt{1-\frac{15}{16}\gamma^{-2}}-1}{3\sqrt{1-\gamma^{-2}}}\right\} = \tfrac{1}{2}\gamma^{-1} + \tfrac{133}{768}\gamma^{-3} + \cdots\end{aligned}$$

where the former equation speaks to the non-relativistic limit $\beta \downarrow 0$, and the latter to the ultra-relativistic limit $\gamma^{-1} \downarrow 0$. In the latter limit, the smallness of γ^{-1} implies that of α: double expansion of (477)—use $\beta = \sqrt{1-\gamma^{-2}}$—gives[288]

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{a^2}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)32\gamma^8\Big\{(\gamma\alpha)^2 - 5(\gamma\alpha)^4 + \cdots\Big\} \\ &\sim \frac{a^2}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)32\gamma^8\frac{(\gamma\alpha)^2}{[1+(\gamma\alpha)^2]^5}\end{aligned}$$

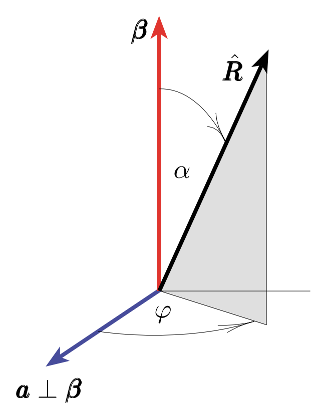
CASE $\boldsymbol{a} \perp \boldsymbol{\beta}$

This is the "least favorable case" in the sense that it is perpendicularity that minimizes P: reading from (476) we have (use $1+\gamma^2\beta^2 = \gamma^2$)

$$P = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{a^2}{c^3}\cdot\begin{cases}\gamma^6 & \text{when} \quad \boldsymbol{a} \parallel \boldsymbol{\beta} \\ \gamma^4 & \text{when} \quad \boldsymbol{a} \perp \boldsymbol{\beta}\end{cases}$$

Working from (475) we find that the angular distribution in the special case at hand can be described

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{1}{4\pi c^3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{(1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta})^3}\Big\{\boldsymbol{a}\cdot\boldsymbol{a} - \frac{1}{\gamma^2}\Big(\frac{\hat{\boldsymbol{R}}\cdot\boldsymbol{a}}{1-\hat{\boldsymbol{R}}\cdot\boldsymbol{\beta}}\Big)^2\Big\} \\ &= \frac{1}{4\pi}\frac{e^2}{4\pi}\frac{a^2}{c^3}\frac{1}{(1-\beta\cos\alpha)^3}\Big\{1 - \frac{1}{\gamma^2}\frac{\sin^2\alpha\cos^2\varphi}{(1-\beta\cos\alpha)^2}\Big\} \qquad (478)\end{aligned}$$



[288] PROBLEMS 79 & 80.

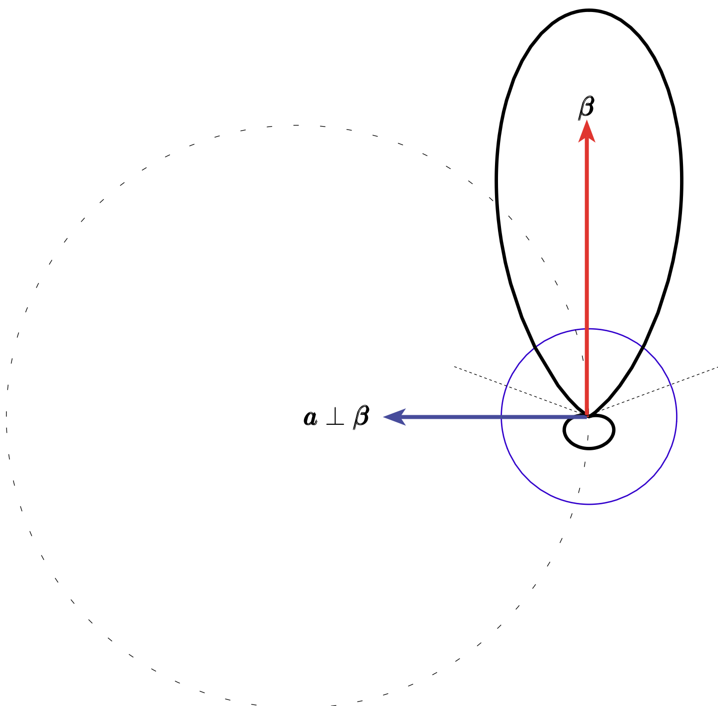
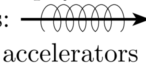


FIGURE 137: *A charge traces a circular orbit (large dashed circle) with constant speed. The figure shows a cross section of the resulting radiation pattern, which is now* not *a figure of revolution. The short dotted lines on left and right indicate the radiation-free nodal directions, which in a 3-dimensional figure would look like dimples on the cheeks of an ellipsoid. The small blue circle sets the scale, here as in Figure 136. The figure was extracted from* (478) *with* $\varphi = 0$ *and, as drawn, refers to the specific case* $\beta = 0.4$.

where the diagram at the bottom of the preceding page indicates the meanings of the angles α and φ. Shown above is a cross section of the associated radiation pattern. Notice that the nodal directions do *not* lie fore and aft: both are tipped forward, and stand in an angular relationship to $\boldsymbol{\beta}$ that can be extracted from Figure 135:

$$\tan(\text{angle between } \boldsymbol{\beta} \text{ and node}) = a/\beta$$

The $D(\alpha)$-factor has now enhanced the leading lobe of the radiation pattern, and attenuated the trailing lobe . . . giving rise to the "**synchrotron searchlight**," in which connection one might also look back again to Figure 128.

The radiative process just described is of major astrophysical importance (arising when electrons spiral about magnetic field lines: ) and sets a limit on the energy which can be achieved by particle accelerators of toroidal geometry (whence the linear design of SLAC: today many of the toroidal accelerators scattered about the world are *dedicated* to the production of synchrotron radiation—serve, in effect, as fancy "lightbulbs"). It is therefore not surprising that the properties of synchrotron radiation have been studied very closely—initially by Julian Schwinger, who asks (for example) "What are the distinguishing spectral and polarization characteristics of the radiation seen by an observer who looks into the synchrotron beam as it sweeps past?" For a detailed account of the theory see Chapters 39–40 in J. Schwinger *et al*, *Classical Electrodynamics* (1998).

Synchrotron radiation would lead also to the

RADIATIVE COLLAPSE OF THE BOHR ATOM

if quantum mechanical constraints did not intervene. To study the details of this topic (which is of mainly historical interest) we look specifically to the Bohr model of hydrogen. In the ground state the electron is imagined to pursue a circular orbit of radius[289]

$$R = \frac{\hbar^2}{me^2} = 5.292 \times 10^{-9}\text{cm}$$

with velocity

$$v = \frac{e^2}{\hbar} = \frac{1}{137}c = 2.188 \times 10^8\text{cm/sec}$$

The natural time characteristic of the system is

$$\tau = \frac{R}{v} = \frac{\hbar^3}{me^4} = 2.419 \times 10^{-17}\text{sec}$$

Reproduced below is the 3rd paragraph (§1) of Bohr's original paper ("On the constitution of atoms and molecules," Phil. Mag. **26**,1 (1913)):

> *"Let us now, however, take the effect of energy radiation into account, calculated in the ordinary way from the acceleration of the electron. In this case the electron will no longer describe stationary orbits. W will continuously increase, and the electron will approach the nucleus describing orbits of smaller and smaller dimensions, and with greater and greater frequency; the electron on the average gaining in kinetic energy at the same time as the whole system loses energy. This process will go on until the dimensions of the orbit are of the same order of magnitude as the dimensions of the electron or those of the nucleus. A simple calculation shows that the energy radiated out during the process considered will be enormously great compared with that radiated out by ordinary molecular processes.*

To make his model work Bohr simply/audaciously assumed the (classical) physical ideas thus described to be "microscopically inoperative." But I want

[289] See, for example, QUANTUM MECHANICS (1967), Chapter 2, pages 138–139. For the duration of the present discussion I adopt rationalized units: $e^2/4\pi \rightarrow e^2$.

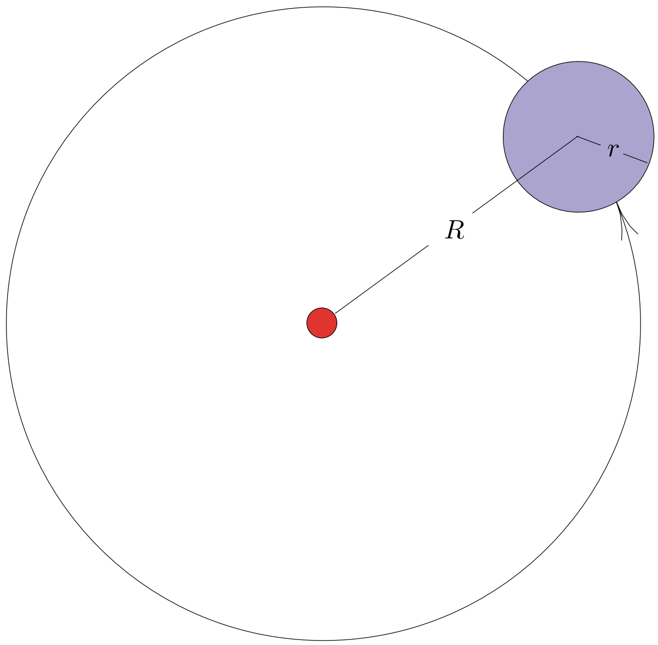


FIGURE 138: *Bohr atom, in which the nuclear proton and orbital electron have been assigned their classical radii. We study the "collapse" of the system which would follow from classical radiation theory if quantum mechanics did not intervene.*

here to pursue the issue—to inquire into the details of the "simple calculation" to which Bohr is content merely to allude. We ask: *How much energy would be released by the radiative collapse of a Bohr atom, and how long would the process take*?

If the electron and proton were literally point particles then, clearly, the energy released would be infinite . . . which is unphysical. So (following Bohr's own lead) let us assume the electron and proton to have "classical radii" given by

$$r = e^2/2mc^2 \quad \text{and} \quad r_p = r/1836.12 \ll r$$

respectively, and the collapse "proceeds to contact." The elementary physics of Keplerean systems[290] leads then to the conclusion that the energy released can be described

$$\begin{aligned} E = \tfrac{1}{2}e^2\Big\{\frac{1}{r+r_p} - \frac{1}{R}\Big\} \sim \tfrac{1}{2}e^2\Big\{\frac{1}{r} - \frac{1}{R}\Big\} &= \tfrac{1}{2}e^2\Big\{\frac{2mc^2}{e^2} - \frac{me^2}{\hbar^2}\Big\} \\ &= mc^2\Big\{1 - \frac{1}{2}\Big(\frac{e^2}{\hbar c}\Big)^2\Big\} \sim mc^2 \end{aligned}$$

[290] See, for example, H. Goldstein, *Classical Mechanics* (2$^{\text{nd}}$ edition 1980), page 97.

The atom radiates at a rate given initially (Larmor's formula) by

$$P = \frac{2}{3}\frac{e^2}{c^3}a^2$$

with

$$a = \frac{v^2}{R} = \left(\frac{1}{137}\frac{e}{\hbar}\right)^2 mc^2$$

and has therefore a lifetime given in first approximation by

$$\begin{aligned}\mathcal{T} = \frac{E}{P} &= mc^2 \Big/ \frac{2}{3}\frac{e^2}{c^3}\left(\frac{1}{137}\frac{e}{\hbar}\right)^4 (mc^2)^2 \\ &= \tfrac{3}{2}(137)^5\,\tau \\ &= (7.239\times 10^{10})\,\tau \\ &= 1.751\times 10^{-6}\text{sec}\end{aligned}$$

Despite the enormous accelerations experienced by the electron, the radiation rate is seen thus to be "small": the orbit shrinks in a gentle spiral and the atom lives for a remarkably long time (10^{10} revlolutions corresponds, in terms of the earth-sun system, to roughly the age of the universe!)...but not long enough. The preceding discussion is, of course, declared to be "naively irrelevant" by the quantum theory (which, in the first instance, means: by Bohr) ...which is seen now to be "super-stabilizing" in some of its corollary effects. It can, in fact, be stated quite generally that <u>the stability of matter is an intrinsically quantum mechanical phenomenon</u>, though the "proof" of this "meta-theorem" is both intricate and surprisingly recent.[291]

5. Collision-induced radiation. In many physical contexts charges move freely except when experiencing *abrupt scattering processes*, as illustrated in the figure on the facing page. We expect the *energy radiated per scatter* to be given in leading approximation by

$$E_{\text{per scatter}} = \frac{2}{3}\frac{e^2}{4\pi c^3}\left(\frac{\Delta v}{\tau}\right)^2\tau$$

where $\Delta v \equiv v_{\text{out}} - v_{\text{in}}$ and where τ denotes the characteristic duration of each scattering event. Suppose we had a *confined population of N such charges*, and that each charge experiences (on average) n collisions per unit time. We expect to have $\tau \sim 1/v$ and $n \sim v$. The rough implication is that the population should radiate at the rate

$$P \sim NnE_{\text{per scatter}} \sim (\Delta v)^2 v^2$$

If we could show that Δv ($\sim$ momentum transfer per collision) is v-independent we would (by $v^2 \sim$ temperature) have established the upshot of *Newton's law of cooling*. The point I want to make is that <u>radiative cooling is a (complicated) *radiative process*</u>. The correct theory is certainly quantum mechanical (and probably system-dependent), but the gross features of the process appear to be within reach of classical analysis. A much more careful account of the radiation produced by impulsive scattering processes can be found in Chapter 37 of the Schwinger text cited on page 392.

[291] See F. J. Dyson & A. Lenard, "Stability of matter. I," J. Math. Phys. **8**, 423 (1967) and subsequent papers.

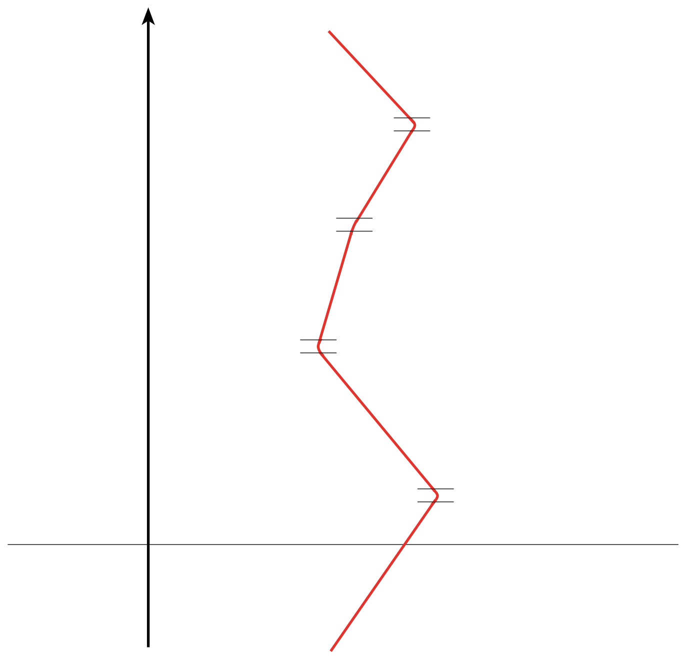


FIGURE 139: *Worldline of a charged particle subject to recurrent scattering events. Brackets mark the intervals during which the particle is experiencing non-zero acceleration.*

We have concentrated thus far mainly on single-source radiative processes, though the theory of cooling invited us to contemplate the radiation produced by random *populations* of accelerated charges. And we will want later to study the radiation produced when multiple sources act in concert (as in an antenna). But there are some important aspects and manifestations of single-source radiation theory which remain to be discussed, and it is to these that I now turn.

6. The self-interaction problem. We know that charges feel—and accelerate in response to—*impressed* electromagnetic fields. But **do charges feel their own fields?**. . . as (say) a motorboat may interact with the waves generated by its own former motion? Thought about the dynamics of a free charge at rest makes it appear semi-plausible that *charges do not feel their own Coulomb fields*. But the situation as it pertains to *radiation* fields is much less clear . . . for when a charge "radiates" it (by definition) "mails energy/momentum to infinity" and thus acquires a debt which (by fundamental conservation theorems) must somehow be paid. One might suppose that the responsibility for payment would fall to the agency which stimulated the charge to accelerate. But theoretical/observational arguments will be advanced which suggest that there is a sense in which **accelerated charges do feel—and recoil from—their own radiative acts**.

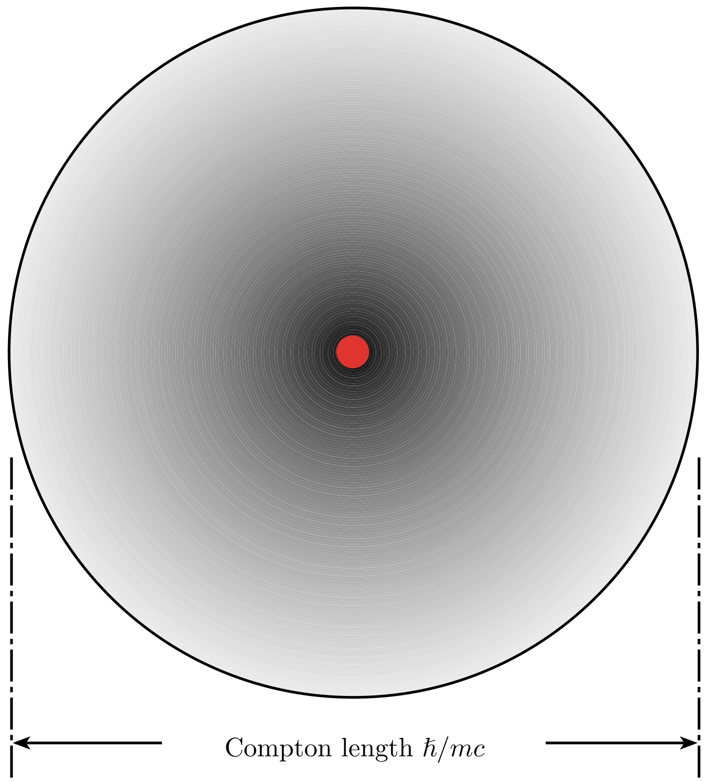


FIGURE 140: *The "classical electron"* • *is not, as one might expect, larger than but much* <u>*smaller*</u> *than the "quantum electron." A photon with wavelength* $\lambda = e^2/2mc^2$ *short enough to permit one to see the* • *would carry energy* $E = h\nu = hc/\lambda = (hc/e^2)2mc^2 = 137 \cdot 2mc^2$ *enough to create 137 electron-positron pairs . . . and in the clutter the intended object of the measurement process would be lost!*

The point at issue is made complicated by at least three interrelated circumstances. The first stems from the fact that the structural properties which distinguish "radiation fields" become manifest only in the "far zone," but *it is in the "near zone" that (in a local theory like electrodynamics) any particle/self-field interaction must occur*. The second derives from the truism that "to describe the motorboat-wake interaction one must know something about the geometry of motorboats": similarly, *to study the electrodynamical self-interaction problem one must be prepared to make assumptions concerning the "structure oif charged particles."* Classical theory speaks of "point particles" and—in the next breath—of "charged balls" of classical radius $e^2/2mc^2$, but (as Abraham/Lorentz/Poincaré discovered: see again page 382) seems incapable of generating a seriously-intended electron model. Which is hardly surprising, for electrons (and charged particles generally) are <u>quantum mechanical objects</u>. In this connection it is illuminating to note that the "quantum radius" of a mass point is (irrespective of its charge) given by $\hbar/mc$. But

$$\text{"classical radius"} \equiv \frac{e^2}{mc^2} = \frac{e^2}{\hbar c}\cdot\frac{\hbar}{mc} = \frac{\text{"quantum radius"}}{137}$$

. . . so the "classical electron" is much *smaller* than the "quantum electron."[292] Which brings us to the third complicating circumstance (Figure 140): *we seek a classical theory of processes which are buried so deeply within the quantum regime as to make the prospects of a formally complete and self-consistent theory seem extremely remote*. From this point of view the theory described below—imperfect though it is—acquires a semi-miraculous quality.

Limited success in this area was first achieved (1904) by M. Abraham, who argued *non-relativistically—from energy conservation*. We have

$$\boldsymbol{F} + \boldsymbol{F}_{\mathsf{R}} = m\boldsymbol{a} \quad \text{where} \quad \begin{cases} \boldsymbol{F} \equiv \text{impressed force} \\ \boldsymbol{F}_{\mathsf{R}} \equiv \text{self-force, the nature of which we seek to determine} \end{cases}$$

$\boldsymbol{F}$ may act to change the energy of the (charged) particle, but we semi-expect $\boldsymbol{F}_{\mathsf{R}}$ to conform to the **energy balance condition**

$$(\text{work on particle by } \boldsymbol{F}_{\mathsf{R}}) + (\text{energy radiated}) = 0$$

Drawing upon Larmor's formula (474) we are led thus to write (on a typical time interval $t_1 \leqslant t \leqslant t_2$)

$$\int_{t_1}^{t_2} \boldsymbol{F}_{\mathsf{R}}\cdot\boldsymbol{v}\,dt + \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\underbrace{\int_{t_1}^{t_2}\boldsymbol{a}\cdot\boldsymbol{a}\,dt}\, = 0$$

Integration by parts gives

$$= \boldsymbol{a}\cdot\boldsymbol{v}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\dot{\boldsymbol{a}}\cdot\boldsymbol{v}\,dt$$

If it may be assumed (in consequence of periodicity or some equivalent condition) that

$$\boldsymbol{a}\cdot\boldsymbol{v}\Big|_{t_1}^{t_2} = 0$$

then

$$\int_{t_1}^{t_2}\Big\{\boldsymbol{F}_{\mathsf{R}} - \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\dot{\boldsymbol{a}}\Big\}\cdot\boldsymbol{v}\,dt = 0$$

[292] Nor is this fact special to electrons. Since m enters identically on left and right, it pertains also to protons, to *every* particle species.

This suggests—but does not strictly entail—that $\boldsymbol{F}_{\mathsf{R}}$ may have the form

$$\boldsymbol{F}_{\mathsf{R}} = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\dddot{\boldsymbol{x}} \tag{479.1}$$

More compactly,

$$= m\tau\dddot{\boldsymbol{x}} \tag{479.2}$$

where the parameter τ can be described

$$\begin{aligned}\tau \equiv \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{mc^3} &= \frac{4}{3}\Big(\frac{e^2}{8\pi mc^2}\Big)\frac{1}{c}\\ &= \frac{4}{3}\frac{\text{classical particle radius}}{c}\\ &\sim \begin{cases}\text{time required for light to transit from}\\ \text{one side of the particle to the other}\end{cases}\end{aligned}$$

The non-relativistic motion of a charged particle can—on the basis of the assumptions that led to (479)—be described

$$\boldsymbol{F} + m\tau\dddot{\boldsymbol{x}} = m\ddot{\boldsymbol{x}} \tag{480.1}$$

or again

$$\boldsymbol{F} = m(\ddot{\boldsymbol{x}} - \tau\dddot{\boldsymbol{x}}) \tag{480.2}$$

. . . which is the so-called "**Abraham-Lorentz equation**." This result has several remarkable features:

- It contains—which is uncommon in dynamical contexts—an allusion to the 3rd derivative. This, by the way, seems on its face to entail that *more than the usual amount of initial data is required to specify a unique solution*.
- The Abraham-Lorentz equation contains *no overt allusion to particle structure* beyond that latent in the definition of the parameter τ.
- The "derivation" is susceptible to criticism at so many points[293] as to have the status of hardly more than a heuristic plausibility argument. It is, in this light, interesting to note that the work of 75 years (by Sommerfeld, Dirac, Rohrlich and many others) has done much to "clean up the derivation," to expose the "physical roots" of (480) . . . but has at the same time *shown the Abraham-Lorentz equation to be essentially correct as it stands* . . . except that
- The Abraham-Lorentz equation (480) is *non-relativistic*, but this is a formal blemish which (see below) admits easily of rectification.

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[293] Most critically, the argument *draws upoon the Larmor formula—a "far field result"—to obtain information about "near field physics."* The first of the "complicating circumstances" mentioned on page 396 is not only not illuminated/resolved, it is not even addressed.

We recall from page 192 that the 4-acceleration of a moving point can be described

$$a(\tau) \equiv \tfrac{d^2}{d\tau^2}x(\tau) = \begin{pmatrix} \frac{1}{c}\gamma^4(\boldsymbol{a}\cdot\boldsymbol{v}) \\ \gamma^2\boldsymbol{a} + \frac{1}{c^2}\gamma^4(\boldsymbol{a}\cdot\boldsymbol{v})\,\boldsymbol{v} \end{pmatrix}$$

where $\boldsymbol{v}$ and $\boldsymbol{a}$ are "garden variety" kinematic 3-variables: $\boldsymbol{v} \equiv d\boldsymbol{x}/dt$ and $\boldsymbol{a} \equiv d\boldsymbol{v}/dt$. We know also (page 192/193) that

$$(u,a) = c^2 \tag{481.1}$$

$$(u,a) = 0 \tag{481.2}$$

and can sho by direct computation that

$$(a,a) = -\gamma^4\Big\{(\boldsymbol{a}\cdot\boldsymbol{a}) + \tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\Big\} \tag{481.3}$$

while a somewhat more tedious computation gives

$$\begin{aligned} b(\tau) &\equiv \tfrac{d}{d\tau}a(\tau) \\ &= \gamma^3\begin{pmatrix} \frac{1}{c}\,\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + (\boldsymbol{a}\cdot\boldsymbol{a}) + 4\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big] \\ \dot{\boldsymbol{a}} + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a} + \frac{1}{c^2}\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + (\boldsymbol{a}\cdot\boldsymbol{a}) + 4\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big]\boldsymbol{v} \end{pmatrix} \end{aligned} \tag{482}$$

where $\qquad \dot{\boldsymbol{a}} \equiv \frac{d}{dt}\boldsymbol{a} = \dddot{\boldsymbol{x}}$

A final preparatory computation gives

$$(u,b) = \gamma^4\Big\{(\boldsymbol{a}\cdot\boldsymbol{a}) + \tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\Big\} = -(a,a) \tag{481.4}$$

We are in position also to evaluate (a,b) and (b,b), but have no immediate need of such information . . . so won't.[294] Our immediate objective is to proceed from $\boldsymbol{F}_\mathsf{R} = \frac{2}{3}(e^2/4\pi)\frac{1}{c^3}\dddot{\boldsymbol{x}}$ to its "most natural" relativistic counterpart—call it K^μ_R. It is tempting to set $K_\mathsf{R} = \frac{2}{3}(e^2/4\pi)\frac{1}{c^3}b$, but such a result would—by (481.4)—be inconsistent with the general requirement (see again page ???) that $(K,u) = 0$. We are led thus—tentatively—to set

$$K_\mathsf{R} = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\,b_\perp \tag{483}$$

$$\begin{aligned} b_\perp &\equiv b - \frac{(b,u)}{(u,u)}u \\ &= b + \frac{(a,a)}{c^2}u \\ &= \gamma^3\begin{pmatrix} \frac{1}{c}\,\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big] \\ \dot{\boldsymbol{a}} + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a} + \frac{1}{c^2}\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + 3\frac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big]\boldsymbol{v} \end{pmatrix} \end{aligned}$$

in which connection we note that

$$\overset{\downarrow}{=} \begin{pmatrix} 0 \\ \boldsymbol{F}_\mathsf{R} \end{pmatrix} \quad \text{in the non-relativistic limit (as required)}$$

[294] PROBLEM 81.

Now, the spatial part of Minkowski's equation $K^\mu = md^2x/d\tau^2$ can (see again (288) page 197) be written $(1/\gamma)\boldsymbol{K} = \frac{d}{dt}(\gamma m\boldsymbol{v})$, and in this sense it is (not $\boldsymbol{K}$ but) $(1/\gamma)\boldsymbol{K}$ which one wants to call the "relativistic force." We are led thus from (483) to the conclusion that the **relativistic self-force**

$$\boldsymbol{\mathfrak{F}}_{\mathsf{R}} = \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\gamma^2\Big\{\dot{\boldsymbol{a}} + 3\tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a} + \tfrac{1}{c^2}\gamma^2\big[(\dot{\boldsymbol{a}}\cdot\boldsymbol{v}) + 3\tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})^2\big]\boldsymbol{v}\Big\} \tag{484.1}$$

This result was first obtained (1905) by Abraham, who however argued not from relativity but from a marginally more physical refinement of the "derivation" of (479). The "argument from relativity" was first accomplished by M. von Laue (1909). The pretty notation

$$\begin{aligned}\boldsymbol{\mathfrak{F}}_{\mathsf{R}} &= \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\gamma^4\Big\{\boldsymbol{g} + \tfrac{1}{c^2}\boldsymbol{v}\times(\boldsymbol{v}\times\boldsymbol{g})\Big\} \\ \boldsymbol{g} &\equiv \dot{\boldsymbol{a}} + 3\tfrac{1}{c^2}\gamma^2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a}\end{aligned} \tag{484.2}$$

was introduced into the modern literature by David Griffiths,[295] but was reportedly original to Abraham.[296]

All modern self-interaction theories[297] hold (483)—which can be notated

$$\begin{aligned}K^\mu_{\mathsf{R}} &= \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\Big\{\frac{d^3x^\mu}{d\tau^3} + \frac{1}{c^2}(a^\alpha a_\alpha)\frac{dx^\mu}{d\tau}\Big\} \\ a^\alpha &\equiv \frac{d^2x^\alpha}{d\tau^2}\end{aligned}$$

—to be *exact* (so far as classical theory allows). Which is surprising, for *we have done no new physics*, addressed none of the conceptual difficulties characteristic of this topic. We note with surprise also that *we can, in the relativistic regime, have* $\boldsymbol{\mathfrak{F}}_{\mathsf{R}} \neq \boldsymbol{0}$ *even when* $\dot{\boldsymbol{a}} = \boldsymbol{0}$.

To study the physical implications of the results now in hand we retreat (in the interest of simplicity) to the non-relativistic case: (480). If (also for simplicity) we assume $\boldsymbol{F}$ to be $\boldsymbol{x}$-independent (*i.e.*, to be some arbitrarily prescribed function of t alone) then the Abraham-Lorentz equation (480) reads

$$\dddot{\boldsymbol{x}} - \tfrac{1}{\tau}\ddot{\boldsymbol{x}} = -\tfrac{1}{m\tau}\boldsymbol{F}(t) \tag{485}$$

and entails

$$\ddot{\boldsymbol{x}}(t) = e^{t/\tau}\Big\{\underset{\uparrow\text{constant of integration}}{\boldsymbol{a}} - \tfrac{1}{m\tau}\int_0^t e^{-s/\tau}\boldsymbol{F}(s)\,ds\Big\} \tag{486.1}$$

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[295] "Dumbbell model for the classical radiation reaction," AJP **46** 244 (1978).
[296] PROBLEM 82.
[297] For references see the Griffiths paper just cited.

Successive integrations give

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v} + \int_0^t \ddot{\boldsymbol{x}}(s)\,ds \tag{486.2}$$

and

$$\boldsymbol{x}(t) = \boldsymbol{x} + \int_0^t \dot{\boldsymbol{x}}(s)\,ds \tag{486.3}$$

where $\boldsymbol{v}$ and $\boldsymbol{x}$ are additional constants of integration.[298]

In the $\boxed{\text{FORCE-FREE CASE } \boldsymbol{F}(t) \equiv \mathbf{0}}$ equations (486) promptly give

$$\boldsymbol{x}(t) = \boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}\tau^2 e^{t/\tau}$$

This entails $\dot{\boldsymbol{x}}(t) = \boldsymbol{v} + \boldsymbol{a}\tau e^{t/\tau}$, which is <u>asymptotically infinite unless $\boldsymbol{a} = \mathbf{0}$</u>. So we encounter right off the bat an instance of the famous **run-away solution problem**, which bedevils all theories of self-interaction. It is dealt with by conjoining to (485) the stipulation that

Run-away solutions are to be considered
"unphysical" . . . and discarded. (487)

One (not immediately obvious) effect of the *asymptotic side-condition* (487) is to reduce to its familiar magnitude the amount of initial data needed to specify a particular particle trajectory.

To gain some sense of the practical effect of (487) we look next to the $\boxed{\text{CASE OF AN IMPULSIVE FORCE } \boldsymbol{F}(t) \equiv m\tau\boldsymbol{A}\delta(t-t_0)}$. Immediately

$$\ddot{\boldsymbol{x}}(t) = \begin{cases} e^{t/\tau}\boldsymbol{a} & : \quad t < t_0 \\ e^{t/\tau}\big[\boldsymbol{a} - \boldsymbol{A}e^{-t_0/\tau}\big] & : \quad t > t_0 \end{cases}$$

Thbe requirement—(487)—that $\ddot{\boldsymbol{x}}(t)$ remain asymptotically finite entails that the adjustable constant $\boldsymbol{a}$ be set equal to $\boldsymbol{A}e^{-t_0/\tau}$. Then

$$\ddot{\boldsymbol{x}}(t) = \begin{cases} \boldsymbol{A}e^{(t-t_0)/\tau} & : \quad t < t_0 \\ \mathbf{0} & : \quad t > t_0 \end{cases} \tag{488}$$

The situation is illustated in Figure 141. The most striking fact to emerge is that <u>the particle starts to accelerate before it has been kicked</u>! This is an instance of the famous **preacceleration** phenomenon. It is not an artifact of the δ-function, not a consequence of the fact that we are working at the moment in the non-relativistic approximation . . . but a systemic feature of the classical self-interaction problem. Roughly, preacceleration may be considered to arise

[298] PROBLEM 83.

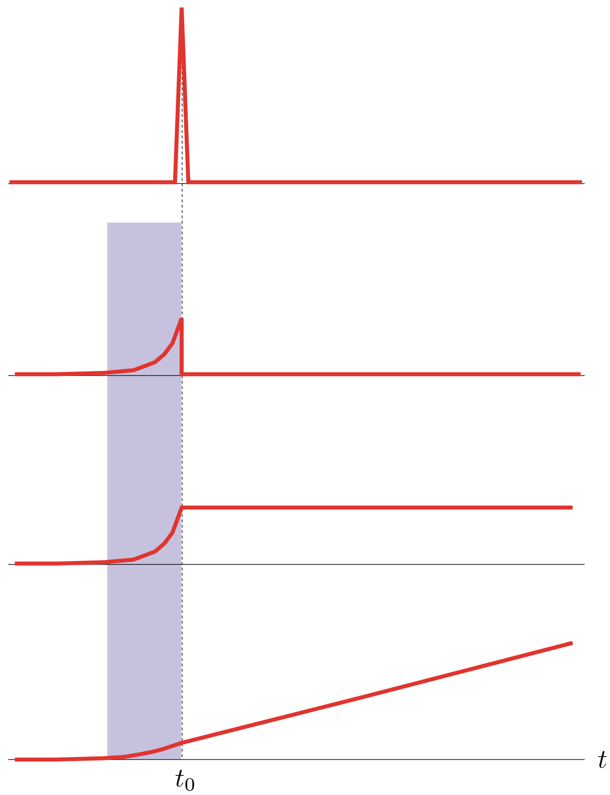

FIGURE 141: *Graphs of (reading from top to bottom) the impulsive force* $\boldsymbol{F}(t) \equiv m\tau\boldsymbol{A}\delta(t-t_0)$ *and of the resulting acceleration* $\ddot{\boldsymbol{x}}(t)$*, velocity* $\dot{\boldsymbol{x}}(t)$ *and position* $\boldsymbol{x}(t)$*. The shaded rectangle identifies the "preacceleration interval."*

because "the leading edge of the extended classical source makes advance contact with the force field." The characteristic preacceleration time is—consistently with this picture—small, being given by τ ($\sim 10^{-24}$ seconds for an electron). On its face, preacceleration represents a *microscoptic violation of causality* . . . and so it is, but the phenomenon lies so deep within the quantum regime as to be (or so I believe) *classical unobservable in every instance.* Preacceleration is generally considered to be (not a physical but) a merely "mathematical phenomenon," a symptom of an attempt to extend classical physics beyond its natural domain of applicability.

We may "agree not to be bothered" by the preacceleration "phenomenon." But preacceleration comes about as a forced consequence of implementation of the asymptotic condition (487) . . . and the fact that the equation of motion (485) cannot stand on its own feet, but must be propped up by such a side condition, *is* bothersome. <u>Can one modify the equation of motion so as to make the make the asymptotic condition *automatic*</u>?. . . so that "run-away solutions" simply do not arise? The question provokes the following formal manipulation. Let (485) be written

$$(1 - \tau D)m\ddot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)$$

or again

$$m\ddot{\boldsymbol{x}}(t) = \frac{1}{1 - \tau D}\boldsymbol{F}(t) \tag{489}$$

where $D \equiv \frac{d}{dt}$. Recalling $\frac{1}{\lambda} = \displaystyle\int_0^\infty e^{-\lambda\theta}\, d\theta$, we presume to write

$$\frac{1}{1 - \tau D} = \int_0^\infty e^{-(1-\tau D)\theta}\, d\theta$$

even though D is here not a number but a differential operator (this is heuristic mathematics in the noble tradition of Heaviside). Then

$$m\ddot{\boldsymbol{x}}(t) = \int_0^\infty e^{-\theta} e^{\theta\tau D}\boldsymbol{F}(t)\, d\theta$$

But $e^{\theta\tau D}\boldsymbol{F}(t) = \boldsymbol{F}(t + \theta\tau)$ by Taylor's theorem, so

$$= \int_0^\infty \boldsymbol{F}(t + \theta\tau)\, d\theta \tag{490}$$

Notice that, since $c \uparrow \infty$ entails $\tau \downarrow 0$, we can use $\displaystyle\int_0^\infty e^{-\theta\, d\theta} = 1$ to recover Newton's $m\ddot{\boldsymbol{x}}(t) = \boldsymbol{F}(t)$ in the non-relativistic limit. Equation (490) states that $\ddot{\boldsymbol{x}}(t)$ is determined by a <u>weighted average of future force values</u>, and therefore provides a relatively sharp and general characterization of the preacceleration phenomenon—encountered thus far only in connection with a single example. Returning to that example . . . insert $\boldsymbol{F}(t) \equiv m\tau\boldsymbol{A}\delta(t - t_0)$ into (490) and obtain

$$\ddot{\boldsymbol{x}}(t) = \int_0^\infty \boldsymbol{A}\delta(t - t_0 + \theta\tau)\tau d\theta = \begin{cases} \boldsymbol{A}e^{(t-t_0)/\tau} & : \quad t < t_0 \\ \\ \boldsymbol{0} & : \quad t > t_0 \end{cases}$$

We have recovered (488), but by an argument that is *free from any explicit reference to the asymptotic condition*. In (490) we have a formulation of the Abraham-Lorentz equation (480) in which the "exotic" features have been translocated into the force term . . . but we have actually come out ahead: we have managed to describe the dynamics of a self-interacting charge by means of an <u>integrodifferential equation of motion</u> that *stands alone, without need of a side condition* such as (487). The general solution of (490) has, by the way, the familiar number of adjustable constants of integration, so <u>standard initial data serves to identify particular solutions</u>.

If in place of the "integral representation of $1/(1 - \tau D)$" we use

$$\frac{1}{1 - \tau D} = 1 + \tau D + (\tau D)^2 + \cdots$$

then in place of (490) we obtain

$$\begin{aligned} m\ddot{\boldsymbol{x}}(t) &= \boldsymbol{F}(t) + \tau\boldsymbol{F}'(t) + \tau^2\boldsymbol{F}''(t) + \cdots \\ &= \text{Newtonian force} + \text{Radiative corrections} \end{aligned} \tag{491}$$

Equations (490) and (491) are equivalent. The latter masks preacceleration (acausality), but makes explicit the Newtonian limit.[299]

Having thus exposed the central issues, I must refer my readers to the literature for discussion of the technical details of modern self-interaction theory: this is good, deep-reaching physics, which has engaged the attention of some first-rate physicists and very much merits close study.[300] I turn now to discussion of some of the observable physical consequences of self-interaction:

7. Thomson scattering. An electron in a microwave cavity or laser beam experiences a Lorentz force of the form

$$\begin{aligned} \boldsymbol{F}(t) &= e(\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{B})\cos\omega t \\ &\downarrow \\ &= e\boldsymbol{E}\cos\omega t \quad \text{in the non-relativistic limit} \end{aligned}$$

For such a *harmonic driving force* (486.1) becomes

$$\ddot{\boldsymbol{x}}(t) = e^{\Omega t}\Big\{\boldsymbol{a} - \tfrac{e}{m}\boldsymbol{E}\Omega\underbrace{\int_0^t e^{-\Omega s}\cos\omega s\,ds}\Big\}$$

where $\Omega \equiv \frac{1}{\tau} = \frac{3}{2}\frac{4\pi}{e^2}mc^3$. But

$$= \frac{e^{-\Omega s}}{\Omega^2+\omega^2}\Big[-\Omega\cos\omega s + \omega\sin\omega s\Big]_0^t$$

so

$$= \frac{e}{m}\boldsymbol{E}\frac{\Omega^2\cos\omega t - \Omega\omega\sin\omega t}{\Omega^2+\omega^2} + e^{\Omega t}\Big\{\boldsymbol{a} - \frac{e}{m}\boldsymbol{E}\frac{\Omega^2}{\Omega^2+\omega^2}\Big\}$$

The asymptotic condition (487) requires that we set $\{\text{etc.}\} = \mathbf{0}$, so after some

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[299] For a much more elaborate discussion of the ideas sketched above see CLASSICAL RADIATION (1974), pages 600–605.

[300] F. Rohrlich's *Classical Charged Particles* (1965), Chapters 2 & 6 and J. D. Jackson's *Classical Electrodynamics* (3rd edition 1998), Chapter 16 are good places to start. See also T. Erber, "The classical theories of radiation reaction," Fortschritte der Physik **9**, 343 (1961) and G. N. Plass, "Classical electrodynamic equations of motion with radiative reaction," Rev. Mod. Phys. **33**, 37 (1961) . . . which are excellent general reviews and provide good bibliographies. Students should also not neglect to examine the classics: Dirac (1938), Wheeler-Feynman (1945).

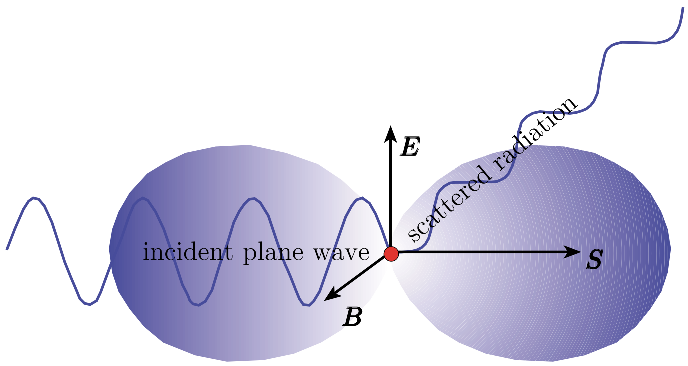


FIGURE 142: *A monochromatic plane wave is incident upon a free electron •, which is stimulated to oscillate ↕ and therefore to radiate in the characteristic sine-squared pattern. The electron drinks energy from the incident beam and dispatches energy in a variety of other directions: in short, it* scatters *radiant energy. Scattering by this classical mechanism—by free charges—is called* **Thomson scattering**.

elementary algebra we obtain

$$\ddot{\boldsymbol{x}}(t) = \frac{1}{\sqrt{1+(\omega/\Omega)^2}}\frac{e}{m}\boldsymbol{E}\cos(\omega t + \delta) \tag{492}$$

where the phase shift

$$\delta = \arctan(\omega/\Omega)$$

is the disguise now worn by the preacceleration phenomenon. We note in passing that

$$\downarrow$$
$$= \tfrac{e}{m}\boldsymbol{E}\cos\omega t \quad \text{in the non-relativistic limit: } \Omega \gg \omega$$

It is upon (492) that the classical theory of the scattering of electromagnetic radiation by free electrons—"Thomson scattering"—rests. We inquire now into the most important details of this important process.

Using (492) in conjunction with the Larmor formula (474) we conclude that the energy radiated per period by the harmonically stimulated electron (see the preceding figure) can be described

$$\begin{aligned}\int_0^T P\,dt &= \frac{2}{3}\Big(\frac{e^2}{4\pi}\Big)\frac{1}{c^3}\Big(\frac{e}{m}E\Big)^2\frac{1}{1+(\omega/\Omega)^2}\int_0^T \cos^2\omega t\,dt \quad \text{with} \quad T \equiv 2\pi/\omega\\ &= \Big(\frac{cE^2\pi}{\omega}\Big)\cdot\frac{8\pi}{3}\Big(\frac{e^2}{4\pi mc^2}\Big)^2\frac{1}{1+(\omega/\Omega)^2}\end{aligned}$$

On the other hand, we know from work on page 305 that the (time-averaged energy flux or) *intensity* of the incident plane wave can be described $I = \frac{1}{2}cE^2$ so the energy incident (per period) upon an area A becomes

$$ITA = \tfrac{1}{2}cE^2(2\pi/\omega)A = \left(\frac{cE^2\pi}{\omega}\right)\cdot A$$

We conclude that

> A free electron absorbs (only to re-radiate) energy from an incident monochromatic wave as though it had a cross-sectional area given by

$$\sigma_{\text{Thomson}} = \tfrac{8\pi}{3}(\text{classical electron radius})^2 \cdot \frac{1}{1+(\omega/\Omega)^2}$$

The final factor can and should be dropped: it differs from unity only if

$$\hbar\omega \gg \hbar\Omega = \tfrac{3}{2}\big(\tfrac{4\pi\hbar c}{e^2}\big)mc^2 = 205mc^2$$

and this carries us so far into the relativistic regime that we must expect our classical results long since to have become meaningless. *Neglect of the factor amounts to neglect of the self-interaction*: it entails $\delta = \arctan(\omega/\Omega) \to \frac{\pi}{2}$ and causes the **Thomson scattering cross-section**

$$\sigma_{\text{Thomson}} = \tfrac{8\pi}{3}\big[e^2/4\pi mc^2\big]^2 \tag{493}$$

to become ω-independent. Thomson scattering—which in the respect just noted is quite atypical—may be considered to comprise the classical limit of Compton scattering, the relativistic *quantum* process diagramed below. The radiation

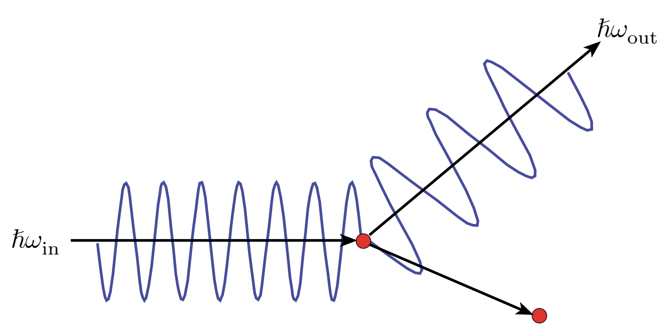


FIGURE 143: *In view of the fact that* **Compton scattering** *yields scattered photons that have been* <u>*frequency-shifted*</u> *it is remarkable that no frequency shift is associated with the Thomson scattering process.*

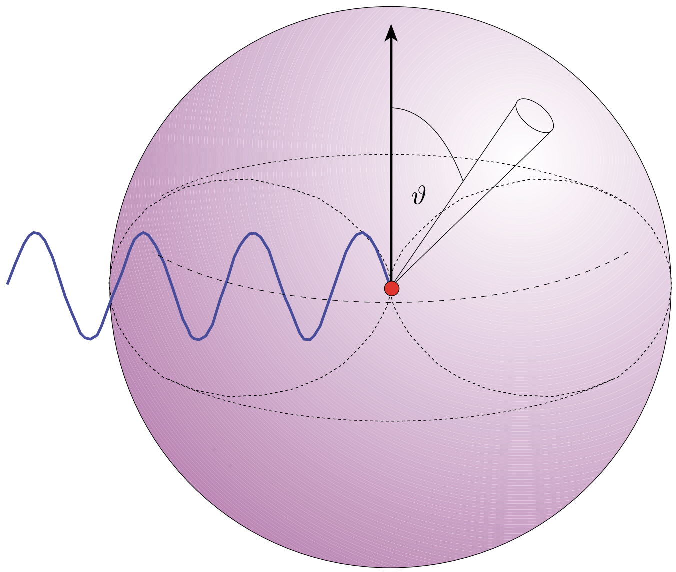


FIGURE 144: *Representation of the axially-symmetric sine-squared character of the Thomson scattering pattern. I invite the reader to consider what would be the pattern if the incidentg radiation were elliptically polarized.*

field generated by a harmonically stimulated free electron has the structure illustrated in Figure 126. The **differential Thomson cross-section** (Figure 144) is readily seen to have the sine-squared structure

$$\frac{d\sigma}{d\Omega}\bigg|_{\text{Thomson}} = \big[e^2/4\pi mc^2\big]^2 \sin^2\vartheta$$

8. Rayleigh scattering. Let our electron—formerly free—be considered now to be attached to a spring, part of a "classical molecule." If the spring force is written $\boldsymbol{f} = -m\omega_0^2\boldsymbol{x}$ then the Abraham-Lorentz equation (480) becomes

$$\ddot{\boldsymbol{x}} - \tau\dddot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} = \frac{e}{m}\boldsymbol{E}\cos\omega t \tag{494}$$

We expect the solution of (494) to have (after transcients have died out) the form

$$\boldsymbol{x}(t) = \boldsymbol{\mathfrak{X}}\cos(\omega t + \delta)$$

with $\boldsymbol{\mathfrak{X}} \parallel \boldsymbol{E}$, and will proceed on the basis of that assumption—an assumption which, by the way,

- renders the asymptotic condition (487) superfluous
- entails $\dddot{\boldsymbol{x}} = -\omega^2\dot{\boldsymbol{x}}$.

Our initial task, therefore, is to describe the solution

$$x(t) = \mathfrak{X}e^{i(\omega t-\delta)}$$

of

$$\begin{aligned}\ddot{x} + 2b\dot{x} + \omega_0^2 x &= \tfrac{e}{m}Ee^{i\omega t}\\ b &\equiv \tfrac{1}{2}\tau\omega^2\end{aligned}$$

But this is precisely the *harmonically driven damped oscillator* problem—painfully familiar to every sophomore—the only novel feature being that the "radiative damping coefficient" b is now ω-dependent. Immediately

$$\begin{aligned}\underbrace{(-\omega^2 + 2ib\omega + \omega_0^2)}\mathfrak{X}e^{-i\delta} &= \tfrac{e}{m}E\\ &= \sqrt{(\omega_0^2-\omega^2)^2 + 4b^2\omega^2}\,\exp\Big\{i\tan^{-1}\frac{2b\omega}{\omega_0^2-\omega^2}\Big\}\end{aligned}$$

which gives

$$\begin{aligned}\mathfrak{X}(\omega) &= \frac{(e/m)E}{\sqrt{(\omega_0^2-\omega^2)^2+4b^2\omega^2}}\\ &= \frac{eE}{m\omega_0^2}\frac{1}{\sqrt{(1-\xi^2)^2+k^2\xi^6}} \equiv \frac{eE}{m\omega_0^2}\mathfrak{X}(\xi,k)\\ \delta(\omega) &= \tan^{-1}\frac{2b\omega}{\omega_0^2-\omega^2}\\ &= \tan^{-1}\frac{k\xi^3}{1-\xi^2} \equiv \delta(\xi,k)\end{aligned}$$

where

$$\xi \equiv \omega/\omega_0 \qquad \text{and} \qquad k \equiv \tau\omega_0$$

are dimensionless parameters. It is useful to note that k is, in point of physical fact, typically quite small:

$$\begin{aligned}k &= \frac{\text{period of optical reverberations within the classical electron}}{\text{period of molecular vibrations}}\\ &\sim \frac{e^2/mc^3}{\hbar^3/me^4} = \Big(\frac{e^2}{\hbar c}\Big)^3\\ &= \Big(\frac{1}{137}\Big)^3 = 3.89\times10^{-7}\end{aligned}$$

Precisely the argument that led to (493) now leads to the conclusion that the **Rayleigh scattering cross-section** can be described[301]

$$\begin{aligned}\sigma_{\text{Rayleigh}}(\omega) &= \sigma_0\cdot\frac{\omega^4}{(\omega_0^2-\omega^2)^2+4b^2\omega^2} \qquad (495)\\ &= \sigma_0\frac{\xi^4}{(1-\xi^2)^2+k^2\xi^6}\end{aligned}$$

$$\sigma_0 \equiv \sigma_{\text{Thomson}} = \tfrac{8\pi}{3}\big[e^2/4\pi mc^2\big]^2$$

[301] PROBLEM 84.

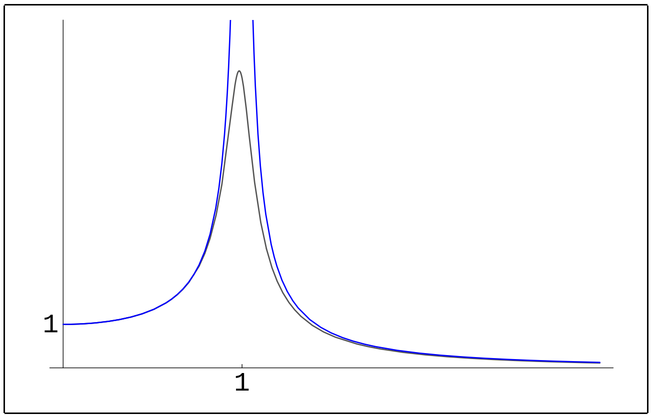


FIGURE 145: *Graphs of* $\mathcal{X}(\xi, k)$ *in which, for clarity,* k *has been assigned the artificially large values* $k = 0.15$ *and* $k = 0.05$. *An easy calculation shows that the resonant peak stands just to the left of unity:*

$$\frac{\partial}{\partial\xi}\mathcal{X}(\xi, k) = 0 \quad at \quad \xi = \left[\frac{\sqrt{1+6k^2}-1}{3k^2}\right]^{\frac{1}{2}} = 1 - \tfrac{3}{4}k^2 + \tfrac{63}{32}k^4 - \cdots$$

and that

$$\mathcal{X}_{\max} = k^{-1} + \tfrac{9}{8}k - \tfrac{189}{128}k^3 + \cdots$$

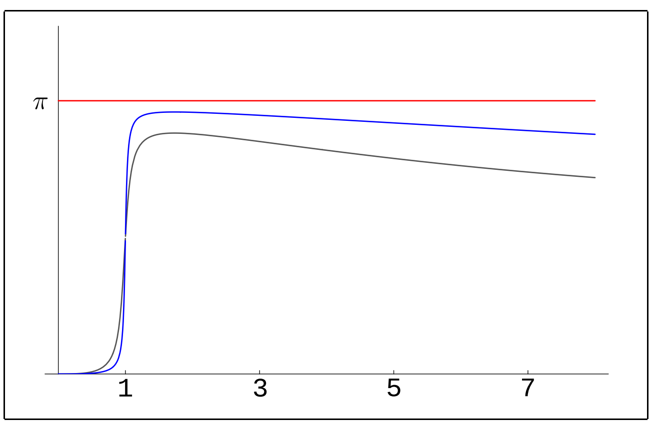


FIGURE 146: *Graphs of* $\delta(\xi, k)$ *in which* k *has been assigned the same artificially large values as described above. As* k *becomes smaller the phase jump becomes steeper,* δ *approaches* π *more closely, and hangs there longer before—at absurdly/unphysically high frequencies* $\omega \gg \Omega$—*dropping to* $\frac{\pi}{2}$:

$$\lim_{\xi\uparrow\infty} \tan^{-1}\frac{k\xi^3}{1-\xi^2} = \lim_{\xi\uparrow\infty}\tan^{-1}(-k\xi) = \tfrac{\pi}{2}$$

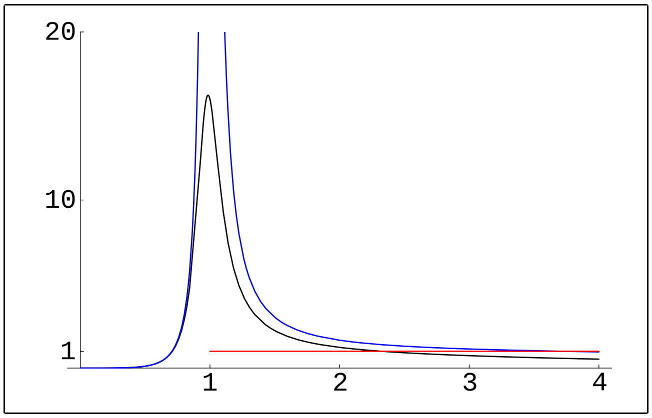


FIGURE 147: *Graphs of the Rayleigh distribution function. In* (495) *I have set* $\sigma_0 = 1$ *and have assigned to* k *the artificially large values* $k = 0.25$ *and* $k = 0.10$. *The red line at unity has been inserted to emphasize the high-frequency asymptote. The resonant peak lies in the very near neighborhood of* $\xi \equiv \omega/\omega_0 = 1$ *and its height becomes infinite when self-interactive effects are turned off*: $k \downarrow 0$. *The physical short of it*: *The apparent size of a "classical molecule" depends upon the color of the light in which it is viewed*.

What we have learned is that **Rayleigh scattering**—energy absorption and reemission by a monochromatically stimulated and self-interactively damped "classical molecule" (charged particle on a spring)—is frequency-dependent. Looking to the qualitative details of that ω-dependence (Figure 147), we find it natural to distinguish three regimes:

LOW-FREQUENCY REGIME $\xi \equiv \omega/\omega_0 \ll 1$ so with *Mathematica*'s aid we expand about $\xi = 0$, obtaining

$$\frac{\xi^4}{(1-\xi^2)^2 + k^2\xi^6} = \xi^4 + 2\xi^6 + 3\xi^8 + (4-k^2)\xi^{10} + (5-4k^2)\xi^{12} + \cdots$$

Thus are we led to the so-called "4^{th} power law"

$$\sigma_{\text{Rayleigh}}(\omega) \sim \sigma_0(\omega/\omega_0)^4 \quad : \quad \omega \ll \omega_0 \tag{496}$$

The accuracy of the approximation is evident in Figure 148.

It is a familiar fact that (if we may allow ourselves to speak classically in such a connection) slight conformational/dynamical adjustments of atomic/molecular state can result in the emission (or from the absorption) of visible light: $[\Delta E \approx \hbar\Delta\omega_0] = \hbar\omega$. From this we infer that the characteristic atomic/molecular vibrational frequencies ω_0 are themselves $\gg$ than the frequencies

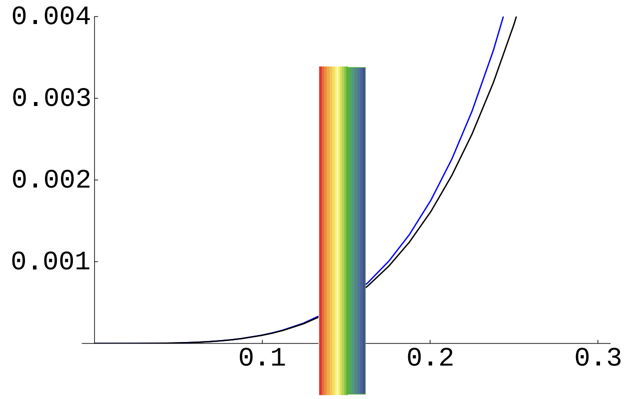


FIGURE 148: *Graph—based upon* (495)*—of* σ_{Rayleigh} *with* $\xi \ll 1$, *compared with the scattering cross-section asserted by the* 4^{th} *power law* (496). *In both cases I have set* $\sigma_0 = 1$, *and in the former case I have taken* $k = 0.00001$. *Naive arguments developed in the text suggest that atomic/molecular rotational/vibrational frequencies* ω_0 *are typically* $\gg$ *than the frequencies present in the visible spectrum.*

characteristic of visible light,[302] and that the scattering of sunlight by air is therefore a "low-frequency phenomenon."[303]

RESONANCE REGIME Here $\xi \sim 1$ (*i.e.*, $\omega \sim \omega_0$) $\Rightarrow \sigma \sim \sigma_{\max}$ and provides a classical intepretation of the phenomenon of *resonance florencence*. Let (495) be written

$$\begin{aligned} \sigma_{\text{Rayleigh}} &= \sigma_0 \frac{\xi^4}{(1+\xi)^2(1-\xi)^2 + k^2\xi^6} \\ &\approx \tfrac{1}{4}\sigma_0 \frac{1}{(\xi-1)^2 + (\frac{1}{2}k)^2} \end{aligned} \tag{497}$$

For a comparison of the exact Rayleigh distribution function with its resonant approximation (497), see Figure 149. The nearly Gaussian appearance of the appoximating function leads us to observe that

$$\int_{-\infty}^{+\infty} \tfrac{1}{4} \frac{1}{(\xi-\xi_0)^2 + (\frac{1}{2}k)^2}\, d\xi = \tfrac{\pi}{2}k^{-1} \quad : \quad \text{all } \xi_0$$

and on the basis of that information to introduce the definition

$$L(\xi-\xi_0, k) \equiv \tfrac{1}{2\pi} \frac{k}{(\xi-\xi_0)^2 + (\frac{1}{2}k)^2} \quad : \quad k > 0 \tag{498}$$

---

[302] For the former we might borrow $\omega_0 = 2\pi(me^4/\hbar^3) = 2.60 \times 10^{17}$Hz from the Bohr theory of hydrogen (see again page 392). For visible light one has $4.0 \times 10^{14}\text{Hz} < \omega < 7.5 \times 10^{14}\text{Hz}$.

[303] PROBLEM 85.

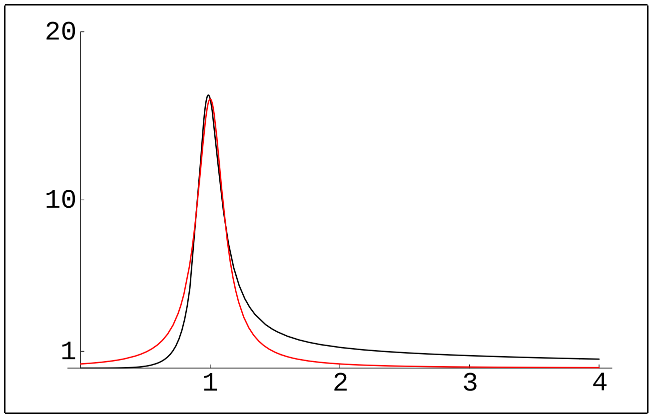


FIGURE 149: *Comparison of the exact Rayleigh cross-section with its resonant approximation (497). In constructing the figure I have assigned k the unphysically large value $k = 0.25$. The fit—already quite good—becomes ever better as k gets smaller.*

We will soon (in §9) have unexpected occasion to inquire more closely into properties of the "Lorenz distribution function" $L(\xi, k)$,[304] but for the moment are content to observe that in this notation

$$\sigma \approx (\pi/2k)\sigma_0 \cdot L(\xi - 1, k) \quad \text{at resonance: } \omega \sim \omega_0$$

and that $L(\xi, k)$ assumes its maximal value at $\xi = 0$: $L(0, k) = \frac{2}{\pi}k^{-1}$ so

$$\sigma_{\max} = \sigma_0/k^2 = (\sigma_0/\tau^2)/\omega_0^2 = (\sigma_0/\tau^2)/(2\pi\nu_0)^2 \tag{499.1}$$

where ν_0 is the literal *frequency* of the resonant radiation and (below) $\lambda_0 = c/\nu_0$ its *wavelength*. But (look back again to pages 398 and 406 for the definitions of τ and σ_0)

$$\sigma_0/\tau^2 = \tfrac{8\pi}{3}\big[e^2/4\pi mc^2\big]^2\Big/\big[\tfrac{2}{3}e^2/4\pi mc^3\big]^2 = 6\pi c^2$$

so

$$\sigma_{\max} = 6\pi(c/2\pi\nu_0)^2 = \tfrac{3}{2\pi}\lambda_0^2 \tag{499.2}$$
$$\sim \begin{cases} \text{cross-sectional area of the smallest object} \\ \text{visible in radiation of resonant frequency} \end{cases}$$

Radiation of resonant frequency, when incident upon a "gas" made of such "classical molecules," is scattered profusely (the gas becomes"florescent," and

[304] Also—and with better reason—called the "Cauchy distribution function." See Abramowitz & Stegun, *Handbook of Mathematical Functions* (1964), page 930.

loses its transparency). Classically, we expect a molecule to possess a *variety* of normal modes . . . a variety of "characteristic frequencies," and resonance florescence to occur at each. Notice that if we were to neglect the self-interaction (formally: let $\tau \downarrow 0$ in (499.1)) then the resonant scattering cross-section would become infinite: $\sigma_{\max} \uparrow \infty$. Here as in (for example) the elementary theory of forced damped oscillators, it is damping that accounts for finiteness at resonance.

HIGH-FREQUENCY REGIME If $\xi \gg 1$ then (495) becomes

$$\sigma_{\text{Rayleigh}} = \sigma_0 \cdot \frac{1}{1 + k^2\xi^2}$$

But $k\xi = (\tau\omega_0)(\omega/\omega_0) = \omega/\Omega \ll 1$ except when—as previously remarked—ω is so large as to render the classical theory meaningless. So the factor $(1 + k^2\xi^2)^{-1}$ can/should be abandoned. The upshot: *Rayleigh scattering reverts to Thomson scattering at frequencies $\omega \gg$ the molecular resonance frequency ω_0.* Physically, the charge is stimulated so briskly that it does not feel its attachment to the slow spring, and responds like a *free* particle. It was to represent this fact that the red asymptote was introduced into Figure 147.

9. Radiative decay. Suppose now that the incident light beam is abruptly switched off. We expect the oscillating electrona to radiate its energy away, coming finally to rest. This is the process which, as explained below, gives rise to the classical theory of spectral line shape. The radiative relaxation of a harmonically bound classical electron is governed by

$$\ddot{\boldsymbol{x}} - \tau\dddot{\boldsymbol{x}} + \omega_0^2\boldsymbol{x} = \boldsymbol{0} \tag{500}$$

which is just the homogeneous counterpart of (494). Borrowing $\tau = k/\omega_0$ from page 408 and multiplying by ω_0 we obtain

$$\omega_0\ddot{\boldsymbol{x}} - k\dddot{\boldsymbol{x}} + \omega_0^3\boldsymbol{x} = \boldsymbol{0}$$

which proves more convenient for the purposes at hand. Looking for solutions of the form $e^{i\omega t}$ we find that ω must be a root of the cubic polynomial

$$ik\omega^3 - \omega_0\omega + \omega_0^3 = 0$$

Mathematica provides complicated closed-form descriptions of those roots, which when expanded in powers of the dimensionless parameter k become

$$\begin{aligned}
\omega_1 &= +\omega_0 + i\tfrac{1}{2}\omega_0 k - \tfrac{5}{8}\omega_0 k^2 - i\omega_0 k^3 + \cdots \\
\omega_2 &= -\omega_0 + i\tfrac{1}{2}\omega_0 k + \tfrac{5}{8}\omega_0 k^2 - i\omega_0 k^3 - \cdots \\
\omega_3 &= -i\omega_0\big\{k^{-1} + k - 2k^3 + 7k^5 - \cdots\big\}
\end{aligned}$$

The root ω_3 we abandon as an unphysical artifact because

$$e^{i\omega_3 t} = \exp\Big[\omega_0\big\{k^{-1} + k - \cdots\big\}t\Big] \quad \text{very rapidly blows up}$$

That leaves us with two linearly independent solutions

$$e^{-\omega_0(\frac{1}{2}k-k^3+\cdots)\,t}\cdot e^{\pm\,i\,\omega_0(1-\frac{5}{8}\omega_0k^2+\cdots)\,t}$$

and with the implication that

$$\boldsymbol{x}(t)=\boldsymbol{\mathfrak{X}}\,e^{-\frac{1}{2}\omega_0k\,t}\cos\big[(\omega_0-\tfrac{5}{8}\omega_0k^2)\,t\big]$$

is *in excellent approximation*[305] a particular solution of (500), and that so also is the function got by $\cos\mapsto\sin$. In a standard notation

$$=\boldsymbol{\mathfrak{X}}\,e^{-\frac{1}{2}\Gamma\,t}\cos\big[(\omega_0-\Delta\omega)\,t\big] \tag{501}$$

where

$$\begin{aligned}\Gamma&\equiv\omega_0k &&\text{describes the \textit{damping coefficient}}\\ \Delta\omega&\equiv\tfrac{5}{8}\omega_0k^2 &&\text{describes a small \textit{downward frequency shift}}\end{aligned}$$

A function of the familiar design (501) is plotted in Figure 150.

Notice that it is *self-interaction*, as described by the small dimensionless parameter k, that is responsible both for the slow attenuation $e^{-\frac{1}{2}\Gamma\,t}$ and for the slight frequency shift $\Delta\omega$, and that attenuation causes the electronic oscillation (whence also the resulting radiation) to be <u>not quite monochromatic</u>. Turning to the Fourier transform tables (which in this instance serve better than *Mathematica*) we find[306]

$$e^{-\beta y}\cos\alpha y=(\beta/\pi)\int_0^\infty\Big\{\frac{1}{(x-\alpha)^2+\beta^2}+\frac{1}{(x+\alpha)^2+\beta^2}\Big\}\cos yx\,dx$$

The implication is that (501) can be expressed

$$\boldsymbol{x}(t)=\boldsymbol{\mathfrak{X}}\int_0^\infty S(\omega)\cos\omega t\,d\omega \tag{502.1}$$

$$\begin{aligned}S(\omega)\equiv\frac{\Gamma}{2\pi}\Big\{&\frac{1}{[\omega-(\omega_0-\Delta\omega)]^2+(\frac{1}{2}\Gamma)^2}\\ &+\frac{1}{[\omega+(\omega_0-\Delta\omega)]^2+(\frac{1}{2}\Gamma)^2}\Big\}\end{aligned}$$

The second term is small even for $\omega=0$ and dies rapidly as ω increases. We therefore *abandon* that term, and work in the good approximation that

$$S(\omega)\approx\frac{\Gamma}{2\pi}\frac{1}{[\omega-(\omega_0-\Delta\omega)]^2+(\frac{1}{2}\Gamma)^2} \tag{502.2}$$

---

[305] How excellent? *Mathematica* supplies

$$\Big\{\omega_0\tfrac{d^2}{dt^2}-k\tfrac{d^3}{dt^3}+\omega_0^3\Big\}e^{-\frac{1}{2}\omega_0k\,t}e^{\pm\,i(\omega_0-\frac{5}{8}\omega_0k^2)\,t}$$
$$=0+0k+0k^2-i2\omega_0^3k^3+\tfrac{103}{64}\omega_0^3k^4+i\tfrac{105}{64}\omega_0^3k^5-\cdots$$

[306] A. Erdélyi *et al* (editors), *Tables of Integral Transforms* (1954), Volume I, Table 1.2#13 (page 8) and Table 1.6#19 (page 21).

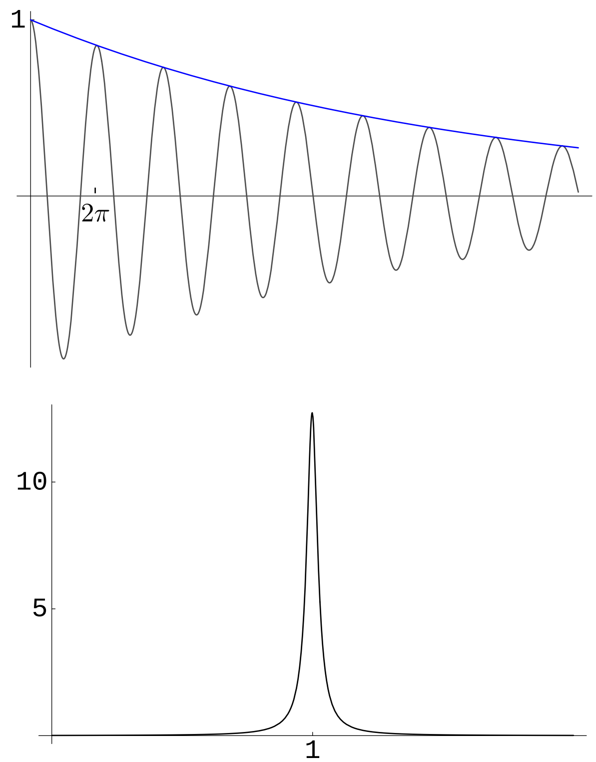


FIGURE 150: *Above: diagram of the motion of a charge-on-a-spring (Rayleigh's "classical molecule") that, because it experiences periodic acceleration, slowly radiates away its initial store of energy. The figure derives from* (501) *with* $\omega_0 = 1$ *and* $k = 0.05$. *The modulating exponential factor* $e^{-\frac{1}{2}\Gamma t}$ *is shown in blue. The Fourier transform of that curve (below) can be interpreted as a description what would be seen by a physicist who examines the emitted radiation with the aid of a spectroscope. The "spectral line" has a "Lorentzian" profile.*

At (502.2) we encounter once again—but this time in the frequency domain—precisely the Lorentz distribution

$$S(\omega) \approx L(\omega - [\omega_0 - \Delta\omega], \Gamma)$$

first encountered at (498), and the basis for the statement that

$$\text{Classical line shape is Lorentzian} \tag{503}$$

We digress to acquire familiarity with some of the basic properties of the Lorentz distribution function $L(x, \Gamma) \equiv \frac{\Gamma}{2\pi}[x^2 + (\frac{1}{2}\Gamma)^2]^{-1}$. Figure 151 shows the

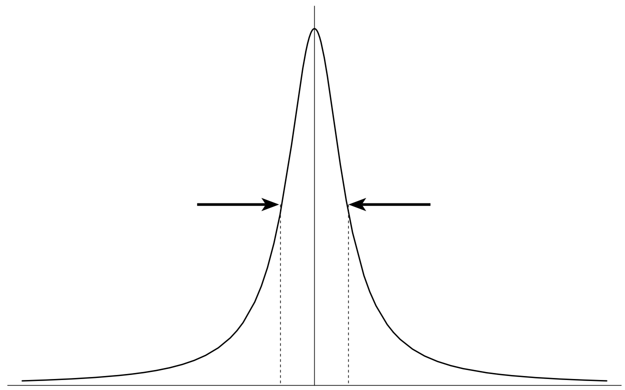


FIGURE 151: *Characteristic shaped of what physicists usually call the "Lorentz distribution" but mathematicians know as the "Cauchy distribution." Arrows mark the half-max points, and Γ is shown in the text to be the distance between those points.*

characteristic shape of the Lorentz distribution. It is elementary that

$$L(x,\Gamma) \leqslant L_{\max} = L(0,\Gamma) = \tfrac{2}{\pi\Gamma}$$

and that

$$L(x,\Gamma) = \tfrac{1}{2}L_{\max} \quad\Longrightarrow\quad x = \pm\tfrac{1}{2}\Gamma$$

so the parameter Γ can be interpreted

$$\Gamma = \text{width at half-max} \tag{504}$$

On casual inspection (Figure 152) the graphs of the Lorentz and Gaussian (or "normal") distributions appear quite similar, though the former has a noticeably sharper central peak and relatively wide hips. Richard Crandall's "The Lorentz distribution is a pig—too fat!" might seem uncharitable . . . until one looks to the *moments* of the two distributions. For the Gaussian the sequence

$$\langle x^0\rangle, \langle x^1\rangle, \langle x^2\rangle, \langle x^3\rangle, \langle x^4\rangle, \langle x^5\rangle, \langle x^6\rangle, \langle x^7\rangle, \langle x^8\rangle, \ldots$$

proceeds unremarkably

$$1,\ \ 0,\ \ \tfrac{1}{2}a^2,\ \ 0,\ \ \tfrac{3}{4}a^4,\ \ 0,\ \ \tfrac{15}{8}a^6,\ \ 0,\ \ \tfrac{105}{16}a^2, \ldots$$

but in the case of the Lorentz distribution even the *definition* of the moments is a bit problematic (as *Mathematica* is quick to remind us): if we proceed from the definition $\langle x^n\rangle \equiv \lim_{z\uparrow\infty}\int_{-z}^{+z} x^n L(x,\Gamma)\,dx$ we obtain

$$1,\quad 0,\quad \infty,\quad 0,\quad \infty,\quad 0,\quad \infty,\quad 0,\quad \infty, \ldots$$

So wide are the hips of the Lorentz distribution that (in particular)

$$\Delta x \equiv \sqrt{\langle (x-\langle x\rangle)^2\rangle} = \infty$$

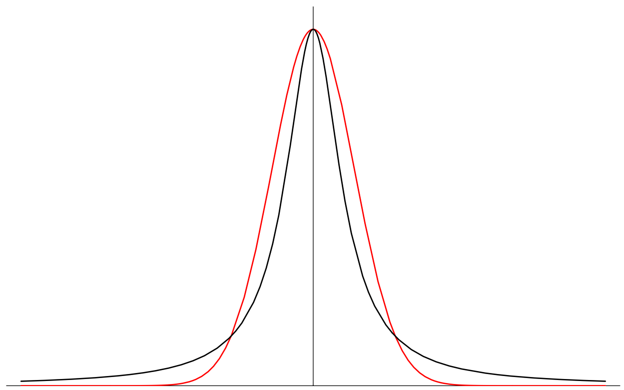


FIGURE 152: *The Lorentz distribution* $L(x,\Gamma) \equiv \frac{\Gamma}{2\pi}[x^2 + (\frac{1}{2}\Gamma)^2]^{-1}$ *has here been superimposed upon the Gaussian distribution*

$$G(x,a) \equiv \tfrac{1}{a\sqrt{\pi}}e^{-(x/a)^2}$$

of the same height (set $a = \frac{\sqrt{\pi}}{2}\Gamma$*). The Lorentz distribution is seen to have a relatively sharp peak, but relatively broader flanks.*

The standard descriptor of the "width" of the distribution is therefore not available: to provide such information one is *forced* to adopt (504). It is remarkable that, of two distributions that—when plotted—so nearly resemble one another,

- one is arguably "the best behaved in the world," and
- the other one of the worst behaved.[307]

And it is in that light remarkable that in some other respects the Lorentz distribution is quite unexceptional: for example, it leads straightforwardly to a representation of the δ-function

$$\delta(x - x_0) = \lim_{\Gamma\downarrow 0} L(x - x_0, \Gamma) = \lim_{\epsilon\downarrow 0} \frac{\epsilon/\pi}{(x - x_0)^2 + \epsilon^2}$$

that often proves useful in applications. Returning now to the physics ...

The classical theory of spectral line shape marks an interesting point in the *history* of physics, but leads to results which are of enduring interest only as zeroth approximations to their quantum counterparts. As such, they are

[307] It was known to Poisson already in 1824 that what came to be called the "Cauchy distribution" is a distribution to which the fundamental "central limit theorem" does not pertain. Cauchy himself entered the picture only in 1853—the year of Lorentz' birth. My source here has been the footnote that appears on page 183 of S. M. Stigler's *The History of Statistics* (1986).

remarkably good. To illustrate the point: Reading from (501) we see that our "classical molecule" has a

$$\text{characteristic lifetime} = 2/\Gamma$$

while its

$$\text{spectral linewidth} = \Gamma/2$$

Evidently

$$(\text{linewidth})\cdot(\text{lifetime}) = 1 \tag{505}$$

Quantum mechanically, spectral line shape arises in first approximation (*via* $E = h\nu = \hbar\omega$) from an instance of the *Heisenberg uncertainty principle*, according to which

$$\Delta E \cdot (\text{lifetime}) \gtrsim \hbar$$

But $\Delta E = \hbar \cdot (\text{linewidth})$ so we are, in effect, led back again to the classical relation (505). Similar parallels could be drawn from the quantum theory of electromagnetic scattering processes.[308]

10. Concluding remarks. Classical radiation theory, though latent in Maxwell's equations, is a subject of which Maxwell himself knew nothing. Its development was stimulated by Hertz' experimental production/detection of electromagnetic waves—a development which Maxwell anticipated, but did not live long enough to see—and especially by the technological effort which attended the invention of radio. It is a subject of which we have only scratched the surface: we have concentrated on the radiation produced by individual accelerated charges, and remain as innocent as babies concerning the fields produced by the currents that flow in the antenna arrays that several generations of radio engineers have worked so ingeniously to devise.

The subject leads, as we have witnessed, to mathematical relationships notable for their complexity. But those intricate relationships among $\boldsymbol{E}$'s, $\boldsymbol{B}$'s, the elements of $S^{\mu\nu}$ . . . sprang from relatively simple properties of the potentials A^{μ}. Indeed, the work of this entire chapter (chapter in the text, chapter in the history of pure/applied physics) can be viewed as an exercise in applied potential theory. It is curious that—in electrodynamics most conspicuously, but also elsewhere in physics—*it appears to be the spooks who speak the language of God, and is in any event certainly the spooks who coordinate our effort to account for and describe the complexity evident in the observable/tangible world of direct experience.*

Our progress thus far has (in 418 pages and $\sim$ 60 hours) taken us in a fairly direct path from the "beginning" or our subject to within sight of its "end" . . . from a discussion of first principles and historical roots into the realm where

[308] See, for example, W. Heitler, *Quantum Theory of Radiation* (1954).

electrodynamics shows an ever-stronger tendency to break down. Along the way, electrodynamics gave birth to special relativity (who has long since left home to lead an independent existence elsewhere) . . . and as we take leave of the lady she is clearly once again pregnant (with quantum mechanics, elementary particle physics, general relativity, . . .). Her best years—if no longer as a dancer, then as a teacher of dance—lie still ahead. But that is another story for another day. In the pages that follow we will be *backtracking*—discussing miscellaneeous issues that, for all their theoretical/technological importance, were judged to be peripheral to our initial effort.

8

DISTRIBUTED CHARGE SYSTEMS

Introduction. We have recently been studying *solutions* of Maxwell's equations —solutions in the complete absence of sources (Chapter 5) and solutions in the presence of but a single point source (Chapters 6 & 7). But in many physical problems and most technological applications one has interest in the fields generated by (static or dynamic) *populations* of charged particles; *i.e.*, by spatially *distributed sources*.

One might suppose that such problems could be solved by application of the principle of superposition . . . but the "application" is more easily talked about than done, and it is not at all straightforward: it inspired much of the mathematical invention for which the period 1775–1875 is remembered. And there are (as always) unexpected physical complications. For example: the presence of conductive materials gives rise to "*induced* charges," which join the unknowns of the problem.

We will look first to the electro*static* problem—to the description of the description of the electrostatic potential set up by an arbitrarily constructed blob of charge. Information of the sort we now seek would comprise our point of departure if se sought (say) to construct an account of the Bohr orbits around a structured nucleus, or (in gravitational terms) of the motion of a satellite around the inhomogenous earth.

1. Multipole representation of a static source. Let $\rho(\boldsymbol{x})$ describe a t-independent (or "static") charge distribution. The resulting electromagnetic field has no magnetic component ($\boldsymbol{B} = \boldsymbol{0}$), and its t-independent electric component (see again page 25) can be described

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}) &= -\boldsymbol{\nabla}\varphi(\boldsymbol{x}) \\ \varphi(\boldsymbol{x}) &= \tfrac{1}{4\pi}\iiint \rho(\boldsymbol{x})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}|}\,d^3x \end{aligned} \tag{506}$$

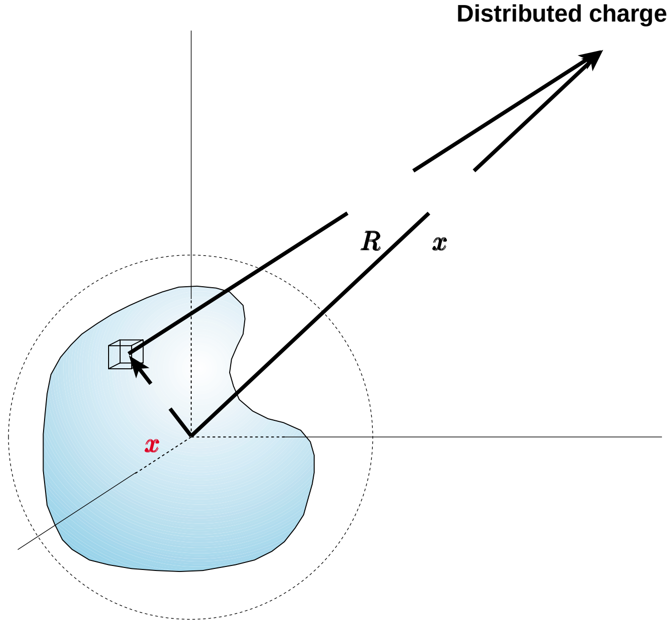


FIGURE 153: *We use* $\boldsymbol{x}$ *to describe the constituent elements of a distributed charge, and* $\boldsymbol{x}$ *to describe the location of a typical field point. The vector* $\boldsymbol{R}(\boldsymbol{x},\boldsymbol{x}) \equiv \boldsymbol{x} - \boldsymbol{x}$ *stretches from the former to the latter, and has length* $R(\boldsymbol{x},\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}|$. *We proceed in the assumption that* $r \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}} > a$, *where*

$$a \equiv \begin{cases} \text{radius of a mental sphere large enough} \\ \text{to enclose the entire distributed charge} \end{cases}$$

The integral $\iiint$ derives from, and expresses, the principle of superposition—as anticipated. But our goal now is to see what we can do to *sharpen* the very general result described above. We want to learn to distinquish the relevant features of (506) from the less relevant, so that by discarding the latter we can *simplify our computational life*.

Let us suppose that the source, though distributed, is "localized" in the sense that $\underline{\rho(\boldsymbol{x}) \equiv 0 \text{ for } \boldsymbol{x} \text{ exterior to a sphere of sufficiently large but finite radius } a}$,[309] and let us agree that our ultimate objective—what we are presently getting in position to do—is to describe the electrostatic potential at points *external* to that sphere (see Figure 153). Writing

$$\begin{aligned} R(\boldsymbol{x},\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}| &= \sqrt{(\boldsymbol{x}-\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{x})} \\ &= \sqrt{r^2 - 2rr\cos\vartheta + r^2} \end{aligned}$$

[309] This weak assumption serves merely to exclude "infinite line charges" and similar (unphysical) abstractions.

with $r \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$ and $r \equiv \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$, we note that the dimensionless ratios x/r, y/r, z/r are in every instance less than unity. It becomes therefore natural to contemplate <u>expanding $1/R(\boldsymbol{x},\boldsymbol{x})$ in powers of those ratios</u>. To that end ... we recall that according to Taylor's theorem

$$f(x+x) = e^{x\frac{\partial}{\partial x}} f(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!} x^n f^{(n)}(x)$$

In the multivariate case we expect therefore to have

$$\begin{aligned} f(x+x, y+y, z+z) &= e^{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}} f(x,y,z) \\ &= \Big\{1 + \big[x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y} + z\tfrac{\partial}{\partial z}\big] \\ &\qquad + \tfrac{1}{2}\big[x^2\tfrac{\partial^2}{\partial x^2} + 2xy\tfrac{\partial^2}{\partial x\partial y} + 2xz\tfrac{\partial^2}{\partial x\partial z} \\ &\qquad\quad + y^2\tfrac{\partial^2}{\partial y^2} + 2yz\tfrac{\partial^2}{\partial y\partial z} + z^2\tfrac{\partial^2}{\partial z^2}\big] + \cdots\Big\} f(x,y,z) \end{aligned}$$

which when applied in particular to the $\boldsymbol{x}$-dependence of $1/R(\boldsymbol{x},\boldsymbol{x})$ gives

$$\begin{aligned} \frac{1}{|\boldsymbol{x}-\boldsymbol{x}|} &= \frac{1}{r} + \frac{1}{r^3}\cdot\big[xx + yy + zz\big] \\ &\quad + \frac{1}{r^5}\cdot\tfrac{1}{2}\big[x^2(3x^2 - r^2) + 6xyxy + 6xzxz \\ &\qquad + y^2(3y^2 - r^2) + 6yzyz + z^2(3z^2 - r^2)\big] + \cdots \end{aligned}$$

In a fairly natural (and quite useful) condensed notation we have

$$\begin{aligned} &= r^{-1} + r^{-3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\quad + r^{-5}\tfrac{1}{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\cdot\begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \cdots \end{aligned}$$

Feeding this expansion back into (506) we obtain

$$\begin{aligned} \varphi(\boldsymbol{x}) &= \tfrac{1}{4\pi}\Big\{r^{-1}q + r^{-3}\boldsymbol{p}\cdot\boldsymbol{x} + r^{-5}\tfrac{1}{2}\boldsymbol{x}\cdot\mathbb{Q}\boldsymbol{x} + \cdots\Big\} \\ &= \tfrac{1}{4\pi}\Big\{r^{-1}q + r^{-2}\boldsymbol{p}\cdot\hat{\boldsymbol{x}} + r^{-3}\tfrac{1}{2}\hat{\boldsymbol{x}}\cdot\mathbb{Q}\hat{\boldsymbol{x}} + \cdots\Big\} \end{aligned} \tag{508}$$

where

$$q \equiv \iiint \rho(\boldsymbol{x})\, d^3x \tag{508.0}$$

$\equiv$ so-called "monopole moment scalar" or **total charge**

$$\boldsymbol{p} \equiv \iiint \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rho(\boldsymbol{x})\, d^3x \tag{508.1}$$

$\equiv$ so-called "**dipole moment vector**"

$$\mathbb{Q} \equiv \iiint \begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix} \rho(\boldsymbol{x})\, d^3x \tag{508.2}$$

$\equiv$ so-called "**quadrupole moment** (tensor or) **matrix**"

In higher order we lose the advantages of matrix notation ... might appear in 3rd order to have to write something like

$$\tfrac{1}{4\pi} r^{-4} \tfrac{1}{3!} \sum_{\substack{a,b,c \\ a+b+c=3}} W_{abc}\, \hat{x}^a \hat{x}^b \hat{x}^c \quad \text{with} \quad W_{abc} \equiv \iiint \underbrace{M_{abc}(x,y,z)}_{\text{complicated cubic}} \rho(\boldsymbol{x})\, d^3x$$

but will soon be in position to proceed in a more orderly manner. As will emerge, it is the lowest-order terms that are of highest practical importance, so (508) is in fact quite useful as it stands: it will be useful also as a benchmark against which to test more general formulæ as they become available. Several comments are now in order:

1. The objects q, $\boldsymbol{p}$, $\mathbb{Q}$, ... are called "scalar," "vector," "tensor,"... in recognition of how they respond to rotations of the Cartesian frame: they are, in short, *tensorial with respect to the rotation group* $O(3)$, as one could demonstrate without difficulty.

2. q is the 0th moment of the charge distribution $\rho(\boldsymbol{x})$, $\boldsymbol{p}$ is assembled from the 1st moments, $\mathbb{Q}$ is assembled from the 2nd moments, *etc*. Not surprisingly, *if one possessed the moments of all orders then one could reconstruct the* $\rho(\boldsymbol{x})$ *which generated those moments.*[310]

3. $\mathbb{Q}$ is (like the energy/momentum tensor $\mathbb{S}$: see again page 215) *symmetric* and *traceless*. These properties are, moreover, preserved under coordinate

---

[310] Usually, not always. The program would fail if, for example (see again page 416), the distribution were Lorentzian

$$\rho(\boldsymbol{x}) \sim \frac{1}{x^2 + y^2 + z^2 + a^2}$$

But such a distribution cannot be enclosed within a sphere of finite radius.

rotation. From symmetric tracelessness it follows that $\mathbb{Q}$ contains (not 9, as one would otherwise expect, but) only 5 adjustable constants (degrees of freedom). Symmetry alone assures that $\mathbb{Q}$ can always be rotated to diagonal form

$$\mathbb{Q} \xrightarrow[\text{properly chosen rotation}]{} \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix}$$

and tracelessness requires that the eigenvalues sum to zero: $Q_1 + Q_2 + Q_3 = 0$.

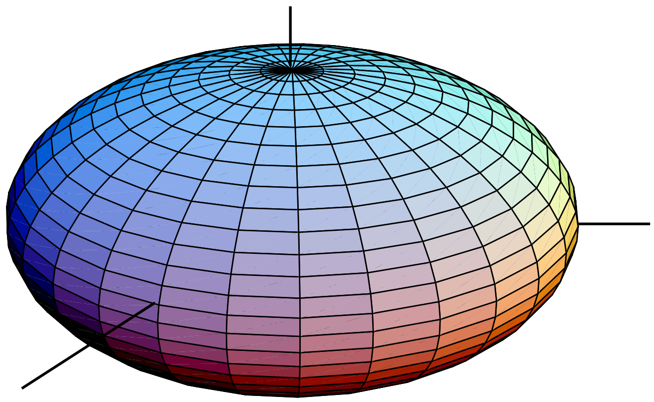


FIGURE 154: *Oblate spheroidal distribution, symmetric about the z-axis. Spinning bodies (stars, planets, atomic nuclei) commonly possess this shape, at least in leading approximation.*

If, as is quite commonly the case, $\rho(\boldsymbol{x})$ is symmetric about the z-axis (see the figure) then $\mathbb{Q}$ acquires the structure

$$\begin{pmatrix} -\frac{1}{2}Q & 0 & 0 \\ 0 & -\frac{1}{2}Q & 0 \\ 0 & 0 & Q \end{pmatrix}$$

In such specialized contexts it is common (among nuclear physicists and others) to speak of "*the* quadrupole moment," the reference being to Q.

4. What is the origin of the monopole/dipole/. . . **multipole** terminology? The answer has little/nothing to do with electrostatics *per se*, much to do with the meaning of n^{th} derivative. Look, for example, a 1-dimensional model of the situation in hand: suppose it to be the case that

$$\varphi(x) = \int \rho({\color{red}x})F(x - {\color{red}x})\, d{\color{red}x}$$

where $F(\bullet)$ is some prescribed differentiable function (*not* necessarily the $\frac{1}{x-x}$ encountered in (506)) and where x remains "small" throughout the range of integration. We expect then to have

$$\varphi(x) = \sum_{n=0}^{\infty}(-)^n \tfrac{1}{n!}\cdot \underbrace{\int \rho(x)x^n dx}_{n^{\text{th}}\text{ moment}} \cdot F^{(n)}(x)$$

where $F^{(0)}(x)$, $F^{(1)}(x)$, $F^{(2)}(x)$, $F^{(3)}(x)$, ... acquire meaning from the following scheme:

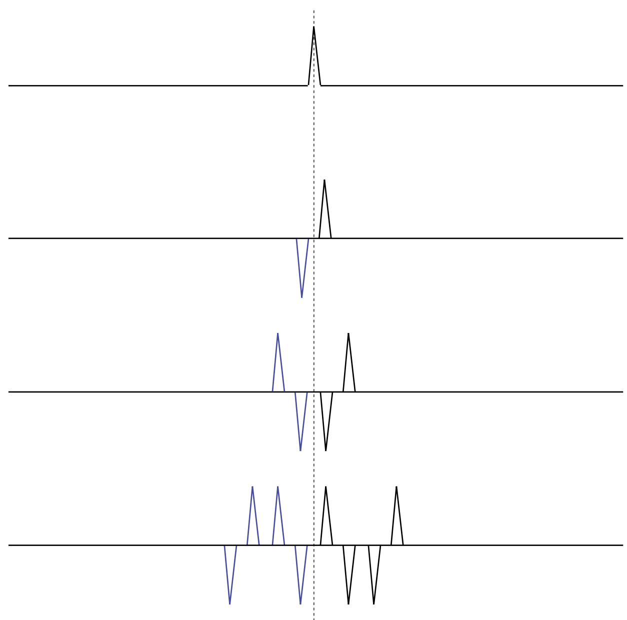


FIGURE 155: *Representation of the mechanism by which iteration of*

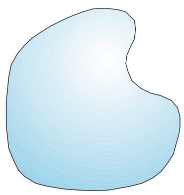
$$F^{(1)}(x) = \lim_{\epsilon\downarrow 0}\int \frac{\delta\big(\xi-(x+\frac{1}{2}\epsilon)\big)-\delta\big(x-(x-\frac{1}{2}\epsilon)\big)}{\epsilon}F(\xi)\,d\xi$$

gives rise to successive derivatives of $F(x)$. Notice that 2^n spikes contribute to the construction of $F^{(n)}(x)$. This is the source of the "di/quadu/octo... 2^n-tuple pole" terminology.

In several dimensions one encounters only this new circumstance: one can displace a sign-reversed monopole in several directions to create a dipole, can displace a sign-reversed dipole in several directions to create a quadrupole, *etc.*

5. We are led thus to the principle that an arbitrary localized distribution $\rho(\boldsymbol{x})$ can be represented as the superposition of

- an appropriately selected monopole +
- an appropriately selected dipole +
- an appropriately selected quadrupole + *etc*:

 $= \text{monopole} + \text{dipole} + \text{quadrupole} + \cdots$

6. Looking back again to (508) we notice that at sufficiently remote field points one can drop all but the monopole term ($\rho(\boldsymbol{x})$ looks like a point charge). At less remote points one can drop all terms subsequent to the dipole term. *High order multipole terms depend upon such high powers of* $1/r$ *that they are of quantitative importance only in the near zone.*

Equation (508) carries us a long way toward our goal, as stated on page 422. But there remains a good deal of meat to be gnawed from the bone.

2. Electrostatic potential of a dipole. Consider the two-charge configuration (no *net* charge) shown in Figure 156. The associated electrostatic potential can be described

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}q\left\{\frac{1}{\sqrt{r^2 - 2ra\cos\vartheta + a^2}} - \frac{1}{\sqrt{r^2 + 2ra\cos\vartheta + a^2}}\right\} \tag{509.1}$$

$$= \tfrac{1}{4\pi}(q/r)\left\{\left[1 - 2\tfrac{a}{r}\cos\vartheta + \left(\tfrac{a}{r}\right)^2\right]^{-\frac{1}{2}} - \left[1 + 2\tfrac{a}{r}\cos\vartheta + \left(\tfrac{a}{r}\right)^2\right]^{-\frac{1}{2}}\right\}$$

$$= \tfrac{1}{4\pi}\frac{2qa\cos\vartheta}{r^2}\left\{1 + \frac{5\cos2\vartheta - 1}{4}\left(\tfrac{a}{r}\right)^2 + \frac{63\cos4\vartheta - 28\cos2\vartheta + 29}{64}\left(\tfrac{a}{r}\right)^4 + \cdots\right\} \tag{509.2}$$

This describes, as a power series in a/r, the potential of a *physical* dipole. Proceeding now to the <u>double limit</u>

$$a \downarrow 0 \text{ and } q \uparrow \infty \text{ in such a way that } p \equiv 2aq \text{ remains constant}$$

we obtain

$$\downarrow$$
$$= \tfrac{1}{4\pi}\frac{p\cos\vartheta}{r^2} = \tfrac{1}{4\pi}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{x}}}{r^2} = \tfrac{1}{4\pi}\frac{\boldsymbol{p}\cdot\boldsymbol{x}}{r^3} \tag{510}$$

Notice that the dipole potential φ would simply vanish if q were held *constant* during the compression process $a \downarrow 0$. Equipotentials derived from (509) and (510) are shown in Figure 157.

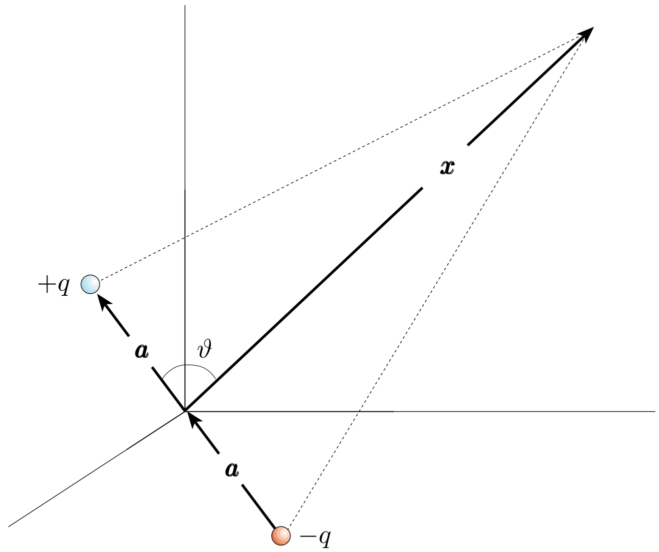


FIGURE 156: *Notation used in the text to describe the field of a physical dipole* •–•. *A "mathematical dipole" results in the idealized limit* $a \downarrow 0$, $q \uparrow \infty$ *with* $p \equiv 2aq$ *held constant.*

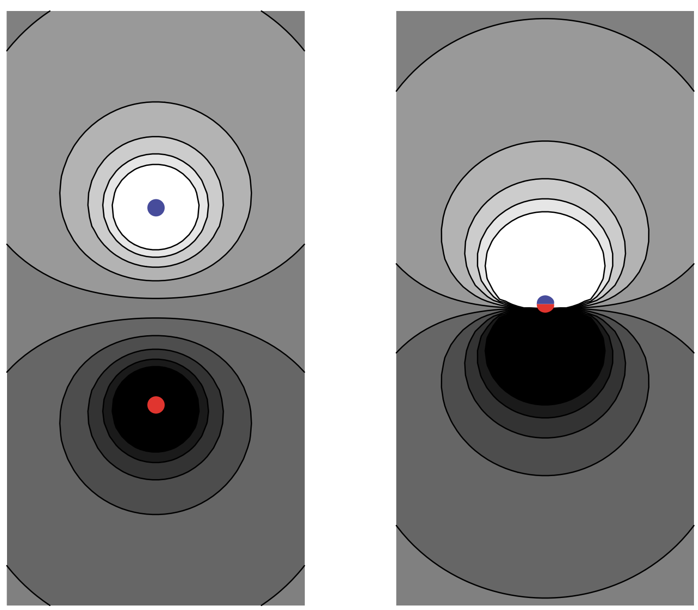


FIGURE 157: *Central cross section of the equipotentials of a physical dipole (on the left) and of an idealized dipole (on the right).*

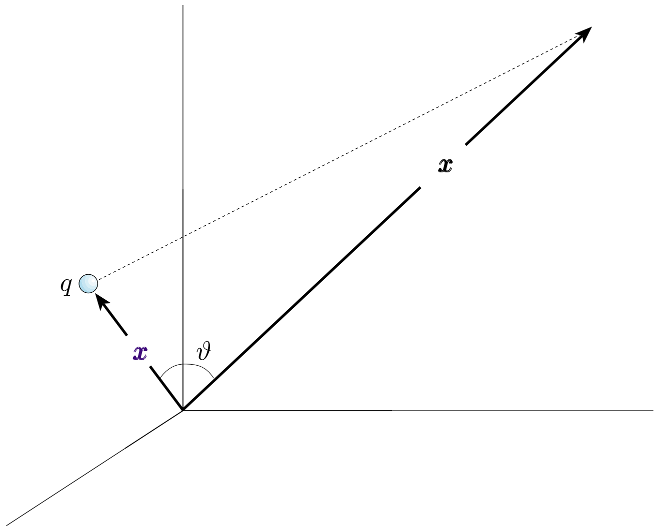


FIGURE 158: *Notation used in the text to describe the field of an "eccentric monopole," i.e., of an isolated charge (or charge element) that is arbitrarily positioned with respect to the coordinate origin. The length of* $\boldsymbol{x}$ *is* r*, the length of* $\boldsymbol{x}$ *is* r.

3. Electrostatic potential of an eccentric monopole. In what might at first sight appear to be a step backward, but will soon be revealed to be a long step forward, we look now to the potential of the primitive system shown above; *i.e.*, to the Coulomb potential of an eccentrically-positioned charge. This we do by systematic elaboration of methods borrowed from the preceding section. Immediately (which is to say: by the Law of Cosines)

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}q\frac{1}{\sqrt{r^2 - 2rr\cos\vartheta + r^2}} \tag{511}$$

which—in preparation for implementation of our plan, which is to proceed by power series expansion—we will write

$$= \begin{cases} \tfrac{1}{4\pi}q\tfrac{1}{r}\cdot\dfrac{1}{\sqrt{1-2\left(\frac{r}{r}\right)\cos\vartheta + \left(\frac{r}{r}\right)^2}} & : \quad \text{adapted to the case } r < r \\[2ex] \tfrac{1}{4\pi}q\tfrac{1}{r}\cdot\dfrac{1}{\sqrt{1-2\left(\frac{r}{r}\right)\cos\vartheta + \left(\frac{r}{r}\right)^2}} & : \quad \text{adapted to the case } r > r \end{cases}$$

Thus do we acquire interest in the objects $P_n(w)$ that arise as coefficients from the series

$$\frac{1}{\sqrt{1-2wt+t^2}} = \sum_{n=0}^{\infty} P_n(w)\,t^n \tag{512.1}$$

Mathematica supplies

$$\left.\begin{aligned}
P_0(w) &= 1\\
P_1(w) &= w\\
P_2(w) &= \tfrac{1}{2}(3w^2-1)\\
P_3(w) &= \tfrac{1}{2}(5w^3-3w)\\
P_4(w) &= \tfrac{1}{8}(35w^4-30w^2+3)\\
P_5(w) &= \tfrac{1}{8}(63w^5-70w^3+15w)\\
&\ \ \vdots
\end{aligned}\right\}\tag{512.2}$$

These are precisely the **Legendre polynomials**, the properties of which were first described (1784) by A. M. Legendre (1752–1833) and are summarized in every mathematical handbook.[311] Graphs of some low-order Legendre polynomials are shown in Figure 159.

Returning with this information to (511) we have

$$\varphi(\boldsymbol{x}) = \begin{cases} \frac{1}{4\pi}q\frac{1}{r}\cdot\displaystyle\sum_{n=0}^{\infty}\left(\frac{{\color{blue}r}}{r}\right)^n P_n(\cos\vartheta) & \text{in the far zone}\\[2ex] \frac{1}{4\pi}q\frac{1}{{\color{blue}r}}\cdot\displaystyle\sum_{n=0}^{\infty}\left(\frac{r}{{\color{blue}r}}\right)^n P_n(\cos\vartheta) & \text{in the near zone}\end{cases}\tag{513}$$

in which connection it becomes pertinent to notice that (ask *Mathematica*)

$$\left.\begin{aligned}
P_0(\cos\vartheta) &= 1\\
P_1(\cos\vartheta) &= \cos\vartheta\\
P_2(\cos\vartheta) &= \tfrac{1}{4}(3\cos2\vartheta+1)\\
P_3(\cos\vartheta) &= \tfrac{1}{8}(5\cos3\vartheta+3\cos\vartheta)\\
P_4(\cos\vartheta) &= \tfrac{1}{64}(35\cos4\vartheta+20\cos2\vartheta+9)\\
P_5(\cos\vartheta) &= \tfrac{1}{128}(63\cos5\vartheta+35\cos3\vartheta+30\cos\vartheta)\\
&\ \ \vdots
\end{aligned}\right\}\tag{512.3}$$

Looking specifically/explicitly to the far zone we have

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}\Big\{r^{-1}q + r^{-2}q{\color{blue}r}P_1(\cos\vartheta) + r^{-3}q{\color{blue}r}^2P_2(\cos\vartheta)+\cdots\Big\}\tag{514}$$

which must comprise the <u>multipole expansion—correct to all orders—of an eccentrically placed monopole</u>. How does this result compare with what (508) has to say in such a specialized situation? Setting $\rho({\color{red}\boldsymbol{x}}) = q\delta({\color{red}\boldsymbol{x}}-{\color{blue}\boldsymbol{x}})$ and working

[311] See, for example, W. Magnus & F. Oberhettinger, *Formulas & Theorems for the Functions of Mathematical Physics* (1949), pages 50–59; J. Spanier & K. B. Oldham, *An Atlas of Functions* (1987), Chapter 21; M. Abramowitz & Irene Stegun, *Handbook of Mathematical Functions* (1964), Chapter 22. For discussion of how the principal properties of the Legendre polynomials are established see pages 471–475 in CLASSICAL ELECTRODYNAMICS (1980).

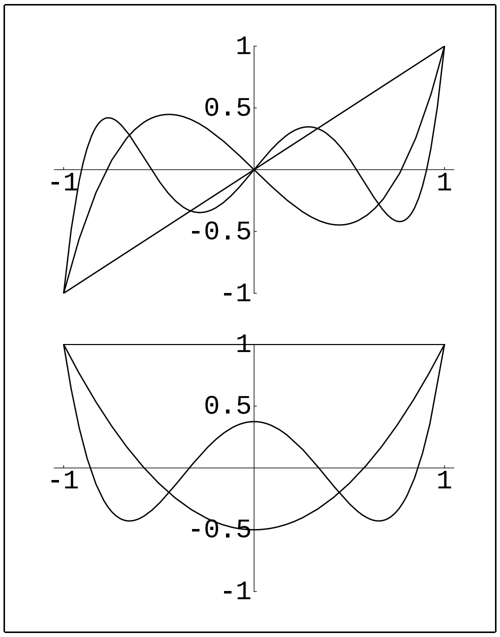


FIGURE 159: *Graphs of Legendre polynomials of low odd order (above) and low even order (below). Order can in each case be determined by counting the number of zero-crossings. The* $P_n(w)$ *are orthogonal in the sense*

$$\int_{-1}^{+1} P_m(w)P_n(w)\,dw = \tfrac{2}{2m+1}\delta_{mn}$$

and provide a natural basis within the space of functions defined on the interval $\big[-1,+1\big]$.

from (508), we find that

$$\begin{aligned} q &\equiv \iiint q\delta(\boldsymbol{x}-\boldsymbol{x})\,d^3x = q \\ &\qquad\qquad = qP_0(\cos\vartheta) \;\; : \;\; \text{monopole terms agree trivially} \end{aligned}$$

$$\begin{aligned} \boldsymbol{p} &\equiv \iiint \begin{pmatrix} x \\ y \\ z \end{pmatrix} q\delta(\boldsymbol{x}-\boldsymbol{x})\,d^3x \\ &= q\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{so } \boldsymbol{p}\cdot\hat{\boldsymbol{x}} = qr\cos\vartheta \;\; \text{by definition of } \vartheta \\ &\qquad\qquad = qrP_1(\cos\vartheta) \;\; : \;\; \text{dipole terms agree} \end{aligned}$$

and finally that

$$
\begin{aligned}
\mathbb{Q} &\equiv \iiint \begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix} q\delta(\boldsymbol{x} - \boldsymbol{x})\, d^3x \\
&= q \begin{pmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3yx & 3y^2 - r^2 & 3yz \\ 3zx & 3zy & 3z^2 - r^2 \end{pmatrix} \\
&\Downarrow \\
\tfrac{1}{2}\hat{\boldsymbol{x}}\cdot\mathbb{Q}\hat{\boldsymbol{x}} &= q\Big\{\tfrac{3}{2}(\boldsymbol{x}\cdot\hat{\boldsymbol{x}})^2 - \tfrac{1}{2}r^2\Big\} \\
&= qr^2\tfrac{1}{2}(\cos^2\vartheta - 1) \\
&= qr^2 P_2(\cos\vartheta)
\end{aligned}
$$

So though (508) and (514) look quite different, they do in fact say exactly the same thing. Which is gratifying, but . . .

Equation (514) says in its complicated way what we could say quite simply if we were to reposition our coordinate system (place the origin *at* the solitary charge), so is of relatively little interest in itself. It acquires profound interest, however, when put to its intended use:

4. Representation of an arbitrary potential by superimposed spherical harmonics. The idea is to apply (514) to each constituent element $\rho(\boldsymbol{x})\, d^3x$ of our distributed charge. To implement the idea we introduce spherical coordinates in the usual way

$$
\boldsymbol{x} = r\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad \boldsymbol{x} = r\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}
$$

where evidently θ signifies *co*latitude (North and South poles are coordinated $\theta = 0$ and $\theta = \pi$, respectively). Then

$$
\cos\vartheta = \hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}} = \cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi - \phi)
$$

and

$$
d^3x = r^2 \sin\theta\, dr d\theta d\phi
$$

so (514) supplies

$$
\begin{aligned}
\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}r^{-1}\sum_{n=0}^{\infty}\iiint \big(\tfrac{r}{r}\big)^n P_n\big(\cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi - \phi)\big)& \\
\cdot\,\rho(r,\theta,\phi)r^2\sin\theta\, dr d\theta d\phi& \qquad (515)
\end{aligned}
$$

Thumbing through the mathematical handbooks, we discover the wonderful identity[312]

$$
\begin{aligned}
&P_n\big(\cos\theta\cos\theta+\sin\theta\sin\theta\cos(\phi-\phi)\big) &(516.1)\\
&\quad= P_n(\cos\theta)P_n(\cos\theta)+2\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta)\cos m(\phi-\phi)
\end{aligned}
$$

Here

$$
\begin{aligned}
P_n^m(w) &\equiv (-)^m(1-w^2)^{\frac{1}{2}m}\big(\tfrac{d}{dw}\big)^m P_n(w) \quad : \quad m=0,1,2,\ldots,n\\
P_n(w) &\equiv (-)^n\tfrac{1}{2^n n!}\big(\tfrac{d}{dw}\big)^n(1-w^2)^n
\end{aligned}
$$

defines the so-called *associated Legendre functions*, the first few of which are displayed below:[313]

$$
\begin{aligned}
P_0(w)\equiv P_0^0(w) &= 1 && = 1\\[1em]
P_1(w)\equiv P_1^0(w) &= w && = \cos\theta\\
P_1^1(w) &= -\sqrt{1-w^2} && = -\sin\theta\\[1em]
P_2(w)\equiv P_2^0(w) &= \tfrac{1}{2}(3w^2-1) && = \tfrac{1}{4}(3\cos 2\theta+1)\\
P_2^1(w) &= -3w\sqrt{1-w^2} && = -\tfrac{3}{2}\sin 2\theta\\
P_2^2(w) &= -3(w^2-1) && = -\tfrac{3}{2}(\cos 2\theta-1)\\[1em]
P_3(w)\equiv P_3^0(w) &= \tfrac{1}{2}(5w^3-3w) && = \tfrac{1}{8}(5\cos 3\theta+3\cos\theta)\\
P_3^1(w) &= -\tfrac{3}{2}(5w^2-1)\sqrt{1-w^2} && = -\tfrac{3}{8}(5\sin 3\theta+\sin\theta)\\
P_3^2(w) &= 15\,w(1-w^2) && = -\tfrac{15}{4}(\cos 3\theta-\cos\theta)\\
P_3^3(w) &= -15(1-w^2)\sqrt{1-w^2} && = -\tfrac{15}{4}(\sin 3\theta-3\sin\theta)
\end{aligned}
$$

I have written these out to demonstrate that, while $P_n^m(w)$ is a polynomial only if m is even, the associated Legendre functions are in all cases simple

---

[312] Magnus & Oberhettinger,[311] page 55; P. Morse & H. Feshbach, *Methods of Theoretical Physics* (1953), page 1274. Identities of the frequently-encountered design

$$f(x+y)=\sum_n g_n(x)g_n(y)$$

are called "addition formulæ."

[313] Use *Mathematica* to reproduce/extend the list. The commands are

```
LegendreP[n,m,w]  and  LegendreP[n,m,Cos[θ]]//TrigReduce
```

combinations of elementary functions—nothing to become nervous about. If we now write

$$\cos m(\phi - \phi) = \frac{e^{im(\phi-\phi)} + e^{-im(\phi-\phi)}}{2}$$

and accept the convention[314] that

$$P_n^m(w) \quad \text{and} \quad P_n^{-m}(w) \quad \text{are two names for the same thing}$$

then (516.1) becomes

$$\begin{aligned} P_n\big(\cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi-\phi)\big) & \\ = \sum_{m=-n}^{m=+n} C_n^m \cdot P_n^m(\cos\theta)e^{-im\phi}\cdot P_n^m(\cos\theta)e^{+im\phi} & \qquad (516.2) \\ C_n^m \equiv \frac{(n-|m|)!}{(n+|m|)!} & \end{aligned}$$

in which the (θ,ϕ)-variables and (θ,ϕ)-variables have been fully disentangled, placed in nearly identical "piles." Further simplifications become possible when one reflects upon the orthogonality properties of $e^{im\phi}$ and $P_n^m(w)$. Familiarly

$$\int_0^{2\pi} e^{-im\phi}e^{+im\phi} = 2\pi\,\delta_{mm}$$

Less familiarly—but as the handbooks inform us, and as (even in the absence of explicit proof) we are readily convinced by a little *Mathematica*-assisted experimentation—

$$\int_{-1}^{+1} P_n^m(w)P_n^m(w) = \tfrac{2}{2n+1}C_n^m\,\delta_{nn} \quad : \quad 0 \leqslant m \leqslant \text{lesser of } n \text{ and } n$$

So we construct

$$\mathcal{Y}_n^m(w,\phi) \equiv (-)^m\sqrt{\tfrac{2n+1}{4\pi}\tfrac{(n+|m|)!}{(n-|m|)!}}\,P_n^m(w)e^{im\phi} \quad : \quad m = 0,\pm1,\pm2,\ldots,\pm n$$

$\uparrow$ a convention, fairly standard to the literature, and honored by *Mathematica*

which are orthonormal in the sense

$$\int_0^{2\pi}\int_{-1}^{+1}[\mathcal{Y}_n^m(w,\phi)]^*\mathcal{Y}_n^m(w,\phi)\,dwd\phi = \delta^{mm}\delta_{nn}$$

Or—more suitably for the matter at hand—

$$Y_n^m(\theta,\phi) \equiv \mathcal{Y}_n^m(\cos\theta,\phi)$$

---

[314] Beware! The designers of *Mathematica* adopted at this point an alternative convention.

which are precisely the celebrated **spherical harmonics**, orthonormal on the surface of the sphere

$$\int_0^{2\pi}\int_0^{\pi}[Y_n^m(\theta,\phi)]^* Y_n^m(\theta,\phi)\,\sin\theta d\theta d\phi = \delta^{mm}\delta_{nn}$$

just as the functions $E_m(\phi) \equiv \frac{1}{\sqrt{2\pi}}e^{im\phi}$ were seen above to be orthonormal on the surface of the circle. The functions $Y_n^m(\theta,\phi)$ are relatively more complicated than the functions $E_m(\phi)$ not so much because they have an extra argument as because *the surface of a sphere is a topologically more complicated place than the surface of a circle* (or—more aptly—than the surface of a torus). *Mathematica*, upon the command `SphericalHarmonicY[n,m,θ,ϕ]`, produces the following explicit list of low-order spherical harmonics:

$$Y_0^0(\theta,\phi) = \sqrt{\tfrac{1}{4\pi}}$$

$$\begin{aligned} Y_1^{-1}(\theta,\phi) &= +\sqrt{\tfrac{3}{8\pi}}e^{-i\phi}\sin\theta \\ Y_1^{0}(\theta,\phi) &= \sqrt{\tfrac{3}{4\pi}}\cos\theta \\ Y_1^{+1}(\theta,\phi) &= -\sqrt{\tfrac{3}{8\pi}}e^{+i\phi}\sin\theta \end{aligned}$$

$$\begin{aligned} Y_2^{-2}(\theta,\phi) &= +\sqrt{\tfrac{15}{32\pi}}e^{-2i\phi}\sin^2\theta \\ Y_2^{-1}(\theta,\phi) &= +\sqrt{\tfrac{15}{8\pi}}e^{-i\phi}\cos\theta\sin\theta \\ Y_2^{0}(\theta,\phi) &= +\sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta-1) \\ Y_2^{+1}(\theta,\phi) &= -\sqrt{\tfrac{15}{8\pi}}e^{+i\phi}\cos\theta\sin\theta \\ Y_2^{+2}(\theta,\phi) &= +\sqrt{\tfrac{15}{32\pi}}e^{+2i\phi}\sin^2\theta \end{aligned}$$

There are $2n+1 = 1, 3, 5, \ldots$ of the things of order $n = 0, 1, 2, \ldots$

By this point (516.2) has assumed the form

$$\begin{aligned} &P_n\big(\cos\theta\cos\theta + \sin\theta\sin\theta\cos(\phi-\phi)\big) \\ &\qquad = \sum_{m=-n}^{m=+n}\tfrac{4\pi}{2n+1}[Y_n^m(\theta,\phi)]^*Y_n^m(\theta,\phi) \end{aligned} \tag{516.3}$$

which when introduced into (515) gives

$$\varphi(\boldsymbol{x}) = \tfrac{1}{4\pi}r^{-1}\sum_{n=0}^{\infty}\tfrac{4\pi}{2n+1}\sum_{m=-n}^{m=+n}Q_n^m\,\frac{Y_n^m(\theta,\phi)}{r^n} \tag{517}$$

where $$Q_m^n \equiv \iiint[Y_n^m(\theta,\phi)]^*\rho(r,\theta,\phi)r^{n+2}\sin\theta\,drd\theta d\phi$$

defines the **multipole moments of the charge distribution**:

$$
\begin{array}{ccccccccc}
 & & & & Q_0^0 & & & & \\
 & & & Q_1^{-1} & Q_1^0 & Q_1^{+1} & & & \\
 & & Q_2^{-2} & Q_2^{-1} & Q_2^0 & Q_2^{+1} & Q_2^{+2} & & \\
 & & & & \vdots & & & & \\
Q_n^{-n} & \ldots\ldots\ldots\ldots & & Q_n^{-1} & Q_n^0 & Q_n^{+1} & & \ldots\ldots\ldots\ldots & Q_n^{+n}
\end{array}
$$

To remove any element of the mystery from the situation let us look to some of the illustrative specifics:

$$
\begin{aligned}
Q_0^0 &= \iiint [Y_0^0(\theta,\phi)]^* \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{1}{4\pi}} \iiint \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{1}{4\pi}}\, q
\end{aligned}
\tag{518$)_0^0$}
$$

$$
\begin{aligned}
Q_1^0 &= \iiint [Y_1^0(\theta,\phi)]^* \rho(r,\theta,\phi) r^3 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{3}{4\pi}} \iiint r\cos\theta \cdot \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{3}{4\pi}} \iiint z \cdot \rho(\boldsymbol{x})\, d^3x \\
&= \sqrt{\tfrac{3}{4\pi}}\, p_3
\end{aligned}
\tag{518$)_1^0$}
$$

$$
\begin{aligned}
Q_1^{-1} &= \iiint [Y_1^{-1}(\theta,\phi)]^* \rho(r,\theta,\phi) r^3 \sin\theta \, dr d\theta d\phi \\
&= +\sqrt{\tfrac{3}{8\pi}} \iiint r(\cos\phi - i\sin\phi)^* \sin\theta \cdot \rho(r,\theta,\phi) r^2 \sin\theta \, dr d\theta d\phi \\
&= +\sqrt{\tfrac{3}{8\pi}} \iiint (x+iy) \cdot \rho(\boldsymbol{x})\, d^3x \\
&= +\sqrt{\tfrac{3}{8\pi}}\, (p_1 + ip_2)
\end{aligned}
\tag{518$)_1^{-1}$}
$$

$$
Q_1^{+1} = -\sqrt{\tfrac{3}{8\pi}}\, (p_1 - ip_2) \tag{518$)_1^{+1}$}
$$

$$
\begin{aligned}
Q_2^0 &= \iiint [Y_2^0(\theta,\phi)]^* \rho(r,\theta,\phi) r^4 \sin\theta \, dr d\theta d\phi \\
&= \sqrt{\tfrac{5}{16\pi}} \iiint (3z^2 - r^2) \cdot \rho(\boldsymbol{x})\, d^3x \\
&= \sqrt{\tfrac{5}{16\pi}}\, Q_{33}
\end{aligned}
\tag{518$)_2^0$}
$$

$$\begin{aligned}
Q_2^{-1} &= \iiint [Y_2^{-1}(\theta,\phi)]^* \rho(r,\theta,\phi)\, r^4 \sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{8\pi}} \iiint r^2(\cos\phi + i\sin\phi)\cos\theta\sin\theta \cdot \rho(r,\theta,\phi) r^2 \sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{8\pi}} \iiint (x+iy)z \cdot \rho(\boldsymbol{x})\, d^3x \\
&= +\sqrt{\tfrac{15}{8\pi}}\tfrac{1}{3}(Q_{13}+iQ_{23}) && (518)_2^{-1}\\
Q_2^{+1} &= -\sqrt{\tfrac{15}{8\pi}}\tfrac{1}{3}(Q_{13}-iQ_{23}) && (518)_2^{+1}\\
Q_2^{-2} &= \iiint [Y_2^{-2}(\theta,\phi)]^* \rho(r,\theta,\phi)\, r^4 \sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{32\pi}} \iiint r^2 \underbrace{(\cos 2\phi + i\sin 2\phi)}_{=\cos^2\phi - \sin^2\phi + 2i\cos\phi\sin\phi} \sin^2\theta \cdot \rho(r,\theta,\phi) r^2 \sin\theta\, dr d\theta d\phi \\
&= +\sqrt{\tfrac{15}{32\pi}} \iiint (x^2 - y^2 + 2ixy) \cdot \rho(\boldsymbol{x})\, d^3x \\
&= +\sqrt{\tfrac{15}{32\pi}}\tfrac{1}{3}(Q_{11}-Q_{22}+2iQ_{12}) && (518)_2^{-2}\\
Q_2^{+2} &= +\sqrt{\tfrac{15}{32\pi}}\tfrac{1}{3}(Q_{11}-Q_{22}-2iQ_{12}) && (518)_2^{+2}
\end{aligned}$$

Here the notations p_a and Q_{ab} have been taken from (508) on page 424. The point is that same physical information is folded (if in a different way) into the designs of Q_1^m, Q_2^m, ... as was folded into the designs of $\boldsymbol{p}$, $\mathbb{Q}$, ...: equations (517) and (508) are saying the same thing, but in different ways.

Were we to pursue the mathematical side of this subject we would want to establish that & how the spherical harmonics $Y_n^m(\theta,\phi)$ spring spontaneously into being when one undertakes to

solve $\nabla^2\varphi = 0$ in spherical coordinates by separation of variables

A little *Mathematica*-assisted experimentation[315] may serve to convince the reader—even in the absence of the formal demonstration—that

$$\nabla^2\{r^p\, Y_n^m(\theta,\phi)\} = 0 \quad \text{if and only if } p = n \text{ or } p = -(n+1)$$

[315] Enter the commands

```
<<Calculus`VectorAnalysis`
```

and `SetCoordinates[Spherical[r,θ,φ]]`

and then test

```
Laplacian[r^p SphericalHarmonicY[n,m,θ,φ]]
```

with various values of m, n and p.

Solutions of the first type blow up as $r \uparrow \infty$: at (517) we find $\varphi(\boldsymbol{x})$ described as a linear combination of solutions of the second type. Looking to the mathematics of the situation from a somewhat different angle ...

$$\varphi(r,\theta,\varphi) = \sum_{m,n} A_{mn}\big\{\big(\tfrac{r}{a}\big)^n\, Y_n^m(\theta,\phi)\big\}$$

describes a solution of Laplace's equation,[316] and so also does

$$\psi(r,\theta,\varphi) = \tfrac{a}{r}\sum_{m,n} A_{mn}\big\{\big(\tfrac{a}{r}\big)^n\, Y_n^m(\theta,\phi)\big\}$$

To say the same thing another way: if $f(x,y,z)$ is a solution of Laplace's equation $\nabla^2 f = 0$ then so also is

$$F(x,y,z) \equiv \tfrac{a}{r} f\big(\tfrac{a^2}{r^2}x, \tfrac{a^2}{r^2}y, \tfrac{a^2}{r^2}z\big)$$

Transformations of the form

$$\boldsymbol{x} \xrightarrow[\text{inversion}]{} \boldsymbol{x} = \tfrac{a^2}{r^2}\boldsymbol{x}$$

are called "inversions in the sphere of radius a" by geometers (they send interior points to exterior points and *visa versa*, subject to the rule $rr = a^2$), and are self-inversive in the sense

$$\boldsymbol{x} \xrightarrow[\text{inversion}]{} \tfrac{a^2}{r^2}\boldsymbol{x} = \tfrac{r^2}{a^2}\boldsymbol{x} = \boldsymbol{x}$$

Transformations of the form

$$f(\boldsymbol{x}) \xrightarrow[\text{Kelvin inversion}]{} f(\boldsymbol{x}) \equiv \tfrac{a}{r} f\big(\tfrac{a^2}{r^2}\boldsymbol{x}\big) \tag{519}$$

acquire their name from the fact that it was William Thompson (Lord Kelvin) who first noticed (1847) that they send "harmonic functions" (solutions of Laplace's equation) into harmonic functions: they are readily seen to be self-inversive in the sense that

$$(\text{Kelvin inversion})^2 = \text{identity transformation}$$

Rotation of the charge distribution (equivalently: counter rotation of the Cartesian frame) would clearly result in an altered set of coefficients Q_n^m that refer to an altered set of spherical harmonics:

$$\begin{aligned}\varphi(\boldsymbol{x}) &= \tfrac{1}{4\pi} r^{-1}\sum_{n=0}^{\infty}\tfrac{4\pi}{2n+1}\sum_{m=-n}^{m=+n} Q_n^m\,\frac{Y_n^m(\theta,\phi)}{r^n}\\ \Big\downarrow&\,\text{rotation}\\ &= \tfrac{1}{4\pi} r^{-1}\sum_{n=0}^{\infty}\tfrac{4\pi}{2n+1}\sum_{m=-n}^{m=+n} Q_n^m\,\frac{Y_n^m(\theta,\phi)}{r^n}\end{aligned}$$

316 Here and below: a is a constant "length" of arbitrary value, introduced for a dimensional reason.

Were we to pursue the theory of spherical harmonics we would certainly want to explore the details of the now-fairly-evident fact that the harmonics of given order n are rotationally induced to fold among themselves

$$\begin{pmatrix} Y_n^{+n}(\theta,\phi) \\ \vdots \\ Y_n^{0}(\theta,\phi) \\ \vdots \\ Y_n^{-n}(\theta,\phi) \end{pmatrix} = \begin{pmatrix} \\ \\ (2n+1)\times(2n+1) \text{ matrix} \\ \\ \\ \end{pmatrix} \begin{pmatrix} Y_n^{+n}(\theta,\phi) \\ \vdots \\ Y_n^{0}(\theta,\phi) \\ \vdots \\ Y_n^{-n}(\theta,\phi) \end{pmatrix}$$

in a why that provides a (2n+1)-dimensional representation of the 3-dimensional rotation group $O(3)$. When those details are approached algebraically (instead off function-theoretically) it is found to make sense to speak also of cases

$$n = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots$$

that give rise to even-dimensional matrix representations of $O(3)$, and that those have indispensible applications to the quantum theory of fractional spin. While electrostatics served historically to inspire the initial development of the theory of spherical harmonics, and does exploit some of the more superficial elements of that theory, it is the *quantum theory of angular momentum* (equivalently: the representation theory of $O(3)$) that first motivated people to explore (in order to exploit) the riches hidden in the deeper nooks and crannies of the theory of spherical harmonics. And it is because the theory is most naturally developed in connection with its quantum mechanical applications[317] that I am content not to pursue it further here.[318]

5. A geophysical application. Though initially formulated in the language of electrostatics, our results pertain also—obviously and quite usefully—to gravitostatics . . . for reasons having to do with the structural similarity of the statements

$$\begin{aligned} \frac{e}{4\pi r} &= \text{electrostatic potential of a point charge } e \\ -\frac{GM}{r} &= \text{gravitostatic potential of a point mass } M \end{aligned}$$

Evidently the gravitational potential exterior to a sphere[319] containing a blob $\rho(\boldsymbol{x})$ of matter—the earth is the "blob" of greatest interest to geophysicists—can

[317] See, for example, David Griffiths, *Introduction to Quantum Mechanics* (1995), Chapter 4 or J. Powell & B. Crasemann, *Quantum Mechanics* (1961), Chapter 7.

[318] In 1980 I had not so much self-control: the missing details are sketched on pages 486–510 of CLASSICAL ELECTRODYNAMICS.

[319] A *mental* sphere, of radius a, commonly identified with the maximal radius of the geosphere ($\sim 6.378 \times 10^3$ km).

be described

$$\begin{aligned}
V(\boldsymbol{x}) &= -G\iiint \rho(\boldsymbol{x})\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}|}\,d^3x \\
&= -Gr^{-1}\sum_{n=0}^{\infty}\tfrac{4\pi}{2n+1}\sum_{m=-n}^{m=+n} Q_n^m\,\frac{Y_n^m(\theta,\phi)}{r^n} \\
&\qquad\qquad Q_m^n \equiv \iiint [Y_n^m(\theta,\phi)]^*\rho(r,\theta,\phi)r^{n+2}\sin\theta\,dr d\theta d\phi \\
&= -GM\tfrac{1}{r}\Big\{1+\tfrac{1}{r}\boldsymbol{P}\cdot\hat{\boldsymbol{x}}+\tfrac{1}{r^2}\tfrac{1}{2}\hat{\boldsymbol{x}}\cdot\mathbb{G}\hat{\boldsymbol{x}}+\cdots\Big\}
\end{aligned}$$

where

$$M \equiv \iiint \rho(\boldsymbol{x})\,d^3x = \text{monopole moment} = \textbf{total mass}$$

$$\begin{aligned}
\boldsymbol{P} \equiv \tfrac{1}{M}\iiint \rho(\boldsymbol{x})\,d^3x &= \frac{\text{dipole moment vector}}{M} \\
&= \textbf{center of mass}\text{ coordinates}
\end{aligned}$$

$$\mathbb{G} \equiv \tfrac{1}{M}\iiint \|3x^ix^j - r^2\delta^{ij}\|\rho(\boldsymbol{x})\,d^3x = \frac{\text{quadrupole moment matrix}}{M}$$

Note that *<u>the dipole term drops away</u> if one places the origin at the center of mass.*[320] Dominant interest shifts therefore to the quadrupole term, which "MacCullagh's formula"

$$V(\boldsymbol{x}) = -GM\tfrac{1}{r}\Big\{1-\frac{A-C}{2Mr^2}(3\sin^2\underset{\uparrow\text{signifies latitude}}{\psi}-1)+\cdots\Big\}$$

serves to relate to the geometrical parameters (A and C) that describe the idealized oblate sphereoidal figure of the gravitating body (see again Figure 154). Higher moments provide information about

- irregularities in the figure of the body
- inhomogeneities of the mass distribution.

Notice that (see again the formula that serves at the top of the page to define the coefficients Q_n^m) the higher moments depend most strongly upon details near the *surface* of the body, and are of quantitative significance only in the near zone: far away the body "looks like a monopole":

$$\begin{aligned}&\downarrow\\ &= -GM\tfrac{1}{r} \quad : \quad r \gg a\end{aligned}$$

For the earth the Q_n^m have been measured through at least $n = 8$, and in the post-Sputnik era satellites have been used to fill in an "island" of higher

[320] That would be a natural thing to do, but a *conventional* thing to do (something one might elect *not* to do) ...and should not be confused with the physical fact that—because Nature provides no "negative mass"—*gravitational dipoles do not exist.*

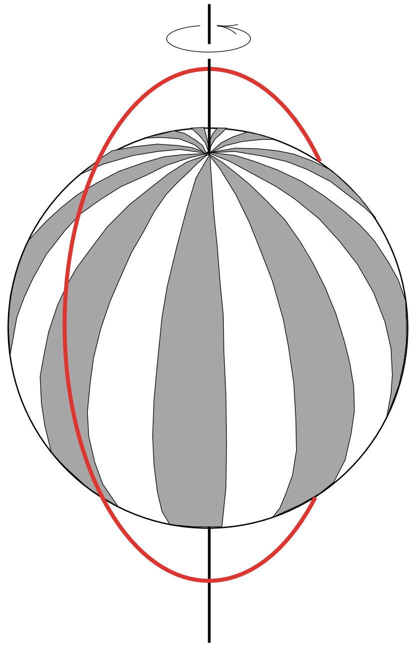


FIGURE 160: *Polar orbit of a satellite in polar orbit. Resolving the spherical harmonics into their real/imaginary parts*

$$Y_n^{\pm m}(\theta,\phi) = C_n^m(\theta,\phi) \pm iS_n^m(\theta,\phi)$$

we observe that $S_n^0(\theta,\phi)$ and $C_n^0(\theta,\phi)$ are ϕ-independent: they vanish on circles parallel to the equator, thus partitioning the surface of the sphere into "zones," so are called "zonal harmonics." At the other extreme, the nodes of

$$C_n^n(\theta,\phi) \sim \cos n\phi \sin^n \theta \quad \textit{and} \quad S_n^n(\theta,\phi) \sim \sin n\phi \sin^n \theta$$

partition the sphere into sectors (bounded by great arcs of constant longitude); such functions are called "sectoral harmonics," while spherical harmonics with $0 < m < n$ are called "tesseral harmonics." Some sectors have been painted on the earth, and rotate with the earth (because they are taken here to refer to a property of the earth).

(m,n)-values—this by the pretty method that I now sketch. The period T of a satellite in circular orbit can, in leading approximation, be described

$$T = 2\pi\sqrt{\frac{a^3}{GM}}\big(\tfrac{r}{a}\big)^{\frac{3}{2}}$$

which in the case of the earth becomes

$$= 84.5\big(\tfrac{r}{a}\big)^{\frac{3}{2}} \text{ minutes}$$

The satellite will be in *resonance* with the sectoral harmonics Y_n^n of the earth's gravitational field if $T = T_n$, where T_n is the time it takes for the rotating earth to replace one of the sectors of Y_n^n by the next. The sidereal day is 1436.07 minutes long, so

$$T_n = \frac{1436.07 \text{ minutes}}{n} = \begin{cases} 89.75 \text{ minutes} & : \quad n = 16 \\ 95.74 \text{ minutes} & : \quad n = 15 \\ 102.58 \text{ minutes} & : \quad n = 14 \\ 110.47 \text{ minutes} & : \quad n = 13 \\ 119.67 \text{ minutes} & : \quad n = 12 \end{cases}$$

and to achieve synchrony in those cases (solve $84.5x^{\frac{3}{2}} = T_n$ for x) we must set

$$\text{orbital radius} = \begin{cases} 1.0410a & : \quad \text{resonance with } Y_{16}^{16} \text{ mode} \\ 1.0868a & : \quad \text{resonance with } Y_{15}^{15} \text{ mode} \\ 1.1380a & : \quad \text{resonance with } Y_{14}^{14} \text{ mode} \\ 1.1956a & : \quad \text{resonance with } Y_{13}^{13} \text{ mode} \\ 1.2611a & : \quad \text{resonance with } Y_{12}^{12} \text{ mode} \end{cases}$$

If $n \gtrsim 16$ the satellite burns up in the atmosphere (or its orbit becomes subterranean!), while if $n \lesssim 12$ then r becomes so large that the $(1/r^n)$-factor makes the effects of resonance unobservably small. The case $n = 15$ seems to be nearly optimal, and indeed: scientists active in the field[321] have been able by this means to estimate the values of Q_{15}^{15}, Q_{17}^{15}, Q_{19}^{15} and Q_{21}^{15}. Since high moments probe progressively more *superficial* properties of $\rho(\boldsymbol{x})$, one might hope from such orbital data to extract information about the earth's *crust and crust-mantle interface*. The technique extends in principle to planetary bodies other than the earth. And microphysical analogs do come to mind: an atom with nuclear charge Ze has orbital radii given typically by (see again page 392)

$$R = \frac{\hbar^2}{mZe^2}$$

which *gets smaller when m is increased*. One therefore expects that *the properties of μ-mesonic atoms might provide information about <u>the surface properties of complex nuclei</u>*.

6. Harmonic polynomials & Maxwell's theory of poles. While the theory of spherical harmonics has much to do with the representation of rotations in 3-space, it has—contrary to the impression conveyed by some of the preceding material—only incidentally to do with spherical coordinates. Important aspects of the theory are, in fact, brought most simply/naturally into view by the adoption of a Cartesian perspective . . . as I undertake now to demonstrate:

---

[321] See R. D. Eberst, "Earth satellites and the gravitational potential" and D. G. King-Hele & H. Heller, "Equations for the 15$^{\text{th}}$-order harmonics in the geopotential," Nature Physical Science **235**, 130 (1972). Also A. E. Roy, *Orbital Motion* §10.4 (1978) and H. F. R. Schöyer & K. F. Walker, *Rocket Propulsion and Space Flight Dynamics* §18.6 (1979).

Introduce the (rotationally invariant!) monomial $T(\boldsymbol{x}) \equiv \boldsymbol{a}\cdot\boldsymbol{x}$ and notice that, by quick calculation,

$$\nabla^2 T^n = n(n-1)T^{n-2}\,\boldsymbol{a}\cdot\boldsymbol{a}$$

Dismissing as trivial the cases $n=0$ and $n=1$, we conclude that <u>the n^{th} powers of $T(\boldsymbol{x})$ will be harmonic iff $\boldsymbol{a}$ is null</u>. But $\boldsymbol{a}\cdot\boldsymbol{a}=0$ entails that $\boldsymbol{a}$ be complex: $\boldsymbol{a}=\boldsymbol{b}+i\boldsymbol{c}$ with $b^2-c^2=0$ and $\boldsymbol{b}\cdot\boldsymbol{c}=0$. If $\boldsymbol{a}\cdot\boldsymbol{a}=0$ is formulated

$$a_3 = \sqrt{-(a_1^2+a_2^2)} = i\sqrt{(a_1+ia_2)(a_1-ia_2)}$$

then it becomes fairly natural to introduce complex parameters

$$\begin{aligned} u &\equiv \sqrt{a_1+ia_2} \\ v &\equiv \sqrt{a_1-ia_2} \end{aligned}$$

in terms of which we can write

$$\left.\begin{aligned} a_1 &= \tfrac{1}{2}(u^2+v^2) \\ a_2 &= \tfrac{1}{2i}(u^2-v^2) \\ a_3 &= iuv \end{aligned}\right\} \tag{520}$$

which provide a (u,v)-parameterized description of the set of all null 3-vectors $\boldsymbol{a}$. In this notation

$$\begin{aligned} T^n(\boldsymbol{x}) &= \tfrac{1}{2^n}\big[(u^2+v^2)x + \tfrac{1}{i}(u^2-v^2)y + 2iuvz\big]^n \\ &= \tfrac{1}{2^n}\big[u^2(x-iy) + 2iuvz + v^2(x+iy)\big]^n \\ &= \begin{cases} \text{polynomial of degree } \ \ n \text{ in variables } (x,y,z) \\ \text{polynomial of degree } 2n \text{ in parameters } (u,v) \end{cases} \end{aligned}$$

To emphasize the latter point of view we write

$$= \tfrac{1}{2^n}\sum_{m=-n}^{m=+n} u^{n-m}v^{n+m}H_n^m(\boldsymbol{x})$$

This, since harmonic for *all* values of u and v, entails that the polynomials $H_n^m(\boldsymbol{x})$ are *individually* harmonic:

$$\nabla^2 H_n^m = 0$$

Arguing from

$$
\begin{aligned}
T\cdot T^n &= \tfrac{1}{2^{n+1}}\big[u^2(x-iy)+2iuvz+v^2(x+iy)\big]\sum_m u^{n-m}v^{n+m}H_n^m\\
&= \tfrac{1}{2^{n+1}}\sum_m\Big\{\quad u^{(n+1)-(m-1)}v^{(n+1)+(m-1)}(x-iy)H_n^m\\
&\qquad\qquad + u^{(n+1)-m}\quad\; v^{(n+1)+m}\qquad\;\; (2iz)H_n^m\\
&\qquad\qquad + u^{(n+1)-(m+1)}v^{(n+1)+(m+1)}(x+iy)H_m^m\Big\}\\
&= T^{(n+1)}\\
&= \tfrac{1}{2^{n+1}}\sum_m u^{(n+1)-m}v^{(n+1)+m}H_{n+1}^m
\end{aligned}
$$

we obtain a relation

$$
H_{n+1}^m = (x-iy)H_n^{m+1} + 2izH_n^m + (x+iy)H_n^{m-1}
$$

from which—sprouting from the "seed"

$$
H_0^m(\boldsymbol{x}) \equiv \begin{cases} 1 & : \quad m=0\\ 0 & : \quad m=\pm1,\pm2,\dots\end{cases}
$$

—the **harmonic polynomials** $H_n^m(\boldsymbol{x})$ can be computed recursively: thus

$$
\begin{aligned}
H_0^0 &= 1\\
\\
H_1^{-1} &= x-iy\\
H_1^0 &= 2iz\\
H_1^{+1} &= x+iy\\
\\
H_2^{-2} &= (x-iy)^2\\
H_2^{-1} &= 4i(x-iy)z\\
H_2^0 &= 2x^2+2y^2-4z^2 = 2(r^2-3z^2)\\
H_2^{+1} &= 4i(x+iy)z\\
H_2^{+2} &= (x+iy)^2\\
&\;\;\vdots
\end{aligned}
$$

The harmonic polynomials are <u>regular at the origin but blow up at ∞</u>. Kelvin inversion (519) permits us, however, to construct from them a population of (non-polynomial) functions

$$
J_n^m(\boldsymbol{x}) \equiv \tfrac{1}{r}H_n^m(\tfrac{1}{r^2}\boldsymbol{x})
$$

which are assuredly *also harmonic* and, though singular at the origin, are regular at ∞. Reading from the preceding list are led thus to the Kelvin transform of that list:

$$
\begin{aligned}
J_0^0 &= r^{-1} && = +\tfrac{1}{r}\\[1em]
J_1^{-1} &= r^{-3}\cdot(x-iy) && = -1(\partial_x - i\partial_y)\tfrac{1}{r}\\
J_1^{0} &= r^{-3}\cdot 2iz && = -2(i\partial_z)\tfrac{1}{r}\\
J_1^{+1} &= r^{-3}\cdot(x+iy) && = -1(\partial_x + i\partial_y)\tfrac{1}{r}\\[1em]
J_2^{-2} &= r^{-5}\cdot(x-iy)^2 && = +\tfrac{1}{3}\cdot 1(\partial_x - i\partial_y)^2\tfrac{1}{r}\\
J_2^{-1} &= r^{-5}\cdot 4(x-iy)iz && = +\tfrac{1}{3}\cdot 4(\partial_x - i\partial_y)(i\partial_z)\tfrac{1}{r}\\
J_2^{0} &= r^{-5}\cdot 2(r^2-3z^2) && = +\tfrac{1}{3}\cdot 6(i\partial_z)^2\tfrac{1}{r}\\
J_2^{+1} &= r^{-5}\cdot 4(x+iy)iz && = +\tfrac{1}{3}\cdot 4(\partial_x + i\partial_y)(i\partial_z)\tfrac{1}{r}\\
J_2^{+2} &= r^{-5}\cdot(x+iy)^2 && = +\tfrac{1}{3}\cdot 1(\partial_x + i\partial_y)^2\tfrac{1}{r}\\
&\ \vdots && \ \vdots
\end{aligned}
$$

That the harmonic functions $J_n^m(\boldsymbol{x})$ can be described by the highly patterned formulæ on the right was discovered by Maxwell, who in the general case would have us write

$$J_n^{\pm m} = (-)^n \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)}\tbinom{2n}{n-m}(\partial_x \pm i\partial_y)^m(i\partial_z)^{n-m}\tfrac{1}{r}$$

where now $m = 0, 1, 2, \ldots, n$.

We are by now not surprised to discover that if we *at this point* use

$$x \pm iy = r\sin\theta\cdot e^{\pm i\phi} \quad \text{and} \quad z = r\cos\theta$$

to pass from Cartesian to spherical coordinates, then the functions J_n^m turn out to differ only numerical factors from the functions $r^{-(n+1)}Y_n^m(\theta,\phi)$. The detailed result can be expressed in several ways:

$$
\begin{aligned}
Y_n^{\pm m}(\theta,\phi) &= (-)^n(i)^{n+m}\tfrac{1}{2^n n!}\sqrt{\tfrac{2n+1}{4\pi}(n-m)!(n+m)!}\cdot r^{n+1}J_n^{\pm m}(\boldsymbol{x})\\
\big(\tfrac{1}{r}\big)^{n+1}Y_n^{\pm m}(\theta,\phi) &= \underbrace{(-)^n\sqrt{\tfrac{2n+1}{4\pi}\tfrac{1}{(n-m)!(n+m)!}}(\partial_x \pm i\partial_y)^m(\partial_z)^{n-m}}_{\equiv\ \mathcal{D}_n^{\pm m}}\ \tfrac{1}{r}
\end{aligned}
$$

From the latter we conclude that

$$\equiv \mathcal{D}_n^{\pm m}$$

is a *differential operator natural to the theory of spherical harmonics*.

Which brings us back again to very nearly our point of departure. We established at (10.2) on page 12 that the function $\frac{1}{r} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ is harmonic except at the origin, where it blows up, but in a very interesting way:

$$\nabla^2 \tfrac{1}{r} = -4\pi\delta(\boldsymbol{x})$$

Application of $\mathcal{D}_n^{\pm m}$ gives

$$\downarrow$$

$$\nabla^2 \big(\tfrac{1}{r}\big)^{n+1} Y_n^{\pm m}(\theta,\phi) = -4\pi \mathcal{D}_n^{\pm m}\delta(\boldsymbol{x}) \tag{521}$$

which shows that a similar remark pertains to the functions $Y_n^m(\theta,\phi)/r^{n+1}$, except that these possess *singularities of higher order*, the latter being described by fancy *derivatives* of δ-functions. When, as at (517), we display $\varphi(\boldsymbol{x})$ as a weighted superposition of the functions that appear on the left, we are in effect claiming that $\rho(\boldsymbol{x})$ is equivalent to an identically weighted superposition of the singular functions ("distributions") that appear on the right side of (521):

$$\varphi(\boldsymbol{x}) = \sum_{n=0}^{\infty} \tfrac{1}{2n+1} \sum_{m=-n}^{m=+n} Q_n^m \, \frac{Y_n^m(\theta,\phi)}{r^{n+1}}$$

strength of $\mathcal{D}_n^{\pm m}\delta(\boldsymbol{x})$ singularity

number of n^{th}-order singularities

And we remarked already on page 426 the sense in which structured singularities can be interpreted to refer to constellations of "poles." We have arrived thus at the essence of Maxwell's "theory of poles."

It is hard to let go of this beautiful subject. I allow myself the luxury of one parting shot: It is an immediate implication of (520) that

$$\boldsymbol{a}^{*}\!\cdot \boldsymbol{a} = \tfrac{1}{2}(u^*u + v^*v)$$

The expression on the right is invariant under linear transformations

$$\begin{pmatrix} u \\ v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \mathbb{U}\begin{pmatrix} u \\ v \end{pmatrix}$$

provided $\mathbb{U}$ is *unitary* (inverse = conjugate transpose). Such transformations, by (520), induce linear transformations

$$\boldsymbol{a} \longrightarrow \boldsymbol{a} = \mathbb{R}\boldsymbol{a}$$

which, since norm-preserving, must describe 3-dimensional rotations. From this germ of an idea one gains direct access to the rich subject matter to which I allude at the end of §4.[322]

[322] Some of the details are developed in my "Algebraic theory of spherical harmonics" (Seminar Notes 1996). An excellent source is A. Erdélyi *et al*, *Higher Transcendental Functions* (1953), Volume 2, Chapter 11.

The material described above—fruit of the genius mainly of Maxwell and his friends, and of the generation that preceded them—takes Laplace's equation

$$\nabla^2\varphi = 0$$

as its point of departure, but analogous methods are important in a variety of other contexts. Look, for example, to the heat 1-dimensional equation

$$(\partial_x^2 - \partial_t)\varphi(x,t) = 0$$

It is clear that $e^{xz + tz^2}$ describes a z-parameterized family of solutions. Taylor expansion in z

$$\begin{aligned} e^{xz + tz^2} &= 1 + xz + \tfrac{1}{2}(x^2 + 2t)z^2 \\ &\qquad\qquad + \tfrac{1}{6}(x^3 + 6xt)z^3 \\ &\qquad\qquad + \tfrac{1}{24}(x^4 + 12x^2t + 12t^2)z^4 + \cdots \\ &\equiv \sum_{n=0}^{\infty} v_n(x,t)\tfrac{1}{n!}z^n \end{aligned}$$

gives rise to a population of "heat polynomials," analogous to the harmonic polynomials encountered on page 444.[323] And corresponding to the Kelvin transformation (519) one has the (nearly inversive) **Appell transformation** (1892)

$$\varphi(x,t) \xrightarrow[\text{Appell transformation}]{} \psi(x,t) \equiv \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \cdot \varphi(\tfrac{x}{t}, -\tfrac{1}{t})$$

where the exponential factor is itself a solution—the so-called "fundamental solution"—of the heat equation. We have seen that the Kelvin transformation contributes importantly to the theory of harmonic functions. Just so the Appell transformation: I have shown elsewhere that it is an object central to the theory of the conformal group, and that in a quantum mechanical application it serves as the bridge that links the standard formalism to the Feynman formalism.[324]

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[323] See D. V. Widder, *The Heat Equation* (1975), pages 8–14.

[324] "Appell, Galilean & Conformal Transformations in Classical/Quantum Free Particle Dynamics" (RESEARCH NOTES 1976).