1. A linearly polarized electromagnetic wave, polarized in the $\hat{\boldsymbol{x}}$ direction, is traveling in the $\hat{\boldsymbol{z}}$-direction in a dielectric medium of refractive index n_1. The wave is normally reflected from the surface of a conductor of conductivity σ (the conductor occupies the x–y plane). Assume that $\mu = \mu_0$ for both the dielectric and the conductor.

(a) Find the phase change undergone by the electric field vector of the wave after reflection, assuming the refractive index of the conductor is $n_2 = n_1(1 + i\zeta)$, where $\zeta > 0$.

Without loss of generality, we define the z-axis to lie along the direction of the incoming wave and the x-axis to lie along the polarization vector of the incoming wave. Thus, we can take the incoming, transmitted and reflected waves to be,

$$\vec{\boldsymbol{E}} = E_0\hat{\boldsymbol{x}}\, e^{i(kz-\omega t)}\,, \qquad \vec{\boldsymbol{E}}' = E_0'\hat{\boldsymbol{x}}\, e^{i(k'z-\omega t)}\,, \qquad \vec{\boldsymbol{E}}'' = E_0''\hat{\boldsymbol{x}}\, e^{-i(kz+\omega t)}\,, \tag{1}$$

respectively, where $k = n_1\omega/c$, $k' = n_2\,\omega/c$, and the corresponding indices of refraction are

$$n_i = \sqrt{\frac{\epsilon_i\mu_i}{\epsilon_0\mu_0}} = \sqrt{\frac{\epsilon_i}{\epsilon_0}}\,, \quad \text{for } i = 1, 2. \tag{2}$$

In the notation of Section 7.3 of Jackson, the case of normal incidence of the wave corresponds to $\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}' = \hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$, $\hat{\boldsymbol{k}}'' = -\hat{\boldsymbol{z}}$, $\vec{\boldsymbol{E}}_0 = E_0\hat{\boldsymbol{x}}$, $\vec{\boldsymbol{E}}_0' = E_0'\hat{\boldsymbol{x}}$ and $\vec{\boldsymbol{E}}_0'' = E_0''\hat{\boldsymbol{x}}$. Using the last two equations of eq. (7.37) of Jackson, it then follows that

$$E_0 + E_0'' = E_0'\,, \qquad n_1(E_0 - E_0'') = n_2 E_0'\,.$$

Eliminating E_0', we can immediately solve for E_0''/E_0,

$$\frac{E_0''}{E_0} = \frac{n_1 - n_2}{n_1 + n_2} = -\,\frac{i\zeta}{2 + i\zeta} = -\,\frac{\zeta(\zeta + 2i)}{\zeta^2 + 4} = \left|\frac{E_0''}{E_0}\right| e^{i\alpha}\,, \tag{3}$$

after using $n_2 = n_1(1 + i\zeta)$, where the phase α is given by[1]

$$\tan\alpha = \frac{\mathrm{Im}(E_0''/E_0)}{\mathrm{Re}(E_0''/E_0)} = \frac{2}{\zeta}\,. \tag{4}$$

Note that the complex number E_0''/E_0 given in eq. (3) lies in the third quadrant of the complex plane. Consequently, eq. (4) yields $\alpha = \tan^{-1}(2/\zeta) - \pi$, where the principal value of the arctangent satisfies $0 \le \tan^{-1}(2/\zeta) < \pi$ for $\zeta > 0$.

Using the reflected wave given in eq. (1),

$$\vec{\boldsymbol{E}}'' = E_0''\hat{\boldsymbol{x}}\, e^{-i(kz+\omega t)} = \left|\frac{E_0''}{E_0}\right| E_0\hat{\boldsymbol{x}}\, e^{i\alpha} e^{-i(kz+\omega t)}\,.$$

Hence, the relative phase of the incident and reflected wave at the interface ($z = 0$) is[2]

$$\phi_{\rm rel} = \alpha = \tan^{-1}(2/\zeta) - \pi\,. \tag{5}$$

[1]In the standard convention, we take $-\pi < \alpha \le \pi$ as the principal value of the argument of E_0''/E_0.

[2]The relative phase $\phi_{\rm rel}$ is defined modulo 2π, so feel free replace $-\pi$ with π in eq. (5) if so inclined.

(b) How is ζ related to the conductivity σ in the limit of high frequency (i.e., in the limit of $\omega \gg \sigma/\epsilon_0$)?

In light of eq. (7.57) of Jackson, a medium of "normal" (i.e., real) dialectric constant ϵ_1 and conductivity σ can be described as having a complex dialectric constant given by

$$\epsilon_2 = \epsilon_1 + \frac{i\sigma}{\omega}\,. \tag{6}$$

The corresponding indices of refraction are given in eq. (2). Note that we have identified $\text{Re}\,\epsilon_2 = \text{Re}\,\epsilon_1$ since in part (a) we have been given the relation between the indices of refraction, which implies that $\text{Re}\,n_2 = \text{Re}\,n_1$.

In the limit of $\omega \gg \sigma/\epsilon_0$, we can use eq. (6) to approximate,

$$n_2 = n_1(1 + i\zeta) = \sqrt{\frac{\epsilon_2}{\epsilon_0}} = \sqrt{\frac{\epsilon_1}{\epsilon_0} + \frac{i\sigma}{\omega\epsilon_0}} \simeq \sqrt{\frac{\epsilon_1}{\epsilon_0}}\left[1 + \frac{i\sigma}{2\omega\epsilon_1}\right]\,.$$

Thus, we identify

$$\zeta = \frac{\sigma}{2\omega\epsilon_1}\,.$$

2. Consider a conducting fluid with conductivity σ. The inertial frame K' is defined to be the reference frame that is attached to the fluid, and the corresponding charge density is denoted by ρ'. Assume that in reference frame K', Ohm's law ($\vec{\boldsymbol{J}}' = \sigma\vec{\boldsymbol{E}}'$) is satisfied. The inertial frame K' moves with velocity $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}$ with respect to the laboratory frame K of the observer.

(a) Show that a suitable covariant generalization of Ohm's law is given by:

$$J^\alpha - \frac{1}{c^2}(u_\beta J^\beta)u^\alpha = \frac{\sigma}{c}F^{\alpha\beta}u_\beta\,, \tag{7}$$

where u^α is the four-velocity of the fluid.

Eq. (7) is a covariant equation. That is, both sides of eq. (7) transform as a Lorentz four-vector. Thus, if this equation is valid in one inertial reference frame then it must be valid in *all* inertial reference frames.

In the reference frame K', the fluid is at rest. Hence, the four-vector velocity in K' is $u'^\alpha = (c\,;\vec{\boldsymbol{0}})$. Thus, $u'_\beta J'^\beta = cJ'^0$ and $F'^{\alpha\beta}u_\beta = cF'^{\alpha 0}$. It follows that the $\alpha = 0$ component of eq. (7) is:

$$J'^0 - J'^0 = \sigma F'^{00}\,, \tag{8}$$

which is a valid equation since $F'^{\alpha\beta}$ is an antisymmetric tensor so that $F'^{00} = 0$.

Next, we examine the case where $\alpha = i \in \{1,2,3\}$. Then, in reference frame K', eq. (7) yields

$$J'^i = \sigma F'^{i0}\,. \tag{9}$$

Recalling that $F'^{i0} = E'^{i}$ where E'^{i} is the ith component of the electric field in reference frame K', it follows that Ohm's law, $\vec{\boldsymbol{J}}' = \sigma\vec{\boldsymbol{E}}'$, is satisfied. Thus, eq. (7) must be the correct generalization of Ohm's law to an arbitrary inertial reference frame.

Note that $J^\alpha = (\sigma/c)F^{\alpha\beta}u_\beta$ is *not* a viable candidate for Ohm's law. Although this equation also yields $\vec{\boldsymbol{J}}' = \sigma\vec{\boldsymbol{E}}'$ when $\alpha = i$, the same equation also implies that $J'^{0} = c\rho' = 0$, which is not true in general. To understand the structure of eq. (7), simply multiply eq. (7) by u_β. Since $F^{\alpha\beta}$ is antisymmetric under the interchange of $\alpha \leftrightarrow \beta$ and the quantity $u_\alpha u_\beta$ is symmetric under the interchange of $\alpha \leftrightarrow \beta$, it follows that $F^{\alpha\beta}u_\alpha u_\beta = 0$. Likewise, multiplying the left-hand side of eq. (7) by $u_\alpha u_\beta$ yields:

$$u_\alpha\Big[J^\alpha - \frac{1}{c^2}(u_\beta J^\beta)u^\alpha\Big] = u_\alpha J^\alpha - u_\beta J^\beta = 0\,, \tag{10}$$

after using $u_\alpha u^\alpha = c^2$. This result explains the form of the left-hand side of eq. (7).

(b) Suppose that the fluid is uncharged ($\rho' = 0$). Using the result of part (a), deduce the form for Ohm's law as viewed in the laboratory frame. That is, express $\vec{\boldsymbol{J}}$ as a function of the electromagnetic fields $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ in the laboratory frame. Note that the charge density ρ in the laboratory frame is not zero. Find an expression for ρ as a function of $\vec{\boldsymbol{E}}$ in the laboratory frame.

If $\rho' = 0$, then it follows that $u'_\beta J'^{\beta} = 0$ in reference frame K'. But, $u_\beta J^\beta$ is a Lorentz-invariant quantity. Thus, it follows that $u_\beta J^\beta = 0$ in all inertial reference frames. Hence, if $\rho' = 0$ then eq. (7) simplifies to

$$J^\alpha = \frac{\sigma}{c}F^{\alpha\beta}u_\beta\,, \qquad \text{when } \rho' = 0. \tag{11}$$

We now evaluate eq. (11) in the laboratory frame. In reference frame K, the velocity and current four-vectors are given by $u^\alpha = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}})$ and $J^\alpha = (c\rho\,;\,\vec{\boldsymbol{J}})$, where $\gamma \equiv (1-\beta^2)^{-1/2}$. For $\alpha = 0$, eq. (11) yields:

$$\rho = \frac{\gamma\sigma}{c}\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,, \tag{12}$$

after using $J^0 = c\rho$, $F^{0i} = -F^{i0} = -E^i$ and $u_i = -u^i = -\gamma v^i = -\gamma c\beta^i$. For $\alpha = i \in \{1,2,3\}$,

$$J^i = \frac{\sigma}{c}\big(F^{i0}u_0 + F^{ij}u_j\big)\,. \tag{13}$$

Using $F^{ij} = -\epsilon^{ijk}B^k$, it follows that:

$$F^{ij}u_j = \gamma\epsilon^{ijk}B^k v^j = \gamma c(\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})^i\,. \tag{14}$$

Hence, eq. (13) yields

$$\vec{\boldsymbol{J}} = \gamma\sigma\big[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\big]\,. \tag{15}$$

We can check the above results by noting that when $\rho' = 0$,

$$0 = u_\beta J^\beta = \gamma(c^2\rho - \vec{\boldsymbol{v}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}) = \gamma c(c\rho - \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}})\,. \tag{16}$$

Consequently, eq. (16) yields

$$c\rho = \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{J}}\,. \tag{17}$$

Using eqs. (15) and (17),

$$c\rho = \gamma\sigma\big[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}\big] \cdot \vec{\boldsymbol{\beta}} = \gamma\sigma\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}}\,, \tag{18}$$

in agreement with eq. (12).

(c) Suppose that $\rho' \neq 0$. Show that the form for Ohm's law as viewed in the laboratory frame now takes the following form:

$$\vec{\boldsymbol{J}} = \gamma\sigma\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}\right] + c\rho\vec{\boldsymbol{\beta}}\,. \tag{19}$$

Note that $\vec{\boldsymbol{J}}$ and ρ are not separately determined by the electromagnetic fields alone. Verify that when $\rho' = 0$, eq. (19) reproduces your results of part (b).

When $\rho' \neq 0$, one must employ eq. (7). Using $u^\alpha = (\gamma c\,;\,\gamma\vec{\boldsymbol{v}})$ and $J^\alpha = (c\rho\,;\,\vec{\boldsymbol{J}})$, it follows from eq. (7) that

$$c\rho - \gamma^2\big(c\rho - \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{J}}\big) = \gamma\sigma\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}}\,, \tag{20}$$

$$\vec{\boldsymbol{J}} - \gamma^2\big(c\rho - \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{J}}\big)\vec{\boldsymbol{\beta}} = \gamma\sigma\big(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}\big)\,. \tag{21}$$

Using eq. (20), $\gamma^2\big(c\rho - \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{J}}\big) = c\rho - \gamma\sigma\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}}$. Inserting this result into eq. (21) yields

$$\vec{\boldsymbol{J}} = \gamma\sigma\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}\right] + c\rho\vec{\boldsymbol{\beta}}\,. \tag{22}$$

To show that eq. (22) reduces to eq. (15) when $\rho' = 0$, one can evaluate $u_\beta J^\beta$ in reference frames K and K', respectively. Since $u_\beta J^\beta$ is a Lorentz-invariant quantity, the two results must coincide. In particular,

$$u_\beta J^\beta = \begin{cases} \gamma c\big(c\rho - \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{J}}\big)\,, & \text{in reference frame } K, \\ c^2\rho'\,, & \text{in reference frame } K'. \end{cases} \tag{23}$$

Hence, it follows that

$$\gamma c\rho' = \gamma^2\big(c\rho - \vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{J}}\big) = c\rho - \gamma\sigma\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}}\,, \tag{24}$$

after using eq. (20) in the final step above. Using eq. (24) to eliminate $c\rho$ in eq. (22), we end up with

$$\vec{\boldsymbol{J}} = \gamma\sigma\big[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\big] + \gamma c\rho'\vec{\boldsymbol{\beta}}\,. \tag{25}$$

If $\rho' = 0$, then eq. (25) reduces to the result [eq. (15)] obtained in part (b).

<u>*REMARK:*</u> Alternatively, if $\rho' = 0$ then $u_\beta J^\beta = 0$, and it follows that $c\rho = \gamma\sigma\vec{\boldsymbol{\beta}} \cdot \vec{\boldsymbol{E}}$ [cf. eq. (12)]. Inserting this into eq. (22), we end up with eq. (15).

An alternative technique for solving Problem 2

Since J^μ is a four-vector, it transforms under a Lorentz boost as:

$$c\rho' = \gamma\bigl(c\rho - \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}}\bigr)\,, \tag{26}$$

$$\vec{\boldsymbol{J}}' = \vec{\boldsymbol{J}} + \frac{(\gamma-1)}{\beta^2}\bigl(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}}\bigr)\vec{\boldsymbol{\beta}} - \gamma c\vec{\boldsymbol{\beta}}\rho\,, \tag{27}$$

whereas the electric field vector transforms as

$$\vec{\boldsymbol{E}}' = \gamma\bigl(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\bigr) = \frac{\gamma^2}{\gamma+1}\vec{\boldsymbol{\beta}}\bigl(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}\bigr)\,. \tag{28}$$

Using $\vec{\boldsymbol{J}}' = \sigma\vec{\boldsymbol{E}}'$, it follows that

$$\vec{\boldsymbol{J}} + \frac{(\gamma-1)}{\beta^2}\bigl(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}}\bigr)\vec{\boldsymbol{\beta}} - \gamma c\vec{\boldsymbol{\beta}}\rho = \sigma\gamma\bigl[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\bigr] - \frac{\gamma^2}{\gamma+1}\sigma\vec{\boldsymbol{\beta}}\bigl(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}\bigr)\,. \tag{29}$$

Taking the dot product of eq. (29) with $\vec{\boldsymbol{\beta}}$, the resulting expression simplifies to

$$\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}} - c\beta^2\rho = \sigma\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}\left(1 - \frac{\gamma\beta^2}{\gamma+1}\right)\,. \tag{30}$$

Noting that

$$\beta^2 = \frac{\gamma^2-1}{\gamma^2}\,, \tag{31}$$

it follows that

$$1 - \frac{\gamma\beta^2}{\gamma+1} = 1 - \frac{\gamma}{\gamma+1}\left(\frac{\gamma^2-1}{\gamma^2}\right) = \frac{1}{\gamma}\,. \tag{32}$$

Hence,

$$\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}} = c\beta^2\rho + \frac{\sigma}{\gamma}\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}\,. \tag{33}$$

Inserting eq. (33) back into eq. (29) results in

$$\vec{\boldsymbol{J}} = \sigma\gamma\bigl[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\bigr] - \sigma\left[\frac{\gamma^2}{\gamma+1} + \frac{\gamma-1}{\gamma\beta^2}\right]\bigl(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}\bigr) + c\rho\vec{\boldsymbol{\beta}}\,. \tag{34}$$

Using eq. (31),

$$\frac{\gamma^2}{\gamma+1} + \frac{\gamma-1}{\gamma\beta^2} = \frac{\gamma^2}{\gamma+1} + \frac{\gamma}{\gamma+1} = \gamma\,. \tag{35}$$

Hence,

$$\vec{\boldsymbol{J}} = \gamma\sigma\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - (\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}\right] + c\rho\vec{\boldsymbol{\beta}}\,, \tag{36}$$

which reproduces eq. (22).

We can rewrite eq. (36) by using eq. (33) to eliminate $\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}$. We then obtain

$$\vec{\boldsymbol{J}} + \gamma^2\bigl(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}}\bigr)\vec{\boldsymbol{\beta}} - c\rho\vec{\boldsymbol{\beta}}(1+\beta^2\gamma^2) = \gamma\sigma\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\right]\,. \tag{37}$$

Using $1 + \beta^2\gamma^2 = \gamma^2$, the above equation reduces to:

$$\vec{\boldsymbol{J}} - \gamma^2\vec{\boldsymbol{\beta}}(c\rho - \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}}) = \gamma\sigma\left[\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}\right]. \tag{38}$$

Finally, multiplying eq. (33) by γ^2 and using $\beta^2\gamma^2 = \gamma^2 - 1$, we can rewrite this equation as

$$c\rho - \gamma^2(c\rho - \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{J}}) = \gamma\sigma\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{E}}. \tag{39}$$

We can now recognize eqs. (38) and (39) as the space and time components, respectively, of the covariant equation:

$$u_\alpha\left[J^\alpha - \frac{1}{c^2}(u_\beta J^\beta)u^\alpha\right] = u_\alpha J^\alpha - u_\beta J^\beta = 0\,, \tag{40}$$

Thus we have proven that eq. (40) is the covariant generalization of Ohm's law.

3. A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency ω and angle ϑ_0 with respect to the z-axis as shown in Fig. 1. That is, the time-dependence of the azimuthal angle is $\varphi_0(t) = \varphi_0 - \omega t$. Electromagnetic radiation is emitted by the precessing dipole.

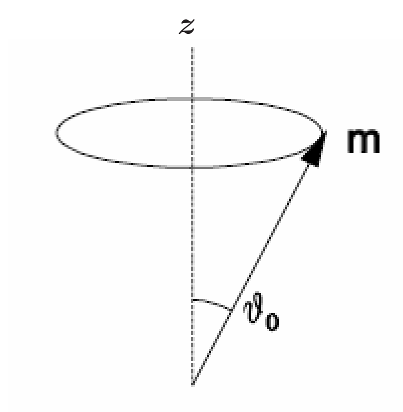

Figure 1: A magnetic dipole $\vec{\boldsymbol{m}}$ undergoes precessional motion with angular frequency ω and angle ϑ_0 with respect to the z-axis.

(a) Write out an explicit expression for the time-dependent magnetic dipole vector $\vec{\boldsymbol{m}}$ in terms of its magnitude m_0, the angles ϑ_0 and φ_0 and the time t. Show that $\vec{\boldsymbol{m}}$ consists of the sum of a time-dependent term and a time-independent term. Verify that the time-dependent term can be written as $\mathrm{Re}(\vec{\boldsymbol{\mu}}\, e^{-i\omega t})$, for some suitably chosen complex vector $\vec{\boldsymbol{\mu}}$.

In light of Fig. 1, the magnetic dipole moment vector is given by:

$$\vec{\boldsymbol{m}} = m_0\left[\hat{\boldsymbol{x}}\sin\vartheta_0\cos(\varphi_0 - \omega t) + \hat{\boldsymbol{y}}\sin\vartheta_0\sin(\varphi_0 - \omega t) + \hat{\boldsymbol{z}}\cos\vartheta_0\right] \tag{41}$$

$$= \mathrm{Re}\left[m_0\sin\vartheta_0\, e^{i\varphi_0}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})e^{-i\omega t}\right] + m_0\cos\vartheta_0\,\hat{\boldsymbol{z}}\,.$$

Thus, we can write the time-dependent term of $\vec{\boldsymbol{m}}$ as $\mathrm{Re}(\vec{\boldsymbol{\mu}}\, e^{-i\omega t})$, where

$$\vec{\boldsymbol{\mu}} = m_0 \sin\varphi_0\, e^{i\varphi_0}(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}})\,. \tag{42}$$

(b) Compute the angular distribution of the time-averaged radiated power, with respect to the z-axis defined in the above figure.

The angular distribution of the time-averaged power is given by eq. (9.21) of Jackson in SI units,

$$\frac{dP}{d\Omega} = \frac{1}{2}\,\mathrm{Re}\big[r^2\,\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^{\,*}\big]\,.$$

The magnetic and electric fields of the magnetic dipole are given by eqs. (9.35) and (9.36) of Jackson. Keeping only the leading terms of $\mathcal{O}(1/r)$, we see that

$$\vec{\boldsymbol{H}} = -\frac{1}{Z_0}\vec{\boldsymbol{E}}\times\hat{\boldsymbol{n}}\,,$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. It follows that

$$\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}^{\,*} = -\frac{1}{Z_0}\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{E}}\times(\vec{\boldsymbol{E}}^{\,*}\times\hat{\boldsymbol{n}}) = \frac{1}{Z_0}\Big[|\vec{\boldsymbol{E}}|^2 - |\vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}}|^2\Big] = \frac{1}{Z_0}|\vec{\boldsymbol{E}}|^2\,,$$

since $\vec{\boldsymbol{E}}\cdot\hat{\boldsymbol{n}} = 0$ (due to the transverse nature of electromagnetic radiation). Hence,

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|\vec{\boldsymbol{E}}|^2\,, \tag{43}$$

where the leading $\mathcal{O}(1/r)$ term of eq. (9.36) of Jackson, applied to the complex magnetic moment vector $\vec{\boldsymbol{\mu}}$, yields

$$\vec{\boldsymbol{E}} = -\frac{Z_0}{4\pi}k^2(\vec{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}})\frac{e^{ikr}}{r}\,. \tag{44}$$

Inserting this result into eq. (43), we end up with

$$\frac{dP}{d\Omega} = \frac{Z_0}{32\pi^2}\,k^4|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}|^2\,. \tag{45}$$

The squared magnitude of the cross product above is easily computed,

$$|\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}|^2 = (\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}})\cdot(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\mu}}^{\,*}) = |\vec{\boldsymbol{\mu}}|^2 - |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\mu}}|^2\,,$$

since $\hat{\boldsymbol{n}}$ is a unit vector. Explicitly, $\vec{\boldsymbol{\mu}}$ is given by eq. (42) and

$$\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}\,.$$

Hence, it follows that

$$|\vec{\boldsymbol{\mu}}|^2 = 2m_0^2\sin^2\vartheta_0\,, \qquad |\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\mu}}| = m_0\sin\vartheta_0\sin\theta\,,$$

and

$$|\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}}|^2 = m_0^2 \sin^2\vartheta_0 (2 - \sin^2\theta) = m_0^2 \sin^2\vartheta_0 (1 + \cos^2\theta)\,.$$

Thus, the angular distribution of the time-averaged radiated power is given by[3]

$$\frac{dP}{d\Omega} = \frac{Z_0 m_0^2 \sin^2\vartheta_0}{32\pi^2}\, k^4 (1 + \cos^2\theta)\,. \tag{46}$$

An alternative technique for computing the time-averaged radiated power

Instead of evaluating eq. (45), which requires the complex magnetic moment $\vec{\boldsymbol{\mu}}$ given in eq. (42), one can instead employ the result of problem 9.7(a) of Jackson,

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^4} |\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}|^2\,, \tag{47}$$

where $\ddot{\vec{\boldsymbol{m}}} \equiv d^2\vec{\boldsymbol{m}}/dt^2$, and $\vec{\boldsymbol{m}}$ is the time-dependent magnetic dipole moment given in eq. (41). Note that eq. (47) yields the time dependent power distribution, so to recover the results obtained in problem 1(b), we must time-average over one cycle.

For convenience, we rewrite eq. (41) here:

$$\vec{\boldsymbol{m}} = m_0 \Big[\hat{\boldsymbol{x}} \sin\vartheta_0 \cos(\varphi_0 - \omega t) + \hat{\boldsymbol{y}} \sin\vartheta_0 \sin(\varphi_0 - \omega t) + \hat{\boldsymbol{z}} \cos\vartheta_0\Big]$$

Taking two time derivatives, we obtain:

$$\ddot{\vec{\boldsymbol{m}}} = -m_0\omega^2 \Big[\hat{\boldsymbol{x}} \sin\vartheta_0 \cos(\varphi_0 - \omega t) + \hat{\boldsymbol{y}} \sin\vartheta_0 \sin(\varphi_0 - \omega t)\Big]\,. \tag{48}$$

Next, we compute the square of the cross product,

$$|\ddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}|^2 = \ddot{\vec{\boldsymbol{m}}} \cdot \ddot{\vec{\boldsymbol{m}}} - (\hat{\boldsymbol{n}} \cdot \ddot{\vec{\boldsymbol{m}}})^2\,,$$

after using the fact that $\hat{\boldsymbol{n}}$ is a unit vector,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}} \sin\theta\cos\phi + \hat{\boldsymbol{y}} \sin\theta \sin\phi + \hat{\boldsymbol{z}} \cos\theta\,. \tag{49}$$

Using eqs. (48) and (49), it follows that

$$\ddot{\vec{\boldsymbol{m}}} \cdot \ddot{\vec{\boldsymbol{m}}} = m_0^2 \omega^4 \sin^2\vartheta_0\,,$$

and

$$\begin{aligned}\hat{\boldsymbol{n}} \cdot \ddot{\vec{\boldsymbol{m}}} &= -m_0\omega^2 \sin\vartheta_0 \sin\theta \Big[\cos\phi \cos(\varphi - \omega t) + \sin\phi \sin(\varphi_0 - \omega t)\Big] \\ &= -m_0\omega^2 \sin\vartheta_0 \sin\theta \cos(\omega t - \varphi_0 + \phi)\,.\end{aligned}$$

[3]To obtain the angular distribution of the time-averaged radiated power in gaussian units, one must replace $Z_0 \to 4\pi/c$ and $m_0 \to m_0 c$ in eq. (46).

Hence,

$$|\dddot{\vec{\boldsymbol{m}}} \times \hat{\boldsymbol{n}}|^2 = m_0^2\omega^4 \sin^2\vartheta\Big[1 - \sin^2\theta\cos^2(\omega t - \varphi_0 + \phi)\Big]\,.$$

Inserting the above result into eq. (47) and using $\omega = kc$, we end up with

$$\frac{dP(t)}{d\Omega} = \frac{Z_0 m_0^2 \sin^2\vartheta}{16\pi^2}\,k^4\Big[1 - \sin^2\theta\cos^2(\omega t - \varphi_0 + \phi)\Big]\,. \tag{50}$$

Time-averaging over one cycle, $\langle\cos^2(\omega t - \varphi_0 + \phi)\rangle = \frac{1}{2}$. Since $1 - \frac{1}{2}\sin^2\theta = \frac{1}{2}(1 + \cos^2\theta)$, we recover eq. (46). One can also check that the total power obtained by integrating eq. (50) over solid angles is time-independent and coincides with eq. (52).

(c) Compute the total power radiated.

Integrating eq. (46) over solid angles,

$$\int d\Omega(1 + \cos^2\theta) = 2\pi\int_{-1}^{1}(1 + \cos^2\theta)\,d\cos\theta = \frac{16\pi}{3}\,. \tag{51}$$

Hence,

$$P = \frac{Z_0 m_0^2 k^4 \sin^2\vartheta_0^2}{6\pi}\,. \tag{52}$$

(d) What is the polarization of the radiation measured by an observer located along the positive z-axis far from the precessing dipole? How would your answer change if the observer were located in the x–y plane?

The polarization is determined from the electric field given in eq. (44). Thus, we must evaluate $\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}}$,

$$\begin{aligned}\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\mu}} &= \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ m_0\sin\vartheta_0 e^{i\varphi_0} & -im_0\sin\vartheta_0 e^{i\varphi_0} & 0 \end{pmatrix} \\ &= im_0 e^{i\varphi_0}\sin\vartheta_0\cos\theta(\hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}) - im_0\sin\vartheta_0\sin\theta\, e^{i(\varphi_0 - \phi)}\,\hat{\boldsymbol{z}}\,. \end{aligned} \tag{53}$$

The polarization depends on the location of the observer. If the observer is located on the positive z-axis then $\theta = 0$. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}$, which corresponds to right-circularly polarized light [cf. p. 300 of Jackson]. If the observer is located in the x–y plane, the $\theta = \frac{1}{2}\pi$. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{z}}$, which corresponds to linearly polarized light in the z-direction.

REMARK: If $\theta = \pi$, then $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{E}} \propto \hat{\boldsymbol{x}} - i\hat{\boldsymbol{y}}$, which corresponds to left-circularly polarized light. For any other value of $\theta \neq 0, \frac{1}{2}\pi$ or π, the radiation is elliptically polarized.