This is a take home exam. You may refer to the textbook by Jackson, any material linked to the class website, and any second classical electromagnetism textbook of your choosing. Reference to integrals or other mathematical facts, and any personal handwritten notes are also OK. However, you should *not* collaborate with anyone else. The point value of each problem is indicated in the square brackets below; use these values as a guide to manage your time during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.*

There is no need to rederive results that have been previously obtained in the textbook, the class notes or the class handouts. But if you make use of any previously derived result, please cite the source of the result.

Completed exams should be delivered in class on Thursday February 22, 2024.

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1. [30] A linearly polarized electromagnetic wave, polarized in the $\hat{\boldsymbol{x}}$ direction, is traveling in the $\hat{\boldsymbol{z}}$-direction in a dielectric medium of refractive index n_1. The wave is normally reflected from the surface of a conductor of conductivity σ (the conductor occupies the x–y plane). Assume that $\mu = \mu_0$ for both the dielectric and the conductor.

(a) Find the phase change undergone by the electric field vector of the wave after reflection, assuming the refractive index of the conductor is $n_2 = n_1(1 + i\zeta)$, where $\zeta > 0$.

(b) How is ζ related to the conductivity σ in the limit of high frequency (i.e., in the limit of $\omega \gg \sigma/\epsilon_0$)?

2. [30] Consider a conducting fluid with conductivity σ. The inertial frame K' is defined to be the reference frame that is attached to the fluid, and the corresponding charge density is denoted by ρ'. Assume that in reference frame K', Ohm's law ($\vec{\boldsymbol{J}}' = \sigma \vec{\boldsymbol{E}}'$) is satisfied. The inertial frame K' moves with velocity $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}$ with respect to the laboratory frame K of the observer.

(a) Show that a suitable covariant generalization of Ohm's law is given by:

$$J^\alpha - \frac{1}{c^2}(u_\beta J^\beta)u^\alpha = \frac{\sigma}{c} F^{\alpha\beta} u_\beta \,, \tag{1}$$

where u^α is the four-velocity of the fluid.

HINT: Verify that eq. (1) reduces to Ohm's law in reference frame K'. Is this observation sufficient to provide an answer to part (a)? Explain why $J^\alpha = (\sigma/c)F^{\alpha\beta}u_\beta$ is not a viable candidate for Ohm's law by considering the quantity $J^\alpha u_\alpha$.

(b) Suppose that the fluid is uncharged ($\rho' = 0$). Using the result of part (a), deduce the form for Ohm's law as viewed in the laboratory frame. That is, express $\vec{\boldsymbol{J}}$ as a function of the electromagnetic fields $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ in the laboratory frame. Note that the charge density ρ

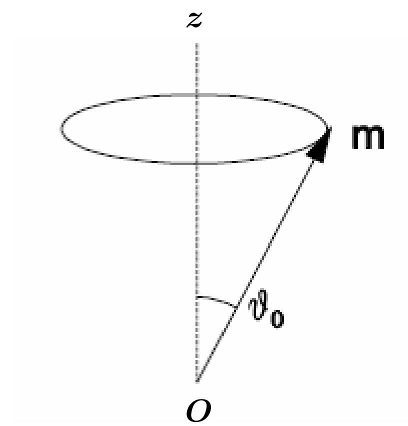
in the laboratory frame is not zero. Find an expression for ρ as a function of $\vec{\boldsymbol{E}}$ in the laboratory frame.

(c) Suppose that $\rho' \neq 0$. Show that the form for Ohm's law as viewed in the laboratory frame now takes the following form:

$$\vec{\boldsymbol{J}} = \gamma\sigma\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}\right] + c\rho\vec{\boldsymbol{\beta}}\,. \tag{2}$$

Note that $\vec{\boldsymbol{J}}$ and ρ are not separately determined by the electromagnetic fields alone. Verify that when $\rho' = 0$, eq. (2) reproduces your results of part (b).

3. [40] A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency ω and angle ϑ_0 with respect to the z-axis as shown below. (The origin of the coordinate system is labeled by $\boldsymbol{O}$.) That is, the time-dependence of the azimuthal angle is $\varphi_0(t) = \varphi_0 - \omega t$.



Electromagnetic radiation is emitted by the precessing dipole.[1]

(a) Write out an explicit expression for the time-dependent magnetic dipole vector $\vec{\boldsymbol{m}}$ in terms of its magnitude m_0, the angles ϑ_0 and φ_0 and the time t. Show that $\vec{\boldsymbol{m}}$ consists of the sum of a time-dependent term and a time-independent term. Verify that the time-dependent term can be written as $\mathrm{Re}(\vec{\boldsymbol{m}}\, e^{-i\omega t})$, for some suitably chosen complex vector $\vec{\boldsymbol{m}}$.

(b) Compute the angular distribution of the time-averaged radiated power measured by an observer located at the point $\vec{\boldsymbol{x}}$ with respect to the coordinate system defined in the above figure.

(c) Compute the total power radiated.

(d) What is the polarization of the radiation measured by an observer located along the positive z-axis far from the precessing dipole? How would your answer change if the observer were located in the x–y plane?

[1] Radiation from pulsars is believed to be due to this mechanism.