(c) For a single charge $q$ rotating about the origin in the $x$-$y$ plane in a circle of radius $R$ at constant angular speed $\omega_0$, calculate the $l = 0$ and $l = 1$ multipole moments by the methods of parts a and b and compare. In method b express the charge density $\rho_a(x)$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

9.2 A radiating quadrupole consists of a square of side $a$ with charges $\pm q$ at alternate corners. The square rotates with angular velocity $\omega$ about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of radiation, and the total radiated power, all in the long-wavelength approximation. What is the frequency of the radiation?

9.3 Two halves of a spherical metallic shell of radius $R$ and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos \omega t$. In the long-wavelength limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

9.4 Apply the approach of Problem 9.1b to the current and magnetization densities of the particle of charge $q$ rotating about the origin in the $x$-$y$ plane in a circle of radius $R$ at constant angular speed $\omega_0$. The motion is such that $\omega_0 R \ll c$.

(a) Find $(J_x)_n$, $(J_y)_n$, and $(J_z)_n$ in terms of cylindrical coordinates for all $n$. Also determine the components of the orbital “magnetization,” $(\mathbf{x} \times \mathbf{J}_n)/2$, and its divergence [which plays the role of a magnetic charge density for magnetic multipoles, as in $M_{\text{mn}}$ (9.172)].

(b) What long-wavelength magnetic multipoles $(l, m)$ occur and at what frequencies? [Remember that the multipole order $l$ does not necessarily equal the harmonic number $n$.]

(c) Use linear superposition to generalize your argument to the four charges rotating in Problem 9.2 at radius $R = a/\sqrt{2}$. What harmonics occur, and what magnetic multipoles at each harmonic? Is there a magnetic multipole contribution at the $E2$ frequency of Problem 9.2? Is it significant relative to the $E2$ radiation?

9.5 (a) Show that for harmonic time variation at frequency $\omega$ the electric dipole scalar and vector potentials in the Lorenz gauge and the long-wavelength limit are

$$\Phi(x) = \frac{e^{ikr}}{4\pi\epsilon_0 r^2} \mathbf{n} \cdot \mathbf{p}(1 - ikr)$$

$$\mathbf{A}(x) = -i \frac{\mu_0 \omega}{4\pi} \frac{e^{ikr}}{r} \mathbf{p} \quad [\text{this is (9.16)}]$$

where $k = \omega/c$, $\mathbf{n}$ is a unit vector in the radial direction, $\mathbf{p}$ is the dipole moment (9.17), and the time dependence $e^{-i\omega t}$ is understood.

(b) Calculate the electric and magnetic fields from the potentials and show that they are given by (9.18).

9.6 (a) Starting from the general expression (9.2) for $\mathbf{A}$ and the corresponding expression for $\Phi$, expand both $R = |x - x'|$ and $t' = t - R/c$ to first order in $|x'|/r$ to obtain the electric dipole potentials for arbitrary time variation

$$\Phi(x, t) = \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{r^2} \mathbf{n} \cdot \mathbf{p}_{\text{ext}} + \frac{1}{cr} \mathbf{n} \cdot \frac{\partial \mathbf{p}_{\text{ext}}}{\partial t} \right]$$

$$\mathbf{A}(x, t) = \frac{\mu_0}{4\pi r} \frac{\partial \mathbf{p}_{\text{ext}}}{\partial t}$$
where \( \mathbf{p}_{\text{ret}} = \mathbf{p}(t' = t - r/c) \) is the dipole moment evaluated at the retarded time measured from the origin.

(b) Calculate the dipole electric and magnetic fields directly from these potentials and show that

\[
\mathbf{B}(x, t) = \frac{\mu_0}{4\pi} \mathbf{n} \times \left( -\frac{1}{cr} \frac{\partial \mathbf{p}_{\text{ret}}}{\partial t} - \frac{1}{c^2 r} \mathbf{n} \times \frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2} \right)
\]

\[
\mathbf{E}(x, t) = \frac{1}{4\pi \varepsilon_0} \left\{ \left( 1 + \frac{r}{c} \frac{\partial}{\partial t} \right) \left[ \frac{3n(n \cdot \mathbf{p}_{\text{ret}})}{r^3} - \mathbf{p}_{\text{ret}} \right] + \frac{1}{c^2 r} \mathbf{n} \times \left( \mathbf{n} \times \frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2} \right) \right\}
\]

(c) Show explicitly how you can go back and forth between these results and the harmonic fields of (9.18) by the substitutions \(-i\omega \leftrightarrow \partial/\partial t \) and \( \mathbf{p}^{\text{elec}} - \text{rot} \leftrightarrow \mathbf{p}_{\text{ret}}(t') \).

9.7 (a) By means of Fourier superposition of different frequencies or equivalent means, show for a real electric dipole \( \mathbf{p}(t) \) that the instantaneous radiated power per unit solid angle at a distance \( r \) from the dipole in a direction \( \mathbf{n} \) is

\[
\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2} \left\| \mathbf{n} \times \frac{d^2 \mathbf{p}}{dt^2} (t') \right\| \times \mathbf{n} \]

where \( t' = t - r/c \) is the retarded time. For a magnetic dipole \( \mathbf{m}(t) \), substitute \((1/c)\mathbf{m} \times \mathbf{n} \) for \((\mathbf{n} \times \dot{\mathbf{p}}) \times \mathbf{n} \).

(b) Show similarly for a real quadrupole tensor \( Q_{\alpha\beta}(t) \) given by (9.41) with a real charge density \( \rho(x, t) \) that the instantaneous radiated power per unit solid angle is

\[
\frac{dP(t)}{d\Omega} = \frac{Z_0}{576\pi^2 c^4} \left\| \mathbf{n} \times \frac{d^2 \mathbf{Q}}{dt^3} (n, t') \right\| \times \mathbf{n} \]

where \( \mathbf{Q}(n, t) \) is defined by (9.43).

9.8 (a) Show that a classical oscillating electric dipole \( \mathbf{p} \) with fields given by (9.18) radiates electromagnetic angular momentum to infinity at the rate

\[
\frac{dL}{dt} = \frac{k^3}{12\pi \varepsilon_0} \text{Im}[\mathbf{p}^* \times \mathbf{p}]
\]

(b) What is the ratio of angular momentum radiated to energy radiated? Interpret.

(c) For a charge \( e \) rotating in the \( x-y \) plane at radius \( a \) and angular speed \( \omega \), show that there is only a \( z \) component of radiated angular momentum with magnitude \( dL_z/dt = e^2 k^3 \omega^2/6\pi \varepsilon_0 \). What about a charge oscillating along the \( z \) axis?

(d) What are the results corresponding to parts a and b for magnetic dipole radiation?

Hint: The electromagnetic angular momentum density comes from more than the transverse (radiation zone) components of the fields.

9.9 (a) From the electric dipole fields with general time dependence of Problem 9.6, show that the total power and the total rate of radiation of angular momentum through a sphere at large radius \( r \) and time \( t \) are

\[
P(t) = \frac{1}{6\pi \varepsilon_0 c^3} \left( \frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2} \right)^2
\]

\[
\frac{dL_{\text{em}}}{dt} = \frac{1}{6\pi \varepsilon_0 c^3} \left( \frac{\partial \mathbf{p}_{\text{ret}}}{\partial t} \times \frac{\partial^2 \mathbf{p}_{\text{ret}}}{\partial t^2} \right)
\]

where the dipole moment \( \mathbf{p} \) is evaluated at the retarded time \( t' = t - r/c \).
the angular distribution of radiation, and the total power radiated. Assume that \(ka \ll 1\).

**9.12** An almost spherical surface defined by

\[ R(\theta) = R_0[1 + \beta P_2(\cos \theta)] \]

has inside of it a uniform volume distribution of charge totaling \(Q\). The small parameter \(\beta\) varies harmonically in time at frequency \(\omega\). This corresponds to surface waves on a sphere. Keeping only lowest order terms in \(\beta\) and making the long-wavelength approximation, calculate the nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated.

**9.13** The uniform charge density of Problem 9.12 is replaced by a uniform density of intrinsic magnetization parallel to the \(z\) axis and having total magnetic moment \(M\). With the same approximations as above calculate the nonvanishing radiation multipole moments, the angular distribution of radiation, and the total power radiated.

**9.14** An antenna consists of a circular loop of wire of radius \(a\) located in the \(x-y\) plane with its center at the origin. The current in the wire is

\[ I = I_0 \cos \omega t = \Re I_0 e^{-i\omega t} \]

(a) Find the expressions for \(\mathbf{E}, \mathbf{H}\) in the radiation zone without approximations as to the magnitude of \(ka\). Determine the power radiated per unit solid angle.

(b) What is the lowest nonvanishing multipole moment \((Q_{l,m}\) or \(M_{l,m}\))? Evaluate this moment in the limit \(ka \ll 1\).

**9.15** Two fixed electric dipoles of dipole moment \(p\) are located in the \(x-y\) plane a distance \(2a\) apart, their axes parallel and perpendicular to the plane, but their moments directed oppositely. The dipoles rotate with constant angular speed \(\omega\) about a \(z\) axis located halfway between them. The motion is nonrelativistic \((\omega a/c \ll 1\).

(a) Find the lowest nonvanishing multipole moments.

(b) Show that the magnetic field in the radiation zone is, apart from an overall phase factor,

\[ \mathbf{H} = \frac{cpa}{2\pi} k^3[(\hat{x} + i\hat{y})\cos \theta - \hat{z}\sin \theta e^{i\phi}] \cos \theta \frac{e^{ikr}}{r} \]

(c) Show that the angular distribution of the radiation is proportional to \((\cos^2 \theta + \cos^4 \theta)\) and the total time-averaged power radiated is

\[ P = \frac{4}{15\pi\varepsilon_0} ck p^2 a^2 \]

*Hint:* Problem 6.21 is relevant.

**9.16** A thin linear antenna of length \(d\) is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in the figure.

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**Problem 9.16**

(a) Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

(b) Determine the total power radiated and find a numerical value for the radiation resistance.
9.17 Treat the linear antenna of Problem 9.16 by the multipole expansion method.

(a) Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long-wavelength approximation.

(b) Compare the shape of the angular distribution of radiated power for the lowest nonvanishing multipole with the exact distribution of Problem 9.16.

(c) Determine the total power radiated for the lowest multipole and the corresponding radiation resistance using both multipole moments from part a. Compare with Problem 9.16b. Is there a paradox here?

9.18 A qualitative understanding of the result for the reactance of a short antenna whose radiation fields are described by the electric dipole fields of Section 9.2 can be achieved by considering the idealized dipole fields (9.18).

(a) Show that the integral over all angles at fixed distance \( r \) of \( \varepsilon_0 |E|^2 - \mu_0 |H|^2 \) is

\[
\int \left[ \varepsilon_0 |E|^2 - \mu_0 |H|^2 \right] d\Omega = \frac{1}{2 \pi \varepsilon_0} \frac{|p|^2}{r^6}
\]

(b) Using (6.140) for the reactance, show that the contribution \( X_a \) to the reactance from fields at distances \( r > a \) is

\[
X_a = -\frac{\omega}{6 \pi \varepsilon_0} \frac{|p|^2}{|I|^2 a^3}
\]

where \( I \) is the input current.

(c) For the short center-fed antenna of Section 9.2 show that \( X_a = -d^3/24\pi \varepsilon_0 \omega a^3 \), corresponding to an effective capacitance \( 24\pi \varepsilon_0 a^3/d^2 \). With \( a = d/2 \), \( X_a \) gives only a small fraction of the total negative reactance of a short antenna. The fields close to the antenna, obviously not dipole in character, contribute heavily. For calculations of reactances of short antennas, see the book by Schelkunoff and Friis.

9.19 Consider the excitation of a waveguide in Problem 8.19 from the point of view of multipole moments of the source.

(a) For the linear probe antenna calculate the multipole moment components of \( p, m, Q_{\alpha\beta}, Q_{\alpha}^{M} \) that enter (9.69).

(b) Calculate the amplitudes for excitation of the \( TE_{10} \) mode and evaluate the power flow. Compare the multipole expansion result with the answer given in Problem 8.19b. Discuss the reasons for agreement or disagreement. What about the comparison for excitation of other modes?

9.20 (a) Verify by direct calculation that the static tangential electric field (3.186) in a circular opening in a flat conducting plane, when inserted into the defining equation (9.72) for the electric dipole moment \( p_{\text{eff}} \), leads to the expression (9.75).

(b) Determine the value of \( i\mu_0 \omega m_{\text{eff}} \) given by (9.72) with the static electric field in part a.

(c) Use the static normal magnetic field (5.132) for the corresponding magnetic boundary problem with a circular opening to compute via (9.74) the magnetic dipole moment \( m_{\text{eff}} \) and compare with (9.75).

(d) Comment on the differences between the results of parts b and c and the use of the definitions (9.72) in a consistent fashion. [See Section 9 of the article, Diffraction Theory, by C. J. B. Bouwkamp in Reports on Progress in Physics, Vol. 17, ed. A. C. Strickland, The Physical Society, London (1954).]