"ELECTRODYNAMICS" IN 2-DIMENSIONAL SPACETIME ‡

Nicholas Wheeler, Reed College Physics Department
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Introduction. We imagine ourselves—accurately enough, in some evidently quite good approximation—to inhabit a world of three spatial dimensions. But as physicists we frequently pretend otherwise; in order the better to expose the operative principles of (say) classical mechanics, or of quantum mechanics, we quite standardly have recourse to the pretense that the world has only one spatial dimension, or two. By means of this commonplace device we reduce the extraneous clutter present in our exploratory calculations, cast points of principle in stark relief, and gain access to some valuable diagramatic opportunities and notational simplifications. The idea seems unproblematic on its face, and its practical utility has been abundantly demonstrated.

In the electrodynamical literature one encounters, however, only pale shadows of what might be called the "Method of Dimensional Reduction." True, some isolated sub-topics—such, for example, as potential theory—can usefully be (and frequently are) studied in their 2-dimensional formulations. And one frequently encounters reference to such idealized structures as "infinite line charges," the intended effect of which is to introduce such a high order of symmetry as effectively to reduce the number of independent spatial variables, just as invocation of the "steady state" serves to reduce the effective number of spacetime variables. But the analytical devices brought into play even in such cases (such, for example, as Gauss' Law) remain rooted in 3-dimensional theory.

We have no difficulty divining the meaning of the author who asks us to "Think of the Newtonian motion of a mass point in an n-dimensional world," but when he asks us to "Think now of the motion of an electromagnetic field

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in such a world" we are at a loss; Maxwell's equations

$$\begin{aligned}
\nabla\cdot\mathbf{E} &= \rho && \text{GAUSS} && (1.1)\\
\nabla\times\mathbf{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{E} &= \tfrac{1}{c}\mathbf{j} && \text{AMPERE/MAXWELL} && (1.2)\\
\nabla\cdot\mathbf{B} &= 0 && \text{NO MAGNETIC MONOPOLES} && (1.3)\\
\nabla\times\mathbf{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{B} &= \mathbf{0} && \text{FARADAY} && (1.4)
\end{aligned}$$

seem so critically dependent upon cross products and curls for their very meaning as to wilt when transported to any environment other than the 3-space in which we found them. Which becomes the more surprising when one takes note of the fact that special relativity—a child (or is she the mother?) of electrodynamics—thrives in 2-dimensional spacetime; indeed, so habituated did physicists become to thinking 2-dimensionally of relativity that several aspects of the theory (such, for example, as Thomas precession) which are absent in the 2-dimensional case went for a long time unnoticed.

There are, of course, notational alternatives to (1). For example, one can—to make manifest the Lorentz covariance of (1) or to facilitate discussion of other formal matters—write

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{2.1}$$

$$\partial_\mu G^{\mu\nu} = 0 \tag{2.2}$$

where I have honored all the standard conventions ($x^0 \equiv ct$, $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, μ ranges on $\{0,1,2,3\}$, summation on repeated indices is understood), where $F^{\mu\nu}$ and $G^{\mu\nu}$ are the elements of

$$\mathbb{F} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ +E_1 & 0 & -B_3 & +B_2 \\ +E_2 & +B_3 & 0 & -B_1 \\ +E_3 & -B_2 & +B_1 & 0 \end{pmatrix} \tag{3.1}$$

and

$$\mathbb{G} \equiv \begin{pmatrix} 0 & +B_1 & +B_2 & +B_3 \\ -B_1 & 0 & -E_3 & +E_2 \\ -B_2 & +E_3 & 0 & -E_1 \\ -B_3 & -E_2 & +E_1 & 0 \end{pmatrix} \tag{3.2}$$

and where $J^0 \equiv \rho$, $\mathbf{J} \equiv \frac{1}{c}\mathbf{j}$. One can, within such a framework, easily imagine a theory which retains (2.1) but in place of (3.1) writes

$$\mathbb{F} \equiv \begin{pmatrix} 0 & -E_1 \\ +E_1 & 0 \end{pmatrix} \qquad \text{or} \qquad \mathbb{F} \equiv \begin{pmatrix} 0 & -E_1 & -E_2 \\ +E_1 & 0 & -B_3 \\ +E_2 & +B_3 & 0 \end{pmatrix}$$

But what, were one to elect to proceed in such a manner, is one to make of (3.2), which involves quite a different population of field variables? How are the $n-1$ variables $E_1, E_2, \ldots, E_{n-1}$ which enter into the evident n-dimensional generalization of (3.1) to be distributed among (or, for $n < 4$, squeezed into) the $\frac{1}{2}(n-1)(n-2)$ E-locations available in the associated generalization of (3.2)?

So subtle is the structure of Maxwellian electrodynamics that it seems on its face to provide only equivocal guidance to the resolution of such questions. It is our habit to suppose that the source field $J^\mu(x)$—subject, as Maxwell's equations mandate, to the charge conservation condition

$$\partial_\mu J^\mu = 0 \tag{4}$$

—is extrinsically "given," and that our canonical assignment is to describe (subject to prescribed initial data) the spacetime-dependence of six fields: the three components of $\mathbf{E}(x)$ together with the three components of $\mathbf{B}(x)$. But the field equations (1) $\sim$ (2) are eight in number, and conceal therefore some redundancy. The nature of that redundancy is somewhat illuminated by the observation that the field equations (2) can be recast[1]

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{5.1}$$
$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{5.2}$$

and that the latter equations become automatic upon introduction by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6}$$

of the 4-potential $A_\mu(x)$; the surviving field equation then reads

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu$$

Maxwellian electrodynamics appears in this light to be a 4-field theory. But the construction (6) is invariant under

$$A_\mu \xrightarrow[\text{gauge}]{} A_\mu + \partial_\mu \varphi \quad : \quad \varphi(x) \text{ arbitrary}$$

so the A_μ-fields themselves cannot be counted among the "observables" of the theory. Within the population of gauge-equivalent potentials A_μ there always exists a member that conforms to the "Lorentz gauge condition"

$$\partial_\mu A^\mu = 0 \tag{7}$$

[1] We write $F_{\mu\nu} \equiv g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}$ and take $g_{\mu\nu}$ to refer to the discovered metric structure of spacetime:

$$\|g_{\mu\nu}\| \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad : \quad \text{Lorentz metric}$$

I will have things to say later about *how* one gets from (2) to (5).

Imposition of such a condition yields this simplified quartet of field equations

$$\Box A^\nu = J^\nu \tag{8}$$

but subjects the potentials A_μ to what is, in effect, a *constraint*. And within the limits set by that constraint some slop remains, for not only (6) but also (7) are invariant under

$$A_\mu \xrightarrow[\text{gauge}]{} A_\mu + \partial_\mu\chi \quad \text{provided} \quad \Box\chi = 0$$

My intent in these remarks has been to establish that questions of the form

- how many degrees of freedom has the electromagnetic field?
- what are the observables of the theory?[2]
- how (consistently with relativity) does one write non-redundant field equations?

lack obvious answers even within standard Maxwellian electrodynamics. That circumstance might by itself serve to inspire interest in simplified "models" of the Maxwellian theory, but it serves at the same time to render obscure the principles that should inform the construction of such models.

1. Formal origin of the 2-dimensional theory. Recently I had occasion to remark[3] that Maxwell's equations can, in notations standard to the exterior calculus, be formulated

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{F} = \mathbf{J} \qquad \text{and} \qquad \boldsymbol{*}\mathbf{d}\ \ \mathbf{F} = \mathbf{0} \tag{9}$$

and in this connection to observe (in response to a question posed by David Griffiths) that (9) lends natural meaning to the notion of a "$2p$-dimensional electrodynamics." The elements of that notion are elaborated in the essay just cited where, however, they figure only incidentally to a densely detailed discussion of a variety of other topics. My objective today will be to provide an account—stripped of all the encrustations of inessential formalism—of the simplest instance of such a theory (case $p = 1$). Of the exterior calculus itself we will today have no need; it is only to remove the element of notational mystery that might otherwise attach to (9), and to render intelligible certain remarks of a "numerological" nature, that I now digress to provide review the bare essentials of this physicist's version of that elegant subject.

To speak of an "n-dimensional p-form" is in more familiar terminology to speak of a totally antisymmetric n-dimensional tensor of covariant rank p; we agree to write

$$\mathbf{A} \prec A_{i_1 \cdots i_p}$$

[2] I remark in this connection that in point of fact one observes $\mathbf{E}$ and $\mathbf{B}$ never directly, but only *via* their effects. . . which is to say: *via* their entry into the construction of the stress-energy tensor $S^{\mu\nu}$. So the question becomes: how many degrees of freedom has $S^{\mu\nu}$? For related discussion see p. 326 of my ELECTRODYNAMICS (1972) or E. Katz, "Concerning the number of independent variables of the classical electromagnetic field," AJP **33**, 306 (1965).

[3] See especially §4 of " Electrodynamical Applications of the Exterior Calculus" (1996).

to symbolize the statement that "$\mathbf{A}$ is the p-form with components $A_{i_1\cdots i_p}$." 0-forms are scalars, 1-forms are covariant vectors, antisymmetry comes first into play at $p=2$ and entails that p-forms with $p>n$ are an impossibility. Writing $\dim\mathbf{A}$ to signify the number of independently specifiable components of $\mathbf{A}$, we have

$$\dim\mathbf{A} = \binom{n}{p}$$

and observe—the point is elementary, but will acquire major importance—that p-forms and $(n-p)$-forms have identical dimension: $\binom{n}{p} = \binom{n}{n-p}$.

Indices are manipulated (i.e., raised and lowered) with the aid of the second rank metric tensor g_{ij} and its contravariant companion g^{ij}; one has $g^{i\alpha}g_{\alpha j} = \delta^i{}_j$ and $g \equiv \det\|g_{ij}\| \neq 0$, and in the electrodynamic application assumes the metric tensor to possess the familiar Lorentzian structure

$$\|g_{ij}\| = \begin{pmatrix} +1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

The "Hodge star operator" $*$ sets up a correspondence between p-forms and $(n-p)$-forms as follows:

$$\text{if} \qquad \mathbf{A} \prec A_{i_1\cdots i_p} \qquad \text{then} \qquad {*\mathbf{A}} \prec \tfrac{1}{p!}\sqrt{g}\,\epsilon_{i_1\cdots i_{n-p}\alpha_1\cdots\alpha_p}A^{\alpha_1\cdots\alpha_p}$$

and can be shown to possess this important property:

$$**\mathbf{A} = (-)^{p(n-p)}\mathbf{A} \sim \mathbf{A}$$

Here $\epsilon_{i_1\cdots i_n}$ is the familiar n-dimensional Levi-Civita tensor; it is a totally antisymmetric tensor density of weight $W=-1$, and possesses therefore the wonderful property that it transforms by numerical invariance: it is given in all coordinate systems by

$$\epsilon_{i_1\cdots i_n} = \begin{cases} +1 & \text{if } i_1 i_2\cdots i_n \text{ is an even permutation of } 12\cdots n \\ -1 & \text{if } i_1 i_2\cdots i_n \text{ is an odd permutation of } 12\cdots n \\ \ \ 0 & \text{otherwise; i.e., if any index is repeated} \end{cases}$$

The $\sqrt{g}$ (which transforms as a scalar density of weight $W=+1$) is introduced into the definition of $*\mathbf{A}$ in order to compensate for the weight of $\epsilon_{i_1\cdots i_n}$, which it does at this cost: it introduces i-factors whenever the metric is negative-definite ($g<0$). I will write $\star$ in place of $*$ when it is my intention that the $\sqrt{g}$ should be omitted; in relativistic applications this will, in fact, be my standard practice.

The multiply-indexed objects $\partial_j A_{i_1\cdots i_p}$ do, in general, *not* transform as the components of a tensor. It is this fact which, in tensor analysis, serves principally to motivate the introduction of the "covariant derivative." The exterior calculus proceeds alternatively from the observation that the act of summing over the signed permutations of $\{j; i_1 \cdots i_p\}$ serves to cancel out all the tensoriality-destroying terms present in $\partial_j A_{i_1\cdots i_p}$; it proceeds, in other words, from the observation that the *antisymmetric part* of $\partial_j A_{i_1\cdots i_p}$ *is* tensorial. If $\mathbf{A}$ is a p-form, then the "exterior derivative" of $\mathbf{A}$—denoted $\mathbf{dA}$—is a $(p{+}1)$-form, defined as follows:

$$\mathbf{dA} \prec \tfrac{1}{p!}\delta_{i_1 i_2\cdots i_{p+1}}{}^{\beta\alpha_1\cdots\alpha_p}\partial_\beta A_{\alpha_1\cdots\alpha_p}$$

Here $\delta^{i_1 i_2\cdots i_p}{}_{j_1 j_2\cdots j_p}$ is the "generalized Kronecker tensor" of rank $2p$; it is totally antisymmetric in all superscripts, totally antisymmetric in all subscripts, and (since assumed to be weightless) transforms by numerical invariance; it can in all coordinate systems be described

$$\delta^{i_1 i_2\cdots i_p}{}_{j_1 j_2\cdots j_p} = \begin{vmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} & \cdots & \delta^{i_1}{}_{j_p} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} & \cdots & \delta^{i_2}{}_{j_p} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_p}{}_{j_1} & \delta^{i_p}{}_{j_2} & \cdots & \delta^{i_p}{}_{j_p} \end{vmatrix}$$

Evidently $\mathbf{d}$ annihilates n-forms, while from $\partial_i\partial_j = \partial_j\partial_i$ it follows that $\mathbf{ddA} = \mathbf{0}$ in all cases. The very powerful CONVERSE OF THE POINCARÉ LEMMA asserts that if $\mathbf{A}$ is a p-form $(p \geq 1)$ such that $\mathbf{dA} = \mathbf{0}$ then there exists a $(p-1)$-form $\mathbf{B}$ such that $\mathbf{A} = \mathbf{dB}$. $\mathbf{B}$ is determined only up to a "gauge transformation"

$$\mathbf{B} \longrightarrow \mathbf{B}' = \mathbf{B} + \mathbf{dG} \qquad \text{where } \mathbf{G} \text{ is an arbitrary } (p-2)\text{-form}$$

and—this is the amazing part!—can (under weak hypotheses, and up to gauge) be described explicitly:

$$B_{i_1\cdots i_{p-1}}(x) = \int_0^1 A_{\alpha i_1\cdots i_{p-1}}(\tau x)x^\alpha\tau^{p-1}\,d\tau$$

where $x \equiv (x^1, x^2, \ldots, x^n)$.

The operators $*$ and $\mathbf{d}$ collectively exhaust the "operational alphabet" of the exterior calculus, and since

$$** \sim \mathbf{1} \qquad \text{and} \qquad \mathbf{dd} = \mathbf{0}$$

the only "words" constructable from them are snippets from the string

$$\cdots *\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\mathbf{d}*\cdots$$

The 1st-order differential operators available to the theory are four in number:

$$\left.\begin{array}{rcl} \mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow \phantom{(n-{}}(p+1)\text{-forms} \\ *\mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow (n-p-1)\text{-forms} \\ \mathbf{d}* & : & \text{sends } p\text{-forms} \longrightarrow (n-p+1)\text{-forms} \\ *\mathbf{d}* & : & \text{sends } p\text{-forms} \longrightarrow \phantom{(n-{}}(p-1)\text{-forms} \end{array}\right\}$$

The available 2[nd]-order differential operators are also four in number:

$$\left.\begin{array}{rcll} \mathbf{d}{*}\mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow & (n-p)\text{-forms} \\ {*}\mathbf{d}{*}\mathbf{d} & : & \text{sends } p\text{-forms} \longrightarrow & p\text{-forms} \\ \mathbf{d}{*}\mathbf{d}{*} & : & \text{sends } p\text{-forms} \longrightarrow & p\text{-forms} \\ {*}\mathbf{d}{*}\mathbf{d}{*} & : & \text{sends } p\text{-forms} \longrightarrow & (n-p)\text{-forms} \end{array}\right\}$$

Associations of the form

$$\begin{array}{ccc} \text{grad} & \longleftrightarrow & \mathbf{d} \\ \text{curl} & \longleftrightarrow & {*}\mathbf{d} \\ \text{div} & \longleftrightarrow & {*}\mathbf{d}{*} \end{array}$$

are recommended thus to our attention. They entail

$$\begin{array}{ccc} \text{curl}\cdot\text{grad} & \longleftrightarrow & {*}\mathbf{d}\cdot\mathbf{d} = \mathbf{0} \\ \text{div}\cdot\text{curl} & \longleftrightarrow & {*}\mathbf{d}{*}\cdot{*}\mathbf{d} = \mathbf{0} \end{array}$$

and thus give much-generalized meaning to a pair of familiar identites—identites which are, in fact, the source of all the "potentials" encountered in physical applications.

Returning now to the physics that motivated the preceding digression. . .

The $\mathbf{F}$ that appears in the exterior formulation (9) of Maxwell's equations is understood to be a 4-dimensional 2-form (and $\mathbf{J}$ therefore a 4-dimensional 1-form); specifically

$$\mathbf{F} \prec F_{\mu\nu} \quad \text{with} \quad \|F_{\mu\nu}\| = \begin{pmatrix} 0 & +E_1 & +E_2 & +E_3 \\ -E_1 & 0 & -B_3 & +B_2 \\ -E_2 & +B_3 & 0 & -B_1 \\ -E_3 & -B_2 & +B_1 & 0 \end{pmatrix}$$

and therefore

$$\star\mathbf{F} \prec \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \equiv G_{\mu\nu} \quad \text{with} \quad \|G_{\mu\nu}\| = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ +B_1 & 0 & -E_3 & +E_2 \\ +B_2 & +E_3 & 0 & -E_1 \\ +B_3 & -E_2 & +E_1 & 0 \end{pmatrix}$$

Evidently $\|F^{\mu\nu}\| = \mathbb{F}$ and $\|G^{\mu\nu}\| = \mathbb{G}$ where $\mathbb{F}$ and $\mathbb{G}$ are the matrices defined at (3). Observe that $F_{\mu\nu}$ is "elecrical" if it contains an $_0$ among its subscripts, and "magnetic" otherwise.

"n-dimensional electrodynamics" proceeds from the assumptions that the field equations (9) are to be retained as they stand, but the stipulation that $\mathbf{F}$ be a 2-form is to be abandoned; in place of the latter, one stipulates that $\mathbf{F}$ is a p-form *and so also is* $*\mathbf{F}$. Immediately, $p = n - p$, which forces $n = 2p$ to be even:

$$\mathbf{F} \text{ and } *\mathbf{F} \text{ are both } p\text{-forms, with } p = \frac{\text{dimension of spacetime}}{2}$$

The "electromagnetic field" acquires therefore a total of $\binom{2p}{p}$ independently specifiable components, of which

$$\begin{aligned}\tbinom{2p-1}{p-1} &= \tfrac{1}{2}\tbinom{2p}{p} \text{ components are "electrical"}\\ \tbinom{2p-1}{p} &= \tfrac{1}{2}\tbinom{2p}{p} \text{ components are "magnetic"}\end{aligned}$$

We have in our terminology preserved here the notion that $F_{i_1\cdots i_p}$ is "elecrical" if it contains an $_0$ among its subscripts, and "magnetic" otherwise, and we need only to look to the Pascal triangle to see how it comes about that

$$\tbinom{2p-1}{p-1} = \tbinom{2p-1}{p} \quad \text{and} \quad \tbinom{2p-1}{p-1} + \tbinom{2p-1}{p} = \tbinom{2p}{p}$$

Evidently

- in 2-space $\mathbf{F}$ is a 1-form, with 2-components, of which 1 is "electrical" and 1 "magnetic;"
- in 4-space $\mathbf{F}$ is a 2-form, with 6-components, of which 3 are "electrical" and 3 "magnetic" (this is the *physical* situation);
- in 6-space $\mathbf{F}$ is a 3-form, with 20-components, of which 10 are "electrical" and 10 "magnetic;"
- in 8-space $\mathbf{F}$ is a 4-form, with 70-components, of which 35 are "electrical" and 35 "magnetic;" etc.

"Electrodynamics" becomes, by this account, "impossible in odd-dimensional spacetime."

2. Fundamentals of the 2-dimensional theory. In the simplest case, $\mathbf{F}$ becomes a 2-dimensional 1-form; the "electromagnetic field" becomes, that is to say, a real *vector* field

$$\mathbf{F} \prec F_\mu \quad \text{which admits of natural display} \quad \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} \equiv \begin{pmatrix} E \\ B \end{pmatrix} \tag{10.1}$$

Then—since the metric is taken to have the Lorentzian structure

$$\|g_{\mu\nu}\| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

—we have

$$\begin{pmatrix} F^0 \\ F^1 \end{pmatrix} = \begin{pmatrix} +E \\ -B \end{pmatrix}$$

giving

$$\star\mathbf{F} \prec \epsilon_{\mu\alpha}F^\alpha \quad \text{which acquires the display} \quad \begin{pmatrix} G_0 \\ G_1 \end{pmatrix} = \begin{pmatrix} -B \\ -E \end{pmatrix} \tag{10.2}$$

The source term $\mathbf{J}$ is, in 2-dimensional theory, an 0-form (a scalar field):

$$\mathbf{J} \prec J \tag{11}$$

The field equations (9) acquire finally[4] these specific meanings:

$$\begin{aligned} {*}\mathbf{d}{*}\mathbf{F} = \mathbf{J} \quad &\prec \quad -(\partial^0 F_0 + \partial^1 F_1) = J \\ {*}\mathbf{d}\ \mathbf{F} = \mathbf{0} \quad &\prec \quad (\partial_0 F_1 - \partial_1 F_0) = 0 \end{aligned}$$

Equivalently

$$\partial_1 F_1 - \partial_0 F_0 = J \tag{12.1}$$
$$\partial_0 F_1 - \partial_1 F_0 = 0 \tag{12.2}$$

which in "physical" variables read

$$\partial_1 B - \partial_0 E = J \tag{13.1}$$
$$\partial_1 E - \partial_0 B = 0 \tag{13.2}$$

These, I claim, are the "Maxwell-Lorentz equations," as they would appear if spacetime were 2-dimensional. Equations (13) comprise a system of two coupled first-order partial differential equations in two field variables $E(x^0, x^1)$ and $B(x^0, x^1)$. The source function $J(x^0, x^1)$ is assumed to have been prescribed.

It is interesting to observe that (13.1) more nearly resembles Maxwell's modification (1.2) of Ampere's Law than Gauss' Law (1.1), and that (13.2) resembles Faraday's Law (1.4) except that—rather surprisingly—a sign (Lenz' Law) has been reversed.

The absence of a statement corresponding to Gauss' Law is, perhaps, *not* surprising: Gauss's Law insures that the field due to an isolated point charge "falls off geometrically," which in 1-dimensional space entails no "fall off" at all.[5] The 2-dimensional theory does, in fact, not recognize the existence of *any* kind of "charge"—neither magnetic *nor* electric—though on the right side of

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[4] Which is to say, after some fairly tedious computation. It helps to know that the action of the generalized "div" operator ${*}\mathbf{d}{*}$ can be described

$${*}\mathbf{d}{*}\mathbf{A} \prec (-)^{p(n-p)} \frac{1}{p!} \frac{1}{\sqrt{g}} \delta_{i_1 \cdots i_{p-1} b}{}^{k_1 \cdots k_p} \partial^b \big(\sqrt{g} A_{k_1 \cdots k_p} \big)$$

For details relating to the derivation of this formula see again the material cited in footnote 3, where it appears as equation (38).

[5] Recall in this connection the electrical fields which in standard theory are associated with "uniformly charged sheets;" the idealization is, of course, unphysical—intended to direct one's attention to the "near zone" of a (necessarily bounded) real charged surface.

(13.1) it does recognize the existence of what we find it natural to call "electrical current."[6] And because the theory contains no "charge," it contains also no analog of (4)—no "charge conservation."

Equations (13) can easily be "decoupled by differentiation," in the manner standard to Maxwellian electrodynamics; one obtains

$$\Box E + \partial_0 J = 0 \tag{14.1}$$

$$\Box B + \partial_1 J = 0 \tag{14.2}$$

where $\Box \equiv g_{\alpha\beta}\partial^\alpha\partial^\beta = \partial_0^2 - \partial_1^2$, and from which it follows that

$$\Box E = \Box B = 0 \qquad \text{at source-free points in spacetime} \tag{15}$$

It is (by the Converse of the Poincaré Lemma) a general implication of (9) that $\mathbf{F}$ can be represented $\mathbf{F} = \mathbf{dA}$ where $\mathbf{A}$ is a $(p-1)$-form, and is susceptible to gauge transformation $\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \mathbf{d}\boldsymbol{\varphi}$ where $\boldsymbol{\varphi}$ is an arbitrary $(p-2)$-form. In 2-dimensional electrodynamics the potential $\mathbf{A}$ becomes an 0-form; we are led to write

$$E = -\partial_0 A \quad \text{and} \quad B = -\partial_1 A \tag{16}$$

Compliance with (13.2) is thus rendered automatic, and the remaining field equation (13.1) becomes

$$\Box A = J \tag{17}$$

Equations (14) can be recovered as corollaries of (16) and (17). Comparing (16) to

$$\mathbf{E} = -\partial_0 \mathbf{A} - \boldsymbol{\nabla}\varphi \quad \text{and} \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

we see that A most closely resembles the familiar *vector* potential.

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[6] It may be useful to place this surprising development in more general context. In $2p$-dimensional electrodynamics $\mathbf{J}$ is a $(p-1)$-form; writing

$$\mathbf{J} \prec J_{i_1 i_2 \ldots i_{p-1}}$$

we associate "charges" with terms of the form $J_{0 i_1 \ldots i_{p-2}}$, which are $\binom{2p}{p-2}$ in number. At $p = 1$ we have $\binom{2}{-1} = 0$ (this is the result returned by *Mathematica*, and it is consistent with a standard convention; see p. 2 of J. Riordan's *Combinatorial Identities* (1968)). We expect on these grounds

- to encounter 1 kind of charge in the 4-dimensional theory;
- to encounter 6 kinds of charge in the 6-dimensional theory;
- to encounter 28 kinds of charge in the 8-dimensional theory;
- to encounter 120 kinds of charge in the 10-dimensional theory.

In Maxwellian electrodynamics one must—as previously remarked—invoke the "Lorentz gauge condition"

$$\ast\mathbf{d}\ast\mathbf{A} = 0 \quad \prec \quad \partial_\mu A^\mu = 0$$

to achieve the analog $\Box A^\mu = J^\mu$ of (17). In 2-dimensional electrodynamics no such condition is preconditional to (17), and in fact the theory is not robust enough to support such a condition, just as (and for the same reason that) it is unable to support the continuity condition

$$\ast\mathbf{d}\ast\mathbf{J} = 0 \quad \prec \quad \partial_\mu J^\mu = 0$$

In 2-dimensional electrodynamics the A-potential is (like the "potential energy function" $U(x, y, z)$ encountered in mechanics) a *scalar*, and is susceptible to "gauging" only in this almost trivial sense:

$$A \xrightarrow[\text{gauge}]{} A + \text{constant}$$

3. Construction of the stress-energy tensor. Great physical importance attaches in Maxwellian electrodynamics to the stress-energy tensor

$$S_{\mu\nu} = F_{\mu\alpha}F^\alpha{}_\nu - \tfrac{1}{4}(F^{\alpha\beta}F_{\beta\alpha})\cdot g_{\mu\nu} \tag{18}$$

of which

$$\mathbb{S} = \begin{pmatrix} S_{00} & S_{01} & S_{02} & S_{03} \\ S_{10} & S_{11} & S_{12} & S_{13} \\ S_{20} & S_{21} & S_{22} & S_{23} \\ S_{30} & S_{31} & S_{32} & S_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B}) & -(\mathbf{E}\times\mathbf{B})_1 & \text{etc.} & \text{etc.} \\ -(\mathbf{E}\times\mathbf{B})_1 & C_{11}+\frac{1}{2}(\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B}) & \text{etc.} & \text{etc.} \\ -(\mathbf{E}\times\mathbf{B})_2 & C_{21} & \text{etc.} & \text{etc.} \\ \text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} \end{pmatrix}$$

(here $C_{ij} = C_{ji} \equiv -E_iE_j - B_iB_j$) provides an explicit matrix description. The Maxwellian stress-energy matrix is manifestly symmetric and traceless, and is *quadratic* in the (undifferentiated) field components. In §7 of some work already twice cited I show how, within the framework afforded by the $2p$-dimensional theory, to construct an S_{ij} that shares those same properties.[7] Here, however, we have interest only in the case $p = 1$.

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[7] Within that context I was amazed to discover that the physically essential condition

$$\text{energy density } S_{00} \geq 0$$

serves in effect to *force* the spacetime metric to possess its familiar Lorentzian structure! The argument casts new light on the (seldom asked) question "Why did God select the Lorentz metric?" but fails to illuminate the question "Why did God set $p = 2$?"

It is entirely natural in the light of our Maxwellian experience to construct (consistently with the general theory to which I have just alluded)

$$\mathbb{S} = \begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(E^2+B^2) & EB \\ EB & \frac{1}{2}(E^2+B^2) \end{pmatrix} \tag{19}$$

and to observe that in consequence of the field equations

$$\begin{aligned} \partial_0\big(\tfrac{1}{2}(E^2+B^2)\big) - \partial_1\big(EB\big) &= -JE \\ \partial_0\big(EB\big) - \partial_1\big(\tfrac{1}{2}(E^2+B^2)\big) &= -JB \end{aligned}$$

In short,

$$\partial^\mu S_{\mu\nu} = -JF_\nu \tag{20}$$

which—except for the odd sign (which can be traced to the odd sign present in (13.2))—very much resembles the $\partial^\mu S_{\mu\nu} = J^\alpha F_{\alpha\nu}$ encountered in Maxwellian electrodynamics. Drawing inspiration from (18) we observe that (19) can be notated

$$\mathbb{S} = \begin{pmatrix} F_0F_0 & F_0F_1 \\ F_1F_0 & F_1F_1 \end{pmatrix} - \tfrac{1}{2}(F_0F_0 - F_1F_1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

giving

$$S_{\mu\nu} = F_\mu F_\nu - \tfrac{1}{2}(F^\alpha F_\alpha)\cdot g_{\mu\nu} \tag{21}$$

The $S_{\mu\nu}$ thus described is manifestly symmetric and traceless:

$$S_{\mu\nu} = S_{\nu\mu} \quad \text{and} \quad S^\alpha{}_\alpha = 0 \tag{22}$$

The individual matrix elements of $\mathbb{S}$ support "physical interpretations" (if we may call them that) which can be described

$$\begin{pmatrix} S^{00} & S^{01} \\ S^{10} & S^{11} \end{pmatrix} = \begin{pmatrix} \text{energy density} & c\cdot\text{momentum density} \\ \frac{1}{c}\cdot\text{energy flux} & \text{momentum flux} \end{pmatrix} \tag{23}$$

in which connection we note that in 2-dimensional electrodynamics

$$\text{energy density } S_{00} = \tfrac{1}{2}(E^2+B^2)$$

is in fact positive semi-definite, and mimics well its Maxwellian counterpart.

Alternative descriptions of $S_{\mu\nu}$ are readily available, and sometimes useful. We can, for example, write

$$S_{\mu\nu} = G_\mu G_\nu - \tfrac{1}{2}(G^\alpha G_\alpha)\cdot g_{\mu\nu}$$

and since

$$F^\alpha F_\alpha = -G^\alpha G_\alpha = E^2 - B^2 \tag{24}$$

we can use this result in combination with (21) to obtain

$$S_{\mu\nu} = \tfrac{1}{2}\big[F_\mu F_\nu + G_\mu G_\nu\big] \tag{25}$$

This elegantly symmetric equation possesses a direct counterpart in Maxwellian electrodynamics,[8] and will later be seen to foreshadow an important internal symmetry of the theory. Drawing finally upon (16)—which is to say, upon

$$F_\mu = -\partial_\mu A \tag{26}$$

—we readily obtain

$$S_{\mu\nu} = (\partial_\mu A)(\partial_\nu A) - \tfrac{1}{2}(\partial^\alpha A)(\partial_\alpha A)\cdot g_{\mu\nu} \tag{27}$$

Since A enters never nakedly but always differentiated, the expression on the right side of (27) is gauge-invariant.

To the extent that $S_{\mu\nu}$ itself enters not nakedly but only *via* (20) into physical arguments, one has to admit the possibility of "gauge transformations" of this somewhat novel form:

$$S_{\mu\nu} \longrightarrow \tilde{S}_{\mu\nu} = S_{\mu\nu} + T_{\mu\nu}$$
$$T_{\mu\nu} \text{ constrained only by } \partial^\mu T_{\mu\nu} = 0$$

In light of this circumstance, the associations (23) acquire seemingly the status of mere conventions (figures of speech), and appear to lose any claim to being statements of physical fact; one has at this point in field theories generally—as in 2-dimensional electrodynamics particularly—to be on guard against falling into what philosophers have called "the fallacy of misplaced concreteness." Suppose, for example, we were to define

$$\mathbb{T} \equiv \begin{pmatrix} \partial_1 T_1 & \partial_1 T_0 \\ \partial_0 T_1 & \partial_0 T_0 \end{pmatrix} \quad : \quad T_0 \text{ and } T_1 \text{ assigned any meanings we please}$$

Then

$$\partial^\mu T_{\mu\nu} = 0 \quad \text{automatically, in all cases}$$

by a quick calculation that hinges on the special structure of the Lorentz metric ($\partial^0 = +\partial_0$ but $\partial^1 = -\partial_1$).

The preceding paragraph began, however, with a conditional "to the extent that..." What are we to make of the circumstance that $S_{\mu\nu}$ *does* appear "nakedly" as a source term in the gravitational field equations? My tentative response would be that we "turned off" gravitational effects when we elected to work in flat spacetime.[9] The stress-energy tensor enters nakedly also into

[8] See p. 299 in my CLASSICAL ELECTRODYNAMICS (1980).

[9] And anyway, we don't at the moment possess a 2-dimensional general relativity. The loss of Gauss' law prepares us to anticipate a parallel loss of Newton's universal law of gravitation, and suggests that "2-dimensional general relativity" may in some respects be as strange as 2-dimensional electrodynamics. It would, however, be a mistake to accept "2-dimensional answers" to questions of the sort we now confront; the deeper issue retains its force irrespective of dimension.

the construction of (for example) the angular momentum tensor (but in 2-dimensions rotation is not an issue). Radiation theory proceeds from an examination of naked $S_{\mu\nu}$ terms, but its predictions remain unphysical in the absence of detection processes—field/particle interactions, which hinge on (20). Since I am able to draw no firm conclusions, let me report why I undertook this discussion.

In January of 1995 I received for review, from the editors of Physical Review, a manuscript entitled "Alternative electromagnetic energy-momentum tensors" in which the author (a professor of electrical engineering at MIT) puts forward the claim that a stress-energy tensor alternative to (18) is, in certain applications, more useful. I found the manuscript to be, in several respects, so idiosyncratic as to be virtually impenetrable; clarity came to me only when I recast the author's argument in language afforded by the 2-dimensional formalism (it was at this point that I gained my first inkling that 2-dimensional electrodynamics might actually be good for something!). The author had, in effect, set

$$T_0 = -\tfrac{1}{2}A\partial_0 A \quad \text{and} \quad T_1 = -\tfrac{1}{2}A\partial_1 A$$

and obtained (I omit the fairly straightforward details)

$$\tilde{S}_{\mu\nu} = \tfrac{1}{2}\big\{JAg_{\mu\nu} - A(\partial_\mu\partial_\nu A) + (\partial_\mu A)(\partial_\nu A)\big\}$$

which differs markly from the $S_{\mu\nu}$ of (27): not only ∂A but also $\partial\partial A$ and J (equivalently $\Box A$, by (17)) enter into the construction of $\tilde{S}_{\mu\nu}$ which, because it contains naked A-terms, has lost the property of gauge invariance. The adjustment $S_{\mu\nu} \longrightarrow \tilde{S}_{\mu\nu}$, if arguably an "improvement" on some grounds, is hardly an æsthetic improvement. Yet the author proposes to call the tensor

$$T_{\mu\nu} = \tilde{S}_{\mu\nu} - S_{\mu\nu}$$

"beauty," his intent being to associate himself with the view put forward long ago by M. Mason & W. Weaver, who in their classic text[10] report that

> *. . . "we do not believe that 'Where?' is a fair or sensible question to ask concerning energy. Energy is a function of configuration, just as beauty of a certain black-and-white design is a function of configuration. We see no more reason or excuse for speaking of a spatial energy density than we would for saying, in the case of a design, that its beauty was distributed over it with a certain density. . . "*

The manifest ugliness of $\tilde{S}_{\mu\nu}$ is, however, a matter of no physical concern (except in contexts where she appears nakedly), for it is clear by derivation—and is anyway susceptible to easy direct confirmation (use $J = \partial_1 B - \partial_0 E = \Box A$) that $\partial^\mu \tilde{S}_{\mu\nu} = -JF_\nu$; i.e., that $\tilde{S}_{\mu\nu}$ and $S_{\mu\nu}$ support the same physics. As Mason & Weaver remark,

> *"All statements are true if they are made about nothing."*

[10] *The Electromagnetic Field* (1929), p. 264.

4. Contact with the Lagrangian physics of strings. The field equation (17) is familiar as the equation of motion of a forced string, about which a great deal is known. This remark serves, in particular, to place at our immediate disposal the rich resources of classical field theory. We find it natural in this light to introduce a Lagrange density of the form

$$\mathcal{L} = \mathcal{L}_{\text{free field}} + \mathcal{L}_{\text{interaction}}$$

with

$$\mathcal{L}_{\text{free field}} \equiv \tfrac{1}{2}g^{\alpha\beta}(\partial_\alpha A)(\partial_\beta A) = \tfrac{1}{2}\big\{(\partial_0 A)^2 - (\partial_1 A)^2\big\} \tag{28.1}$$

and where the structure of

$$\mathcal{L}_{\text{interaction}} \equiv JA \tag{28.2}$$

is so simple as to provide 2-dimensional expression of the "principle of minimal coupling." The resulting field equation

$$\partial_0 \frac{\partial\mathcal{L}}{\partial(\partial_0 A)} + \partial_1 \frac{\partial\mathcal{L}}{\partial(\partial_1 A)} - \frac{\partial\mathcal{L}}{\partial A} = 0 \tag{29}$$

reads

$$\big\{\partial_0^2 - \partial_1^2\big\}A - J = 0$$

which reproduces (17), as was our intent.

Field theory (Noether's theorem, as it pertains to spacetime translations) supplies[11] this general description of the stress-energy tensor:

$$\mathcal{S}_{\mu\nu} = g_{\mu\lambda}\frac{\partial\mathcal{L}}{\partial(\partial_\lambda A)}(\partial_\nu A) - \mathcal{L}\cdot g_{\mu\nu}$$

Working from $\mathcal{L}_{\text{free}}$ we are led thus to write

$$\mathcal{S}_{\mu\nu} = (\partial_\mu A)(\partial_\nu A) - \tfrac{1}{2}(\partial^\alpha A)(\partial_\alpha A)\cdot g_{\mu\nu}$$

which precisely reproduces (27).

Importance attaches in Lagrangian field theory to the circumstance that field equations fix the Lagrange density only to within a "gauge transformation" of the form

$$\mathcal{L} \xrightarrow[\text{gauge}]{} \hat{\mathcal{L}} = \mathcal{L} + \partial_\alpha \Lambda^\alpha \;:\; \Lambda^\alpha \text{ arbitrary functions of fields \& coordinates}$$

Returning with this news to the 2-dimensional single-field system at hand, we construct

$$\hat{\mathcal{L}}_{\text{free}} = \mathcal{L}_{\text{free}} + \partial_0\Lambda^0 + \partial_1\Lambda^1 \quad : \quad \Lambda^0 \text{ and } \Lambda^1 \text{ are arbitrary functions of } A$$

and observe that $\mathcal{L}_{\text{free}} \longrightarrow \hat{\mathcal{L}}_{\text{free}}$ induces

$$\mathcal{S}_{\mu\nu} \longrightarrow \hat{\mathcal{S}}_{\mu\nu} = \mathcal{S}_{\mu\nu} + \mathcal{T}_{\mu\nu}$$

with

$$\mathcal{T}_{\mu\nu} = \left\{g_{\mu 0}\frac{\partial\Lambda^0}{\partial A} + g_{\mu 1}\frac{\partial\Lambda^1}{\partial A}\right\}(\partial_\nu A) - \left\{\frac{\partial\Lambda^0}{\partial A}(\partial_0 A) + \frac{\partial\Lambda^1}{\partial A}(\partial_1 A)\right\}\cdot g_{\mu\nu}$$

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[11] See p. 27 of my RELATIVISTIC CLASSICAL FIELDS (1973).

which is found to entail

$$\|\mathcal{T}_{\mu\nu}\| = \begin{pmatrix} -\frac{\partial\Lambda^1}{\partial A}(\partial_1 A) & +\frac{\partial\Lambda^0}{\partial A}(\partial_1 A) \\ -\frac{\partial\Lambda^1}{\partial A}(\partial_0 A) & +\frac{\partial\Lambda^0}{\partial A}(\partial_0 A) \end{pmatrix} = \begin{pmatrix} -\partial_1\Lambda^1 & +\partial_1\Lambda^0 \\ -\partial_0\Lambda^1 & +\partial_0\Lambda^0 \end{pmatrix}$$

Quick calculation establishes that

$$\partial^\mu \mathcal{T}_{\mu\nu} = 0 \quad : \quad \text{all gauge functions } \Lambda^0(A) \text{ and } \Lambda^1(A)$$

though this pair of equations would have been lost had we allowed the gauge functions to possess any symmetry-breaking *direct* dependence upon the spacetime coordinates x^0 and x^1. We observe that

$$\|\mathcal{T}_{\mu\nu}\| \text{ becomes the } \mathbb{T} \text{ of §3 by notational adjustment: } \begin{aligned} \Lambda^0 &\to +T_0 \\ \Lambda^1 &\to -T_1 \end{aligned}$$

and that to recover "beauty" we have only to set

$$\Lambda^\alpha = -\tfrac{1}{4}g^{\alpha\beta}\partial_\beta(A^2) = -\tfrac{1}{2}A\cdot g^{\alpha\beta}\partial_\beta A$$

This, however, is a step that we are—as well-bred field theorists—strongly disinclined to take... for to write

$$\begin{aligned} \mathcal{L}_\text{free} \longrightarrow \hat{\mathcal{L}}_\text{free} = \mathcal{L}_\text{free} &+ \partial_\alpha\Lambda^\alpha \\ \partial_\alpha\Lambda^\alpha &= -\tfrac{1}{4}\Box A^2 \\ &= -\tfrac{1}{2}(\partial_\alpha A)g^{\alpha\beta}(\partial_\beta A) - \tfrac{1}{2}A\cdot\Box A \end{aligned}$$

is to introduce a gauge term that depends not only upon A but *also upon the first and second derivatives* of A.[12] The situation is made even more bizarre by the observation that

$$= -\mathcal{L}_\text{free} - \tfrac{1}{2}AJ$$

We have come upon a formal tangle which I hope one day to be in position to illuminate. But today is not the day; we have other, more pressing, work to do.

We have proceeded thus far in this field-theoretic discussion by direct appropriation of the physics of strings, as though A were the field of interest. But A is not even a "physical" field (it is a "potential"—susceptible to gauge), and as 2-dimensional electrodynamicists our deeper interest lies in fact elsewhere—in the fields F_μ. The forced string can, from this point of view, be

[12] Of much lesser consequence is the *loss of gauge invariance*, in the sense $A \longrightarrow \hat{A} = A + \text{constant}$; gauge terms—since "merely" gauge terms (devoid of physical implication)—are permitted to violate any symmetry principle they have a mind to.

thought of as a 3-field system, governed by field equations that can, according to (12.1) and (16), be notated

$$\partial^\mu F_\mu = -J \tag{30.1}$$
$$F_\mu = -\partial_\mu A \tag{30.2}$$

We note that our former Lagrangian can in this notation be described

$$\mathcal{L} = \tfrac{1}{2}g^{\alpha\beta}F_\alpha F_\beta + JA$$

but for the purposes now at hand adopt this hybrid construction:

$$\begin{aligned}\mathcal{L}(F_0, F_1, A, \partial A) &= -\tfrac{1}{2}g^{\alpha\beta}F_\alpha F_\beta - g^{\alpha\beta}F_\alpha(\partial_\beta A) + JA \\ &= -\tfrac{1}{2}F_0F_0 + \tfrac{1}{2}F_1F_1 - F_0(\partial_0 A) + F_1(\partial_1 A) + JA\end{aligned} \tag{31}$$

The field equations are then three in number:

$$\begin{aligned}
&\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 F_0)} + \partial_1\frac{\partial\mathcal{L}}{\partial(\partial_1 F_0)} - \frac{\partial\mathcal{L}}{\partial F_0} = 0, \text{ which gives} \quad F_0 = -\partial_0 A\\
&\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 F_1)} + \partial_1\frac{\partial\mathcal{L}}{\partial(\partial_1 F_1)} - \frac{\partial\mathcal{L}}{\partial F_1} = 0, \text{ which gives} \quad F_1 = -\partial_1 A\\
&\partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 A)} + \partial_1\frac{\partial\mathcal{L}}{\partial(\partial_1 A)} - \frac{\partial\mathcal{L}}{\partial A} = 0, \text{ which gives} \quad \partial^\mu F_\mu = -J
\end{aligned}$$

The Lagrangian (31) is seen thus to give rise to the *complete system* of equations (30)—"Ampere's Law" (30.1) *as well as* the definitions (30.2) which render "Faraday's Law" (12.2) automatic; (31) provides the 2-dimensional analog of what in the field-theoretic formulation of Maxwellian electrodynamics is sometimes called[13] the "Schwinger Lagrangian."

Our excursion into rudimentary field theory has positioned us to view with new eyes the (longitudinal) vibrations of a string; we find ourselves in position to say that "the time partial of displacement is very **E**-like, and the space partial very **B**-like." The expressions that familiarly provide descriptions of energy & momentum density & flux on a string have acquired the representation

$$\begin{pmatrix}\text{energy density} & c\cdot\text{momentum density}\\ \frac{1}{c}\cdot\text{energy flux} & \text{momentum flux}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}(E^2+B^2) & -EB\\ -EB & \frac{1}{2}(E^2+B^2)\end{pmatrix}$$

and we are brought close to something very like the "mechanical model" of electromagnetism that for a while figured so prominently in Maxwell's own thought.[14]

[13] See A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (1964), p. 103.

[14] See, for example, Chapter 7 of C. W. F. Everitt, *James Clerk Maxwell: Physicist & Natural Philosopher* (1975).

One small point before I take leave of this topic: the methods of classical field theory, when applied to Maxwellian electrodynamics, tend to veer off track when they encounter implications of the fact that the 4-potential A_μ is defined only to within a gauge transformation. The specific form of the field equations is conditional upon the imposition of a side condition (such, for example, as the Lorentz gauge condition $\partial_\mu A^\mu = 0$), which is by nature not a field equation but an arbitrarily imposed constraint. Constraints, when present in field theories, always call for the introduction of specialized machinery of one description or another.[15] My point is that 2-dimensional electrodynamics is, in this respect, especially simple; no specialized machinery is called for because the theory supports no non-trivial analog of a gauge condition.

5. Lorentz covariance. I have many times, in many connections, had occasion to remark that

> *The covariance group $\mathfrak{G}$ of a given system of physical equations depends critically upon how those equations are conceptualized and/or notated, and upon how the elements of $\mathfrak{G}$ are presumed to act.*

The history of electrodynamics provides an example of what I mean: Lorentz, working from (1), was led in 1904 to what we now call the Lorentz group. About five years later, Bateman & Cunningham—working independently from a variant of (2)—were led to the conformal group, which contains the Lorentz group as a subgroup, but contains also some non-linear (“accelerational”) elements. And in the 1930’s van Danzig, working from a slight reinterpretation of (2), was led to a metric-independent electrodynamics which is unrestrictedly covariant.[16]

In undertaking to discuss the transformational properties of 2-dimensional electrodynamics I acquire therefore an obligation to stipulate that I will concern myself with the theory *as described in preceding pages*. We will be led to the Lorentz group, but an alternative formulation lies readily at hand which would lead to the 2-dimensional conformal group. That remark opens upon a landscape it might, on some other occasion, be interesting to explore, for the 2-dimensional conformal group is in an important respect exceptional; it lacks the “crystaline” quality of its N-dimensional siblings ($N \geq 3$), and embraces

[15] A variety of ingenious special mechanisms have been devised to deal with the contraint problem as it arises within Maxwellian electrodyamics. Proca, for example, has described a field system which contains an additional parameter $\varkappa$ (associated physically with a hypothetical “photon mass”), and in which an equation of the form $\partial_\mu A^\mu = 0$ has joined the population of *field* equations. By design

$$\lim_{\varkappa\downarrow 0}\big\{\text{Proca equations}\big\} = \text{Maxwell equations}$$

For the details, see pp. 89–93 in RELATIVISTIC CLASSICAL FIELDS (1973).

[16] For an account of these developments, and references to related literature, see pp. 179–186 of my CLASSICAL ELECTRODYNAMICS (1980).

the entire "theory of conformal transformations," as supplied by the theory of functions of a complex variable.[17]

Our theory was born of the exterior calculus, and has been developed in language borrowed from ordinary tensor calculus. Its *general* covariance would therefore be assured, save for one detail: we have, since the beginning of §2, steadfastly insisted that the metric tensor $g_{\mu\nu}$—which we have used only to set up a correspondence between covariant and contravariant objects (i.e., to raise/lower indices)—possess the Lorentzian structure

$$||g_{\mu\nu}|| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

But

$$x^\mu \longrightarrow \tilde{x}^\mu = \tilde{x}^\mu(x)$$

induces

$$\begin{aligned} g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu}(\tilde{x}) &= \tfrac{\partial x^\alpha}{\partial \tilde{x}^\mu}\tfrac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta} \\ &\updownarrow \\ \|\tilde{g}_{\mu\nu}(\tilde{x})\| &= \|\tfrac{\partial x^\alpha}{\partial \tilde{x}^\mu}\|^\intercal \|g_{\alpha\beta}\| \|\tfrac{\partial x^\beta}{\partial \tilde{x}^\nu}\| \quad \text{in matrix notation} \end{aligned}$$

To insist that

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is to impose a set of constraints upon the functions $x^\beta(\tilde{x})$. If we restrict our attention to *linear* coordinate transformations

$$\tilde{\boldsymbol{x}} \longrightarrow \boldsymbol{x} = \mathbb{L}\tilde{\boldsymbol{x}} \tag{32}$$

then we have

$$\mathbb{L}^\intercal \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mathbb{L} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

which is precisely the condition that $\mathbb{L}$ be a "Lorentz matrix," and that (32) be a Lorentz transformation. We conclude that 2-dimensional electrodynamics shares the Lorentz covariance of its Maxwellian big sister.

Though not at all unexpected, this result is in one respect a little surprising, for it appears to contradict the previously noted "string-like" character of the 2-dimensional theory. What I have in mind is this: you and I stand before a quiescent stretched string. I stimulate the string, and you write something like

$$\Big\{\big(\tfrac{\partial}{\partial x}\big)^2 - \tfrac{1}{u^2}\big(\tfrac{\partial}{\partial t}\big)^2\Big\}\phi(t,x) = f(t,x)$$

[17] Elaborate discussion of this topic (exclusive of its electrodynamical application) can be found in TRANSFORMATIONAL PHYSICS OF WAVES (1979).

to describe its subsequent vibratory motion. We see a second observer **O** to be going by with speed v. He also has interest in our string, and—taking

$$\left.\begin{aligned} t &= \boldsymbol{t} \\ x &= \boldsymbol{x} + v\boldsymbol{t} \end{aligned}\right\} \tag{33}$$

to describe the (Galilean) relationship between our coordinates and his, finds that he must write

$$\Big\{\big(\tfrac{\partial}{\partial \boldsymbol{x}}\big)^2 + \tfrac{2v}{u^2-v^2}\tfrac{\partial}{\partial \boldsymbol{x}}\tfrac{\partial}{\partial \boldsymbol{t}} - \tfrac{1}{u^2-v^2}\big(\tfrac{\partial}{\partial \boldsymbol{t}}\big)^2\Big\}\boldsymbol{\phi}(\boldsymbol{t},\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{t},\boldsymbol{x})$$

to describe the same physics.[18] Neither **O** nor we are disturbed by the fact that his equation is structurally distinct from ours, for we stand in asymmetric relationships to the "medium;" we are at rest with respect to the string, but he is not. Such a rationale would, however, lose its force if he/we were discussing *waves unsupported by a "medium."*[19] **O** and we busy ourselves, and discover[20] that for him to achieve

$$\Big\{\big(\tfrac{\partial}{\partial \boldsymbol{x}}\big)^2 - \tfrac{1}{\boldsymbol{u}^2}\big(\tfrac{\partial}{\partial \boldsymbol{t}}\big)^2\Big\}\boldsymbol{\phi}(\boldsymbol{t},\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{t},\boldsymbol{x})$$

he must set $\boldsymbol{u} = u$ (call their mutual value c) and, in place of (33), write[21]

$$\left.\begin{aligned} x^0 &= \gamma(\hat{x}^0 + \beta\hat{x}^1) \\ x^1 &= \gamma(\beta\hat{x}^0 + \hat{x}^1) \end{aligned}\right\} \tag{34}$$

with $\beta \equiv v/c$ and $\gamma^2 \equiv 1/\det\left(\begin{smallmatrix} 1 & \beta \\ \beta & 1 \end{smallmatrix}\right)$. But equations (34) are just the Lorentz transformation equations (32)—spelled out now in kinematic detail. These remarks put us in position to make the following observation:

> *2-dimensional electrodynamics is "string-like" except in one detail: the potential A is, since susceptible to gauge adjustment, not physical, and therefore cannot be assigned the status of a "medium." The Lorentz covariance of the theory is enforced by this circumstance.*

In Maxwellian electrodynamics the fields **E** and **B** acquire the strange transformation properties first described by Einstein (1905) because they are components of a tensor $F^{\mu\nu}$ of 2nd rank. Their 2-dimensional counterparts are,

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[18] Here $\boldsymbol{\phi}(\boldsymbol{t},\boldsymbol{x}) \equiv \phi(\boldsymbol{t},\boldsymbol{x}+v\boldsymbol{t})$, and $\boldsymbol{f}(\boldsymbol{t},\boldsymbol{x})$ is defined similarly.

[19] It took physicists a long time to appreciate that the real world—in the instance originally of electromagnetic radiation—gives meaning to such a seemingly paradoxical notion.

[20] For the detailed argument, see the seminar notes *"How Einstein might have achieved relativity already in 1895"*, which appear as an introduction to the material cited in footnote 16.

[21] I revert here to our former notation, writing $x^0 = ct$, $x^1 = x$, $\hat{x}^0 = c\boldsymbol{t}$ and $\hat{x}^1 = \boldsymbol{x}$.

however, components of a tensor F^μ of only 1st rank; they transform, that is to say, like coordinates:

$$\tilde{\boldsymbol{x}} \longrightarrow \boldsymbol{x} = \mathbb{L}\tilde{\boldsymbol{x}} \quad \text{induces} \quad \tilde{\mathbf{F}} \longrightarrow \mathbf{F} = \mathbb{L}\tilde{\mathbf{F}} \tag{35}$$

where $\mathbf{F} \equiv \binom{F^0}{F^1} = \binom{+E}{-B}$. It follows that

$$\left.\begin{array}{r} F^\alpha F_\alpha = -G^\alpha G_\alpha = E^2 - B^2 \\ F^\alpha G_\alpha = 0 \end{array}\right\} \quad \text{are Lorentz invariant} \tag{36}$$

Here $\mathbf{G} \equiv \binom{G_0}{G_1} = \binom{-B}{-E}$ refers to the "dual" field $G_\mu \equiv \epsilon_{\mu\alpha}F^\alpha$, which was first encountered at (10.2).[22]

At source-free points, $\Box A = (\partial_0 + \partial_1)(\partial_0 - \partial_1)A = 0$ entails

$$\text{either } (\partial_0 + \partial_1)A = 0 \text{ or } (\partial_0 - \partial_1)A = 0 \quad ;\text{ i. e.,} \quad B = \pm E \tag{37.1}$$

It follows in either case that

$$E^2 - B^2 = 0 \quad \text{at source-free points} \tag{37.2}$$

We were prepared, in view of (35), to classify fields F_μ as

$$\left.\begin{array}{r} \text{"time-like"} \\ \text{"light-like"} \\ \text{"space-like"} \end{array}\right\} \text{ according as } E^2 - B^2 \text{ is } \left\{\begin{array}{l} > 0 \\ = 0 \\ < 0 \end{array}\right.$$

but see now that—with a literalness to which we are unaccustomed—*every* such field is "light-like" at source-free points. Equations (37) are reminiscent of statements encountered in connection with the Maxwellian theory of plane waves: $\mathbf{E}\perp\mathbf{B}$ and have (in my units not only the same physical dimension but also) the same magnitude.[23] But nothing comparable to a "plane wave assumption" is presently in force.

That stress-energy $S^{\mu\nu}$ is—in the 2-dimensional theory as in Maxwellian theory (as, indeed, it is also in the $2p$-dimensional theory)—a doubly-indexed object comes as no surprise, since Noether's theorem supplies one μ-indexed object per free parameter, and the spacetime translations supply

$$\text{number of free parameters} = \text{dimension of spacetime}$$

Nor is it surprising that $S^{\mu\nu}$ transforms tensorially. My simple point, in a nutshell, is this: the rank (whence also the transformation properties) of the electromagnetic field $\mathbf{F}$ are dimension-dependent, but those of stress-energy are not; they are, that is to say, *universal*.

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[22] The Maxwellian counterparts of (36) can be seen below at (48).

[23] See p. 342 in the text just cited.

Adopting now a slight (first index up) variant

$$\mathbb{S} \equiv \|S^\mu{}_\nu\| = \begin{pmatrix} \frac{1}{2}(E^2+B^2) & EB \\ -EB & -\frac{1}{2}(E^2+B^2) \end{pmatrix} \tag{38}$$

of the notation introduced at (19) in §3, we observe that

$$\det(\mathbb{S}-\lambda\mathbb{I}) = \lambda^2 - \lambda\cdot\operatorname{tr}\mathbb{S} + \tfrac{1}{2}\big[(\operatorname{tr}\mathbb{S})^2 - \operatorname{tr}\mathbb{S}^2\big] \tag{39}$$

But the traces of all powers of $\mathbb{S}$ are manifestly Lorentz invariant; specifically[24]

$$\left.\begin{aligned} \operatorname{tr}\mathbb{S} \;\; &= \quad\;\; S^\alpha{}_\alpha = 0 \\ \operatorname{tr}\mathbb{S}^2 &= S^\alpha{}_\beta S^\beta{}_\alpha = \tfrac{1}{2}(F^\alpha F_\alpha)^2 = \tfrac{1}{2}(E^2-B^2)^2 \end{aligned}\right\} \tag{40}$$

The characteristic equation now becomes

$$\lambda^2 - \sigma^2 = 0 \quad\text{with}\quad \begin{aligned}\sigma &\equiv \tfrac{1}{2}(E^2-B^2) \\ &= \sqrt{-\det\mathbb{S}}\end{aligned} \tag{41}$$

so by the Cayley-Hamilton theorem

$$\mathbb{S}^2 - \sigma^2\,\mathbb{I} = \mathbb{O} \tag{42}$$

From this it follows readily that the traces of higher powers of $\mathbb{S}$ supply no new invariants:

$$\begin{aligned} \operatorname{tr}\mathbb{S}^{2n+1} &= 0 \\ \operatorname{tr}\mathbb{S}^{2n} \;\; &= 2\sigma^{2n} \end{aligned} \quad : \quad n = 0, 1, 2, \ldots \tag{43}$$

The eigenvalues $\lambda = \pm\sigma$ are Lorentz invariant, and vanish (with the consequence that $\mathbb{S}$ becomes singular) at source-free points.

Looking to the factors of (42), we are led—subject to the proviso that $\sigma \neq 0$ —to introduce

$$\mathbb{P}_0 \equiv +\tfrac{1}{2\sigma}\big(\mathbb{S}+\sigma\mathbb{I}\big) \quad\text{and}\quad \mathbb{P}_1 \equiv -\tfrac{1}{2\sigma}\big(\mathbb{S}-\sigma\mathbb{I}\big) \tag{44}$$

and to notice that these comprise a complementary set of orthogonal projection operators

$$\mathbb{P}_0 + \mathbb{P}_1 = \mathbb{I}, \quad \mathbb{P}_0\cdot\mathbb{P}_1 = \mathbb{O}, \quad \mathbb{P}_0\cdot\mathbb{P}_0 = \mathbb{P}_0 \quad\text{and}\quad \mathbb{P}_1\cdot\mathbb{P}_1 = \mathbb{P}_1 \tag{45}$$

in terms of which we achieve this "spectral representation" of $\mathbb{S}$:

$$\mathbb{S} = \lambda_0\mathbb{P}_0 + \lambda_1\mathbb{P}_1 \tag{46}$$

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[24] Though it is easy enough to work directly from (38), it is more elegant—and marginally more informative—to work from (25) with the aid of (36).

Here $\lambda_0 \equiv +\sigma$ and $\lambda_1 \equiv -\sigma$; motivation for the design of my subscripted notation will soon be evident.

The theory of n-dimensional projection matrices supplies the information that $\det\big(\mathbb{P} - \lambda\mathbb{I}\big) = (\lambda - 1)^d \lambda^{n-d}$, where d is the dimension of the subspace upon which $\mathbb{P}$ projects, and $n - d$ is the dimension of the subspace annhilated by $\mathbb{P}$. In the case at hand we have

$$\begin{aligned}
\det\big(\mathbb{P}_0 - \lambda\mathbb{I}\big) &= \det\big(\big\{\tfrac{1}{2\sigma}\mathbb{S} + \tfrac{1}{2}\mathbb{I}\big\} - \lambda\mathbb{I}\big)\\
&= \tfrac{1}{4\sigma^2}\det\big(\mathbb{S} - 2\sigma(\lambda - \tfrac{1}{2})\mathbb{I}\big)\\
&= \tfrac{1}{4\sigma^2}\Big\{\big[2\sigma(\lambda - \tfrac{1}{2})\big]^2 - \sigma^2\Big\} \quad \text{by (41)}\\
&= (\lambda - 1)\cdot\lambda\\
\det\big(\mathbb{P}_1 - \lambda\mathbb{I}\big) &= \lambda\cdot(\lambda - 1) \quad \text{by a similar argument}
\end{aligned}$$

Evidently $\mathbb{P}_0$ and $\mathbb{P}_1$ project onto an orthogonal pair of *vectors*. The implication is that if $J(x) \neq 0$ (i.e., if x is not a source-free point) then written into the structure of the local stress-energy tensor $S^\mu{}_\nu(x)$ is an "eigenbasis," in terms of which it becomes possible to write

$$\boldsymbol{x} = \tilde{x}^0\boldsymbol{e}_0 + \tilde{x}^1\boldsymbol{e}_1 \tag{47.1}$$

where

$$\mathbb{P}_0\boldsymbol{e}_0 = \boldsymbol{e}_0 \quad \text{and} \quad \mathbb{P}_1\boldsymbol{e}_1 = \boldsymbol{e}_1 \tag{47.2}$$

These statements are, I emphasize, *local*; their detailed meaning varies from sourcey point to sourcey point.

The results developed above are so striking, and—algebraically natural though they are—touch upon aspects of electrodynamics that are so unfamiliar, as to inspire the following commentary; we proceed in Lorentzian analogy with the familiar fact that "every real symmetric matrix can be diagonalized by rotation." In 2-dimensional spacetime the most general (proper) Lorentz matrix $\mathbb{L} = \|L^\mu{}_\nu\|$ can be described[25]

$$\mathbb{L} = \exp\big\{\psi\big(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\big)\big\} = \begin{pmatrix}\cosh\psi & \sinh\psi\\ \sinh\psi & \cosh\psi\end{pmatrix}$$

where $\beta = \tanh\psi$ provides the kinematic interpretation of the parameter ψ (which is sometimes called the "rapidity"). If $S_{\mu\nu}$ is symmetric then $\mathbb{S} \equiv \|S_{\mu\nu}\|$ (note that I have, for purposes of the present argument, again lowered the leading index) has necessarily the form $\mathbb{S} = \big(\begin{smallmatrix}a & b\\ b & c\end{smallmatrix}\big)$, and to achieve tracelessness (in the sense $S^\alpha{}_\alpha = 0$) we must set $c = a$. Lorentz transformation sends $S_{\mu\nu} \longrightarrow \tilde{S}_{\mu\nu} = L^\alpha{}_\mu L^\beta{}_\nu S_{\alpha\beta}$, which in matrix notation reads

[25] See pp. 195–197 of the class notes cited in footnote 22.

$$\begin{aligned}\mathbb{S} \longrightarrow \tilde{\mathbb{S}} &= \mathbb{L}^{\mathsf{T}}\,\mathbb{S}\,\mathbb{L}\\ &= \begin{pmatrix}\cosh\psi & \sinh\psi\\ \sinh\psi & \cosh\psi\end{pmatrix}\begin{pmatrix}a & b\\ b & a\end{pmatrix}\begin{pmatrix}\cosh\psi & \sinh\psi\\ \sinh\psi & \cosh\psi\end{pmatrix}\\ &= \begin{pmatrix}a\cosh 2\psi + b\sinh 2\psi & b\cosh 2\psi + a\sinh 2\psi\\ b\cosh 2\psi + a\sinh 2\psi & a\cosh 2\psi + b\sinh 2\psi\end{pmatrix}\\ &\downarrow\\ &= \begin{pmatrix}a\,\mathrm{sech}\,2\psi & 0\\ 0 & a\,\mathrm{sech}\,2\psi\end{pmatrix} \quad \text{if} \quad \tanh 2\psi = -b/a\end{aligned}$$

But $-1 < \tanh 2\psi < +1$. The implication is that $\mathbb{S}$ can be *Lorentz transformed to diagonal form* if and only if $0 \le (b/a)^2 < 1$. To discover the transformation that does the job, we use

$$\tanh 2\psi = \frac{2\tanh\psi}{1+\tanh^2\psi} = \frac{2\beta}{1+\beta^2} = -b/a$$

to obtain

$$\beta = -(a/b)\Big[1 \pm \sqrt{1-(b/a)^2}\,\Big] = -(a/b)\Big[1 \pm \frac{\sqrt{a^2-b^2}}{a}\Big]$$

which in physical variables (i.e., when we take $\mathbb{S}$ to have the specific meaning stated at (38)) becomes

$$\begin{aligned}\beta &= -\frac{E^2+B^2}{2EB}\Big[1 \pm \frac{E^2-B^2}{E^2+B^2}\Big]\\ &= \begin{cases}-B/E\\ -E/B\end{cases} \quad : \quad \text{take whichever conforms to } -1<\beta<+1\end{aligned}$$

Diagonalization—when it can be achieved—accomplishes this result:[26]

$$\tilde{\mathbb{S}} = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix} \quad \text{where, as before, } \sigma \equiv \tfrac{1}{2}(E^2-B^2)$$

But, on the ground that $\beta = \pm 1$ is "unphysical," we conclude that in fact diagonalization of the stress-energy is *not possible at source-free points*. This is a gratifying result, for in the contrary case we would in effect have transformed the stress-energy tensor to extinction. And it is entirely consistent with our discovery that the projectors $\mathbb{P}_0$ and $\mathbb{P}_1$ exist if and only if $J \neq 0$. To establish explicit contact with material developed in the preceding paragraph we have once again to raise the leading index, writing

$$\|\tilde{S}^\mu{}_\nu\| = \begin{pmatrix}\sigma & 0\\ 0 & -\sigma\end{pmatrix}$$

[26] Use $\mathrm{sech}\,2\psi = \sqrt{1-\tanh^2 2\psi}$.

The definitions (44) then give

$$\tilde{\mathbb{P}}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbb{P}}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The Lorentz transformation that serves to diagonalize the stress-energy tensor acquires now a second interpretation: it is the transformation that sends

$$\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \longrightarrow \mathbb{L}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \boldsymbol{e}_0 \quad \text{and} \quad \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \longrightarrow \mathbb{L}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \boldsymbol{e}_1$$

The $\mathbb{L}$ in question can be described

$$\mathbb{L} = \frac{1}{\sqrt{1-\tanh^2\psi}}\begin{pmatrix} 1 & \tanh\psi \\ \tanh\psi & 1 \end{pmatrix}$$

$$\tanh\psi \equiv \beta = \begin{cases} -\frac{B}{E} & \text{if } E^2 > B^2, \text{ i.e., if } \sigma > 0 \\ -\frac{E}{B} & \text{if } E^2 < B^2, \text{ i.e., if } \sigma < 0 \end{cases}$$

It is no surprise that with β (and the associated $\gamma = 1/\sqrt{1-\beta^2}\,$) thus defined we have

$$\boldsymbol{e}_0 = \gamma\begin{pmatrix} 1 \\ \beta \end{pmatrix} \quad \text{and} \quad \boldsymbol{e}_1 = \gamma\begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

Nor is it surprising that we have (in the Lorentzian sense)

$$(\boldsymbol{e}_0, \boldsymbol{e}_0) = +1, \quad (\boldsymbol{e}_1, \boldsymbol{e}_1) = -1 \quad \text{and} \quad (\boldsymbol{e}_0, \boldsymbol{e}_1) = 0$$

It is, however, gratifying (if not really surprising) that $\mathbb{P}_0$ and $\mathbb{P}_1$ can be rendered in terms of such β and γ: inserting (38) into (44) we find

$$\mathbb{P}_0 = \frac{1}{E^2 - B^2}\begin{pmatrix} EE & EB \\ -BE & -BB \end{pmatrix} \quad \text{and} \quad \mathbb{P}_1 = \frac{1}{B^2 - E^2}\begin{pmatrix} BB & BE \\ -EB & -EE \end{pmatrix}$$

which display a pretty symmetry. If E dominates B then we have

$$\mathbb{P}_0 = \gamma^2\begin{pmatrix} 1 & -\beta \\ \beta & -\beta^2 \end{pmatrix} \quad \text{and} \quad \mathbb{P}_1 = \gamma^2\begin{pmatrix} -\beta^2 & \beta \\ -\beta & 1 \end{pmatrix}$$

while if B dominates E then the descriptions of $\mathbb{P}_0$ and $\mathbb{P}_1$ are reversed. In the former case we are gratified to compute

$$\mathbb{P}_0\boldsymbol{e}_0 = \boldsymbol{e}_0, \quad \mathbb{P}_0\boldsymbol{e}_1 = \mathbf{0}, \quad \mathbb{P}_1\boldsymbol{e}_0 = \mathbf{0} \quad \text{and} \quad \mathbb{P}_1\boldsymbol{e}_1 = \boldsymbol{e}_1$$

while in the latter case we find that the roles of $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ are reversed.

To demonstrate that the previous discussion is in no essential respect special to the 2-dimensional case, I conduct my reader now along the first quarter mile of the trail that leads through the corresponding Maxwellian landscape—seldom traveled though the trail happens to be. For *every* 4×4 matrix one has[27]

$$\det(\mathbb{S} - \lambda\mathbb{I}) = \lambda^4 - \Delta_1\lambda^3 + \tfrac{1}{2!}\Delta_2\lambda^2 - \tfrac{1}{3!}\Delta_3\lambda + \tfrac{1}{4!}\Delta_4$$

where

$$\begin{aligned}
\Delta_1 &\equiv T_1\\
\Delta_2 &\equiv T_1^2 - \quad T_2\\
\Delta_3 &\equiv T_1^3 - 3T_1T_2 + 2T_3\\
\Delta_4 &\equiv T_1^4 - 6T_1^2T_2 + 8T_1T_3 + 3T_2^2 - 6T_4 = 4!\det\mathbb{S}
\end{aligned}$$

and $T_k \equiv \operatorname{tr}\mathbb{S}^k$. Taking $\mathbb{S} \equiv \|S^\mu{}_\nu\|$ to be, in particular, the Maxwellian stress-energy tensor (18), one can, with moderate effort,[28] establish (compare (40) & (43)) these trace relations:

$$\begin{aligned}
T_1 &= 0\\
T_2 &= 4\tau^2\\
T_3 &= 0\\
T_4 &= 4\tau^4
\end{aligned}$$

where $\tau^2 \equiv \alpha^2 + \beta^2$ with (compare (36))

$$\left.\begin{aligned}
\alpha &\equiv \tfrac{1}{4}F^\alpha{}_\beta F^\beta{}_\alpha = -\tfrac{1}{4}G^\alpha{}_\beta G^\beta{}_\alpha = \tfrac{1}{2}(E^2 - B^2)\\
\beta &\equiv \tfrac{1}{4}F^\alpha{}_\beta G^\beta{}_\alpha = +\tfrac{1}{4}G^\alpha{}_\beta F^\beta{}_\alpha = \mathbf{E}\cdot\mathbf{B}
\end{aligned}\right\} \tag{48}$$

Clearly, α and β are Lorentz invariant.[29] So also, therefore, are all the T's, whence all the Δ's, whence all the eigenvalues of $\mathbb{S}$. Major simplifications result from the fact that the T's of odd order vanish; we have (compare (39))

$$\begin{aligned}
\det(\mathbb{S} - \lambda\mathbb{I}) &= \lambda^4 - \tfrac{1}{2}T_2\lambda^2 + \left[\tfrac{1}{8}T_2^2 - \tfrac{1}{4}T_4\right]\\
&= \lambda^2 - 2\tau^2\lambda^2 + \tau^4\\
&= \left(\lambda^2 - \tau^2\right)^2
\end{aligned}$$

By the Cayley-Hamilton theorem $\left(\mathbb{S}^2 - \tau^2\,\mathbb{I}\right)^2 = \mathbb{O}$, but in fact one can establish the stronger condition (compare (42))

$$\mathbb{S}^2 - \tau^2\,\mathbb{I} = \mathbb{O}$$

One can proceed, with familiar consequences, to the construction of projection

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[27] See *"Some applications of an elegant formula due to V. F. Ivanoff"* in COLLECTED SEMINARS (1963–1970).

[28] All the details are spelled out on pp. 321 *et seq* in ELECTRODYNAMICS (1972).

[29] It is interesting that while the 4-dimensional theory supplies two such invariants (and the 2-dimensional theory only one), they enter into $\mathbb{S}$-theory only in combination. Elsewhere in Maxwellian electrodynamics they do, however, make separate appearances. We expect the $2p$-dimensional theory to supply a total of p such invariants.

operators $\mathbb{P}_0 \equiv +\frac{1}{2\tau}\big(\mathbb{S}+\tau\mathbb{I}\big)$ and $\mathbb{P}_1 \equiv -\frac{1}{2\tau}\big(\mathbb{S}-\tau\mathbb{I}\big)$, but I shall not; my main point is, I think, established: fine details with the 2-dimensional theory are simpler, and computation correspondingly easier, but to a remarkable degree *the structural essentials of the 2-dimensional theory accurately anticipate and reflect the structural essentials of Maxwellian electrodynamics*.

6. Conservation laws for the free field. We have discussed at length, from several points of view, the $S^{\mu\nu}$ that enters into this ν-indexed pair of local conservation laws:

$$\partial_\mu S^{\mu\nu} = 0 \quad \text{in the absence of sources}$$

My objective here will be to provide unified discussion of a population of similar statements—each rooted in a symmetry of the of the field equations, and each providing 2-dimensional expression of a property already known to attach to the Maxwellian free field. Our principal analytical tool was provided by Emmy Noether (1918), and I digress now to review the practical essentials of her accomplishment.[30]

Given any field system $\varphi \equiv \{\varphi_\alpha(x) : \alpha = 1, 2, \ldots, N\}$ and an associated Lagrangian density $\mathcal{L}(\varphi, \partial\varphi, x)$, Noether concerns herself with the first-order response $\delta_\omega S_{\mathcal{R}}[\varphi_{\text{dynamical}}]$ of the "dynamical action functional"

$$S_{\mathcal{R}}[\varphi_{\text{dynamical}}] \equiv \tfrac{1}{c}\int_{\mathcal{R}} \mathcal{L}(\varphi, \partial\varphi, x)\, dx^0 dx^1$$

$$\varphi \text{ any } \textit{solution} \text{ of the field equations}$$

$$\mathcal{R} \text{ any "bubble" in (2-dimensional) spacetime}$$

to a $\delta\omega \equiv \{\delta\omega^r : r = 1, 2, \ldots, \rho\}$-parameterized infinitesimial map. Such a map is assumed, in the general case, to entail simultaneous

RELOCATION IN SPACETIME

$$x^\mu \longmapsto X^\mu(x;\delta\omega) = x^\mu + \delta_\omega x^\mu$$

$$\delta_\omega x^\mu = \sum_{r=1}^{\rho} \mathcal{X}_r^{\,\mu}(x)\delta\omega^r$$

COMPOUND ADJUSTMENT OF THE FIELD FUNCTIONS[31]

$$\varphi_\alpha(x) \longmapsto \phi_\alpha(X;\delta\omega) = \varphi_\alpha(x) + \delta_\omega\varphi_\alpha(x)$$

$$\delta_\omega\varphi_\alpha(x) = \sum_{r=1}^{\rho} \varPhi_{\alpha r}(\varphi)\delta\omega^r$$

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[30] For more elaborate discussion and all the missing details, see (for example) Chapter I of my ANALYTICAL DYNAMICS OF FIELDS (1996).

[31] To make clear the burden of the adjective "compound" one writes

$$\begin{aligned}\delta_\omega\varphi_\alpha(x) &= \varphi_{\alpha,\mu}\delta_\omega x^\mu + \big\{\varPhi_{\alpha r} - \varphi_{\alpha,\mu}\mathcal{X}_r^{\,\mu}\big\}\delta\omega^r\\ &= \text{contribution from variation of } \textit{argument}\\ &\qquad + \text{contribution from variation of } \textit{functional form}\end{aligned}$$

and GAUGE ADJUSTMENT OF THE LAGRANGIAN DENSITY

$$\mathcal{L}(\varphi,\partial\varphi,x)\longmapsto\mathcal{L}(\varphi,\partial\varphi,x)+\sum_{r=1}^{\rho}\partial_\mu\Lambda_r^\mu(\varphi,x)\delta\omega^r$$

To particularize such a map one assigns particular meaning to the functions

$$\mathcal{X}_r^\mu(x),\quad \varPhi_{\alpha r}(\varphi)\quad\text{and}\quad \Lambda_r^\mu(\varphi,x)$$

Noether's accomplishment was to show that

$$\delta_\omega S_{\mathcal{R}}[\varphi_{\text{dynamical}}]=0\quad(\text{all }\mathcal{R})\quad\Longleftrightarrow\quad\partial_\mu J_r^\mu=0\quad:\quad r=1,2,\ldots,\rho$$

where

$$\begin{aligned}J_r^\mu&=J_r^\mu(\varphi,\partial\varphi,x)\\&\equiv\frac{\partial\mathcal{L}}{\partial\varphi_{\alpha,\mu}}\Big\{\varPhi_{\alpha r}-\varphi_{\alpha,\sigma}\mathcal{X}_r^\sigma\Big\}+\mathcal{L}\mathcal{X}_r^\mu+\Lambda_r^\mu\end{aligned}\tag{49}$$

So much for general background... in connection with which it is well to take note of several points. Energy, momentum, angular momentum, etc. are eminently useful constructs, even in contexts in which they happen not to be conserved. Similarly, a "Noetherean current" J^μ lays claim to our interest in direct proportion to the strength of our interest in the underlying symmetry—even when it happens *not* to be the case that $\partial_\mu J^\mu=0$. Indeed, in such cases it often proves instructive to ask "How did the anticipated symmetry come to be broken?" It is important to notice also that, while Noether's line of argument may recommend a J^μ to our attention, the associated conservation law holds *by virtue of the equations of motion, of which it is a corollary*.

What do the methods outlined above have to teach us about the dynamics of free fields in 2-dimensional electromagnetism? We look first, for mainly methodological reasons, to the familiar *translation* map, and confront at once this question: Do we imagine ourselves to be discussing a one-field theory, governed by

$$\begin{aligned}\mathcal{L}(\partial A)&\equiv\tfrac{1}{2}g^{\alpha\beta}(\partial_\alpha A)(\partial_\beta A)\\&=\tfrac{1}{2}\big\{(\partial_0A)^2-(\partial_1A)^2\big\}\end{aligned}\tag{50.1}$$

or a three-field theory, governed by

$$\begin{aligned}\mathcal{L}(F,\partial A)&=-\tfrac{1}{2}g^{\alpha\beta}F_\alpha F_\beta-g^{\alpha\beta}F_\alpha(\partial_\beta A)\\&=-\tfrac{1}{2}F_0F_0+\tfrac{1}{2}F_1F_1-F_0(\partial_0A)+F_1(\partial_1A)\end{aligned}\tag{50.2}$$

Let us, for simplicity, select the former option. To speak of an (infinitesimial) "translation in spacetime" is to have then in mind the 2-parameter map[32]

$$\begin{aligned}x^\mu&\longmapsto x^\mu+\delta\omega^\mu\\A&\longmapsto A\\\mathcal{L}(\partial A)&\longmapsto\mathcal{L}(\partial A)\end{aligned}$$

[32] Note that the parameters are now stripped of their generic index $_r$ and assigned identifiers $_\nu$ more natural to the instance; this is typical. Note also that, since there is only one field, the field identifier $_\alpha$ can be omitted.

The associated structure functions are therefore

$$\mathcal{X}^\mu_\nu(x) = \delta^\mu{}_\nu, \quad \varPhi_\nu(A) = 0 \quad \text{and} \quad \Lambda^\mu_\nu(A, x) = 0$$

which upon introduction into (49) give

$$\begin{aligned}
J^\mu{}_\nu &= -\frac{\partial \mathcal{L}}{\partial A_{,\mu}} A_{,\sigma}\delta^\sigma{}_\nu + \mathcal{L}\delta^\mu{}_\nu \\
&= -g^{\mu\beta}A_{,\beta}A_{,\nu} + \tfrac{1}{2}g^{\alpha\beta}A_{,\alpha}A_{,\beta}\delta^\mu{}_\nu \\
&\downarrow \\
J_{\mu\nu} &= -F_\mu F_\nu + \tfrac{1}{2}(F^\alpha F_\alpha)\cdot g_{\mu\nu} \quad \text{by} \quad F_\mu = -\partial_\mu A \\
&= -S_{\mu\nu} \quad \text{as defined at (21)}
\end{aligned}$$

The translation map has recommended $S_{\mu\nu}$ to our attention, but to establish $\partial^\mu S_{\mu\nu} = 0$ we must appeal to the field equations. And indeed, we have

$$\partial_\mu J^\mu{}_\nu = -\Box A \cdot A_{,\nu} - A_{,\beta}\partial^\beta A_{,\nu} + A_{,\beta}\partial^\beta A_{,\nu} = 0 \quad \text{by} \quad \Box A = 0$$

Had we elected to work alternatively from (50.2) we would have had to adjust slightly our conception of the map

$$\begin{aligned}
x^\mu &\longmapsto x^\mu + \delta\omega^\mu \\
A &\longmapsto A \\
F^\mu &\longmapsto F^\mu \\
\mathcal{L}(\partial A) &\longmapsto \mathcal{L}(\partial A)
\end{aligned}$$

but would have been led by a variant of the same argument to an identical result. All of which is old news, but gratifying. Venturing now into a fresh pasture...

Free-field electrodynamics provides no "natural length." We expect the theory therefore to display scale invariance. To describe a finite *dilation* we write

$$\begin{aligned}
x^\mu &\longmapsto e^\omega x^\mu \\
A &\longmapsto e^{k\omega} A \quad : \quad k \text{ an adjustable constant}
\end{aligned}$$

and are led to the structure functions

$$\mathcal{X}^\mu(x) = x^\mu, \quad \varPhi(A) = kA \quad \text{and} \quad \Lambda^\mu(A, x) = 0$$

Noether directs our attention therefore to

$$\begin{aligned}
D^\mu &\equiv \frac{\partial \mathcal{L}}{\partial A_{,\mu}}\big\{kA - A_{,\sigma}x^\sigma\big\} + \mathcal{L}x^\mu \\
&= kA\partial^\mu A - S^\mu{}_\nu x^\nu
\end{aligned} \tag{51}$$

Bringing $\Box A = 0$, $\partial_\mu S^\mu{}_\nu = 0$ and $S^\mu{}_\mu$ to

$$\partial_\mu D^\mu = kA\cdot\Box A + k\underbrace{(\partial^\mu A)(\partial_\mu A)}_{= F^\mu F_\mu = E^2 - B^2 = 0 \text{ for free fields}} - (\partial_\mu S^\mu{}_\nu)\,x^\nu - S^\mu{}_\mu$$

we discover, rather to our surprise, that

$$\partial_\mu D^\mu = 0 \quad \text{irrespective of the value assigned to } k$$

We have come away with two free-field conservation laws for the price of one:

$$\partial_\mu(A\partial^\mu A) = 0 \quad \text{and} \quad \partial_\mu(S^\mu{}_\nu x^\nu) = 0 \tag{52}$$

—both of which are transparently correct. This development is, as I have remarked, "surprising" because the dilational invariance of

$$S_{\mathfrak{R}}[A] = \tfrac{1}{c}\int_{\mathfrak{R}} \tfrac{1}{2}g^{\alpha\beta}(\partial_\alpha A)(\partial_\beta A)\,dx^0 dx^1$$

clearly entails $k = 0$. And because the dilational symmetry of the Maxwellian free field supplies only the latter of the preceding conservation laws.[33]

In 4-dimensional spacetime the most general (proper) Lorentz matrix can be described

$$\mathbb{L} = \exp\begin{pmatrix} 0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_1 & 0 & \vartheta_3 & -\vartheta_2 \\ \psi_2 & -\vartheta_3 & 0 & \vartheta_1 \\ \psi_3 & \vartheta_2 & -\vartheta_1 & 0 \end{pmatrix}$$

and is evidently a 6-parameter object; the ϑ's generate rotations and give rise *via* Noether's theorem to angular momentum, while the ψ's generate "boosts" and give rise to three constructs which—though their relationship to angular momentum is as close as that of **E** to **B**—remain, so far as I am aware, nameless. The situation in 2-dimensional spacetime is much simpler; we have

$$\mathbb{L} = \exp\begin{pmatrix} 0 & \psi \\ \psi & 0 \end{pmatrix}$$

Rotations and angular momentum have been rendered moot; the boost $\mathbb{L}$ contains only a single parameter, and supports only a single conservation law. Since in 2-dimensional electrodynamics the potential A boosts as a scalar field, we have only to write

$$\begin{aligned} x^0 &\longmapsto x^0 + x^1\delta\psi = x^0 + g^{0\alpha}\epsilon_{\alpha\beta}x^\beta\cdot\delta\psi \\ x^1 &\longmapsto x^1 + x^0\delta\psi = x^1 + g^{1\alpha}\epsilon_{\alpha\beta}x^\beta\cdot\delta\psi \\ A &\longmapsto A \end{aligned}$$

[33] See B. F. Plybon, "Observations on the Bessel-Hagen conservations laws for electromagnetic fields," AJP **42**, 998 (1974).

Therefore

$$\mathfrak{X}^\mu(x) = g^{\mu\alpha}\epsilon_{\alpha\beta}x^\beta, \quad \Phi(A) = 0 \quad \text{and} \quad \Lambda^\mu(A,x) = 0$$

and we are led (after a bit of calculation and an overall sign reversal) from (49) to the Noetherean current

$$K^\mu \equiv S^{\mu\alpha}\epsilon_{\alpha\beta}x^\beta \tag{53}$$

Immediately

$$\partial_\mu K^\mu = (\partial_\mu S^{\mu\alpha})\epsilon_{\alpha\beta}x^\beta + S^{\mu\alpha}\epsilon_{\alpha\mu} = 0 \quad \text{by the symmetry of } S^{\mu\nu}$$

Because (53) has such a distinguished pedigree—it is a child of Noether fathered by Lorentz, an expression of the *Lorentz covariance* of our theory—I linger in an effort to clarify (or at least to make more memorably vivid) its meaning. The equation $\partial_\mu K^\mu$ can be written

$$\tfrac{\partial}{\partial t}K + \tfrac{\partial}{\partial x}(\text{associated flux}) = 0$$

where

$$\begin{aligned} K =&\ \ K^0 = S^{00}x - S^{01}ct \\ \text{associated flux} &= cK^1 \end{aligned}$$

But

$$\begin{aligned} S^{00} &= \text{energy density (call it } \mathcal{E} \equiv \mathcal{M}c^2) \\ S^{01} = S^{10} &= c \cdot \text{momentum density (call it } \wp) \end{aligned}$$

so we have

$$K = \mathcal{E}\,x - c^2\wp t = c^2(\mathcal{M}\,x - \wp t) \tag{54}$$

This construction recalls to mind not only the $\boldsymbol{K} = c^2(\mathcal{M}\boldsymbol{x} - \boldsymbol{\wp}t)$ that arises similarly from Maxwellian electrodynamics, but also the construction[34]

$$\boldsymbol{g} \equiv m\boldsymbol{x} - \boldsymbol{p}\,t$$

that by its conservation ($\dot{\boldsymbol{g}} = \boldsymbol{0}$) expresses the Galilean covariance of the free particle system $L = \frac{1}{2}m\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}$. There are, however, some notable distinctions to be made: in the classical mechanics of a non-relativistic particle, $\boldsymbol{g}$ and angular momentum $\boldsymbol{L}$ are transformation-theoretically disjoint and distinct, whereas in relativistic theory the Lorentz transform properties of $\boldsymbol{K}$ and $\boldsymbol{L}$ are intertwined, and in fact mimic those of $\boldsymbol{E}$ and $\boldsymbol{B}$. And in 2-dimensional spacetime K is, as previously remarked, accidentally deprived of the companionship of a sibling.

The 2-dimensional theory supports, as we have seen, only this relatively impoverished analog

$$A \xrightarrow[\text{gauge}]{} A + \text{constant}$$

[34] See CLASSICAL MECHANICS (1983), p. 170.

of the Maxwellian gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \varphi$. From the associated infinitesimal map $A \longmapsto A + \delta\omega$ we are led to write

$$\mathcal{X}^\mu(x) = 0, \quad \Phi(A) = 1 \quad \text{and} \quad \Lambda^\mu(A, x) = 0$$

The associated Noetherean current

$$\frac{\partial \mathcal{L}}{\partial A_{,\mu}} = \partial^\mu A$$

is (by (26), and to within a sign) just the field F^μ itself. That

$$\partial_\mu F^\mu = -\Box A = 0 \quad \text{in the absence of sources}$$

is hardly news, but it is of some interest that the preceding equations can be considered to have been forced by the *gauge invariance* of the free action functional.

The Maxwellian free field is (see again the beginning of §5) actually *conformally* covariant, and the conformal group contains elements additional to the Lorentz transformations and dilations. It contains, in particular, the subgroup of so-called “Möbius transformations,” which come in as many flavors as do the translations. When the dimension of spacetime passes through the value $n = 2$ the conformal group acquires—abruptly and exceptionally—a much more fluid structure than it possesses at other values of n, but within that fluid richness can be found the 2-dimensional shadow of the n-dimensional conformal groups ($n = 3, 4, \ldots$). The structure of that shadow (and of the general situation) is suggested by the following scheme:

$$\begin{pmatrix} 0 & \text{translation} & \text{translation} & \text{dilation} \\ & 0 & \text{boost} & \text{Möbius} \\ & & 0 & \text{Möbius} \\ & & & 0 \end{pmatrix}$$

That said, and in the absence of all supporting detail, I can report that the Möbius map gives rise to this ν-indexed set of currents:

$$M^{\mu\nu} \equiv S^\mu{}_\alpha x^\alpha x^\nu - \tfrac{1}{2} S^{\mu\nu} x^\alpha x_\alpha \tag{55}$$

One can, in any event, verify readily enough that

$$\partial_\mu M^{\mu\nu} = 0$$

In the notations developed in connection with the interpretation of K we have

$$\begin{aligned} M^0 &\equiv \tfrac{1}{c^2} M^{00} \equiv \text{Möbius density of the } 0^{\text{th}} \text{ kind} \\ &= \tfrac{1}{2} \begin{pmatrix} ct \\ x \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} \mathcal{M} & -\wp/c \\ -\wp/c & \mathcal{M} \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \end{aligned} \tag{56.1}$$

$$\begin{aligned} M^1 &\equiv \tfrac{1}{c^2} M^{01} \equiv \text{Möbius density of the } 1^{\text{st}} \text{ kind} \\ &= \tfrac{1}{2} \begin{pmatrix} ct \\ x \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -\wp/c & \mathcal{M} \\ \mathcal{M} & -\wp/c \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \end{aligned} \tag{56.2}$$

but I cannot claim that these (of which we will later have practical need) speak powerfully to my own intuition. It would, I suspect, prove easier to think the matter through in two dimensions than in four, if one had motivation to do so. And it is, in any event, interesting that even in these exotic respects the 2-dimensional theory exactly mimics Maxwellian electrodynamics.

In 1964 D. M. Lipkin[35] surprised the world with his observation that if

$$\left.\begin{array}{l} Z^0 \equiv \mathbf{E}\cdot\text{curl}\,\mathbf{E}+\mathbf{B}\cdot\text{curl}\,\mathbf{B} \\ \boldsymbol{Z} \;\equiv\; \mathbf{E}\times\partial_0\mathbf{E}+\mathbf{B}\times\partial_0\mathbf{B} \end{array}\right\} \tag{57.1}$$

then $\partial_\mu Z^\mu = 0$ in the absence of sources. When he asked himself "How does 'zilch' transform?" Lipkin was led to the discovery of nine yet additional new free-field conservation laws, for a total of ten. It was promptly pointed out by others that Lipkin's conservation laws can be notated

$$\begin{aligned} \partial_\mu V^{\mu\alpha\beta} &= 0 \\ V^{\mu\alpha\beta} &\equiv (\partial^\mu G^\alpha{}_\lambda)F^{\lambda\beta} - (\partial^\mu F^\alpha{}_\lambda)G^{\lambda\beta} \\ &= V^{\mu\beta\alpha} \\ &\downarrow \\ V^{\mu 00} &= Z^\mu = -(\mathbf{E}\cdot\partial^\mu\mathbf{B} - \mathbf{B}\cdot\partial^\mu\mathbf{E}) \end{aligned} \tag{57.2}$$

and that they are in fact merely the simplest instances of an *infinite set* of free-field conservation laws—all of which involved field *derivatives* (of ascending order), and none of which could be understood to derive *via* any straightforward application of Noether's theorem from a symmetry principle. Getting from (57.1) to (57.2) poses an interesting challenge (see the class notes cited above for indication of how it is done), but upon arrival at (57.2) it becomes instantly clear that and how "zilch" and its companions are to be imported into 2-dimensional electrodynamics; introducing

$$Z^\mu \equiv -(E\cdot\partial^\mu B - B\cdot\partial^\mu E)$$

we have

$$\partial_\mu Z^\mu = 0 \quad \text{by } \Box E = \Box B = 0\text{; the other two terms cancel}$$

This observation opens a magic door; we are inspired to introduce

$$Y^\mu \equiv \tfrac{1}{2}(E\cdot\partial^\mu E + B\cdot\partial^\mu B)$$

and to observe that

$$\partial_\mu Y^\mu = 0$$

[35] "Existence of a new conservation law in electromagnetic theory," J. Math. Phys **5**, 696 (1964). For references to the flurry of activity that quickly followed, see pp. 329–332 in CLASSICAL ELECTRODYNAMICS (1980).

by $\Box E = \Box B = 0$ and the “cross cancellations” made possible by the field equations (13)

$$\partial_0 B = \partial_1 E \quad \text{and} \quad \partial_1 B = \partial_0 E \quad \text{in the absence of sources}$$

Differentiation of (37.2)—i.e., of

$$\tfrac{1}{2}(E^2 - B^2) = 0 \quad \text{in the absence of sources}$$

—supplies on the other hand the information that

$$E \cdot \partial^\mu E = B \cdot \partial^\mu B$$

Evidently $Y^\mu = \frac{1}{2}(U^\mu + V^\mu) = U^\mu = V^\mu$, where

$$U^\mu \equiv E \cdot \partial^\mu E \quad \text{and} \quad V^\mu \equiv B \cdot \partial^\mu B$$

are (in the absence of sources) numerically identical and individually conserved:

$$U^\mu = V^\mu \quad \text{and} \quad \partial_\mu U^\mu = \partial_\mu V^\mu = 0$$

Returning in the light of this development to Z^μ, we are surprised to discover that in fact

$$E \cdot \partial^\mu B \quad \text{and} \quad B \cdot \partial^\mu E \quad \text{are } \textit{individually} \text{ conserved}$$

For we have

$$\begin{aligned} \partial_\mu(E \cdot \partial^\mu B) &= \partial_\mu E \cdot \partial^\mu B & &\text{by } \Box B = 0 \\ &= \partial_\mu \partial_0 A \cdot \partial^\mu \partial_1 A \\ &= A_{00}A_{01} - A_{10}A_{11} \\ &= A_{00}\underbrace{(A_{01} - A_{10})}_{0} & &\text{by } \Box A = 0 \\ \partial_\mu(B \cdot \partial^\mu E) &= 0 \quad \text{by an identical argument} \end{aligned}$$

Our theory supplies, therefore, a quartet of zilch-like expressions

$$Z^{\mu\alpha}{}_\beta \equiv F^\alpha \partial^\mu F_\beta \quad \longleftrightarrow \quad \mathbb{Z}^\mu \equiv \begin{pmatrix} +E\partial^\mu E & +E\partial^\mu B \\ -B\partial^\mu E & -B\partial^\mu B \end{pmatrix}$$

and has led us to the conclusions that

$$\partial_\mu \mathbb{Z}^\mu = \mathbb{O} \quad \text{and} \quad \text{tr}\,\mathbb{Z} = 0$$

Pauli matrices spring to mind, but it seems pointless to pursue that remark in the absence of any intended application. Instead, I look to the curious circumstance that comes to light when one undertakes to confirm by direct argument that $\partial_\mu U^\mu = 0$. We come immediately upon the (nonlinear) "eikonal equation"[36]

$$\partial_\mu E \cdot \partial^\mu E = (E_0)^2 - (E_1)^2 = (E_0 + E_1)(E_0 - E_1) = 0$$

which in the 2-dimensional case entails

$$\frac{E_0}{E_1} = \frac{E_1}{E_0} = \pm 1$$

Hitting the eikonal equation with ∂_0 (alternatively with ∂_1) we obtain

$$E_{01} = \frac{E_0}{E_1} E_{00} \quad \text{and} \quad E_{10} = \frac{E_1}{E_0} E_{11}$$

It follows from these remarks that $E_{00} = E_{11}$; in the 2-dimensional case

$$E \text{ satisfies the eikonal equation} \implies E \text{ satisfies the wave equation}$$

and $\partial_\mu U^\mu = 0$ asserts simply/economically that E (in the absence of sources) simultaneously satisfies *both* equations. Similar remarks pertain, of course, to B. To summarize, zilch finds a natural accommodation within the 2-dimensional formalism, and its inclusion is in fact instructive. But knowledge of the conservation of zilch—here as in Maxwellian theory (but here much more easily) —is gained not be appeal to Noether's theorem but "by inspection." Zilch is (with its differentiated fields) structurally unlike the conserved constructs encountered earlier in this discussion, and its conservation has been attributed to no symmetry; it is difficult to imagine structure functions which, upon insertion into (49), would give rise to zilch. Maybe one should look to this circumstance for clues as to how standard Noetherean analysis might usefully be enlarged upon.

Each of the conservation laws discussed above has the form $\partial_\mu J^\mu = 0$. Each provides covariant local espression of a global statement of the type

$$\tfrac{d}{dt}\mathfrak{J} = 0 \quad \text{with} \quad \mathfrak{J} \equiv \int_{-\infty}^{+\infty} J^0\, dx$$

I look to some concatenated global implications of some of the local results now in hand.[37] The stress energy tensor (translational invariance) supplies

[36] For the purposes of this discussion I allow myself to write E_μ for $\partial_\mu E$ and B_μ for $\partial_\mu B$ when no confusion can result.

[37] For parallel Maxwellian remarks, see CLASSICAL ELECTRODYNAMICS (1980), pp. 324–329. My ultimate source has been some unpublished material by Julian Schwinger; in the late 1970's Schwinger prepared the draft of a projected electrodynamics text, which was sent to me for review. The work was highly original in concept—the mature work of a master—and we were deprived of a valuable resource when (not, I hope, as a result of my comments!) Schwinger decided not to pursue the project.

$$
\begin{aligned}
(\text{conserved total energy}) &= \int (\text{energy density})\, dx \\
&\equiv Mc^2 \;:\; \text{defines the “total mass” of the field} \\
(\text{conserved total momentum}) &= \int (\text{momentum density})\, dx \\
&\equiv P
\end{aligned}
$$

From $\partial_\mu K^\mu = 0$ (Lorentz covariance) we have

$$
Pt - \underbrace{\int (\text{mass density})\, x\, dx}_{\equiv\, MX(t)\;:\;\text{defines “center of mass” of the field}} = \text{constant of the motion}
$$

from which it follows that

$$
\tfrac{d}{dt}X = P/M \;:\; \text{the center of mass moves uniformly}
$$

The definition of "center of mass" conflates information supplied by the 0^{th} and 1^{st} moments of the mass (energy) distribution. In the mechanics of distributed systems (rigid bodies) importance attaches also to the 2^{nd} moments (i.e., to the "moment of inertia" tensor). It is to Schwinger that I owe the realization that valuable information is supplied also by the 2^{nd} moment structure of the electromagnetic field. Such information can be obtained from the conservation laws

$$
\begin{aligned}
\partial_\mu M^{\mu\nu} &= 0 \\
M^{\mu\nu} &\equiv S^\mu{}_\alpha x^\alpha x^\nu - \tfrac{1}{2} S^{\mu\nu} x^\alpha x_\alpha
\end{aligned}
$$

which Schwinger was content simply to pluck from his hat, but which we know reflect the *Möbius invariance* of the field equations.[38] Availing ourselves of a notational device borrowed from probability theory

$$
\begin{aligned}
\langle x^0 \rangle &\equiv \tfrac{1}{M}\int \mathcal{M} x^0\, dx = 1 \\
\langle x^1 \rangle &\equiv \tfrac{1}{M}\int \mathcal{M} x^1\, dx = X = X_{\text{initial}} + (P/M)t \\
\langle x^2 \rangle &\equiv \tfrac{1}{M}\int \mathcal{M} x^2\, dx
\end{aligned}
$$

we have interest in the "mass-localization parameter"

$$
\sigma^2 \equiv \big\langle [x - \langle x \rangle]^2 \big\rangle = \big\{ \langle x^2 \rangle - \langle x \rangle^2 \big\} \tag{58}
$$

and notice it to be (by (56.1)) an implication of $\partial_\mu M^{\mu 0} = 0$ that

[38] They derive, that is to say, from the *conformal* symmetry of the theory—from the circumstance that we are concerned with the classical limit of a theory of *massless* vector bosons (photons).

$$\begin{aligned}\int M^0\,dx &= \text{constant of the motion}\\ &= \tfrac{1}{2}\Big\{Mc^2t^2 - 2t\int x\wp\,dx + M\langle x^2\rangle\Big\}\\ &= \tfrac{1}{2}M\langle x^2\rangle_{\text{initial}}\end{aligned}$$

So we have

$$\langle x^2\rangle = \tfrac{2t}{M}\int x\wp\,dx - c^2t^2 + \langle x^2\rangle_{\text{initial}}$$

To get a handle on the $\int x\wp\,dx$-term (which is proportional to the first moment of the *momentum* distribution) Schwinger observes that by trivial implication of energy conservation $\partial_\mu S^{\mu 0} = 0$ one has

$$\begin{aligned}(x^\alpha x_\alpha)\cdot\partial_\mu S^{\mu 0} = \partial_\mu(x^\alpha x_\alpha\cdot S^{\mu 0}) - 2S^{\mu 0}x_\mu &= 0\\ &\downarrow\\ \tfrac{1}{c}\tfrac{\partial}{\partial t}\big[(c^2t^2 - x^2)\mathcal{M}\big] + \tfrac{\partial}{\partial x}\big[(c^2t^2 - x^2)\tfrac{1}{c}\wp\big] - 2\big[\mathcal{M}ct - \tfrac{1}{c}\wp x\big] &= 0\end{aligned}$$

which from our point of view is simply a corollary of $\partial_\mu M^{\mu 0} = 0$. In any event, upon integrating over space (and abandoning the "surface term" that results from $\int_{-\infty}^{+\infty}\frac{\partial}{\partial x}[\text{etc.}]\,dx$) we obtain

$$2\Big\{Mc^2t - \int x\wp\,dx\Big\} = M\tfrac{d}{dt}\big[c^2t^2 - \langle x^2\rangle\big]$$

With Schwinger we observe that {etc.} is in fact, by the following argument, a constant of the motion:

$$\begin{aligned}\tfrac{d}{dt}\{\text{etc.}\} &= Mc^2 - \int x\tfrac{\partial}{\partial t}(\text{momentum density})\,dx\\ &= Mc^2 + \int x\tfrac{\partial}{\partial x}\underbrace{(\text{momentum flux})}_{=\ \text{energy density }\mathcal{M}c^2}\,dx\\ &= Mc^2 - \int \mathcal{M}c^2\,dx \quad \text{after integrating by parts}\\ &= 0\end{aligned}$$

We find ourselves in position now to write

$$\begin{aligned}\int x\wp\,dx &= A + Mc^2t \quad : \quad \text{serves to define the "action" constant } A\\ \langle x^2\rangle &= c^2t^2 + 2(A/M)t + \langle x^2\rangle_{\text{initial}}\end{aligned}$$

whence

$$\begin{aligned}\sigma^2 &= \big[c^2t^2 + 2(A/M)t + \langle x^2\rangle_{\text{initial}}\big] - \big[(P/M)t + \langle x^1\rangle_{\text{initial}}\big]^2\\ &= \underbrace{\big[1 - (P/Mc)^2\big]}_{a}\cdot(ct)^2 + 2\,\underbrace{\big[(A - PX_0)/Mc\big]}_{b}\cdot(ct) + \sigma_0^2\end{aligned}$$

Necessarily $\sigma^2(t) \geq 0$ (all t); a graph of σ^2 *vs.* t has the form of an upturned parabola with imaginary roots, which entails $a \geq 0$ and $0 \leq b^2 \leq a\sigma_0^2$. Following again in (2-dimensional analogs of) the footsteps of Schwinger, we look in particular to implications of the condition $a = 0$; then $(Mc)^2 = P^2$ or again

$$\text{total energy} = c \cdot |(\text{total momentum})| \tag{59}$$

which looks very "photonic." Moreover, $a = 0 \Rightarrow b = 0$, which can be expressed

$$\int \mathcal{E}\, dx \cdot \int x\wp\, dx = \int x\mathcal{E}\, dx \cdot \int \wp\, dx \tag{60}$$

I conclude with discussion of *how—in more directly physical terms—it comes about* that (59) gives rise to the "cross moment condition" (60). We have, on the one hand,

$$\begin{aligned}\text{total energy} &= c \cdot \Big|\int \wp\, dx\Big| \\ &\leq c \cdot \int |\wp|\, dx \ : \ \text{equality entails } \ \wp = |\wp| \ \text{ (all } x)\end{aligned}$$

But on the other hand $(EB)^2 = \big[\frac{E^2+B^2}{2}\big]^2 - \big[\frac{E^2-B^2}{2}\big]^2 \leq \big[\frac{E^2+B^2}{2}\big]^2$ gives

$$\begin{aligned}&\geq \int |EB|\, dx \ : \ \text{equality entails } E^2 = B^2 \text{ (all } x) \\ &= c \cdot \int |\wp|\, dx\end{aligned}$$

Since the inequalities point both ways, we conclude that (59) comes about if and only if it is—at all spacetime points—simultaneously the case that $E^2 = B^2$ *and* that E and B have opposite signs.[39] In short: the "photonic" condition

$$(59) \text{ entails } E = -B \text{ at all spacetime points}$$

But then

$$\text{energy density} = c \cdot (\text{momentum density}) = E^2 \tag{61}$$

from which the "cross moment condition" (60) follows trivially. We note that $\partial_\mu M^{\mu 1} = 0$ remained unexploited in this discussion; it would presumably come into play if we undertook to describe (not the localization of mass/energy but) the "localization of field *momentum*." In the photonic case we would expect to come out at the same place.

7. Dimensional considerations. It is not at all surprising that structural aspects of 2-dimensional electrodynamics are to some extent prefigured by dimensional necessity; that in some respects the theory mimics—but in other respects

[39] This sharpens the condition $B = \pm E$ obtained at (37). In Maxwellian theory the corresponding conclusion reads $E^2 = B^2$ and $\mathbf{E} \perp \mathbf{B}$.

departs markedly from—the pattern established by its Maxwellian model. My intent here is to survey the simple facts of the matter.

For Lagrangian field theories inscribed on n-space one has

$$\begin{aligned} S &= \int L\,dt \quad \text{with} \quad L = \int\!\cdots\!\int \mathcal{L}\,dx^1\cdots dx^n \\ &\downarrow \\ &= \tfrac{1}{c}\int\!\cdots\!\int \mathcal{L}\,dx^0dx^1\cdots dx^n \quad \text{in relativistic cases} \end{aligned}$$

In all cases

$$[S] = \text{action} \tag{62}$$

$$\begin{aligned} [\mathcal{L}] = \text{energy density} &= \text{energy}/(\text{length})^n \\ &\downarrow \\ &= \text{force} \quad : \quad n = 1 \end{aligned} \tag{63}$$

In theories that contemplate the existence of structureless point "charges" which in the static case interact $e \leftrightarrow e$ by a force that falls off "geometrically" one expects to encounter equations of the form

$$F = e^2/r^{n-1} = eE$$

Then

$$\begin{aligned} [e^2] &= (\text{force})(\text{length})^{n-1} = (\text{energy})(\text{length})^{n-2} \\ &\downarrow \qquad\qquad\qquad\quad\;\; = (\text{energy density})(\text{length})^{2n-2} \\ &= \text{force} \quad : \quad n = 1 \quad (\text{no "geometrical fall-off" possible}) \end{aligned} \tag{64}$$

$$\begin{aligned} [E^2] &= \text{force}/(\text{length})^{n-1} = \text{energy}/(\text{length})^n \\ &\downarrow \qquad\qquad\qquad\quad\;\; = \text{energy density} \\ &= \text{force} \quad : \quad n = 1 \end{aligned} \tag{65}$$

In all cases "charge density" and "current density" stand in the relationship

$$[\text{charge density}] = [(\text{current density})/c\,] \tag{66}$$

Moreover

$$\begin{aligned} \text{charge density} &= \sqrt{(\text{energy density})/(\text{length})^2} \\ &= [\partial E/\partial x] \end{aligned} \tag{67}$$

holds in all cases. Upon the introduction of potentials A by equations of the form $E = \partial A$ one obtains in all cases (i.e., for all values of n)

$$\begin{aligned} [A] &= \sqrt{(\text{energy density})(\text{length})^2} \\ &= (\text{energy density})/(\text{charge density}) \end{aligned}$$

"Charged mass points," when introduced into such a relativistic classical setting, carry with them a

$$\text{natural length} = \left[e^2/mc^2\right]^{\frac{1}{n-2}} = \begin{cases} e^2/mc^2 & : \quad n=3 \\ \text{undefined} & : \quad n=2 \\ mc^2/e^2 & : \quad n=1 \end{cases} \tag{68}$$

We do not expect charged particles to be massless, nor do we expect charge to enter in any way into the *free* field equations; it is, therefore, probably frivilous to observe that $[e^2/E^2] = (\text{length})^{2(n-1)}$, and meaningless to point out that such a "natural length"—which can, by the way, be assigned no "natural value"—is (exceptionally) not available to the case $n = 1$. It is, in all events (which is to say: for all values of n), from the *absence of a natural length* that the free field acquires its dilational invariance, and the presence of coupling to massive particles that serves to destroy that symmetry.

Other possibilities become available to the *quantum* theory of such a system. We find that

$$[\alpha \equiv e^2/\hbar c] = (\text{length})^{n-3}$$

is *dimensionless in the physical case* $n = 3$, but in other cases supplies

$$\begin{aligned} \text{natural length} &= \left[e^2/\hbar c\right]^{\frac{1}{n-3}} \\ &\downarrow \\ &= \sqrt{\hbar c/e^2} \quad : \quad n=1 \quad (\text{vanishes as } \hbar\downarrow 0) \end{aligned} \tag{69}$$

In the absence of sources it would, however, be difficult to justify even such a mass-independent allusion to e^2.

We are in position now to observe that $[\mathcal{L}] = [E^2] = [(\partial A)^2]$, and that in the absence of a natural length it is difficult to see how any ∂E-dependence might be built into $\mathcal{L}$; indeed, if we insist that $\mathcal{L}$ be Lorentz-invariant, that it give rise to linear field equations, and that the implied energy density S^{00} be non-negative, then the Lagrangian density familiar from §4 would appear to exhaust the possibilities. Our "2-dimensional electrodynamics" acquires on those grounds a certain claim to uniqueness.

If we were (maybe with $\psi = e^{\frac{i}{\hbar}S}$ in mind) motivated to write e^{kS} we would again find ourselves with no alternative but to set $k \sim 1/\hbar$. And if we were—as later we will be—motivated to write e^{kA} it would follow from previous remarks that we have no alternative but to set $k \sim 1/\sqrt{\hbar c}$.

Looking finally to the dimensionality of the source term J, we know, whether we work from the structure (13) of the field equations or from the structure (31) of the associated Lagrangian density, that

$$\left.\begin{aligned} [J\,] &= \text{charge density} \\ [c\cdot J\,] &= \text{current density} \end{aligned}\right\} \tag{70}$$

so my J-notation is inconsistent with an established convention. Notice finally that

$$[JA] = \text{energy density} \qquad \Longleftrightarrow \qquad [eA] = \text{energy}$$

8. Motion of a charged particle in an ambient field. To describe the relativistic motion of a mass point m we write

$$K^\mu = \tfrac{d}{d\tau}p^\mu \tag{71}$$

$$p^\mu \equiv mu^\mu \quad \text{with} \quad u^\mu \equiv \tfrac{d}{d\tau}x^\mu = \gamma\binom{c}{\boldsymbol{v}} \tag{72}$$

and require of the Minkowski force K^μ that

$$K \perp u \text{ in the Lorentzian sense:} \quad u_\mu K^\mu = 0 \tag{73}$$

In the electrodynamical application we require that K depend upon the local value of the ambient field, but in such a u-dependent way as to achieve $K \perp u$. Maxwellian theory supplies an antisymmetric field tensor $F^{\mu\nu}$, and so permits such an objective to be achieved by literally the simplest of means; one writes[40]

$$K^\mu \equiv (e/c)F^{\mu\nu}u_\nu \tag{74}$$

and finds that the antisymmetry of $F^{\mu\nu}$ does all the work. In the 2-dimensional theory the electomagnetic field is, however, a *vector* field; some other means must be devised to achieve $K \perp u$.

The simplest procedure would appear to be to write

$$\left.\begin{aligned} K^\mu &= eF^\mu_\perp \\ &= e\big\{F^\mu - \tfrac{1}{c^2}(u_\alpha F^\alpha)u^\mu\big\} \\ &= e\big\{g^{\mu\alpha} - \tfrac{1}{c^2}u^\mu u^\alpha\big\}F_\alpha \\ &= e\tfrac{1}{c^2}\big\{g^{\mu\lambda}g^{\alpha\beta} - g^{\mu\alpha}g^{\lambda\beta}\big\}F_\lambda u_\alpha u_\beta \end{aligned}\right\} \tag{75}$$

The force law (75) is, like its Maxwellian counterpart (74), linear in the field variables, but it is quadratic in the components u^μ of relativistic velocity.[41] It works (in the sense that it achieves $K \perp u$) in consequence of

$$g_{\alpha\beta}u^\beta u^\alpha = u_\alpha u^\alpha = c^2$$

In 1912 G. Nordström explored the possibility that (75) might be made the basis of a scalar theory of gravitation,[42] and it was Nordström who (in collaboration with Einstein) taught us how to extract the most characteristic juice from (75).

Recall from (26) that $F_\mu = -\partial_\mu A$. Introducing this information into (75) we have

$$\begin{aligned} K^\mu &= -e\big\{\partial^\mu A - \tfrac{1}{c^2}\underbrace{(u^\alpha\partial_\alpha A)}_{=\frac{d}{d\tau}A}u^\mu\big\} \\ &= -e\Big\{\partial^\mu A - \tfrac{1}{c^2}\big[\tfrac{d}{d\tau}(Au^\mu) - Aa^\mu\big]\Big\} \quad \text{with} \quad a^\mu \equiv \tfrac{d}{d\tau}u^\mu \end{aligned}$$

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[40] For details see CLASSICAL ELECTRODYNAMICS (1980), pp. 267–276.

[41] In RELATIVISTIC DYNAMICS (1967) I describe (p. 20) a population of force laws K^μ that depend upon ascending powers of u^μ, and of which (75) provides the simplest example. Those laws are, however, of no present utility, since they depend also upon tensor fields of ascending rank.

[42] See A. Pais, *Subtle is the Lord* (1982), pp. 232–235.

Familiarly, $a \perp u$. Nordström's idea was to *abandon* the a^μ-dependent term in the preceding equation; i.e., to adopt this *alternative* force law[43]

$$N^\mu \equiv -e\Big\{\partial^\mu A - \tfrac{1}{c^2}\tfrac{d}{d\tau}(Au^\mu)\Big\} \tag{76}$$

The equation of motion then reads $N^\mu = \frac{d}{d\tau}(mu^\mu)$ which by rearrangement becomes[44]

$$\tfrac{d}{d\tau}\big\{(mc^2 - eA)u^\mu\big\} = -ec^2\partial^\mu A \tag{77.1}$$

Nordström attached physical importance to the observation that the preceding equation can be written

$$\begin{array}{c}\tfrac{d}{d\tau}(m^*u^\mu) = -e\partial^\mu A \quad \text{with} \quad m^* \equiv m - (e/c^2)A \\ \updownarrow \\ \tfrac{d}{dt}(m\boldsymbol{v}) = -\boldsymbol{\nabla}U \quad : \quad \text{Newtonian counterpart}\end{array}$$

My own special interest in (77.1) derives from the circumstance that it lends itself so easily to Lagrangian formulation.

To construct such a formulation of the relativistic dynamics of a particle one is tempted simply to write

$$\left\{\frac{d}{d\tau}\frac{\partial}{\partial u^\mu} - \frac{\partial}{\partial x^\mu}\right\}L(x,u) = 0 \tag{78}$$

and to insure covariance by requiring that $L(x,u)$ be Lorentz invariant. To proceed thus is, however, to proceed at risk of violating the condition

$$g_{\alpha\beta}u^\beta u^\alpha = c^2 \tag{79}$$

[43] Starting with $K \perp u$, we obtain N by abandoning a term that is itself normal to u, so clearly $N \perp u$, as required. A similar argument shows N to be necessarily *acceleration-dependent*—precisely because K isn't. Potentially more damaging is the circumstance that by discarding a gauge-sensitive term we have sacrificed the gauge-invariance of K.

[44] Compare this with the unadjusted equation of motion $K^\mu = \frac{d}{d\tau}(mu^\mu)$ which can, by identical manipulations, be brought to the form

$$\tfrac{d}{d\tau}\big\{(mc^2 - eA)u^\mu\big\} = -ec^2\big\{\partial^\mu A + Aa^\mu\big\} \tag{77.2}$$

On pp. 245–259 in ELECTRODYNAMICS (1972) I present a detailed comparative discussion about what (77.1) and (77.2) have to say concerning the "relativistic harmonic oscillator" $eA = \frac{1}{2}kx^2$. The moral of that tale is that it is meaningless to speak of "the" relativistic generalization of any given non-relativistic system; there are a variety of such generalizations—each with things to recommend it. What one needs is a *principle of choice*.

which has no non-relativistic counterpart, but enters into relativistic kinematics as an inviolable *constraint*—a concomitant of the definition of proper time. Several methods have been devised for dealing with this bothersome detail; the simplest, most frequently encountered and least satisfactory is to proceed as outlined, but with fingers crossed. If, for example, we set

$$L(x,u) = \tfrac{1}{2}(m - e\tfrac{1}{c^2}A)\cdot g_{\alpha\beta}u^\beta u^\alpha \tag{79}$$

then (78) gives

$$\tfrac{d}{d\tau}\big\{(m - e\tfrac{1}{c^2}A)u_\mu\big\} + e\cdot \underbrace{\tfrac{1}{c^2}g_{\alpha\beta}u^\beta u^\alpha}_{=\,1\quad :\quad \text{adjustment made ``by hand''}}\cdot\partial^\mu A = 0$$

which is precisely (77.1).[45] The success of the method appears to be entirely "accidental," but since the list of such accidents includes most relativistic systems of practical interest (and since the method is—when it works—so swiftly efficient) most casual authors are content to overlook the fact that the method is defective in principle. Immune from such criticism is a method which uses the "method of Lagrange multipliers" to accommodate (79) as an explicit constraint. One writes

$$\tilde{L}(x,u;\lambda) = L(x,u) + \tfrac{1}{2}\lambda(u^2 - c^2)$$

and obtains (note that $\omega \equiv \frac{d\lambda}{d\tau}$ is in fact absent from $\tilde{L}$)

$$\begin{aligned}\left\{\frac{d}{d\tau}\frac{\partial}{\partial u^\mu} - \frac{\partial}{\partial x^\mu}\right\}\tilde{L}(x,u;\lambda) &= \frac{d}{d\tau}\left[\frac{\partial L}{\partial u^\mu} + \lambda u_\mu\right] - \frac{\partial L}{\partial x^\mu} = 0\\ \left\{\frac{d}{d\tau}\frac{\partial}{\partial \omega} - \frac{\partial}{\partial \lambda}\right\}\tilde{L}(x,u;\lambda) &\sim u^2 - c^2 = 0\end{aligned}$$

The constraint (79) has joined the population of "equations of motion." The remaining equations can, with clever effort,[46] be brought to the form

$$\left.\begin{aligned}\left\{\frac{d}{d\tau}\frac{\partial}{\partial u^\mu} - \frac{\partial}{\partial x^\mu}\right\}L(x,u) + \text{ strange term } &= 0\\ \text{strange term } &\equiv \frac{1}{c^2}\frac{d}{d\tau}\left\{\left(L - u^\alpha\frac{dL}{du^\alpha}\right)u_\mu\right\}\end{aligned}\right\} \tag{80}$$

We note that the "strange term" vanishes if L is homogeneous of unit degree in u, though in applications it is frequently the "strange term" that does much of the work. Suppose, for example, we were to set

$$L(x,u) = mc^2 - eA \tag{81}$$

[45] I have been unable to concoct a Lagrangian that by similar slight of hand —or, indeed, by any method— yields (77.2).

[46] See pp. 30–31 in RELATIVISTIC DYNAMICS (1967). The argument is due to L. Infeld (1957).

Then (80) gives $e\partial_\mu A + \frac{1}{c^2}\frac{d}{d\tau}\big\{(mc^2 - eA)u_\mu\big\} = 0$, which is precisely (77.1). This method stands, as will emerge, in an interesting relationship to what I have come to regard as the "method of choice," but to describe the latter I must back up a bit:

Relativity promotes (but cannot lay exclusive claim to) the view that a mass point, simply by persisting, traces a curve—a "worldline"—in spacetime.[47] To distinguish one point from another we parameterize the curve, writing

$$\{t(\theta), x^1(\theta), \ldots, x^n(\theta)\}$$

In non-relativistic physics it is physically natural to associate θ with time itself; we obtain then the more primitive notion of a "t-parameterized curve in configuration space:

$$\{x^1(t), \ldots, x^n(t)\}$$

In relativistic physics we take advantage of the fact that spacetime has become a metric space to associate θ with arc length s (or equivalently, with proper time $\tau \equiv s/c$):

$$\{x^0(\tau), x^1(\tau), \ldots, x^n(\tau)\}$$

Those associations are, however, entirely conventional; Hamilton's principle, and the associated Lagrange equations—which serve to set the figure of the worldline—are *structurally stable with respect to arbitrary re-parameterizations of the curve*.[48] Conventionally one proceeds from

$$S[\text{path}] = \int L(x, \dot{x}, t)\, dt$$

but upon arbitrary re-parameterization

$$t \xrightarrow[\text{re-parameterization}]{} \theta = \theta(t) \quad : \quad t = t(\theta)$$

one can use $\dot{x} = dx/dt = \frac{dx}{d\theta}\big/\frac{dt}{d\theta} \equiv \mathring{x}/\,\mathring{t}$ to obtain

$$\begin{aligned} S[\text{path}] &= \int \underbrace{L\big(x(\theta), \mathring{x}/\,\mathring{t}, t(\theta)\big)\cdot\,\mathring{t}}\, d\theta \\ &\qquad\qquad \equiv \tilde{L}\big(t(\theta), x(\theta),\, \mathring{t}(\theta), \mathring{x}(\theta)\big) \\ &= \int \tilde{L}\big(x(\theta), \mathring{x}(\theta)\big)\, d\theta \quad \text{in relativistic notation} \\ &\downarrow \end{aligned}$$

$$\left\{\frac{d}{d\theta}\frac{\partial}{\partial \mathring{x}^\mu} - \frac{\partial}{\partial x^\mu}\right\}\tilde{L}(x, \mathring{x}) = 0 \tag{82}$$

[47] A closed string traces similarly a "worldtube," etc.

[48] The point is seldom noted in classical mechanics (but see CLASSICAL MECHANICS (1983), p. 141 and Appendix A: "Rubber clocks"), more commonly encountered in relativistic mechanics, and central to the relativistic quantum mechanics of strings; see §1.3 of M. B. Green, J. H. Schwarz & E. Witten, *Superstring Theory* (1987).

The formalism acquires much of its distinctive flavor from the circumstance that $\tilde{L}(x,\mathring{x})$ is manifestly *homogeneous of unit degree* in the variables $\mathring{x}^\mu$:

$$\tilde{L}(x,\lambda\mathring{x}) = \lambda\tilde{L}(x,\mathring{x})$$

It is, for example, immediately evident that the formalism possesses no Hamiltonian companion, for the requisite Legendre transformation

$$\tilde{L}(x,\mathring{x}) \xrightarrow[\text{Legendre}]{} \tilde{H}(x,p) = p_\mu\mathring{x}^\mu - \tilde{L}(x,\mathring{x})\Big|_{\mathring{x}=\mathring{x}(x,p)}$$

cannot be executed.[49] By Euler's theorem

$$\tilde{L} = \mathring{x}^\mu\frac{\partial\tilde{L}}{\partial\mathring{x}^\mu}$$

From this it follows that

$$\begin{aligned} 0 = \frac{d}{d\theta}\left\{\tilde{L} - \mathring{x}^\mu\frac{\partial\tilde{L}}{\partial\mathring{x}^\mu}\right\} &= \frac{\partial\tilde{L}}{\partial x^\mu}\mathring{x}^\mu + \frac{\partial\tilde{L}}{\partial\mathring{x}^\mu}\,\mathring{\mathring{x}}^\mu - \mathring{\mathring{x}}^\mu\frac{\partial\tilde{L}}{\partial\mathring{x}^\mu} - \mathring{x}^\mu\frac{d}{d\theta}\frac{\partial\tilde{L}}{\partial\mathring{x}^\mu} \\ &= \mathring{x}^\mu\left\{\frac{\partial\tilde{L}}{\partial x^\mu} - \frac{d}{d\theta}\frac{\partial\tilde{L}}{\partial\mathring{x}^\mu}\right\} \end{aligned} \tag{83}$$

from which we conclude that only n of the $n+1$ equations (82) are independent; one is an implication of the others. To illusrate how these ideas work out in practice, we set

$$\tilde{L}(x,\mathring{x}) = (mc - \tfrac{e}{c}A)\sqrt{g_{\alpha\beta}\mathring{x}^\alpha\mathring{x}^\beta} \tag{84}$$

which is

- manifestly Lorentz invariant
- homogeneous of unit degree in the variables $\mathring{x}^\mu$
- dimensionally correct provided we assume $[\theta] = $ time, which we can do without loss of generality.

Working from (82) we find

$$\begin{aligned} \left\{\frac{d}{d\theta}\frac{\partial}{\partial\mathring{x}^\mu} - \frac{\partial}{\partial x^\mu}\right\}\tilde{L} &= \frac{d}{d\theta}\left\{(mc - \tfrac{e}{c}A)\frac{\mathring{x}_\mu}{\sqrt{\mathring{x}\cdot\mathring{x}}}\right\} + \tfrac{e}{c}(\partial_\mu A)\sqrt{\mathring{x}\cdot\mathring{x}} \\ &= \frac{d\tau}{d\theta}\frac{d}{d\tau}\left\{(mc - \tfrac{e}{c}A)\frac{u_\mu}{c}\right\} + \tfrac{e}{c}(\partial_\mu A)\frac{d\tau}{d\theta}c \\ &= \frac{1}{c^2}\frac{d\tau}{d\theta}\left\{\frac{d}{d\tau}(mc^2 - eA)u_\mu + ec^2\partial_\mu A\right\} = 0 \end{aligned}$$

which once again precisely reproduces (77.1). We are inspired by (83) to notice (and gratified to confirm) that

$$\begin{aligned} &u^\mu\left\{\frac{d}{d\tau}(mc^2 - eA)u_\mu + ec^2\partial_\mu A\right\} \\ &\qquad = (mc^2 - eA)\underbrace{(u^\mu a_\mu)}_{0} - e\tfrac{dA}{d\tau}\underbrace{(u^\mu u_\mu)}_{c^2} + ec^2\underbrace{(u^\mu\partial_\mu)}_{\frac{d}{d\tau}}A = 0 \end{aligned}$$

This is valuable information which other methods do not call to our attention.

---

[49] This remark carries with it some insight into why relativistic quantization always entails the deployment of special apparatus.

We are brought thus to the conclusion that (77.1)—of which

$$\begin{aligned}
\tfrac{d}{d\tau}(mu^\mu) &= -e\underbrace{\left\{\partial^\mu A - \tfrac{1}{c^2}\tfrac{d}{d\tau}(Au^\mu)\right\}}_{\text{Minkowski force }\perp\text{ to }u^\mu\text{, as required}}\\
&\Updownarrow\\
\tfrac{d}{d\tau}(m^*u^\mu) &= -e\partial^\mu A\\
&= eF^\mu \qquad (85.1)\\
m^* \equiv m + \Delta m \quad &\text{with} \quad \Delta m \equiv -\tfrac{1}{c^2}eA \qquad (85.2)
\end{aligned}$$

comprise alternative formulations—provides a description of the *relativistic motion of a charged mass point in the presence of an ambient electromagnetic field* which is, in every formal respect,[50] entirely satisfactory; it is manifestly Lorentz covariant, and admits of Lagrangian formulation (whatever that phrase is imagined to mean). The Δm-term is critical to maintenance of the mandatory $K \perp u$ condition, but

$$\Delta m \downarrow 0 \quad \begin{cases}\text{in the weak-field limit } |eA| \ll mc^2\\ \text{in the non-relativistic approximation } c\uparrow\infty\end{cases}$$

Since A is a scalar field it does, by the way, make frame-independent good sense to speak of a "weak field."

It is notable that A enters nakedly (i.e., undifferentiatedly) into (85); the motion of the particle depends upon the *value* of A (as well as upon the values $F_\mu = -\partial_\mu A$ of its derivatives). Evidently the presence of a single charged particle in the 2-dimensional universe serves to promote the A-field to the status of a "physical" field. We will, as we proceed, want to be on the alert for means to clarify this surprising development.

I bring this discussion to a close with some remarks intented to clarify this question: "To what extent is (85.2) a *forced* implication of (85.1)?" Assuming the * on m^* to signify simply that m^* bears some unknown x-dependence, we have

$$\tfrac{dm^*}{d\tau}u^\mu + m^*a^\mu = eF^\mu \tag{86}$$

and by $u \perp a$ are motivated to resolve the vector on the right into components which are respectively parallel and perpendicular to u^μ:

$$F^\mu = \tfrac{1}{c^2}(F_\alpha u^\alpha)u^\mu + \left\{F^\mu - \tfrac{1}{c^2}(F_\alpha u^\alpha)u^\mu\right\}$$

Equation (86) is resolved thus into a *pair* of equations:

$$\tfrac{dm^*}{d\tau} = e\tfrac{1}{c^2}(F_\alpha u^\alpha) \tag{87.1}$$

$$m^*\tfrac{du^\mu}{d\tau} = e\left\{F^\mu - \tfrac{1}{c^2}(F_\alpha u^\alpha)u^\mu\right\} \tag{87.2}$$

[50] We are, since we don't inhabit 2-dimensional spacetime, precluded form saying "in every observational respect"!

If $F_\mu = -\partial_\mu A$ then (87.1) becomes $\frac{dm^*}{d\tau} = -e\frac{1}{c^2}(u^\alpha\partial_\alpha)A = -e\frac{1}{c^2}\frac{dA}{d\tau}$, which integrates promptly to give

$$
\begin{aligned}
m^* &= m_0 - e\tfrac{1}{c^2}A \\
&\qquad m_0 = m + \text{arbitrary constant of integration}
\end{aligned}
$$

By a curiously round-about mechanism A-gauge has re-entered the theory![51]

9. Interactive dynamics of source and field. We have now in hand a theory that describes

- field motion in the presence of a prescribed source J
- particle motion in the presence of a prescribed field A

but are not yet in position to describe what happens when you place matter and field in the same box and let them simply (but self-consistently) "fight it out." The situation is clouded by the circumstance that J, though a scalar field, enters the field equations (13) as a current-like entity, though matter responds "like a charged particle" to ambient fields. The problem now before us is to *clarify the interaction mechanism*.

The field-particle interaction problem tends—in all of its manifestations, and irrespective of however sharply it may speak to our physical intuition—to be formally awkward; particle-particle interactions pose no such difficulty. Nor, for that matter, do field-field interactions. It is in an effort to simplify analysis of the issue before us that we now ask: "How, in 2-dimensional theory, does an electromagnetic field interact with a *distributed* system of charged matter?" In pursuing the question I borrow from arguments standard to our Maxwellian model.[52]

Fluid mechanics concerns itself with two fields: $\rho(t,\boldsymbol{x})$, a scalar field which describes "mass density," and $\boldsymbol{v}(t,\boldsymbol{x})$, a vector field which describes "fluid flow." Their product is susceptible to a double interpretation:

$$\rho\boldsymbol{v} \equiv \text{“mass current”} \equiv \text{“momentum density”}$$

By the first interpretation one has

$$\tfrac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \tag{88.1}$$

which expresses mass conservation. Newton's law gives rise to the equations

$$\tfrac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j - \sigma_{ij}) = f_i \tag{88.2}$$

where f_i described the "impressed force density" and the "stress tensor" σ_{ij} describes the stresses which fluid elements exert upon contiguous fluid elements;

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[51] For related discussion see pp. 245–247 of ELECTRODYNAMICS (1972), which gives also some references.

[52] See, for example, pp. 301–312 of CLASSICAL ELECTRODYNAMICS (1980).

general principles supply the information that $\sigma_{ij} = \sigma_{ji}$, but beyond that it is the special structure assigned to σ_{ij} that serves to distinguish one fluid type from another. "Dust" arises when one sets $\sigma_{ij} = 0$.

Equations (88) can be written

$$\partial_\mu s^{\mu\nu} = k^\nu \tag{89}$$

where the stress tensor has expanded now to become the "stress-energy tensor"

$$\|s^{\mu\nu}\| \equiv \begin{pmatrix} \rho c^2 & \rho\, c\, v_1 & \rho\, c\, v_2 & \rho\, c\, v_3 \\ \rho v_1 c & \rho v_1 v_1 - \sigma_{11} & \rho v_1 v_2 - \sigma_{12} & \rho v_1 v_3 - \sigma_{13} \\ \rho v_2 c & \rho v_2 v_1 - \sigma_{21} & \rho v_2 v_2 - \sigma_{22} & \rho v_2 v_3 - \sigma_{23} \\ \rho v_3 c & \rho v_3 v_1 - \sigma_{31} & \rho v_3 v_2 - \sigma_{32} & \rho v_3 v_3 - \sigma_{33} \end{pmatrix}$$

and where

$$\|\, k^\nu \,\| \equiv \begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Here I have borrowed only my *notation* from relativity, but a "relativistic fluid dynamics" does result if one assumes the preceding equations to hold—as written—in the local rest frame of the fluid, and to have responded tensorially to the Lorentzian boost that brought them to the lab frame.[53] Since

$$\begin{pmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \text{ and } \|\, u^\nu \,\| = \gamma \begin{pmatrix} c \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \rightarrow \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\text{rest frame}} \qquad \text{are } \perp \text{ in the rest frame}$$

they are, by this construction, $\perp$ in all frames: $k^\nu u_\nu = 0$. These remarks carry with them the implication that to describe mass conservation (which should hold even in the presence of forces) we should write not $\partial_\mu s^{\mu 0} = 0$ (which is violated when $k^0 \neq 0$) but $(\partial_\mu s^{\mu\nu})u_\nu = 0$. Equations (89) are reminiscent of the electromagnetic field equations (2.1): $\partial_\mu F^{\mu\nu} = J^\nu$. The resemblance is,

[53] Fluid dynamicists recognize a distinction between the "Eulerian method" (describe fluid motion relative to the lab frame) and the "Lagrangian method" (ride along with a fluid element). Here we have adopted the latter viewpoint as a momentary device. The idea of going to the rest frame to acquire information about the numbers σ_{ij} and f_i is, of course, familiar from electrodynamics, where it is the construction

$$\big[\mathbf{F} = e\mathbf{E}\big]_{\text{rest frame}} \xrightarrow[\text{boost}]{} \big[\mathbf{F} = e(\mathbf{E} + \tfrac{1}{c}\,\mathbf{v}\times\mathbf{B})\big]_{\text{lab frame}}$$

that gives rise to the Lorentz force law.

however, deceptive, for

- while $s^{\mu\nu}$ is symmetric, $F^{\mu\nu}$ is antisymmetric;
- $k^\nu u_\nu = 0$, but J^ν is subject to no such condition;
- the equations $\partial_\mu s^{\mu\nu} = k^\nu$ hold irrespective of the dimension of spacetime, but $\partial_\mu F^{\mu\nu} = J^\nu$ are special to the 4-dimensional world.

To extract from (89) a manifestly covariant account of the relativistic "fluid dynamics of dust" we have now only to set

$$s^{\mu\nu} = \rho u^\mu u^\nu \tag{90}$$

and the point of this fluid dynamical digression comes at last into view: it puts us in position to speak of the stress-energy tensor *of a particle*, and of its relationship to the stress-energy tensor of the electromagnetic field with which the particle interacts. We are led to think of a "particle" as so much "degenerate dust," and to write

$$S^{\mu\nu}_{\text{particle}}(x) \equiv c\int_{-\infty}^{+\infty} m^\star\big(x(\tau)\big)\dot{x}^\mu(\tau)\,\dot{x}^\nu(\tau)\,\delta\big(x - x(\tau)\big)\,d\tau \tag{91}$$

where the expression on the right *vanishes except on the worldline* $x(\tau)$, and has the dimensionality of "energy density" because $[\delta(\cdot)] = 1/(\text{length})^{n+1}$. We now compute

$$\begin{aligned}\partial_\mu S^{\mu\nu}_{\text{particle}}(x) &= c\int_{-\infty}^{+\infty} m^\star\big(x(\tau)\big)\dot{x}^\nu(\tau)\,\dot{x}^\mu(\tau)\,\tfrac{\partial}{\partial x^\mu}\delta\big(x - x(\tau)\big)\,d\tau\\ &= -c\int_{-\infty}^{+\infty} m^\star\big(x(\tau)\big)\dot{x}^\nu(\tau)\,\underbrace{\dot{x}^\mu(\tau)\,\tfrac{\partial}{\partial x^\mu(\tau)}}_{=\frac{d}{d\tau}}\delta\big(x - x(\tau)\big)\,d\tau\end{aligned}$$

which upon integration by parts yields

$$\begin{aligned}&= \underbrace{-c\int_{-\infty}^{+\infty} \tfrac{d}{d\tau}\big[m^\star\big(x(\tau)\big)\dot{x}^\nu(\tau)\delta\big(x - x(\tau)\big)\big]\,d\tau}_{0}\\ &\qquad + c\int_{-\infty}^{+\infty} \delta\big(x - x(\tau)\big)\tfrac{d}{d\tau}\big[m^\star\big(x(\tau)\big)\dot{x}^\nu(\tau)\big]\,d\tau\end{aligned}$$

The argument thus far is a straightforward adaptation of an argument original to Minkowski (1908);[54] it has made no reference to the dimension of spacetime, and has drawn as yet upon no feature of our 2-dimensional theory. But let us now, as instructed by that theory, set

$$m^\star(x) = m - (e/c^2)A(x)$$

---

[54] For parallel discussion of its application to Maxwellian electrodynamics see p. 337 in ELECTODYNAMICS (1972) or §8.5 in J. Anderson's splendid *Principles of Relativity Physics* (1967)

and draw upon (77.1); we obtain

$$\begin{aligned}\partial_\mu S^{\mu\nu}_{\text{particle}}(x) &= \underbrace{ec\int_{-\infty}^{+\infty}\delta\big(x-x(\tau)\big)\,d\tau}_{\equiv\, J(x)\quad\text{by proposed } \textit{definition}}\cdot F^\nu(x) \\ &= +J(x)F^\nu(x)\end{aligned} \tag{92}$$

At (20) we established on the other hand that

$$\partial_\mu S^{\mu\nu}_{\text{field}}(x) = -J(x)F^\nu(x)$$

So we have achieved

$$\partial_\mu S^{\mu\nu}(x) = 0 \quad\text{with}\quad S^{\mu\nu}(x) \equiv S^{\mu\nu}_{\text{field}}(x) + S^{\mu\nu}_{\text{particle}}(x) \tag{93}$$

which provides an elegant account of the energy-momentum balance maintained locally by the interactive field-particle system. The analysis pertains to any instance of a "Nordström theory," and it pertains in particular to our "2-dimensional electrodynamics."

The equations (91) and (92) that serve to define $S^{\mu\nu}_{\text{particle}}$ and J fall at first sight very strangely upon the eye, but are on second thought entirely natural (see the figure); both are decorations of the a construction $\int\delta\big(x-x(\tau)\big)\,d\tau$ that manages to refer simultaneously to the field-theoretic and to the particulate aspects of the matters at hand (and, as relativity requires, to take a wholistic view of the worldline). The following remarks are intended to expose more clearly the meaning and some of the implications of (91) and (92).

The equations (77) that describe the motion of a charged particle can be written

$$\begin{aligned}&\tfrac{d}{d\tau}p^\mu = eF^\mu \\ &\quad p^\mu \equiv m^\star u^\mu \text{ defines the "momentum 2-vector" of the charged particle}\end{aligned}$$

in which notation (91) becomes[55]

$$S^{\mu\nu}_{\text{particle}}(x^0,x^1) \equiv c\int_{-\infty}^{+\infty} p^\mu(\tau)\,u^\nu(\tau)\,\delta\big(x-x(\tau)\big)\,d\tau$$

We (in our intertial frame) write

$$\begin{aligned}P^\mu(x^0) &= \tfrac{1}{c}\int S^{\mu 0}_{\text{particle}}(x^0,x^1)\,dx^1 \\ &= \int\int_{-\infty}^{+\infty} p^\mu(\tau)\,u^0(\tau)\,\delta\big(x^0-x^0(\tau)\big)\delta\big(x^1-x^1(\tau)\big)\,d\tau dx^1\end{aligned}$$

[55] I allow myself to write $p^\mu(\tau)$ where, owing to the x-dependence which $m^\star$ acquires from $A(x)$, I should more properly write $p^\mu\big(\tau, x(\tau)\big)$.

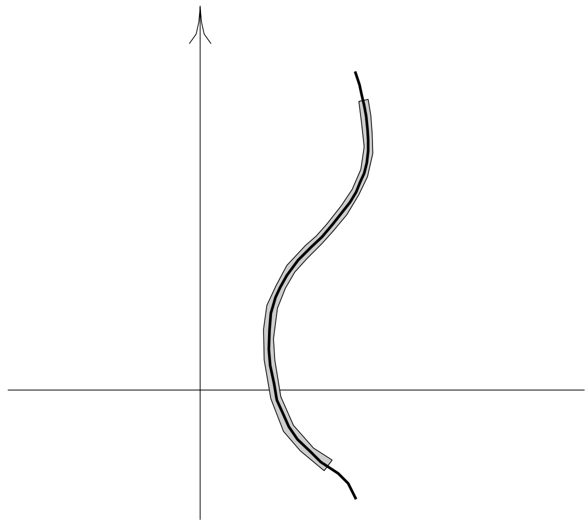

FIGURE 1: *Representation of the construction fundamental to the definitions (91) and (92); i.e., of a field which vanishes except in the immediate vicinity of a worldline $x(\tau)$.*

to describe (at time x^0) the components of "total momentum of the particle, thought of as a field." But a well-known property of the δ-function supplies

$$\delta\big(x^0 - x^0(\tau)\big) = \frac{1}{u^0(\tau_0)}\delta(\tau - \tau_0) \tag{94}$$

where τ_0, defined by $x^0 - x^0(\tau_0) = 0$, is the proper time at which the worldline $x(\tau)$ *punctures the timeslice* to which we have assigned the name x^0. So we have

$$P^\mu(x^0) = \int p^\mu(\tau_0)\delta\big(x^1 - x^1(\tau_0)\big)\,dx^1 = p^\mu(\tau_0) \tag{95}$$

This is, from several points of view, a gratifying result; it illustrates how field theory and particle theory manage, in their respective ways, to say the same thing. And it exposes in explicit detail how it comes about that

$$P^\mu \equiv \int_{\text{timeslice}} S^{\mu 0} d(\text{volume})$$

transforms vectorially, even though $S^{\mu\nu}$ transforms tensorially.[56]

[56] It makes therefore no more sense to speak of the "transformation properties of the isolated components $S^{\mu 0}$" than it does to speak of those of **E** without reference to **B**. The preceding equation loses all of its mystery, by the way, when it is properly notated:

$$P^\mu \equiv \int_{\text{timeslice}} S^{\mu\alpha}\, d\sigma_\alpha$$

$d\sigma_\alpha$ has the meaning supplied by the exterior calculus

Which brings us to J, the sharpened interpretation of which[57] has been our motivating objective throughout this entire discussion. In Maxwellian electrodynamics the analog of (92) reads $J^\nu = ec\int \dot{x}^\nu(\tau)\delta\big(x - x(\tau)\big)\,d\tau$, and a variant of the argument that gave us (95) gives[58]

$$J^\nu(x^0, \boldsymbol{x}) = e\delta\big(\boldsymbol{x} - \boldsymbol{x}(\tau)\big)\begin{pmatrix} c \\ \boldsymbol{v}(\tau)\end{pmatrix}_{\tau=\tau_0}$$

In 2-dimensional electrodynamics, we bring (94) to (92) and obtain

$$\begin{aligned} J(x^0, x^1) &= ec\frac{1}{u^0(\tau)}\delta\big(x^1 - x^1(\tau)\big)\Big|_{\tau=\tau_0} \\ &= \tfrac{1}{\gamma}\cdot e\cdot\delta\big(x^1 - x^1(\tau)\big)\Big|_{\tau=\tau_0} \end{aligned} \tag{96}$$

which has, as was anticipated already at (70), the physical dimension of a “charge density,” and where the leading $\frac{1}{\gamma}$ is an artifact of our having *sectioned* the invariant construction (92). At (85.1) we obtained $\gamma\frac{d}{dt}(m^\star u^\mu) = eF^\mu$ which by (96) becomes

$$\tfrac{d}{dt}(m^\star u^\mu) = \int_{\text{timeslice}} JF^\mu\,dx \tag{97}$$

Finally—drawing inspiration from (31) and (84)—we form

$$\mathcal{L}(x, \mathring{x}; A, \partial A) = \mathcal{L}_{\text{particle}}(x, \mathring{x}) + \mathcal{L}_{\text{field}}(F, \partial A) + \mathcal{L}_{\text{interaction}}(x, A)$$

with

$$\left.\begin{aligned} \mathcal{L}_{\text{particle}}(x, \mathring{x}) &\equiv -mc\sqrt{g_{\alpha\beta}\mathring{x}^\alpha\mathring{x}^\beta} \\ \mathcal{L}_{\text{field}}(F, \partial A) &\equiv -\tfrac{1}{2}g^{\alpha\beta}F_\alpha F_\beta - g^{\alpha\beta}F_\alpha(\partial_\beta A) \\ \mathcal{L}_{\text{interaction}}(x, A) &\equiv e\int\sqrt{g_{\alpha\beta}\mathring{x}^\alpha\mathring{x}^\beta}\,\delta\big(y - x(\theta)\big)\,d\theta\cdot A \end{aligned}\right\} \tag{98}$$

where, since the variable x has been preempted by the particle, I take the field variables to be $y \equiv \{y^0, y^1\}$. The equations of motion of the interactive system are

$$\begin{aligned} &\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A)} - \frac{\partial\mathcal{L}}{\partial A} = -\partial_\mu F^\mu - J(y) = 0 \\ &\qquad\qquad J(y) \equiv e\int\sqrt{g_{\alpha\beta}\mathring{x}^\alpha\mathring{x}^\beta}\,\delta\big(y - x(\theta)\big)\,d\theta \\ &\qquad\qquad\quad\downarrow \\ &\qquad\qquad\quad = ec\int\delta\big(y - x(\tau)\big)\,d\tau \\ &\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu F_\nu)} - \frac{\partial\mathcal{L}}{\partial F_\nu} = F_\mu + \partial_\mu A = 0 \end{aligned}$$

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[57] See again the paragraph which which I introduced this section.

[58] See p. 308 in the class notes cited in footnote 54. Notice that J^ν has the physical dimension $[c\cdot(\text{charge density})]$ of (see again (66)) a “current;” to achieve consistency with my present eccentric conventions one should omit the c-factor that was build into the definition of J^ν.

$$\left\{\frac{d}{d\theta}\frac{\partial}{\partial \mathring{x}^\mu} - \frac{\partial}{\partial x^\mu}\right\}L = \frac{d\tau}{d\theta}\left\{\frac{d}{d\tau}(-m + \tfrac{e}{c}A)u_\mu - ec\partial_\mu A\right\} = 0$$

which precisely reproduce the field equation (30.1), the equation (30.2) that serves to introduce the potential A, the definition (92) of J and the equation (85) that describes the field-driven motion of a solitary charged particle. The derivation of the interactive part of the last of the preceding equations involves integration by parts and other standard trickery which I have omitted.

It is a striking fact that in no respect is the preceding material special to 2-dimensional spacetime. It becomes immediately evident upon perusal of (for example) Anderson's §8-8[59] that we have done no more than to put fresh wine into a very old bottle; we have shown that the theory invented by Nordström to provide a (special) relativistic account of gravitation can, in the 2-dimensional case, be interpreted to provide a natural analog of Maxwellian electrodynamics. That our 2-dimensional theory has, with regard to so many of its details, such a strikingly "Maxwellian" look and feel about it can be attributed now to the circumstance that those qualities attach to *all* instances of Nordström's theory—whatever, the dimension, and whatever may be the intended physical interpretation.

So what is the answer? Does J more nearly resemble ρ or $\frac{1}{c}\mathbf{j}$? The question—which would not have arisen but for the "electromagnetic" interpretation we have attached to a 2-dimensional Nordström theory, and the answer to which pertains necessarily to *all* such theories—made seeming sense when we noticed that (13.1) more nearly resembles Ampere's Law (1.2) than Gauss' Law (1.1). But it can make no transformation-theoretic sense to ask "Does a scalar J more nearly resemble the J^0-component or the J^k-components ($k = 1, 2, 3$) of a 4-vector?" In reference to the Lagrangian formalism, we found that

$$J^\mu A_\mu \quad \text{becomes} \quad JA$$

which has "minimal coupling" structure in both cases; J lost its index because A did. But to ask "How did A come to be a scalar?" is to ask "How did the theory come to be constructed the way it is? Why is it the case that in $2p$-dimensional spacetime $\mathbf{F}$ is a p-form while $\mathbf{A}$ and $\mathbf{J}$ are 1-forms?" The only possible answer is "Because that is the (attractive) hypothesis we have chosen to explore." The absence of an $\mathring{x}$-term from (92) seemed at the time to speak of a "ρ-like" interpretation, but we found subsequent to (98) that *covert* $\mathring{x}$-dependence had actually to be considered present in (92): $c \longleftarrow \sqrt{g^{\alpha\beta}\mathring{x}_\alpha\mathring{x}_\beta}$.[60] The short answer to our question, therefore, is "both. . . and neither."

10. Charged dust. In the preceding section I looked to fluid dynamics in order to obtain a relativistic theory of "dust," from which I could export a conception

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[59] See again footnote 54 for the detailed citation.

[60] It is interesting that in relativity a particle can, in this sense, acquire significant "velocity-dependence" *by just sitting there*!

(91) of the "stress energy tensor of a relativistic particle" and, relatedly, of the "source field" $J(x)$ that can by (92) be associated with such a particle when it carries charge. Anyone who considers the definitions (91) and (92) to be self-evident—or to be, in any event, sufficiently justified by their demonstrated success—is, of course, free to disclaim any interest either in fluids or in dust. Here I take the opposite point of view, and do so for a reason more formal than physical. The Lagrangian (98) acquired its bizarrely hybrid design from the circumstance that it refers to an interactive system of the type

$$\text{field} \Longleftrightarrow \text{particle}$$

It is to achieve symmetry of design that I undertake to replace the solitary particle with a distributed population of semi-autonomous particles; the resulting system

$$\text{field} \Longleftrightarrow \text{field of particles (dust)}$$

might have held interest for Nordström as having to do with "gravitation in a dust-filled universe." Though I will phrase my remarks as though I had physical interest in the (2-dimensional) "electrodynamics of a charged dust cloud," my interest in the problem is in fact (and as I have already indicated) entirely formal, methodological.

Since fluids are, in general, dissipative, one cannot expect to be able to devise a "Lagrangian formulation of fluid dynamics."[61] But dust is a fluid of such elemental simplicity as to suggest that Lagrangian methods may in fact be applicable. I will carry this discussion as far as I can, but readers may not wish to follow in my steps, for my goal continues to elude me, and the literature known to me[62] provides little assistance.

The field variables are taken to be mass density $\rho(x)$, charge/current density $J(x)$ and $u^\mu(x)$. We assume

$$u_\mu(x)u^\mu(x) = c^2 \quad : \quad \text{all x} \tag{99}$$

[61] The situation is not without its curious features; it was fluid dynamics that introduced us (meaning Stokes, Maxwell and their contemporaries) to the stress-energy tensor, but in fluid dynamics that object—which is fundamental to Lagrangian field theory, and springs there from Noether's theorem—is deprived of any kind of "Noetherian basis." It is supplied instead by Newton's laws of motion. And it enters, like the p in $\frac{d}{dt}p = F$, into what is primarily an equation of motion, and only exceptionally/incidentally into the expression of a conservation law.

[62] An accessibly detailed account of this class of problems can be found in Chapter 9: "Relativistic continuum mechanics" of James Anderson's *Principles of Relativity Physics* (1967), but Anderson abandons Lagrangian methods just at the point where I need them. Also helpful (in an eccentric way) are §§30–33 (also §§78–79) of V. Fock's *The Theory of Space, Time and Gravitation* (1959), but Fock—who claims in his introduction to have learned his physics "under the influence of Lenin's 'Materialism and Empirocriticism' "—appears to be philosophically antagonistic to be the Lagrangian method.

and are prepared to assume the x-independence of

$$J(x)/\rho(x) = \text{universal charge/mass ratio: call it } e/m \tag{100}$$

We expect to have

$$\partial_\mu(\rho u^\mu) = 0 \quad : \quad \text{mass conservation} \tag{101}$$

We expect the role of $\frac{d}{d\tau}$ to be taken over by the "substantial derivative" $u^\alpha\partial_\alpha$, and therefore to have

$$u^\alpha\partial_\alpha(\rho u^\mu) = K^\mu \tag{102}$$

Then $K^\mu u_\mu = u^\alpha\partial_\alpha(\rho u^\mu u_\mu) - \rho u^\alpha u^\mu \partial_\alpha u_\mu = c^2 u^\alpha\partial_\alpha\rho - \frac{1}{2}\rho u^\alpha\partial_\alpha c^2$ entails $u^\alpha\partial_\alpha\rho = \frac{1}{c^2}(K^\alpha u_\alpha)$; returning with this information to the equation of motion from which it sprang, we have

$$\begin{aligned} \underbrace{\rho u^\alpha\partial_\alpha u^\mu}_{} &= K^\mu - u^\mu\cdot u^\alpha\partial_\alpha\rho = K^\mu - \tfrac{1}{c^2}(K^\alpha u_\alpha)u^\mu = K^\mu_\perp \\ &= \partial_\alpha(\rho u^\alpha u^\mu) \quad \text{by mass conservation} \end{aligned} \tag{103}$$

In 2-dimensional electrodynamics we are motivated to set

$$K^\mu = JF^\mu = \tfrac{e}{m}\rho F^\mu = -\tfrac{e}{m}\rho\,\partial^\mu A \tag{104}$$

Then

$$\begin{aligned} u^\alpha\partial_\alpha\rho = \tfrac{1}{c^2}(K^\alpha u_\alpha) \quad \text{becomes} \quad (u^\alpha\partial_\alpha)\rho &= -\tfrac{e}{mc^2}\rho\,(u^\alpha\partial_\alpha)A \\ &\downarrow \\ (u^\alpha\partial_\alpha)\log\rho &= -(u^\alpha\partial_\alpha)\underbrace{(eA/mc^2)}_{\text{dimensionless}} \end{aligned} \tag{105}$$

so *along a flowline* we expect to have something like[63]

$$\rho(\theta) = \rho(0)\exp\Big\{-\tfrac{e}{mc^2}\int_0^\theta A(\vartheta)\,d\vartheta\Big\} \tag{106}$$

It follows also from (105) that

$$\begin{aligned} K^\mu_\parallel = \tfrac{1}{c^2}(K^\alpha u_\alpha)u^\mu &= -\tfrac{e}{mc^2}u^\mu\rho\,(u^\alpha\partial_\alpha)A \\ &= -\rho\,u^\alpha\partial_\alpha\big(\tfrac{e}{mc^2}u^\mu A\big) \qquad \text{by } (u^\alpha\partial_\alpha)u^\mu = 0 \\ &= -\partial_\alpha\big(\tfrac{e}{mc^2}\rho A\cdot u^\alpha u^\mu\big) \quad \text{by mass conservation} \end{aligned}$$

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[63] For the occurrence of a similar formula in what appears, at least superficially, to be quite a different physical setting, see p. 463 in CLASSICAL MECHANICS (1983)

Returning with this information to (103)—which upon installation of (104) reads

$$\partial_\alpha(\rho u^\alpha u^\mu) = JF^\mu_\perp \tag{107}$$

—we obtain

$$\partial_\alpha(\rho^\star u^\alpha u^\mu) = JF^\mu \tag{108}$$

$$\rho^\star \equiv \rho - \tfrac{e}{mc^2}\rho A = \rho - \tfrac{1}{c^2}JA \tag{109}$$

Writing

$$S^{\mu\nu}_{\text{charged dust}}(x) \equiv \rho^\star(x)u^\mu(x)u^\nu(x) \tag{110}$$

we have, according to (108),

$$\partial_\mu S^{\mu\nu}_{\text{charged dust}} = JF^\nu \tag{111}$$

By specialization (i.e., by considering charged dust that consists of but a single charged particle) one expects to recover (91) $\longleftarrow$ (110) and (92) $\longleftarrow$ (111), but I will not linger to write out the explicit demonstration.

Nor will I linger to record the fruit of my efforts—thus far unsuccessful as they have been—to obtain the preceding equations from a Lagrangian. It remains unclear to me what to take as my “field variables,” how to resolve equations into “equations of motion” and “conservation laws implied by the equations of motion,” how to manage the constraint (99). The source of my difficulty seems to reside in the circumstance that we are dealing here (as in fluid dynamics generally) with a system of *first order* partial differential equations. I cling, nevertheless, to the conviction that the answer—once it is in hand—will be so simple as (in retrospect) to seem obvious.

11. Is the theory an instance of a gauge theory? Maxwellian electrodynamics managed by intentional design to provide a unified account of the great variety of electrical and magnetic phenomena which were known already before 1850, and gave rise unbidden to an electromagnetic theory of light. Maxwell’s equations—what better evidence that they speak of real stuff?—promptly spawned several life-transforming industries (electrical power generation and distribution, telegraphy, radio). But they created a theoretical problem where none had been before. I allude to the “æther problem,” the resolution of which (invention of special relativity) was found to reside not (as was initially supposed) in specific phenomenological implications of the equations, but in their *formal structure*. Thus did we acquire acquaintance with one of the great symmetry principles of the world. But the tutorial role of electrodynamics did not come then to an end; Maxwell’s creation informed and inspired those who successively invented quantum mechanics, general relativity (gravitational field theory), quantum electrodynamics and the quantum field theory that supports modern theories of elementary particles. At about the same time as I first opened my eyes to the light of day (which is to say: not all that recently) a few theorists gained the first dim perception that electrodynamics had something entirely novel yet to teach us about the symmetry structure of the world, and by the time I had become a graduate student that perception had acquired a sharp cutting edge; "gauge field theory" had come into being.[64] Electrodynamics enjoys a relationship to "physical reality" which is today no less urgently immediate than ever it was, and the gauge theorist intends no denial of that fact when (in a somewhat informal moment) he allows himself to assert that "the electromagnetic field was called into being in order to impart enhanced internal symmetry to field systems that, on their face, have nothing to do with the physics of electrical charge." The question I propose now to explore is this: "Is a physicist who resides in 2-dimensional spacetime likely to say the same thing?" I begin with a review of the simplest elements of gauge field theory.[65]

Let complex scalar fields $\psi(x)$ and $\psi^*(x)$ be required to satisfy the Klein-Gordon equations

$$\Box\psi + \kappa^2\psi = 0 \quad \text{and complex conjugate} \tag{112}$$

where $[\kappa] = 1/\text{length}$.[66] The associated Lagrange density can be written

$$\mathcal{L}(\boldsymbol{\psi}, \partial\boldsymbol{\psi}) = g^{\alpha\beta}\psi^*_\beta\psi_\alpha - \kappa^2\psi^*\psi \tag{113}$$
$$\psi_\alpha \equiv \partial_\alpha\psi$$

and gives rise to the stress-energy tensor

$$S_{\mu\nu} = \{\psi^*_\mu\psi_\nu + \psi_\mu\psi^*_\nu\} - \mathcal{L}g_{\mu\nu} \tag{114}$$

in connection with which we have

$$\begin{aligned}\partial^\mu S_{\mu\nu} &= \{\psi_\nu\Box\psi^* + \psi^*_\nu\Box\psi + (\psi^*_\mu\psi^\mu)_\nu\} - (\psi^*_\alpha\psi^\alpha)_\nu + \psi_\nu\kappa^2\psi^* + \psi^*_\nu\kappa^2\psi \\ &= \psi_\nu\underbrace{(\Box\psi^* + \kappa^2\psi^*)}_{0} + \psi^*_\nu\underbrace{(\Box\psi + \kappa^2\psi)}_{0} = 0\end{aligned}$$

We note in passing that $S_{\mu\nu}$ is in all cases symmetric, and that

$$S^\alpha{}_\alpha = (2-\text{dim})\psi^*_\alpha\psi^\alpha + (\text{dim})\kappa^2\psi^*\psi$$
$$\text{dim} \equiv \text{dimension of spacetime}$$

[64] For an excellent account of the history of this subject, see L. O'Raifeartaigh, *The Dawning of Gauge Theory* (1997).

[65] For a more fulsome introduction to the subject see Chapter 11 in David Griffiths' *Introduction to Elementary Particles* (1987).

[66] In relativistic quantum theory it is natural to set

$$\kappa = mc/\hbar = 1/\text{Compton length}$$

but we have no actual need at present to be so specific. I will, however, honor quantum mechanical convention by calling κ the "mass parameter." To reduce extraneous notational clutter I assign to ψ the (quantum mechanically unnatural) dimension $[\psi] = \sqrt{(\text{energy density})\cdot(\text{length})^2}$.

is in all cases Lorentz invariant, but only very exceptionally (only, that is to say, when spacetime is 2-dimensional and $\kappa = 0$) is the stress-energy tensor traceless.

The system just described will for us play the role of a "field system that, on its face, has nothing to do with the physics of electrical charge..." It was selected because it exhibits an elementary instance of an "internal symmetry;" the action functional $S = \int \mathcal{L}$ is transparently invariant under global phase adjustments

$$\left.\begin{aligned} \psi \ &\longmapsto e^{+i\omega}\cdot\psi \\ \psi^* &\longmapsto e^{-i\omega}\cdot\psi^* \end{aligned}\right\} \tag{115}$$

By Noetherean analysis we are led to the associated conservation law

$$\begin{aligned} \partial_\mu Q^\mu &= 0 \\ Q_\mu &\equiv i\big(\psi^*\psi_\mu - \psi\,\psi^*_\mu\big) \end{aligned} \tag{116}$$

The objective of our gauge theorist is to *admit the possibility that ω may vary (smoothly) from point to point* in spacetime, and thus to achieve "enhancement" in this sense:

$$\text{global} \xrightarrow[\text{objective of gauge field theory}]{\qquad\qquad\qquad} \text{local}$$

But if ω is x-dependent then

$$\begin{aligned} \psi \ &\longmapsto \Psi \ = e^{+i\omega}\cdot\psi \\ \psi^* &\longmapsto \Psi^* = e^{-i\omega}\cdot\psi^* \end{aligned}$$

induces

$$\begin{aligned} \partial_\mu\psi \ &\longmapsto \partial_\mu\Psi \ = e^{+i\omega}\cdot\big(\partial_\mu + i\omega_\mu\big)\psi \\ \partial_\mu\psi^* &\longmapsto \partial_\mu\Psi^* = e^{-i\omega}\cdot\big(\partial_\mu - i\omega_\mu\big)\psi^* \end{aligned}$$

To escape the force of this difficulty, the gauge theorist makes the replacement

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu - ig\cdot a_\mu \tag{117}$$

where a_μ are the components of a vector field (the "gauge field," sometimes called the "compensating field;" g will acquire the interpretation of a coupling constant) and—in order to achieve

$$\begin{aligned} \big(\partial_\mu - ig\cdot a_\mu\big)\psi \ &\longmapsto \big(\partial_\mu - ig\cdot A_\mu\big)\Psi \ = e^{+i\omega}\cdot\big(\partial_\mu - ig\cdot A_\mu + i\omega_\mu\big)\psi \\ &\qquad\qquad\qquad\qquad\quad = e^{+i\omega}\cdot\big(\partial_\mu - ig\cdot a_\mu\big)\psi \\ \big(\partial_\mu + ig\cdot a_\mu\big)\psi^* &\longmapsto \big(\partial_\mu + ig\cdot A_\mu\big)\Psi^* = e^{-i\omega}\cdot\big(\partial_\mu + ig\cdot A_\mu - i\omega_\mu\big)\psi \\ &\qquad\qquad\qquad\qquad\quad = e^{-i\omega}\cdot\big(\partial_\mu + ig\cdot a_\mu\big)\psi \end{aligned}$$

—stipulates that the meaning of the phase transformation will at this point be expanded to include *additive* participation by the gauge field

$$a_\mu \longmapsto A_\mu = a_\mu + \tfrac{1}{g}\cdot\partial_\mu\omega$$

The gauge theorist now abandons the original system, and looks instead to its sibling

$$\begin{aligned}
\mathcal{L}'(\boldsymbol{\psi},\partial\boldsymbol{\psi};a) &\equiv g^{\alpha\beta}(D_\beta\psi^*)(D_\alpha\psi)-\kappa^2\psi^*\psi\\
&= g^{\alpha\beta}(\psi^*_\beta + ig\cdot a_\beta\psi^*)(\psi_\alpha - ig\cdot a_\alpha\psi)-\kappa^2\psi^*\psi\\
&= \mathcal{L}(\boldsymbol{\psi},\partial\boldsymbol{\psi}) + \big\{g\cdot\underbrace{i(\psi^*\psi_\alpha-\psi\,\psi^*_\alpha)}_{=\,Q_\alpha}a^\alpha + g^2(\psi^*\psi)(a_\alpha a^\alpha)\big\}\\
&= \mathcal{L}_{\psi\text{ field}}(\boldsymbol{\psi},\partial\boldsymbol{\psi}) + \mathcal{L}_{\text{interaction}}(\boldsymbol{\psi},\partial\boldsymbol{\psi};a)
\end{aligned}$$

Next he notices that

$$\begin{aligned}
\partial_\mu a_\nu \longmapsto \partial_\mu A_\nu &= \partial_\mu a_\nu + \partial_\mu\partial_\nu\omega\\
&\Downarrow\\
f_{\mu\nu} &\equiv \partial_\mu a_\nu - \partial_\nu a_\mu \quad\text{is gauge invariant: } f_{\mu\nu}=F_{\mu\nu}
\end{aligned}$$

and that the trace-like construction $f^{\alpha\beta}f_{\beta\alpha}$ is also Lorentz invariant. With the intention of launching the gauge field into motion (and with, as will emerge, the effect of "calling electrodynamics into being") our gauge theorist finally forms

$$\begin{aligned}
\mathcal{L}''(\boldsymbol{\psi},\partial\boldsymbol{\psi};a,\partial a) \equiv\ & \mathcal{L}'(\boldsymbol{\psi},\partial\boldsymbol{\psi};a) + \mathcal{L}_{\text{gauge field}}(\,\cdot\,,\partial a)\\
& \qquad\qquad \mathcal{L}_{\text{gauge field}}(\,\cdot\,,\partial a) \equiv \tfrac{1}{4}f^{\alpha\beta}f_{\alpha\beta}\\
=\ & \mathcal{L}_{\psi\text{ field}}(\boldsymbol{\psi},\partial\boldsymbol{\psi}) + \mathcal{L}_{\text{gauge field}}(\,\cdot\,,\partial a) + \mathcal{L}_{\text{interaction}}(\boldsymbol{\psi},\partial\boldsymbol{\psi};a)
\end{aligned}$$

and obtains the following equations of motion:

$$(\partial_\mu - iga_\mu)(\partial^\mu - iga^\mu)\,\psi + \kappa^2\psi = 0 \quad\text{and complex conjugate} \tag{118}$$

$$\partial_\mu f^{\mu\nu} = j^\nu \tag{119.1}$$

$$\begin{aligned}
j^\nu &\equiv g\cdot\big\{Q^\nu + 2g\psi^* a^\nu\psi\big\} \qquad (119.2)\\
&= g\cdot\big\{i\psi^*(\partial^\nu - iga^\nu)\psi + \text{conjugate}\big\}
\end{aligned}$$

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu \tag{119.3}$$

This coupled system is, by explicit design and intent, invariant under gauge transformations of the composite form

$$\left.\begin{aligned}
\psi\ &\longmapsto e^{+i\omega(x)}\cdot\psi\\
\psi^*\ &\longmapsto e^{-i\omega(x)}\cdot\psi^*\\
a_\mu\ &\longmapsto a_\mu + \partial_\mu\varphi \quad\text{with}\quad \varphi\equiv\tfrac{1}{g}\,\omega
\end{aligned}\right\} \tag{120}$$

In the limit $g \downarrow 0$ we recover the equation (112) which was our point of departure, though it is joined now by a companion: $\partial_\mu f^{\mu\nu} = 0$. That

$$\partial_\nu j^\nu = 0 \tag{121}$$

is (by the antisymmetry of $f^{\mu\nu}$) an immediate implication of (119.1), but if one looks to the definition (119.2) of j^ν then it follows also by quick calculation from (118).

As before, let

$$\begin{aligned} n &\equiv \text{ dimension of space} \\ n+1 &\equiv \text{ dimension of spacetime} \end{aligned}$$

Looking back over the work of the preceding paragraph, we see it to be the case that

$$\left.\begin{aligned} [\psi] = [a_\mu] &= \sqrt{(\text{energy density})\cdot(\text{length})^2} \\ [g a_\mu] &= 1/\text{length} \\ [g] &= 1/\sqrt{(\text{energy density})\cdot(\text{length})^4} \end{aligned}\right\} \quad \text{all } n$$

It is, however, an implication jointly of (64) and of (69) that

$$\begin{aligned} [e] &= \sqrt{(\text{energy density})\cdot(\text{length})^{2(n-1)}} \\ &= (\text{length})^{n-3}\sqrt{(\text{energy density})\cdot(\text{length})^4} \\ &= [e^2/\hbar c]\cdot[1/g] \end{aligned}$$

We come thus to the striking conclusion that in *all* cases it makes dimensional good sense to write

$$g = e/\hbar c \tag{122}$$

It would be difficult for an inhabitant of 4-dimensional spacetime not to infer that equations (119) are speaking of Maxwellian electrodynamics, and to make the implied assignments of physical meaning to the field variables $f_{\mu\nu}$. The only odd detail has to do with the structure of the conserved current j^μ, which contains a term that depends nakedly—but in a gauge-invariant way!—upon the gauge field (4-potential) a^μ. This is, to some extent, an artifact of our model; had we taken the Dirac equation (rather than the Klein-Gordon equation) as our starting point then the offending term would have been absent.[67] It is, in any event, the case that the "offending term" is critical to the successful derivation of (121) from (118).

Consider now the conclusions that would be drawn by the gauge theorist who inhabits $(n+1)$-dimensional spacetime. He will agree that the gauge program has called into being a vector field a_μ—a 1-form

$$\boldsymbol{a} \prec \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$$

[67] See p. 199 (also pp. 193 & 195) in CLASSICAL FIELD THEORY (1979).

in the language of §1. And that the gauge field has given rise by (119.3) to an antisymmetric tensor field of rank two—a 2-form:

$$\boldsymbol{a} \longrightarrow \mathbf{d}\boldsymbol{a} = \mathbf{f} \prec \begin{pmatrix} 0 & f_{01} & f_{02} & \dots & f_{0n} \\ & 0 & f_{12} & \dots & f_{1n} \\ & & 0 & \dots & f_{2n} \\ & (-) & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}$$

He will dismiss $\mathbf{df} = \mathbf{0}$ as a triviality, and will agree that $*\mathbf{d}*\mathbf{f} = \mathbf{j}$ (which entails $*\mathbf{d}*\mathbf{j} = \mathbf{0} \prec \partial_\mu j^\mu = 0$; his $\mathbf{j}$ is seen to be a 1-form) serves to capture the substance of (119.1).

Remarkably; Maxwell, we and our gauge theorist all *agree* that equations of the form (9)—which can be rendered

$$*\mathbf{d}*\mathbf{F} = \mathbf{J} \quad \text{with} \quad \mathbf{F} = \mathbf{dA} \text{ invariant under } \mathbf{A} \longrightarrow \mathbf{A} + \mathbf{d}\boldsymbol{\varphi}$$

—serve to describe "electrodynamics" in the presence of a prescribed source. But our gauge theorist has (except in the 4-dimensional case, and for reasons developed already in my introduction) been forced to abandon any claim that the components of $\mathbf{F} \prec F_{\mu\nu}$ can be resolved into "E-fields, and an equal number of B-fields."

Looking now to the specifics of the 2-dimensional situation: we take $\mathbf{F}$ to be a 1-form (a p-form with $p = \frac{\text{spacetime dimension}}{2}$); then $\mathbf{A}$ and $\mathbf{J}$ are 0-forms (scalar fields), with implications which I have been at pains to describe in these pages. Our gauge theorist, on the other hand, takes (because he is invariably forced to take) $\boldsymbol{a}$—whence also $\mathbf{j}$—to be a 1-form, with the consequence that $\mathbf{f}$ is in all cases a 2-form. But a 2-dimensional 2-form has only a single degree of freedom:

$$\mathbf{f} \prec \|f_{\mu\nu} = \epsilon_{\mu\nu} f\| = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}$$

f is a psuedo-scalar field, the dual of a scalar field

And from this it follows[68] that the sourceless Maxwell equations $*\mathbf{df} = \mathbf{0}$ are devoid of content in the 2-dimensional gauge theory. The remaining equations $\partial^\mu f_{\mu\nu} = j_\nu$, when spelled out, give

$$\left.\begin{aligned} \partial_1 f = j_0 \ &: \ \text{compare Gauss' Law (1.1)} \\ \partial_0 f = j_1 \ &: \ \text{compare Maxwell's adjustment (1.2) of Ampere's Law} \end{aligned}\right\} \quad (123)$$

[68] The operative mathematical circumstance here is this:

$$\mathbf{d}(n\text{-dimensional } p\text{-form}) = \begin{cases} (p+1)\text{-form} & \text{if } p = 0, 1, 2, \dots, n-1 \\ 0 & \text{if } p = n \end{cases}$$

of which charge conservation $\partial^0 j_0 + \partial^1 j_1 = 0$ is (use $\partial^0 = \partial_0$ and $\partial^1 = -\partial_1$) an immediate consequence. Writing

$$f = \partial_0 a_1 - \partial_1 a_0 \tag{124}$$

the field equations become

$$\partial_1\partial_0 a_1 - \partial_1\partial_1 a_0 = j_0 \quad \text{and} \quad \partial_0\partial_0 a_1 - \partial_0\partial_1 a_0 = j_1$$

Imposition of the Lorentz gauge condition $\partial^\mu a_\mu = \partial_0 a_0 - \partial_1 a_1 = 0$ brings the preceding equations to the form

$$\Box a_0 = j_0 \quad \text{and} \quad \Box a_1 = j_1 \tag{125}$$

And if we introduce

$$\mathcal{L} = \tfrac{1}{2}(\partial_0 a_1 - \partial_1 a_0)^2 - (j_0 a_0 - j_1 a_1) \tag{126}$$

then we are led back to the field equations $-\partial_1 f + j_0 = 0$ and $\partial_0 f - f_1 = 0$. In some respects this “gauge theorists version of 2-dimensional electrodynamics” possesses a more pronouncedly “Maxwellian feel” than does the theory which I introduced at (12); David Griffiths has remarked that it can, in fact, be considered to comprise an *instance* of Maxwellian electrodynamics, for it results from (1) upon imposition of these specialized assumptions:

$$\mathbf{E} = \begin{pmatrix} f(x^0, x^1) \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{B} = \mathbf{0}\,, \ \rho = j^0(x^0, x^1) \text{ and } \tfrac{1}{c}\mathbf{j} = \begin{pmatrix} j^1(x_0, x_1) \\ 0 \\ 0 \end{pmatrix}$$

That such stringent conditions, once imposed, can self-consistently persist is not immediately obvious, but is made obvious by the very *existence* of the (self-consistent) gauge theory. The latter theory does, however, possess some odd features; it makes no provision for any analog of “magnetism.” And it draws upon the services of a potential which is (uniquely, in my experience) more complicated than the thing it describes. It might be interesting on another occasion to explore more detailed ramifications of the gauge theory, which would appear to possess its own kind of “tutorial potential.” But my immediate objective in this discussion has now been achieved: we have established that the theory implicit in (12) is *not* a gauge theory. And it has been brought vividly to our attention that the equations

$$\boldsymbol{*}\mathbf{d}\boldsymbol{*}\mathbf{F} = \mathbf{J} \qquad \text{and} \qquad \boldsymbol{*}\mathbf{d}\ \ \mathbf{F} = \mathbf{0} \tag{9}$$

admit of a great *variety of alternative realizations*, of which we have addressed only two; to select one realization over another is to emphasize some of the structural elements of Maxwellian electrodynamics at—inevitably—the expense of others. It is interesting that Lorentz covariance is a shared feature of all such theories.

12. Phenomenological models of materials. It was Lorentz who taught us to do our serious electrodynamics in a vacuum; Maxwell himself was a creature of the Victorian laboratory, obligated to explore *simultaneously and in tandum*

- the dynamics of the electromagnetic field and
- the (gross) electromagnetic properties of (simple bulk) materials (in the weak-field approximation).

"Maxwell's equations," as they came from his desk, more nearly resembled[69]

$$\begin{aligned} \tfrac{1}{\epsilon_0}\boldsymbol{\nabla}\cdot\mathbf{D} &= \rho \\ \mu_0\boldsymbol{\nabla}\times\mathbf{H} - \tfrac{1}{c\epsilon_0}\tfrac{\partial}{\partial t}\mathbf{D} &= \tfrac{1}{c}\mathbf{j} \\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\mathbf{B} &= \mathbf{0} \end{aligned}$$

than (1), and are indeterminate in the absence of "constitutive relations" that describe $\mathbf{D}$ and $\mathbf{H}$ in terms of $\mathbf{E}$ and $\mathbf{B}$. In the simplest case one has

$$\mathbf{D} = \epsilon\mathbf{E} \quad\text{and}\quad \mathbf{H} = \tfrac{1}{\mu}\mathbf{B}$$

where ϵ and μ are constants characteristic of the material (sealing wax, whale oil, ...); in vacuum $\epsilon \to \epsilon_0$, $\mu \to \mu_0$ and one recovers (1).

To phrase the issue a bit more abstractly, we expect in the presence of electromagnetically active matter to write equations

$$\ast\mathbf{d}\ast\mathbf{G} = \mathbf{J} \qquad\text{and}\qquad \ast\mathbf{d}\;\mathbf{F} = \mathbf{0} \tag{127}$$

$\mathbf{G}$ a material-dependent function of $\mathbf{F}$; becomes $\mathbf{F}$ in vacuum

and on this basis to understand (9) as a "special case." In §9 of some material previously cited[70] I discuss how the exterior calculus can be used to construct an orderly survey of "all possible" constitutive relations. Here I am content to limit my remarks to the simplest rudiments of this subject, as it relates to 2-dimensional electrodynamics.

In place of (13) we expect to have

$$\left.\begin{aligned} \mu_0\partial_1 H - \tfrac{1}{\epsilon_0}\partial_0 D &= J \\ \partial_1 E - \partial_0 B &= 0 \end{aligned}\right\} \tag{128.1}$$

to which we conjoin the relations

$$D = \epsilon E \quad\text{and}\quad H = \tfrac{1}{\mu}B \tag{128.2}$$

---

[69] I take here certain gross liberties. For more careful discussion see §7.3 in D. Griffiths, *Introduction to Electrodynamics* (1981).

[70] See again footnote 3.

Interior to such 1-dimensional material (but in the absence of J) we have

$$\tfrac{\mu_0}{\mu}\partial_1 B - \tfrac{\epsilon}{\epsilon_0}\partial_0 E = 0$$
$$\partial_1 E - \partial_0 B = 0$$

These can be "uncoupled by differentiation;" hitting the former with ∂_0 and the latter with ∂_1 (and then reversing that procedure) we obtain equations that can be written

$$\left.\begin{aligned}(\partial_x^2 - \tfrac{1}{u^2}\partial_t^2)E = 0\\ (\partial_x^2 - \tfrac{1}{u^2}\partial_t^2)B = 0\end{aligned}\right\} \tag{129}$$

$$u \equiv \sqrt{\tfrac{\epsilon_0\mu_0}{\epsilon\,\mu}}\cdot c \le c \text{ under natural conditions}$$

Formal problems famously arise in connection with determination of the correct *Lorentz covariant* description of electromagnetic fields in the presence of media (especially media in differential motion).[71] Our toy theory would appear to provide a useful laboratory within which to explore such problems.

13. Remarks concerning solutions of the field equations. The analytical problem posed by Maxwellian electrodynamics is (in Lorentz gauge, and in the presence of a prescribed source J^μ) to solve a quartet of uncoupled equations

$$\Box A^\mu = J^\mu \tag{8}$$

subject to prescribed initial and boundary conditions. The 2-dimensional theory provides a single instance

$$\Box A = J \tag{17}$$

of such an equation, and puts one in position to appropriate all that is known concerning the physics of forced strings. My purpose here will be not to review the considerable body of theory relating to the latter topic,[72] but to examine only a few topics illustrative of the "tutorial potential" of the 2-dimensional theory.

In the absence of sources we have (see again (13))

$$\left.\begin{aligned}\partial_1 B - \partial_0 E = 0\\ \partial_1 E - \partial_0 B = 0\end{aligned}\right\} \tag{130}$$

which was seen at (15) to entail

$$\Box F_\mu = 0 \quad \text{where again} \quad \begin{pmatrix}F_0\\F_1\end{pmatrix} \equiv \begin{pmatrix}E\\B\end{pmatrix} \tag{131}$$

[71] Such problems provided the principal motivation for the work to which I alluded in the preceding footnote..

[72] For elaborately detailed discussion of that topic see ANALYTICAL METHODS OF PHYSICS (1981), pp. 221–433.

If we had interest in "running wave solutions" of (131) we would find it natural to write[73]

$$E(x,t) = \mathcal{E}\, e^{i[k_0 x \pm \omega_0 t]} \quad \text{and} \quad B(x,t) = \mathcal{B}\, e^{i[k_1 x \pm \omega_1 t]}$$

and to insist that $(\omega, k)_0$ and $(\omega, k)_1$ be (possibly independent) solutions of

$$k^2 - (\omega/c)^2 = 0 \tag{132}$$

The field equations (130) serve, however, to restrict our seeming options; we find ourselves obligated to insist that $(\omega, k)_0$ and $(\omega, k)_1$ refer to the *same* solution of (132), and moreover that

$$\mathcal{B} = \pm\mathcal{E} \quad \text{according as} \quad \omega/c = \pm k$$

The running wave solutions of (131) have therefore the form

$$\mathbf{F}(x,t) = \boldsymbol{\mathcal{F}} \cdot e^{i[kx \pm \omega t]} \prec \begin{pmatrix} \mathcal{E} \\ \pm\mathcal{E} \end{pmatrix} e^{i[kx \pm \omega t]}$$

The components of F_μ must, in other words, be synchronized. The amplitude of the magnetic component of the field must, moreover, be correlated with the amplitude of the electrical component, and that correlation is of such a nature as to achieve

$$\text{momentum density} \equiv -EB \gtrless 0 \text{ according as the wave runs right or left}$$

This result is consonant with the "antiparallelism condition" that was obtained near the end of §6 by more general means; i.e., without specific reference to running waves. The preceding argument captures the main features of the argument that in Maxwellian electrodynamics[74] gives rise to the "transverse plane waves" so characteristic of that theory. In 2-dimensional theory it would, of course, be senseless to speak of "plane waves," and doubly senseless to speak of "transverse plane waves with $E \perp B$;" it is of interest that the role of the latter condition has in 2-dimensional theory been taken over by an antiparallelism condition.

In 2-dimensions—exceptionally—the wave operator *factors* within the field of complex numbers[75]

$$\Box = (\partial_0 + \partial_1)(\partial_0 - \partial_1) \tag{133}$$

From this it follows that every solution of $\Box A = 0$ can be described

$$A(x^0, x^1) = f(x^1 - x^0) + g(x^1 + x^0) \tag{134}$$

---

[73] I find it convenient here to revert to pre-relativistic $\{x, t\}$ notation.

[74] See CLASSICAL ELECTRODYNAMICS (1980), pp. 338–344.

[75] To achieve factorization when $n > 2$ one must—*à la* Dirac—have recourse to the Clifford numbers, and attend carefully to the complications that derive from non-commutivity.

where $f(\bullet)$ and $g(\bullet)$ are arbitrary functions of a single argument. Evidently

$f(x^1 - x^0)$ glides rigidly to the right, with speed c, while
$g(x^1 + x^0)$ glides rigidly to the left

Suppose (allowing ourselves on occasion to adopt the simplified notation $x \leftarrow x^1$) we are given this initial data:

$$\mathsf{A}(x) \equiv A(0,x) \quad \text{and} \quad \mathsf{B}(x) \equiv \partial_0 A(x^0,x)\Big|_{x^0=0} \equiv A_0(0,x) \quad \text{are prescribed}$$

Then

$$\begin{aligned} f\,(x) + g\,(x) &= +\mathsf{A}(x) \\ f'(x) - g'(x) &= -\mathsf{B}(x) \quad \Longrightarrow \quad f\,(x) - g\,(x) = -\int^x \mathsf{B}(y)\,dy \end{aligned}$$

from which we obtain

$$\begin{aligned} f(x) &= \tfrac{1}{2}\Big\{\mathsf{A}(x) - \int^x \mathsf{B}(y)\,dy\Big\} \\ g(x) &= \tfrac{1}{2}\Big\{\mathsf{A}(x) + \int^x \mathsf{B}(y)\,dy\Big\} \end{aligned}$$

giving

$$A(x^0,x^1) = \tfrac{1}{2}\Big\{A(0,x^1-x^0) + A(0,x^1+x^0)\Big\} + \tfrac{1}{2}\int_{x^1-x^0}^{x^1+x^0} A_0(0,y)\,dy \qquad (135)$$

This result, which provides an explicit description of the solution of $\Box A = 0$ that evolves from prescribed initial data, was known to d'Alembert already in 1747, when the theory of partial differential equations was still in its infancy. That the right side of (135) does in fact conform to the prescribed data—and that it serves not only to predict but also to retrodict—is manifest.

We obtained (135) by the most elementary of means. It is a remarkable fact that one can, by a chain of argument which at every link is equally elementary, proceed from (135) to a description *in the n-dimensional case* of the $A(x)$ which satisfies $\Box A = J$ and at the same time conforms to prescribed initial data. One can, in short, extract from (135) an account of the *general theory of the Green's functions of $\Box$ operators*—stripped of the apparatus (Fourier transformation, introduction of hyperspherical coordinates, delicate contour integration) that standardly encumbers that theory. The following remarks are intended to provide an outline of the program to which I allude; details are developed elsewhere.

STEP ONE: ESTABLISH CONTACT WITH THE NOTATIONS OF GREEN. We notice that the integral in (135) can be described

$$\tfrac{1}{2}\int_{x^1-x^0}^{x^1+x^0} A_0(0,y)\,dy = \int_{-\infty}^{+\infty} \underbrace{\Delta^0(x^0,x^1-y)}\,A_0(0,y)\,dy$$
$$\equiv \tfrac{1}{2}\Big\{\theta\big(y-(x^1-x^0)\big)-\theta\big(y-(x^1+x^0)\big)\Big\}$$

where $\Delta^0(x^0,x^1-y)$ is simply a "switch," installed to turn the integrand on and off, automatically. We observe, moreover, that

$$\partial_0\Delta^0(x^0,x^1-y) = \tfrac{1}{2}\Big\{\delta\big(y-(x^1-x^0)\big)+\delta\big(y-(x^1+x^0)\big)\Big\}$$

and that (135) can, in these notations, be written

$$A(x^0,x^1) = \int_{-\infty}^{+\infty}\Big\{A(0,y)\partial_0\Delta^0(x^0,x^1-y) + \Delta^0(x^0,x^1-y)A_0(0,y)\Big\}\,dy \tag{136.1}$$

Setting $A(0,y) = 0$ and $A_0(0,y) = \delta(y)$ we see that $\Delta^0(x^0,x^1)$ is itself the solution of $\Box\Delta = 0$ that evolves from the especially simple initial data

$$\Delta^0(0,x^1) = 0 \quad\text{and}\quad \partial_0\Delta^0(x^0,x^1)\Big|_{x^0=0} = \delta(x^1)$$

For a diagramatic representation of the functional structure of $\Delta(x^0-y^0,x^1-y^1)$ see Figure 2.

STEP TWO: TAKE SOURCES INTO ACCOUNT. The idea is to write

$$A = A^{\text{free}} + A^{\text{forced}}$$

where A^{free} is the solution of $\Box A = 0$ that evolves from the prescribed initial data (as described already by (136.1)), and A^{forced} is the solution of $\Box A = J$ that is initally null. Writing

$$A^{\text{forced}}(x^0,x^1) = \iint \Delta(x^0-u,x^1-y)J(u,y)\,dudy \tag{136.2}$$

we are led to impose upon $\Delta(\bullet,\bullet)$ the requirement

$$\Box\Delta(x^0-u,x^1-y) = \delta(x^0-u)\delta(x^1-y) \tag{137}$$

It can be shown by elementary means[76] that a simple "truncation process"

$$\Delta^0(x^0,x^1) \xrightarrow[\text{truncation}]{} \Delta(x^0,x^1) \equiv \theta(x^0)\cdot\Delta^0(x^0,x^1)$$

yields a $\Delta(\bullet,\bullet)$ with all the required properties: it satisfies (137), and when inserted into (136.2) entails $A(0,x^1) = 0$ provided the $\iint$ is taken to range on the spacetime region bounded by the timeslices identified in Figure 3.

[76] See §3 of "Formal theory of singular functions" (1997) for details.

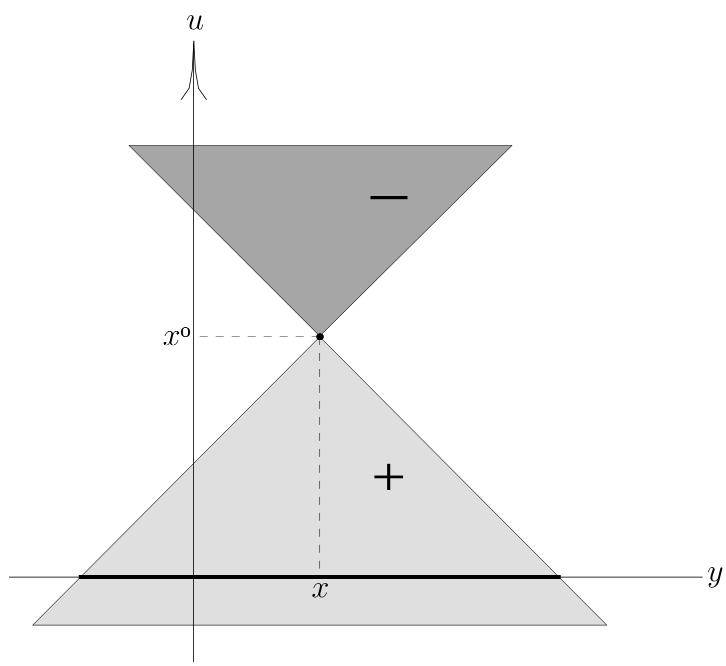

FIGURE 2: *The homogeneous Green's function* $\Delta(x^0-u, x^1-y)$ *has the constant value* $+\frac{1}{2}$ *on the interior of the lightcone that extends backward from the field point* (x^0, x^1), *and the constant value* $-\frac{1}{2}$ *on the forward lightcone. It vanishes exterior to the lightcone. Initial data* $A(0, y)$ *and* $A_0(0, y)$ *is spread like peanut butter on the timeslice* $x^0 = 0$, *but only the data interior to the cone (represented by a heavy line in the figure) contributes according to (136.1) to the value assumed by* $A^{\text{free}}(x^0, x^1)$.

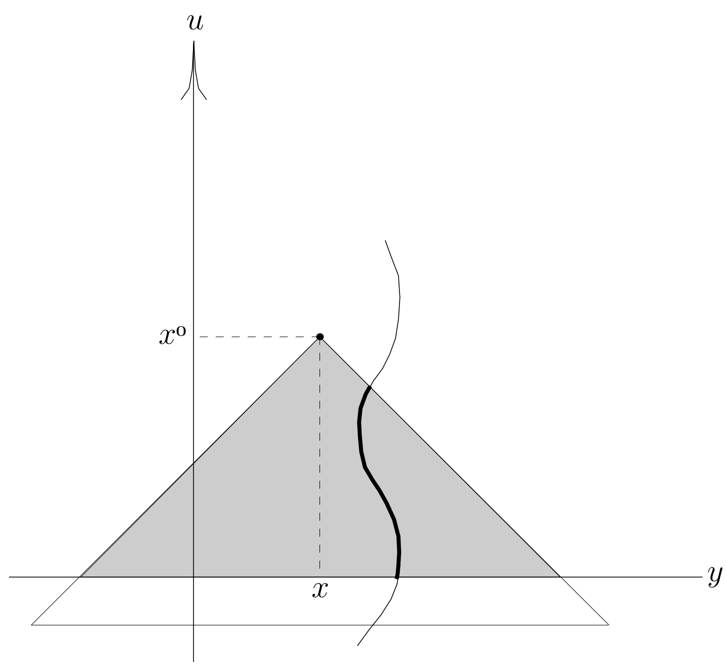

FIGURE 3: *"Truncation" of $\Delta(x^0 - u, x^1 - y)$ (elimination of the forward sector) yields the retarded solution of the inhomogeneous wave equation (137). The curve represents the activity of a source (here—for diagramatic convenience—a point source). Only the source data interior to the shaded region (bounded below by the timeslice on which the initial data is inscribed, and above by the timeslice that contains the fieldpoint) contributes according to (136.2) to the value assumed by $A^{\text{forced}}(x^0, x^1)$. Superposition gives the function*

$$A(x^0, x^1) = A^{\text{free}}(x^0, x^1) + A^{\text{forced}}(x^0, x^1)$$

which satisfies $\Box A = J$ and at the same time satisfies the prescribed initial conditions.

STEP THREE: DIMENSIONAL GENERALIZATION. d'Alembert has supplied us with motivation to introduce a homogeneous Green's function $\Delta^0(x^0-u, x^1-y)$ which we agree at this point to notate $\Delta^0_1(x^0-y^0, x^1-y^1)$ in order to emphasize that it is the $n=1$ member of a *population* of functions

$$\Delta^0_n(x^0-y^0, x^1-y^1, \dots, x^n-y^n)$$

which satisfy

$$\big\{\partial_0^2-\partial_1^2-\partial_2^2-\cdots-\partial_n^2\big\}\Delta^0_n=0$$

$$n \equiv \text{dimension of space}$$

We have

$$\begin{aligned}
&\Delta^0_1(x^0-y^0, x^1-y^1)\\
&\qquad=\tfrac{1}{2}\Big\{\theta\big(y^1-[x^1-(x^0-y^0)]\big)-\theta\big(y^1-[x^1+(x^0-y^0)]\big)\Big\}\\
&\qquad=\tfrac{1}{2}\varepsilon(x^0-y^0)\cdot\theta\big(\underbrace{(x^0-y^0)^2-(x^1-y^1)^2}_{\equiv\,\sigma\,\equiv\,(x^0-y^0)^2-r^2}\big)
\end{aligned}$$

In the 1930's Marcel Reisz, in a famous application of the fractional calculus, managed to invert what Jacques Hadamard, a decade earlier, had called the "method of (dimensional) descent;" Reisz—building upon the circumstance that (consistently with the result just established in the case $n=1$) the variables $\{x^0, x^1, \dots, x^n; y^0, y^1, \dots, y^n\}$ enter into the structure of Δ^0_n only in lumped combination

$$\Delta^0_n=\Delta^0_n(\sigma)\quad\text{with}\quad\sigma\equiv(x^0-y^0)^2-r^2$$
$$r^2\equiv(x^1-y^1)^2+\cdots+(x^n-y^n)^2$$

—showed that the functions $\Delta^0_n(\sigma)$ are interlinked by a network of very simple relations[77]

$$\begin{array}{ccccccccccc}
\Delta_1 & & \longrightarrow & & \Delta_3 & & \longrightarrow & & \Delta_5 & \longrightarrow & \Delta_7\\
& \searrow & & \nearrow & & \searrow & & \nearrow & \searrow & \nearrow & \quad\dots\text{etc.}\\
& & \Delta_2 & & \longrightarrow & & \Delta_4 & \longrightarrow & \Delta_6 & &
\end{array}$$

where the horizontal arrows refer to the action of $\frac{1}{\pi}\frac{\partial}{\partial\sigma}=-\frac{1}{2\pi r}\frac{\partial}{\partial r}$ and the diagonal arrows refer to the action of the semi-differentiation operator $\big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\big)^{\frac{1}{2}}$. Writing

$$\Delta^0_1(\sigma)=\pm\tfrac{1}{2}\theta(\sigma)$$

(take $+$ in the backward lightcone, $-$ in the forward cone), the fractional calculus supplies

$$\Delta^0_2(\sigma)=\pm\tfrac{1}{2\pi}\sigma^{-\frac{1}{2}}\theta(\sigma)$$

[77] For details see §7 of "Construction & physical application of the fractional calculus" (1997) and additional sources cited there.

When n is odd, Riesz' construction gives

$$\begin{aligned}
\Delta_1^0(\sigma) &= \pm\tfrac{1}{2}\theta(\sigma)\\
&\downarrow\\
\Delta_3^0(\sigma) = \left(\tfrac{1}{\pi}\tfrac{\partial}{\partial\sigma}\right)^1 \Delta_1^0(\sigma) &= \pm\tfrac{1}{2\pi}\delta(\sigma)\\
\Delta_5^0(\sigma) = \left(\tfrac{1}{\pi}\tfrac{\partial}{\partial\sigma}\right)^2 \Delta_1^0(\sigma) &= \pm\tfrac{1}{2\pi^2}\delta'(\sigma)\\
&\vdots\\
\Delta_{2n+1}^0(\sigma) = \left(\tfrac{1}{\pi}\tfrac{\partial}{\partial\sigma}\right)^n \Delta_1^0(\sigma) &= \pm\tfrac{1}{2\pi^n}\delta^{(n)}(\sigma)
\end{aligned}$$

When n is even the situation is significantly more complicated and qualitatively distinct, but for the simplest of reasons; Riesz' construction gives

$$\begin{aligned}
\Delta_2^0(\sigma) &= \pm\tfrac{1}{2\pi}\sigma^{-\frac{1}{2}}\theta(\sigma)\\
&\downarrow\\
\Delta_4^0(\sigma) = \left(\tfrac{1}{\pi}\tfrac{\partial}{\partial\sigma}\right)^1 \Delta_2^0(\sigma) &= \pm\tfrac{1}{2\pi^2}\Big\{\sigma^{-\frac{1}{2}}\delta(\sigma) - \tfrac{1}{2}\sigma^{-\frac{3}{2}}\theta(\sigma)\Big\}\\
\Delta_6^0(\sigma) = \left(\tfrac{1}{\pi}\tfrac{\partial}{\partial\sigma}\right)^2 \Delta_2^0(\sigma) &= \pm\tfrac{1}{2\pi^3}\Big\{\sigma^{-\frac{1}{2}}\delta'(\sigma) - \sigma^{-\frac{3}{2}}\delta(\sigma) + \tfrac{3}{4}\sigma^{-\frac{5}{2}}\theta(\sigma)\Big\}\\
&\vdots\\
\Delta_{2n+2}^0(\sigma) = \left(\tfrac{1}{\pi}\tfrac{\partial}{\partial\sigma}\right)^n \Delta_2^0(\sigma) &= \pm\tfrac{1}{2\pi^{n+1}}\sum_{p=0}^{n}\tbinom{n}{p}\big[\sigma^{-\frac{1}{2}}\big]^{(p)}\big[\theta(\sigma)\big]^{(n-p)}
\end{aligned}$$

where $\big[\sigma^{-\frac{1}{2}}\big]^{(p)} = (-)^p\frac{(2p)!}{2^{2p}p!}\sigma^{-\frac{1}{2}-p}$ and $\big[\theta(\sigma)\big]^{(n-p)} = \delta^{(n-p-1)}(\sigma) : 0 \le p < n$. From these results we learn that

- In all cases, $\Delta_n^0(\sigma)$ vanishes outside the lightcone;
- $\Delta_{\text{odd}\ge 3}^0(\sigma)$ is singular on the lightcone, but vanishes inside;
- $\Delta_{\text{even}}^0(\sigma)$ is singular on the lightcone, but—owing to the presence of a "dangling θ-function"—fails to vanish inside (as also does $\Delta_1^0(\sigma)$); it follows that radiative events in odd-dimensional spacetimes have persistent local effects. This is in sharp contrast to the situation in spacetimes of even dimension $n+1 \ge 4$.

Such commentary is of only incidental interest to our friend, the one-dimensional physicist, who simply wants to get on with his 2-dimensional electrodynamical work. But it does serve to illustrate the "tutorial (2-torial?) potential" of the 2-dimensional formalism, and to make vivid the sense in which Δ_1^0 serves as a "seed" from which the entire theory can be harvested; it underscores a sense in which the physical case $n = 3$ (Lienard-Wiechert) is "special," and it draws attention to an analytical consequence (θ instead of δ) of the fact that in 1-space "to fall off geometrically" means "to fall off not at all." Though phrased in terms of Δ_n^0, the discussion pertains straightforwardly to the truncated functions Δ_n. Less obviously, it pertains also to the theory of Klein-Gordon Green's functions.

I turn now from dynamics to consideration of the *static* solutions of the field equations (13), which for present purposes read

$$\begin{aligned}\partial_1 B &= J\\ \partial_1 E &= 0\end{aligned}$$

Once again, we find it convenient to adopt a simplified notation: $x \longleftarrow x^1$. Immediately[78]

$$\begin{aligned}B(x) &= \int^x J(y)\,dy + \text{constant}\\ E(x) &= \text{constant}\end{aligned}$$

I elect not to restrict my attention to "fields interior to a box," and am forced therefore by the physical requirement that

$$\begin{aligned}\text{total field energy} \equiv \int_{-\infty}^{+\infty} &\mathcal{E}(x)\,dx < \infty\\ &\mathcal{E}(x) = \tfrac{1}{2}(E^2 + B^2)\end{aligned}$$

to set the constant value of E equal to zero. In 2-dimensional theory a static field can, on such grounds, be argued to be not "electrostatic" but "*magneto*static." Suppose the source to be a

$$\begin{aligned}\text{point source} \quad : \quad J(x) = &g \cdot \delta(x-a)\\ &g \equiv \text{"magnetic charge"} \;\; : \;\; [g] = [B]\end{aligned}$$

Then

$$B(x) = \begin{cases}\text{constant} & : \;\; x < a\\ \text{constant} + g & : \;\; a < x\end{cases}$$

which in all non-trivial cases describes a field of infinite energy. To avoid the latter problem[79] we assemble a

$$\text{"magnetic dipole"} \quad : \quad J(x) = g \cdot \big\{\delta(x+a) - \delta(x-a)\big\}$$

Then

$$B(x) = \begin{cases}\text{constant} & : \;\; x < -a\\ \text{constant} + g & : \;\; -a < x < +a\\ \text{constant} & : \;\; +a < x\end{cases}$$

which has finite energy

$$\mathrm{E}(a) = \int \mathcal{E}(x)\,dx = g^2 a$$

[78] Here consequences of a point just remarked ("to fall off geometrically" is, in the one-dimensional case, to fall off not at all) become starkly evident.

[79]

provided we assign a null value to the constant. From

$$-\tfrac{\partial}{\partial a}\mathrm{E}(a) = -g^2$$

we learn that *opposite magnetic charges attract one another with a force that is independent of separation*. Note that the field energy is confined to the *interior* of such a structure (and acquires an electrical component when the dipole and observer are in relative motion). Note also that such a structure can be assigned an

$$\text{effective mass } m = g^2 a/c^2$$

that gets larger as the size parameter a increases; this is the reverse of the situation made familiar by our 3-dimensional experience, where $m = e^2/ac^2$. Note finally that, while in 3-dimensional theory it makes sense to speak of the size of a massive monopole, in 2-dimensional theory the simplest structure to which one can assign a natural size is a dipole.

14. Formal theory of blackbody radiation. Though I take my motivation from the case $n = 1$, I find amusing (and moderately instructive) a pattern evident in the details that would remain invisible were n not allowed to assume arbitrary values.

Planck's objective was to describe the functional structure of the "spectral density function" $u(\nu, T)$ which is characteristic of equilibrated radiation at temperature T. Dimensionally

$$\begin{aligned}[\text{spectral density}] &= \frac{\text{energy}}{(\text{volume})\cdot(\text{frequency})}\\ &= \text{action density}\end{aligned}$$

If we assume the "photon"[80] to be massless and uncharged, then the only variables available to us in constructing such a theory (I for the moment hold Planck's constant in reserve) are ν, c and kT. We might, on this basis, expect to obtain

$$u(\nu, T) \sim (\nu/c)^n (kT/\nu)\cdot\varphi(x)$$

where $\varphi(x)$ is some presently undetermined function of the dimensionless variable x. Since it is not possible to form such a variable from the material at hand, one has seemingly no option but to set $\varphi(x) = \text{constant}$. But then

$$\int_0^\infty u(\nu, T)\,d\nu = \infty$$

which is clearly untenable; the services of $\varphi(x)$—the only device available to discriminate against high frequencies—are evidently indispensable.

[80] It serves my expository purposes thoughout this discussion to indulge freely in anachronism.

With Planck we set

$$x \equiv h\nu/kT \quad \text{with} \quad [h] = \text{action}$$

And with Planck (who borrowed the idea from Rayleigh) we require that the “equipartition principle” hold at low frequencies (i.e., for $x \ll 1$):

$$u(\nu, T) \approx (\text{modal density}) \cdot kT$$

where

$$\begin{aligned}(\text{modal density}) &= (\text{polarizational degrees of freedom})\\ &\qquad\times \frac{(\text{area of } n\text{-sphere of radius } \nu)}{c^n}\\ &= 2\cdot(4\pi\nu^2/c^3) \text{ in the physical case } n=3\end{aligned}$$

and where it’s because we are thinking now about “Hertzian oscillators” (which possess both kinetic and potential energy) rather than free-flying “molecules” that we write kT instead of the more familiar $\frac{1}{2}kT$.[81] In the n-dimensional case we expect therefore to have

$$u(\nu, T) = \# \cdot \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\frac{\nu^{n-1}}{c^n}kT\cdot\varphi(\tfrac{h\nu}{kT})$$
$$\# \equiv \text{polarizational degrees of freedom}$$

Writing

$$u(\nu, T) = \# \cdot \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\cdot h(\nu/c)^n \underbrace{\Big[\tfrac{kT}{h\nu}\cdot\varphi(\tfrac{h\nu}{kT})\Big]} \tag{138}$$

Planck is at length led[82] to set

$$\begin{aligned}&= \frac{1}{e^x - 1} \\ &= \tfrac{1}{x} - \tfrac{1}{2} + \tfrac{1}{12}x - \tfrac{1}{720}x^3 + \tfrac{1}{30240}x^5 + \cdots\end{aligned} \tag{139}$$

Then

$$u(T) = \# \cdot \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\cdot \underbrace{\int_0^\infty h(\nu/c)^n\frac{1}{\exp\left\{\frac{hv}{kT}\right\}-1}\,d\nu}_{=\, n!\zeta(n+1)\frac{(kT)^{n+1}}{(2\pi\hbar c)^n}}$$

[81] For discussion of this point see, for example, Max Born, *The Mechanics of the Atom* (1924) §1, or p. 76 in his *Natural Philosophy of Cause & Chance* (1949). I return to the point later in the text.

[82] I return later to review of the most essential steps in his clever chain of argument.

where the integral was supplied by *Mathematica*.[83] We are led thus to the dimensionally-generalized "Steffan-Boltzmann law"

$$u(T) = aT^{n+1} \tag{140}$$

$$a \equiv \# \cdot \underbrace{\frac{2}{(2\sqrt{\pi})^n}\frac{\Gamma(n+1)\zeta(n+1)}{\Gamma(\frac{n}{2})}\cdot k(k/\hbar c)^n}_{A(n)} \tag{141}$$

Mathematica informs us that

$$\begin{aligned} A(1) &= 3.15699 \times 10^{-21}\,\text{Joule/Meter Kelvin}^2 \\ A(2) &= 1.00749 \times 10^{-18}\,\text{Joule/Meter}^2\text{Kelvin}^3 \\ A(3) &= 3.78296 \times 10^{-16}\,\text{Joule/Meter}^3\text{Kelvin}^4 \\ &\vdots \end{aligned}$$

and that $A(n+1)/A(n)$ is a slowly growing function of n:

$$\begin{aligned} A(2)/A(1) &= 319.128/\text{Meter Kelvin} \\ A(3)/A(2) &= 375.485/\text{Meter Kelvin} \\ A(4)/A(3) &= 418.389/\text{Meter Kelvin} \\ &\vdots \\ A(11)/A(10) &= 621.189/\text{Meter Kelvin} \\ A(101)/A(100) &= 1764.03/\text{Meter Kelvin} \\ A(1001)/A(1000) &= 5516.11/\text{Meter Kelvin} \\ &\vdots \end{aligned}$$

It is of formal interest that (141) makes sense even when n is not an integer.

Concerning the function $\#(n)$, all that is known with certainty is that $\#(3) = 2$. One expects intuitively (but with a low level of confidence) to have

$$\#(n) = \begin{cases} 1 & : \quad n = 1 \\ n-1 & : \quad n = 2, 3, 2, \ldots \end{cases}$$

but to secure such a result one must first possess an "electrodynamics in $(n+1)$-dimensional spacetime," and then, working within the framework of such a theory, must develop a theory of plane waves. Theories of the sort I have proposed become available only when n is odd; gauge theories are available for all values of n, but appear to present interpretive problems.

[83] But see **23.2.7** and **27.1.3** in Abramowitz & Stegun.

I look now to the n-dimensional generalization of Boltzmann's celebrated thermodynamic derivation (1884) of (140).[84] We proceed from the assertions that the internal energy U of a boxfull (hypervolume V, with $[V] = (\text{length})^n$) of thermalized radiation can be described

$$U(T,V) = V \cdot u(T) \tag{142}$$

and that the pressure p is given by

$$p = \tfrac{1}{n} \cdot u(T) \tag{143}$$

Thermodynamics[85] supplies the identity

$$\left(\frac{\partial U}{\partial V}\right)_T = T^2 \left(\frac{\partial}{\partial T}\frac{p}{V}\right)_V$$

which in the application at hand becomes

$$T\,\frac{du}{dT} = (n+1)u \quad \text{or again} \quad \frac{du}{u} = (n+1)\frac{dT}{T}$$

from which

$$u(T) = \text{constant} \cdot T^{n+1}$$

follows at once. Planck's accomplishment was, *inter alia*, to provide a unique theoretical evaluation of the constant of integration. To better comprehend (142) and (143)—of which the Steffan-Boltzmann law is seen now to be an almost immediate consequence—it is useful to set up a point-by-point comparison with the corresponding ideal gas relations. Writing

$$U_{\text{radiation}}(T,V) = V \cdot u(T) \quad : \quad U_{\text{gas}}(T,V,N) = Nc_V T$$

it becomes striking that the expression on the left makes no reference to the "number" of "photons" present in the sample, and that for radiation U/V is a *universal* function of T. For gases the striking fact is that U (not U/V) is itself V-independent, though not universal: U depends both on the size N of the sample and (through c_V) upon its specific construction. Kinetic theory supplies

$$U_{\text{gas}} = \text{number of degrees of freedom} \cdot \tfrac{1}{2}kT$$

which for a *monomolecular* gas entails $c_V = \frac{3}{2}k$, giving

$$\begin{aligned} U = \tfrac{3}{2}NkT = \tfrac{3}{2}pV \quad &\text{by the gas law}\\ \downarrow\qquad&\\ p = \tfrac{2}{3}u(T,N) \quad &\text{where } u \equiv U/V \;:\; \text{energy density of gas}\end{aligned}$$

[84] A "true pearl of theoretical physics" in the estimation of Lorentz. See Chapter 3, p. 110 of STATISTICAL PHYSICS (1969), and additional references cited there.

[85] See p. 59 in the notes just cited. This frequently useful identity is sometimes called the "thermodynamic equation of state."

It was by one of the first significant applications of Maxwellian electrodynamics that Boltzmann was led to the conclusion[86] that

$$p = \tfrac{1}{3}u(T) \quad \text{for incoherent radiation}$$

Presumably, "n-dimensional electrodynamics" leads by similar argument to (143), but I myself have *not* constructed the demonstration.[87] That is why, in the title to this section, I allude to a "formal theory..." It is tempting to suppose that the "disappearance of the 2-factor" can be attributed to the following circumstance:

$$\begin{aligned} \text{momentum} &= 2\,\frac{\text{energy}}{\text{velocity}} &&: \quad \text{non-relativistic mass point} \\ \text{momentum} &= \frac{\text{energy}}{\text{velocity}} &&: \quad \text{photon} \end{aligned}$$

Planck had been working on the blackbody radiation problem for the better part of a decade by the time—December 1900, the final month of the 19th Century—he experienced his revolutionary breakthrough. The story of that month's effort cleaves naturally into two parts, both of which involve steps only someone throughly saturated in a problem would think to take. Writing

$$\begin{aligned} u(\nu,T) &= (\text{modal density})\cdot U_\nu(T) \\ &\qquad\qquad U_\nu(T) \equiv \text{energy density per mode} \end{aligned}$$

he first sought a function $U_\nu(T)$ with the properties that

$$U_\nu(T) \sim \begin{cases} kT & h\nu \ll kT \quad \text{(Rayleigh)} \\ h\nu \exp\left\{-\frac{h\nu}{kT}\right\} & h\nu \gg kT \quad \text{(Wien)} \end{cases}$$

To that end he introduced

$$S_\nu(T) \equiv \text{entropy density per mode}$$

and by appropriation of $\frac{\partial S}{\partial U} = \frac{1}{T}$ obtained

$$\frac{\partial S_\nu}{\partial U_\nu} = \begin{cases} +\frac{k}{U_\nu} & h\nu \ll kT \quad \text{(Rayleigh)} \\ -\frac{k}{h\nu}\log\frac{U_\nu}{h\nu} & h\nu \gg kT \quad \text{(Wien)} \end{cases}$$

$$\downarrow$$

$$\frac{\partial^2 S_\nu}{\partial U_\nu^2} = \begin{cases} -\frac{k}{U_\nu^2} & h\nu \ll kT \quad \text{(Rayleigh)} \\ -\frac{k}{h\nu U_\nu} & h\nu \gg kT \quad \text{(Wien)} \end{cases}$$

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[86] See §**11.3** of W. Panofsky & M. Phillips, *Classical Electricity & Magnetism* (1955) for the details.

[87] Nor, for that matter, have I constructed an n-dimensional thermodynamic/kinetic theory of ideal gases; such a project would seem too straightforward to be interesting.

The functions encountered in the final equation are so simple—and so similar—as to have inspired Planck to take an interpolative leap, writing

$$\frac{\partial^2 S_\nu}{\partial U_\nu^2} = -k\frac{1}{U_\nu^2 + h\nu U_\nu} = -\frac{k}{h\nu}\left\{\frac{1}{U_\nu} - \frac{1}{U_\nu + h\nu}\right\}$$

Integrating his way back to his starting point, Planck obtained

$$U_\nu(T) = \frac{h\nu}{\exp\left\{\frac{h\nu}{kT}\right\} - 1} \qquad \text{(Planck)}$$

Thus was born the "Planck distribution," which by interpolative design displayed all the required limit properties. At this point Planck entered the second phase of his project; from the preceding equation—written

$$\frac{\partial S_\nu}{\partial U_\nu} = \frac{1}{T} = -\frac{k}{h\nu}\log\frac{U_\nu}{U_\nu + h\nu}$$

—Planck by a final integration obtained

$$S_\nu = \tfrac{k}{h\nu}\Big\{\big[(U_\nu + h\nu)\log(U_\nu + h\nu) - (U_\nu + h\nu)\big] - \big[U_\nu \log U_\nu - U_\nu\big]\Big\}$$

from which he constructed

$$\begin{aligned} s(\nu, T) &= (\text{modal density}) \cdot S_\nu(T) \\ &= \text{ result of what "counting problem"?} \end{aligned}$$

He was led and length and reluctantly to the conclusion that to resolve the question just posed one had unavoidably to "quantize" the Hertzian oscillators.

I have reviewed the relevant details of this familiar story in order to place myself in position to observe that *at no point is Planck's argument sensitive to the dimension of spacetime*. All dimension-dependent features of the problem (including those responsible for the T^{n+1} in the Steffan-Boltzmann equation) were excised upon removal of the modal density factor. Electrodynamics is in many respects highly dimension-dependent, but it was from dimension-neutral properties of the theory (and from dimension-neutral thermodynamics and statistical mechanics) that quantum mechanics sprang, and from those that it gained its own dimension-neutrality. The discussion has served also to focus attention on this question: "How many degrees of freedom has an n-dimensional photon?"

15. Conclusion and prospects. This essay has grown over-long, but in a sense length itself is, in the present instance, the message: I have been at pains to demonstrate that—contrary to popular belief—it makes good sense to speak of a "2-dimensional electrodynamics," and that such a theory provides simplified analogs of virtually all the points of principle and technique familiar from Maxwellian electrodynamics. The essay could, in principle, have mimiced both the design and the length of any standard electrodynamical text. It is as short as it is only because I have omitted topics that could easily have been included; I have, for instance, omitted any detailed reference to the *internal symmetry* structure of the theory (which precisely duplicates that of Maxwellian theory), and have elected not to include radiation theory (properties of the field produced by—for example—a vibrating dipole).

The parallels are instructive (particularly since 2-dimensional arguments tend to be more transparent than their 4-dimensional counterparts), but so also are the points at which precise parallelism breaks down, for they cast fresh light on what is "special" about Maxwellian electrodynamics, about "the world which is" relative to worlds that might have been.

It would be interesting on some future occasion to look to the quantization of the theory now in hand, to explore the outlines of "2-dimensional QED." In such a context the tutorial advantages of simplification would seem to be too valuable to be wasted.

It is a pleasure to acknowledge my indebtedness to David Griffiths for helpful discussion of several points, and especially for the casual question that stimulated this entire enterprise.