A common textbook example of a radiating system is a configuration of charges fixed relative to each other but in rotation. The charge density is obviously a function of time, but it is not of the form of

$$\rho(\vec{\boldsymbol{x}}, t) = \rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}\,, \tag{1}$$

where $\omega > 0$ is a positive angular frequency.

(a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\vec{\boldsymbol{x}}, t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of eq. (1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{\ell m}(t)$ to form linear combinations that are real before making the connection.

The time-dependent multipole moment $q_{\ell m}(t)$ is given by [cf. eq. (4.3) of Jackson]:

$$q_{\ell m}(t) = \int Y^*_{\ell m}(\theta, \phi)\, r^\ell\, \rho_R(\vec{\boldsymbol{x}}, t)\, d^3x\,, \tag{2}$$

where $\rho_R(\vec{\boldsymbol{x}}, t)$ is the physical (real) time-dependent charge density. The subscript R is employed here to emphasize that $\rho_R(\vec{\boldsymbol{x}}, t)$ is a real quantity. In contrast, $q_{\ell m}(t)$ is not a real quantity if $m \neq 0$, since

$$q^*_{\ell m}(t) = (-1)^m q_{\ell,-m}(t)\,, \tag{3}$$

which follows from the corresponding property of the spherical harmonics,

$$Y^*_{\ell m}(\theta, \phi) = (-1)^m Y_{\ell,-m}(\theta, \phi)\,. \tag{4}$$

Consider the case where $\rho(\vec{\boldsymbol{x}}, t) = \rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}$, and the physical charge distribution is given by $\mathrm{Re}\,\rho(\vec{\boldsymbol{x}}, t)$. We then define the corresponding multipole moments,[1]

$$Q_{\ell m}(t) = \int Y^*_{\ell m}(\theta, \phi)\, r^\ell\, \rho(\vec{\boldsymbol{x}}, t)\, d^3x = Q_{\ell m}\, e^{-i\omega t}\,, \tag{5}$$

where

$$Q_{\ell m} \equiv \int Y^*_{\ell m}(\theta, \phi)\, r^\ell\, \rho(\vec{\boldsymbol{x}})\, d^3x\,. \tag{6}$$

However, the *physical* time-dependent multipole moment is $q_{\ell m}(t)$, which is not the same as $\mathrm{Re}\,[Q_{\ell m}\, e^{-i\omega t}]$. Indeed, $q_{\ell m}(t)$ is complex when $m \neq 0$, whereas $\mathrm{Re}\,[Q_{\ell m}\, e^{-i\omega t}]$ is real.

The central question of this problem is the relation between $q_{\ell m}(t)$ and $Q_{\ell m}$. Given eq. (1), the real physical time-dependent charge distribution for a given harmonic frequency is given by

$$\rho_R(\vec{\boldsymbol{x}}, t) = \mathrm{Re}\left[\rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}\right]\,. \tag{7}$$

[1]Note that because $\rho(\vec{\boldsymbol{x}}, t)$ and $\rho(\vec{\boldsymbol{x}})$ are generally complex, neither $Q_{\ell m}(t)$ nor $Q_{\ell m}$ satisfies a condition analogous to eq. (3).

Inserting this result into eq. (2) yields

$$q_{\ell m}(t) = \int Y^*_{\ell m}(\theta,\phi)\, r^\ell \;\mathrm{Re}\left[\rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}\right]\, d^3x\,. \tag{8}$$

Using eqs. (4) and (8), it follows that[2]

$$\begin{aligned} q_{\ell m}(t) + q^*_{\ell m}(t) &= \int \left[Y^*_{\ell m}(\theta,\phi) + (-1)^m Y^*_{\ell,-m}(\theta,\phi)\right] r^\ell \;\mathrm{Re}\left[\rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}\right]\, d^3x \\ &= \mathrm{Re}\int \left[Y^*_{\ell m}(\theta,\phi) + (-1)^m Y^*_{\ell,-m}(\theta,\phi)\right] r^\ell\, \rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}\, d^3x \\ &= \mathrm{Re}\left[Q_{\ell m}(t) + (-1)^m Q_{\ell,-m}(t)\right]\,, \end{aligned} \tag{9}$$

and

$$\begin{aligned} i[q_{\ell m}(t) - q^*_{\ell m}(t)] &= \int i\left[Y^*_{\ell m}(\theta,\phi) - (-1)^m Y^*_{\ell,-m}(\theta,\phi)\right] r^\ell \;\mathrm{Re}\left[\rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}\right]\, d^3x \\ &= \mathrm{Re}\int i\left[Y^*_{\ell m}(\theta,\phi) - (-1)^m Y^*_{\ell,-m}(\theta,\phi)\right] r^\ell\, \rho(\vec{\boldsymbol{x}})\, e^{-i\omega t}\, d^3x \\ &= \mathrm{Re}\Big\{ i\left[Q_{\ell m}(t) - (-1)^m Q_{\ell,-m}(t)\right]\Big\} = -\,\mathrm{Im}\left[Q_{\ell m}(t) - (-1)^m Q_{\ell,-m}(t)\right]\,. \end{aligned}$$

In the last step above, we noted that for any complex number z, we have $\mathrm{Re}(iz) = -\,\mathrm{Im}\, z$. Thus,

$$q_{\ell m}(t) - q^*_{\ell m}(t) = i\,\mathrm{Im}\left[Q_{\ell m}(t) - (-1)^m Q_{\ell,-m}(t)\right]\,. \tag{10}$$

Combining eqs. (9) and (10) yields

$$q_{\ell m}(t) = \tfrac{1}{2}\left[Q_{\ell m}(t) + (-1)^m Q^*_{\ell,-m}(t)\right]\,. \tag{11}$$

Using eq. (5), we can rewrite the above result as

$$q_{\ell m}(t) = \tfrac{1}{2}\left[Q_{\ell m}\, e^{-i\omega t} + (-1)^m Q^*_{\ell,-m}\, e^{i\omega t}\right]\,. \tag{12}$$

Eq. (12) shows us how to obtain the physical multipole moments for a given harmonic frequency from the $Q_{\ell m}$. Note that $q_{\ell m}(t)$ given by eq. (12) automatically satisfies eq. (3).

For the case of rotating charges, we can choose the z axis to lie along the axis of rotation, which is perpendicular to the plane of rotation. In this case,

$$\rho_R(\vec{\boldsymbol{x}}, t) = \rho_R(r,\theta,\phi - \omega_0 t)\,,$$

where ω_0 is the angular frequency of rotation. Using eq. (2),

$$q_{\ell m}(t) = \int Y^*_{\ell m}(\theta,\phi)\, r^\ell\, \rho_R(r,\theta,\phi-\omega_0 t)\, d^3x\,.$$

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[2]Note that for any complex number z and real number c, we have $c\,\mathrm{Re}\, z = \mathrm{Re}(cz)$.

Defining a new variable $\phi' = \phi - \omega_0 t$,

$$
\begin{aligned}
q_{\ell m}(t) &= \int Y^*_{\ell m}(\theta, \phi' + \omega_0 t)\, r^\ell\, \rho_R(r,\theta,\phi')\, r^2 dr\, d\cos\theta\, d\phi' \\
&= \int Y^*_{\ell m}(\theta, \phi')\, e^{-im\omega_0 t}\, r^\ell\, \rho_R(r,\theta,\phi')\, r^2 dr\, d\cos\theta\, d\phi'\,, \qquad (13)
\end{aligned}
$$

where we have used the fact that $Y_{\ell m}(\theta,\phi) \propto e^{im\phi}$. The time-independent multipole moment is defined by

$$q_{\ell m} = \int Y^*_{\ell m}(\theta,\phi)\, r^\ell\, \rho_R(\vec{\boldsymbol{x}})\, d^3x\,.$$

Hence, eq. (13) yields

$$q_{\ell m}(t) = e^{-im\omega_0 t}\, q_{\ell m}\,. \qquad (14)$$

This result already corresponds to a single harmonic frequency, $\omega = |m|\,\omega_0$, so there is no need to perform a Fourier decomposition. Thus, eq. (12) is applicable and it follows that

$$e^{-im\omega_0 t}\, q_{\ell m} = \tfrac{1}{2}\left[Q_{\ell m}\, e^{-i\omega t} + (-1)^m Q^*_{\ell,-m}\, e^{i\omega t}\right]\,. \qquad (15)$$

Matching the coefficients of the harmonic terms, we first consider separately the cases of positive and negative m. For $m > 0$, it follows that

$$\omega = m\omega_0\,, \qquad Q_{\ell m} = 2q_{\ell m}\,, \qquad Q_{\ell,-m} = 0\,. \qquad (16)$$

whereas for $m < 0$, it follows that

$$\omega = -m\omega_0\,, \qquad Q_{\ell,-m} = 2(-1)^m q^*_{\ell m}\,, \qquad Q_{\ell m} = 0\,. \qquad (17)$$

In light of eq. (3), eqs. (16) and (17) are equivalent to:

$$\omega = |m|\omega_0\,, \qquad Q_{\ell,|m|} = 2q_{\ell,|m|}\,, \qquad Q_{\ell,-|m|} = 0\,, \qquad \text{for } m \neq 0\,. \qquad (18)$$

Finally, the case of $m = 0$ is not particularly interesting, since in this case there is no time dependence. Indeed, eq. (15) yields

$$\omega = 0\,, \qquad q_{\ell 0} = \mathrm{Re}\, Q_{\ell 0}\,, \qquad \text{for } m = 0\,. \qquad (19)$$

(b) Consider the charge density $\rho(\vec{\boldsymbol{x}}, t)$ that is periodic in time with period $T = 2\pi/\omega_0$. By making a Fourier *series* expansion, show that it can be written as

$$\rho(\vec{\boldsymbol{x}}, t) = \rho_0(\vec{\boldsymbol{x}}) + \sum_{n=1}^{\infty} \mathrm{Re}\left[2\rho_n(\vec{\boldsymbol{x}})\, e^{-in\omega_0 t}\right]\,, \qquad (20)$$

where

$$\rho_n(\vec{\boldsymbol{x}}) = \frac{1}{T}\int_0^T \rho(\vec{\boldsymbol{x}}, t)\, e^{in\omega_0 t}\, dt\,.$$

This shows explicitly how to establish connection with eq. (1).

The Fourier series for $\rho_R(\vec{\boldsymbol{x}},t)$ is given by:

$$\rho_R(\vec{\boldsymbol{x}},t) = \sum_{n=-\infty}^{\infty} \rho_n(\vec{\boldsymbol{x}})\, e^{-in\omega_0 t}\,. \tag{21}$$

Multiplying both sides of eq. (21) by $(1/T)e^{im\omega_0 t}$ and integrating from $t=0$ to T, we can use the orthonormality relation,

$$\frac{1}{T}\int_0^T e^{i(m-n)\omega_0 t}\, dt = \delta_{mn}\,,$$

to obtain

$$\rho_n(\vec{\boldsymbol{x}}) = \frac{1}{T}\int_0^T \rho_R(\vec{\boldsymbol{x}},t)\, e^{in\omega_0 t}\, dt\,.$$

We can rewrite eq. (21) by making use of the fact that $\rho_R(\vec{\boldsymbol{x}},t)$ is real. That is $\rho_R(\vec{\boldsymbol{x}},t)^* = \rho_R(\vec{\boldsymbol{x}},t)$, from which it follows that

$$\sum_{n=-\infty}^{\infty} \rho_n(\vec{\boldsymbol{x}})\, e^{-in\omega_0 t} = \sum_{n=-\infty}^{\infty} \rho_n^*(\vec{\boldsymbol{x}})\, e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} \rho_{-n}^*(\vec{\boldsymbol{x}})\, e^{-in\omega_0 t}\,,$$

where we have redefined the summation index by transforming $n \to -n$ at the last step. Hence, it follows that

$$\rho_{-n}(\vec{\boldsymbol{x}}) = \rho_n^*(\vec{\boldsymbol{x}})\,. \tag{22}$$

Inserting this result in eq. (21) then yields,

$$\begin{aligned}\rho_R(\vec{\boldsymbol{x}},t) &= \rho_0(\vec{\boldsymbol{x}}) + \sum_{n=1}^{\infty}\left[\rho_n(\vec{\boldsymbol{x}})\, e^{-in\omega_0 t} + \rho_n^*(\vec{\boldsymbol{x}})\, e^{in\omega_0 t}\right] \\ &= \rho_0(\vec{\boldsymbol{x}}) + \sum_{n=1}^{\infty} \mathrm{Re}\left[2\rho_n(\vec{\boldsymbol{x}})\, e^{-in\omega_0 t}\right]\,. \end{aligned} \tag{23}$$

Comparing with part (a) and eq. (1), when $n \neq 0$ we identify $\rho(\mathbf{x}) = 2\rho_n(\vec{\boldsymbol{x}})$ and $\omega = n\omega_0$. That is, if we focus on a particular harmonic $n \neq 0$, then eqs. (7) and (23) yields

$$\rho_R(\vec{\boldsymbol{x}},t) = \rho_n(\vec{\boldsymbol{x}})e^{-in\omega_0 t} + \rho_n^*(\vec{\boldsymbol{x}})e^{in\omega_0 t} = \tfrac{1}{2}\left[\rho(\vec{\boldsymbol{x}})e^{-i\omega t} + \rho^*(\vec{\boldsymbol{x}})e^{i\omega t}\right]\,. \tag{24}$$

Multiplying eq. (24) by $Y_{\ell m}^*(\theta,\phi)\, r^\ell$ and integrating over all of space, it follows that

$$q_{\ell m}(t) = \tfrac{1}{2}\left[Q_{\ell m}\, e^{-i\omega t} + (-1)^m Q_{\ell,-m}^*\, e^{i\omega t}\right]\,, \qquad \text{for } \omega \neq 0\,,$$

which reproduces eq. (12).

Finally, we note that in the case of $n=0$ (which corresponds to $\omega = 0$), eq. (22) implies that $\rho_0(\vec{\boldsymbol{x}})$ is real. Since $Y_{\ell 0}^*(\theta,\phi)$ is real, it follows from eq. (6) that in the case of zero frequency, $Q_{\ell,0}$ is real. In light of this observation, eq. (12) implies that

$$q_{\ell,0}(t) = Q_{\ell,0} \qquad \text{for } \omega = 0\,. \tag{25}$$

(c) For a single charge q rotating about the origin in the x–y plane in a circle of radius R at constant angular speed ω_0, calculate the $\ell = 0$ and $\ell = 1$ multipole moments by the methods of parts (a) and (b) and compare. In method (b) express the charge density $\rho_n(\vec{\boldsymbol{x}})$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

Consider a single charge q rotating about the origin in the x–y plane in a circle of radius R at constant angular speed ω_0 (by convention, $\omega_0 > 0$). Then, in cylindrical coordinates (r, ϕ, z) we can write

$$\rho_R(\vec{\boldsymbol{x}}, t) = \frac{q}{R}\,\delta(r-R)\delta(z)\delta(\phi-\omega_0 t)\,.$$

First, we use the method of part (a). We compute

$$\begin{aligned} q_{\ell m}(t) &= \int Y^*_{\ell m}(\theta,\phi)\, r^\ell\, \rho_R(\vec{\boldsymbol{x}},t)\, d^3x = \frac{q}{R}\int Y^*_{\ell m}(\theta,\phi)\, r^\ell\, \delta(r-R)\delta(z)\delta(\phi-\omega_0 t)\, r dr\, d\phi\, dz \\ &= q\, R^\ell\, Y^*_{\ell m}(\tfrac{1}{2}\pi, \omega_0 t)\,, \qquad (26) \end{aligned}$$

since $z = 0$ corresponds to a polar angle of $\theta = \frac{1}{2}\pi$. The spherical harmonic given in eq. (26) can be evaluated using[3]

$$P^m_\ell(0) = \begin{cases} (-1)^{(\ell+m)/2}\dfrac{(\ell+m)!}{(\ell-m)!!(\ell+m)!!}\,, & \text{for } \ell+m \text{ even},\\[6pt] 0\,, & \text{for } \ell+m \text{ odd}. \end{cases}$$

Then, by employing the definition of the spherical harmonics given in eq. (3.53) of Jackson, it immediately follows that:

$$Y_{\ell m}(\tfrac{1}{2}\pi, \phi) = \begin{cases} (-1)^{(\ell+m)/2}\left(\dfrac{2\ell+1}{4\pi}\right)^{1/2}\dfrac{[(\ell-m)!\,(\ell+m)!]^{1/2}}{(\ell-m)!!\,(\ell+m)!!}\,e^{im\phi}\,, & \text{for } \ell+m \text{ even}\,,\\[6pt] 0\,, & \text{for } \ell+m \text{ odd}\,. \end{cases} \qquad (27)$$

For $\ell = m = 0$, we have

$$q_{00}(t) = \frac{q}{\sqrt{4\pi}}\,,$$

which is a constant in time, as expected from charge conservation. For $\ell = 1$, we can have $m = -1, 0, +1$. Then,

$$q_{1m}(t) = q\, R\, Y^*_{1m}(\tfrac{1}{2}\pi, \omega_0 t)\,.$$

Inserting the explicit forms for the spherical harmonics,

$$q_{10}(t) = 0\,, \qquad q_{1,\pm 1}(t) = \mp\sqrt{\frac{3}{8\pi}}\, q\, R\, e^{\mp i\omega_0 t}\,. \qquad (28)$$

Using the results of part (a), we identify:

$$q_{1,m} = -mqR\sqrt{\frac{3}{8\pi}}\,, \qquad \text{for } m = -1, 0, +1\,.$$

[3] For example, see eq. (15.96) of George B. Arfken, Hans J. Weber and Frank E. Harris, *Mathematical Methods for Physicists*, 7th edition (Academic Press, Waltham, MA, 2013) p. 746.

Thus, eqs. (18) and (25) yield

$$\omega = \omega_0 \,, \qquad Q_{1,1} = -qR\sqrt{\frac{3}{2\pi}}\,, \qquad Q_{1,0} = Q_{1,-1} = 0\,. \tag{29}$$

Second, we use the method of part (b). We first evaluate the Fourier coefficients,

$$\begin{aligned}
\rho_n(\vec{\boldsymbol{x}}) &= \frac{1}{T}\int_0^T \rho(\vec{\boldsymbol{x}})\, e^{in\omega_0 t}\, dt = \frac{q}{RT}\,\delta(r-R)\delta(z)\int_0^T e^{in\omega_0 t}\delta(\phi-\omega_0 t)\, dt \\
&= \frac{q}{R\omega_0 T}\,\delta(r-R)\delta(z)\, e^{in\phi} = \frac{q}{2\pi R}\,\delta(r-R)\delta(z)\, e^{in\phi}\,,
\end{aligned}$$

after using $\omega_0 T = 2\pi$. Note that we have also used

$$\frac{1}{r}\,\delta(r-R) = \frac{1}{R}\,\delta(r-R)\,, \qquad \delta(\phi-\omega_0 t) = \frac{1}{\omega_0}\,\delta(t-\phi/\omega_0)\,,$$

in performing the integral over t.

We begin by computing the multipole moment Q_{00} defined in eq. (6),

$$Q_{00} = \frac{1}{\sqrt{4\pi}}\int \rho(\vec{\boldsymbol{x}})\, r dr\, d\phi\, dz\,, \tag{30}$$

where $\rho(\vec{\boldsymbol{x}})$ is the complex Fourier coefficient,

$$\rho(\vec{\boldsymbol{x}}) = \begin{cases} \dfrac{q}{2\pi R}\,\delta(r-R)\,\delta(z)\,, & \text{for } n = 0\,, \\ \dfrac{q}{\pi R}\,\delta(r-R)\,\delta(z)\, e^{in\phi}\,, & \text{for } n = 1,2,3,\ldots\,, \end{cases}$$

where we have used eq. (20). That is, for $n = 0$, we identify $\rho(\vec{\boldsymbol{x}}) = \rho_0(\vec{\boldsymbol{x}})$ and for $n = 1, 2, 3, \ldots$, we identify $\rho(\vec{\boldsymbol{x}}) = 2\rho_n(\vec{\boldsymbol{x}})$. Inserting the above result into eq. (30) and identifying the total charge, $Q = \sqrt{4\pi}\, Q_{00}$, we obtain

$$Q = \begin{cases} q\,, & \text{for } n = 0\,, \\ 0\,, & \text{for } n = 1,2,3,\ldots\,. \end{cases}$$

The time-dependent moment is given by $Q_{00}(t) = Q_{00}\, e^{-i\omega t} = Q_{00}\, e^{-in\omega_0 t}$. Thus, we expect $Q = 0$ for $n \neq 0$ due do the conservation of electric charge. The case of $n = 0$ corresponds to the zero frequency, which is the static case. Indeed we obtained $Q = q$ in this case, as this is the total charge of the system.

For $\ell = 1$, we compute,

$$Q_{1,m} = \int Y^*_{1,m}(\theta,\phi)\, r\, \rho(\vec{\boldsymbol{x}})\, r dr\, d\phi\, dz\,.$$

For $n \neq 0$ we obtain,[4]

$$Q_{1,m} = \frac{q}{\pi R}\int Y^*_{1,m}(\theta,\phi)\, r\,\delta(r-R)\,\delta(z)\, e^{in\phi}\, r dr\, d\phi\, dz = \frac{qR}{\pi}\int_0^{2\pi} Y^*_{1,m}(\tfrac{1}{2}\pi,\phi) e^{in\phi}\, d\phi\,,$$

[4]For the case of $n = 0$, simply multiply the expressions for $Q_{1,m}$ given above by $\frac{1}{2}$.

since $z = 0$ corresponds to $\theta = \frac{1}{2}\pi$. Using

$$Y_{1,\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{\pm i\phi}\,, \qquad\qquad Y_{10}(\theta,\phi) = \quad\sqrt{\frac{3}{4\pi}}\,\cos\theta\,,$$

it follows that $Q_{10} = 0$ and

$$Q_{1,\pm 1} = \mp\frac{qR}{\pi}\sqrt{\frac{3}{8\pi}}\int_0^{2\pi} e^{i(n\mp 1)\phi}\,d\phi = \mp qR\sqrt{\frac{3}{2\pi}}\delta_{n,\pm 1}\,.$$

Since n is a positive integer, we conclude that for $n = 1$ (which corresponds to $\omega = \omega_0$),

$$Q_{11} = -qR\sqrt{\frac{3}{2\pi}}\,, \qquad\qquad Q_{10} = Q_{1,-1} = 0\,. \tag{31}$$

For any value of non-negative integer value $n \neq 1$ (which corresponds to $\omega = n\omega_0$), we have $Q_{1m} = 0$ for $m = -1, 0, 1$. These results are equivalent to that of eq. (29), which was obtained based on the method of part (a).

Finally, we briefly examine the higher multipoles. In eq. (26), we found that

$$q_{\ell m}(t) = q\,R^\ell\,Y^*_{\ell m}(\tfrac{1}{2}\pi, \omega_0 t)\,,$$

where $Y^*_{\ell m}(\frac{1}{2}\pi, \omega_0 t) = A_{\ell m}e^{-im\omega_0 t}$ and the coefficients $A_{\ell m}$ were given in eq. (27). Thus, all the multipoles are present. Furthermore, since $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$ and the relevant frequencies are obtained by Fourier analysis by identifying the coefficients of $e^{-i\omega t}$ (where $\omega > 0$ by convention), we conclude that for the ℓth multipole, the contributing frequencies are:

$$\omega = n\omega_0\,, \qquad \text{for } n = 1, 2, \ldots, m, \text{ where } \ell + n \text{ is even}\,.$$

The corresponding values of the $Q_{\ell m}$ are easily computed either by the method of part (a) or part (b) of this problem. The end result is that for a fixed value of n, the only non-zero value is $Q_{\ell n}$ for $\ell + n$ even and n positive.[5]

Applying these results to the quadrupole, we see that only $n = 2$ is possible. Thus, $\omega = 2\omega_0$ and Q_{22} is the only non-vanishing component of Q_{2m} [where $m = -2, -1, 0, +1, +2$]. If I allow for static multipoles (corresponding to $n = 0$ or equivalently to $\omega = 0$), then the allowed non-vanishing static moments are $q_{2\ell,0}$ for $\ell = 0, 1, 2, 3, \ldots$. We have already noted above the case of $\ell = 0$, where q_{00} is proportional to the total static charge. In the case of $\ell = 2$, we also have $q_{20} \neq 0$.

REMARK:

To gain further insight into the case of $\ell = 1$, let us rewrite $q_{1,m}(t)$ in terms of the real physical electric dipole vector [following eq. (4.5) of Jackson],

$$q_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}(p_x \mp ip_y)\,, \qquad\qquad q_{10} = \sqrt{\frac{3}{4\pi}}\,p_z\,. \tag{32}$$

[5]The condition that $\ell + n$ is even arises because $Y_{\ell m}(\frac{1}{2}\pi, \phi) = 0$ if $\ell + m$ is odd.

In light of eq. (28), the components of the electric dipole vector are given by,

$$\vec{\boldsymbol{p}} = qR(\cos\omega_0 t\,,\,\sin\omega_0 t\,,\,0)\,,$$

as expected for a rotating charge q in the x–y plane. One can rewrite this result as:

$$\vec{\boldsymbol{p}} = \mathrm{Re}\left\{qR\,e^{-i\omega_0 t}\,(1\,,\,i\,,\,0)\right\}. \tag{33}$$

Thus, we may define a *complex* electric dipole vector,

$$\vec{\boldsymbol{p}}(t) = \vec{\boldsymbol{p}}\,e^{-i\omega_0 t}\,, \qquad \text{where } \vec{\boldsymbol{p}} = qR\,(1\,,\,i\,,\,0)\,.$$

The physical electric dipole vector is then given by $\mathrm{Re}\,\vec{\boldsymbol{p}}(t) = \mathrm{Re}(\vec{\boldsymbol{p}}\,e^{-i\omega_0 t})$.

The complex electric dipole vector is related to $Q_{1,m}$ in the usual way [cf. eq. (32)]:

$$Q_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}(p_x \mp i p_y)\,, \qquad Q_{10} = \sqrt{\frac{3}{4\pi}}\,p_z\,, \tag{34}$$

where $(p_x\,,\,p_y\,,p_z)$ in eq. (34) is the complex vector, $\vec{\boldsymbol{p}} = qR(1\,,\,i\,,\,0)$, in contrast to eq. (32) where $(p_x\,,\,p_y\,,p_z)$ stands for the real physical electric dipole vector. That is, we must insert $p_x = qR$, $p_y = iqR$ and $p_z = 0$ in eq. (34), which yields

$$Q_{11} = -qR\sqrt{\frac{3}{2\pi}}\,, \qquad Q_{10} = Q_{1,-1} = 0\,,$$

in agreement with our previous results [cf. eqs. (29) and (31)].