The Sokhotski-Plemelj Formula

1. The Sokhotski-Plemelj formula

The Sokhotski-Plemelj formula is a relation between the following generalized functions (also called *distributions*),

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\epsilon}=\mathrm{P}\frac{1}{x}\mp i\pi\delta(x)\,,\tag{1}$$

where $\epsilon>0$ is an infinitesimal quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin, and then integrated over a range of x containing the origin. We shall also assume that $f(x)\to 0$ sufficiently fast as $x\to\pm\infty$ in order that integrals evaluated over the entire real line are convergent. Moreover, all surface terms at $\pm\infty$ that arise when integrating by parts are assumed to vanish.

To establish eq. (1), we shall prove that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x\pm i\epsilon}=\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x}\mp i\pi f(0)\,,\tag{2}$$

where the Cauchy principal value integral is defined as:

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x}\equiv\lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\,dx}{x}+\int_{\delta}^{\infty}\frac{f(x)\,dx}{x}\right\}\,,\tag{3}$$

assuming $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x|\to\infty$.

In these notes, I will provide three different derivations of eq. (2). The first derivation is a mathematically non-rigorous proof of eq. (2), which should at least provide some insight into the origin of this result. A more rigorous derivation starts with a contour integral in the complex plane,

$$\int_C\frac{f(z)\,dz}{z}\,.$$

By defining C appropriately, we will obtain two different expressions for this integral. Setting the two resulting expressions equal yields eq. (2) with the upper sign. Complex conjugating this result yields eq. (2) with the lower sign. Finally, an elegant third proof makes direct use of the theory of distributions. Finally, a useful check is to consider the Fourier transform of eq. (1), as discussed in the Appendix to these notes.

Note that eq. (1) can be generalized as follows,

$$\lim_{\varepsilon\to 0}\frac{1}{x-x_0\pm i\varepsilon}=\mathrm{P}\frac{1}{x-x_0}\mp i\pi\delta(x-x_0)\,,$$

where

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x-x_0}\equiv\lim_{\delta\to 0}\left\{\int_{-\infty}^{x_0-\delta}\frac{f(x)\,dx}{x-x_0}+\int_{x_0+\delta}^{\infty}\frac{f(x)\,dx}{x-x_0}\right\}\,.$$

The derivations presented below are easily modified to encompass the above generalization.

2. A non-rigorous derivation of the Sokhotski-Plemelj formula

We begin with the identity,

$$\frac{1}{x \pm i\epsilon} = \frac{x \mp i\epsilon}{x^2 + \epsilon^2}\,,$$

where ϵ is a positive infinitesimal quantity. Thus, for any smooth function that is non-singular in a neighborhood of the origin,

$$\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x \pm i\epsilon} = \int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} \mp i\epsilon \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+\epsilon^2}\,. \tag{4}$$

The first integral on the right had side of eq. (4),

$$\int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} = \int_{-\infty}^{-\delta} \frac{x f(x)\,dx}{x^2+\epsilon^2} + \int_{\delta}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} + \int_{-\delta}^{\delta} \frac{x f(x)\,dx}{x^2+\epsilon^2}\,. \tag{5}$$

In the first two integrals on the right hand side of eq. (5), it is safe to take the limit $\epsilon \to 0$. In the third integral on the right hand side of eq. (5), if δ is small enough, then we can approximate $f(x) \simeq f(0)$ for values of $|x| < \delta$. Hence, eq. (5) yields,

$$\int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} = \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta} \frac{f(x)\,dx}{x} + \int_{\delta}^{\infty} \frac{f(x)\,dx}{x}\right\} + f(0)\int_{-\delta}^{\delta} \frac{x\,dx}{x^2+\epsilon^2}\,. \tag{6}$$

However,

$$\int_{-\delta}^{\delta} \frac{x\,dx}{x^2+\epsilon^2} = 0\,,$$

since the integrand is an odd function of x that is being integrated symmetrically about the origin, and

$$P\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} \equiv \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta} \frac{f(x)\,dx}{x} + \int_{\delta}^{\infty} \frac{f(x)\,dx}{x}\right\}\,,$$

defines the principal value integral. Hence, eq. (6) yields

$$\int_{-\infty}^{\infty} \frac{x f(x)\,dx}{x^2+\epsilon^2} = P\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x}\,. \tag{7}$$

Next, we consider the second integral on the right hand side of eq. (4). Since ϵ is an infinitesimal quantity, the only significant contribution from

$$\epsilon\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+\epsilon^2}$$

can come from the integration region where $x \simeq 0$, where the integrand behaves like ϵ^{-2}. Thus, we can again approximate $f(x) \simeq f(0)$, in which case we obtain

$$\epsilon\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x^2+\epsilon^2} \simeq \epsilon f(0)\int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \pi f(0)\,, \tag{8}$$

where we have made use of

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+\epsilon^2} = \frac{1}{\epsilon}\tan^{-1}(x/\epsilon)\bigg|_{-\infty}^{\infty} = \frac{\pi}{\epsilon}\,.$$

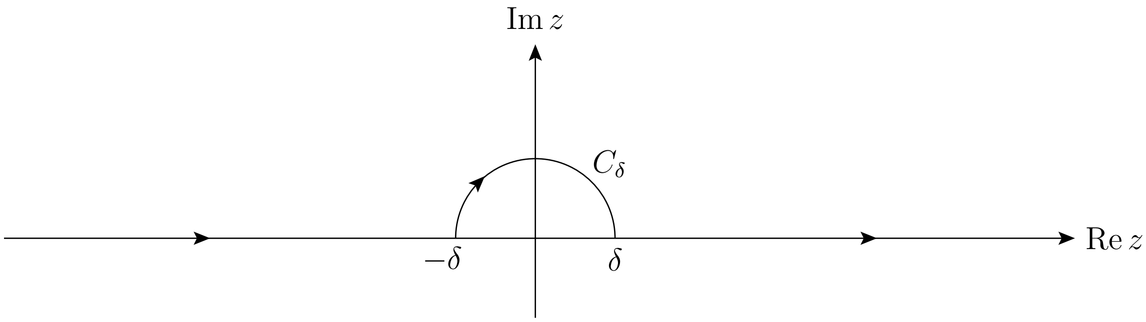
Using the results of eqs. (7) and (8), we see that eq. (4) yields,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x\pm i\epsilon} = P\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} \mp i\pi f(0)\,, \qquad (9)$$

which establishes eq. (2).

2. A more rigorous derivation of the Sokhotski-Plemelj formula

We consider the following path of integration in the complex plane, denoted by C, shown below.



That is, C is the contour along the real axis from $-\infty$ to $-\delta$, followed by a semicircular path C_δ (of radius δ), followed by the contour along the real axis from δ to ∞. The infinitesimal quantity δ is assumed to be positive. Then

$$\int_C \frac{f(x)}{x}\,dx = P\int_{-\infty}^{\infty}\frac{f(x)}{x} + \int_{C_\delta}\frac{f(x)}{x}\,dx\,, \qquad (10)$$

where the principal value integral is defined in eq. (3). In the limit of $\delta\to 0$, we can approximate $f(x)\simeq f(0)$ in the last integral on the right hand side of eq. (10). Noting that the contour C_δ can be parameterized as $x=\delta e^{i\theta}$ for $0\leq\theta\leq\pi$, we end up with

$$\lim_{\delta\to 0}\int_{C_\delta}\frac{f(x)}{x}\,dx = f(0)\lim_{\delta\to 0}\int_{\pi}^{0}\frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\,d\theta = -i\pi f(0)\,.$$

Hence,

$$\int_C \frac{f(x)}{x}\,dx = P\int_{-\infty}^{\infty}\frac{f(x)}{x} - i\pi f(0)\,. \qquad (11)$$

We can also evaluate the left hand side of eq. (11) by deforming the contour C to a contour C' that consists of a straight line that runs from $-\infty+i\varepsilon$ to $\infty+i\varepsilon$, where ε is a positive infinitesimal (of the same order of magnitude as δ). Assuming that $f(x)$ has no singularities in an infinitesimal neighborhood around the real axis, we are free to deform the contour C into C' without changing the value of the integral. It follows that

$$\int_C \frac{f(x)}{x}\,dx = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{f(x)}{x}\,dx = \int_{-\infty}^{\infty} \frac{f(y+i\varepsilon)}{y+i\varepsilon}\,dy\,, \tag{12}$$

where in the last step we have made a change of the integration variable.

Since ε is infinitesimal, we can approximate $f(y+i\varepsilon) \simeq f(y)$.[1] Thus, after relabeling the integration variable y as x, eq. (12) yields

$$\int_C \frac{f(x)}{x}\,dx = \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\,dx\,. \tag{13}$$

Inserting this result back into eq. (11) yields

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\,dx = P\int_{-\infty}^{\infty} \frac{f(x)}{x} - i\pi f(0)\,. \tag{14}$$

Eq. (14) is also valid if $f(x)$ is replaced by $f^*(x)$. We can then take the complex conjugate of the resulting equation. The end result is[2]

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x\pm i\varepsilon}\,dx = P\int_{-\infty}^{\infty} \frac{f(x)}{x} \mp i\pi f(0)\,,$$

in agreement with eq. (9).

3. An elegant derivation of the Sokhotski-Plemelj formula

Starting from the definition of the Cauchy principal value given in eq. (3), we integrate by parts to obtain

$$\int_{-\infty}^{-\delta} \frac{f(x)}{x}\,dx = f(x)\ln|x|\Big|_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx = f(-\epsilon)\ln\epsilon - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx\,,$$

$$\int_{\delta}^{\infty} \frac{f(x)}{x}\,dx = f(x)\ln|x|\Big|_{-\infty}^{-\delta} - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx = -f(\epsilon)\ln\epsilon - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\,,$$

where $f'(x) \equiv df/dx$ and we have assumed that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ so that the surface terms at $\pm\infty$ vanish. Hence,

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} = \lim_{\delta\to 0}\left\{\big[f(-\delta) - f(\delta)\big]\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\right\}. \tag{15}$$

[1] More precisely, we can expand $f(y+i\varepsilon)$ in a Taylor series about $\varepsilon = 0$ to obtain $f(y+i\varepsilon) = f(y) + \mathcal{O}(\varepsilon)$. At the end of the calculation, we may take $\varepsilon \to 0$, in which case the $\mathcal{O}(\varepsilon)$ terms vanish.

[2] Alternatively, we can repeat the above derivation where the contour C_δ is replaced by a semicircle of radius δ in the lower half complex plane, which yields eq. (11) with $-i$ replaced by i. Finally, after deforming the contour of integration to a new contour that consists of a straight line that runs from $-\infty - i\varepsilon$ to $\infty - i\varepsilon$, one obtains eq. (13) with i replaced by $-i$.

Since $f(x)$ is differentiable and well behaved, we can define

$$g(x) \equiv \int_0^1 f'(xt)dt = \frac{f(x)-f(0)}{x}\,,$$

which implies that $g(x)$ is smooth and non-singular and

$$f(x) = f(0) + xg(x)\,. \tag{16}$$

Inserting eq. (16) back into eq. (15) then yields

$$\begin{aligned} \mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} &= \lim_{\delta\to 0}\left\{\left[-2g(x)\delta\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\,dx - \int_{\delta}^{\infty} f'(x)\ln|x|\,dx\right\}\right. \\ &= -\int_{-\infty}^{\infty} f'(x)\ln|x|\,dx\,. \end{aligned}$$

Note that $\ln|x|$ is integrable at $x = 0$, so that the last integral is well-defined. Finally, we integrate by parts and drop the surface terms at $\pm\infty$ (under the usual assumption that $f'(x) \to 0$ sufficiently fast as $x \to \infty$). The end result is

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x} = \int_{-\infty}^{\infty} f(x)\frac{d}{dx}\ln|x|\,dx\,.$$

That is, we have proven that as distributions,

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}\,. \tag{17}$$

We can employ eq. (17) to provide a very elegant derivation of eq. (1). We begin with the definition of the principal value of the complex logarithm,

$$\mathrm{Ln}\ z = \ln|z| + i\arg\ z\,,$$

where arg z is the principal value of the argument (or phase) of the complex number z, with the convention that $-\pi < \arg z \leq \pi$. In particular, for real x and a positive infinitesimal ϵ,

$$\lim_{\epsilon\to 0}\mathrm{Ln}(x \pm i\epsilon) = \ln|x| \pm i\pi\Theta(-x)\,, \tag{18}$$

where $\Theta(x)$ is the Heaviside step function. Differentiating eq. (18) with respect to x immediately yields the Sokhotski-Plemelj formula,[3]

$$\lim_{\varepsilon\to 0}\frac{1}{x \pm i\epsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x)\,, \tag{19}$$

where we have used eq. (17) and

$$\frac{d}{dx}\Theta(-x) = -\frac{d}{dx}\Theta(x) = -\delta(x)\,.$$

[3]The derivative of the complex logarithm is $d\,\mathrm{Ln}\ z/dz = 1/z$ for $z \neq 0$.

Appendix: Fourier transforms of distributions

Eqs. (17) and (19), which we repeat below

$$\frac{d}{dx}\ln|x| = \mathrm{P}\frac{1}{x}\,, \tag{20}$$

$$\lim_{\varepsilon\to 0}\frac{1}{x\pm i\varepsilon} = \mathrm{P}\frac{1}{x}\mp i\pi\delta(x)\,, \tag{21}$$

are only meaningful when multiplied by a test function $f(x)$ and integrated over a region of the real line that includes the point $x=0$. In the theory of *tempered distributions*, test functions must be infinitely differentiable and vanish at $\pm\infty$ faster than any inverse power of x. Clearly, e^{ikx} does not satisfy this requirement for a test function. Nevertheless, one can define Fourier transforms of tempered distributions by using the well known property of the Fourier transform,

$$\int_{-\infty}^{\infty}\widetilde{f}(k)g(k)\,dk = \int_{-\infty}^{\infty} f(k)\widetilde{g}(k)\,dx\,, \tag{22}$$

where

$$\widetilde{f}(k)\equiv\int_{-\infty}^{\infty} f(x)e^{ikx}\,dx\,.$$

If $f(x)$ is a tempered distribution and $g(x)$ is a test function, then if follows that $\widetilde{g}(x)$ exists and is well defined. The Fourier transform of $f(x)$, denoted by $\widetilde{g}(k)$, is *defined* via eq. (22).

One can now check the validity of eqs. (20) and (21) by computing their Fourier transforms. To compute the Fourier transform of eq. (20), we make use of the property of Fourier transforms that

$$\int_{-\infty}^{\infty}\frac{df(x)}{dx}e^{ikx}\,dx = -ik\widetilde{f}(k)\,.$$

Hence,

$$\int_{-\infty}^{\infty}\frac{d}{dx}\ln|x|\,e^{ikx}\,dx = -ik\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx\,. \tag{23}$$

The calculation of the right-hand side of eq. (23) is rather involved, since it only exists in the sense of distributions. One can show that[4]

$$\int_{-\infty}^{\infty}\ln|x|\,e^{ikx}\,dx = -\pi\left[\mathrm{Pf}\frac{1}{|k|}+2\gamma\delta(k)\right]\,, \tag{24}$$

where γ is the Euler-Mascheroni constant, and the distribution $\mathrm{Pf}(1/|k|)$ is defined as

$$\int_{-\infty}^{\infty} f(k)\,\mathrm{Pf}\frac{1}{|k|}\,dk \equiv \int_{-\infty}^{-1}\frac{f(k)}{|k|}\,dk + \int_{-1}^{1}\frac{f(k)-f(0)}{|k|}\,dk + \int_{1}^{\infty}\frac{f(k)}{|k|}\,dk\,, \tag{25}$$

for any valid test function $f(k)$.

[4]See, e.g., Ram P. Kanwal, *Generalized Functions: Theory and Applications*, Third edition (Birkhäuser, Boston, 2004) pp. 153–154 and pp. 160–161. There are two typographical errors on these pages. In eq. (6.4.33d), $1/u$ should be $1/|u|$ and in the last term in eq. (6.4.57), $-i(u-i0)^{-1}$ should be $+i(u-i0)^{-1}$. Eq. (24) is a consequence of the corrected eq. (6.4.57).

Inserting the result of eq. (24) into eq. (23) and using $k\delta(k)=0$ and[5]

$$k\left(\mathrm{Pf}\frac{1}{|k|}\right)=\frac{k}{|k|}=\mathrm{sgn}(k)\,,\tag{26}$$

the end result is given by,

$$\int_{-\infty}^{\infty}\frac{d}{dx}\ln|x|\,e^{ikx}\,dx=i\pi\,\mathrm{sgn}(k)\,.\tag{27}$$

Next, we consider

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x}\,dx=\mathrm{P}\int_{-\infty}^{\infty}\frac{\cos(kx)}{x}\,dx+i\mathrm{P}\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx\,.\tag{28}$$

Since $\cos(kx)/x$ is an odd function of x (i.e., it changes sign under $x\to -x$), it immediately follows from the definition of the Cauchy principle value that

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{\cos(kx)}{x}\,dx=0\,.\tag{29}$$

Next, we observe that $\lim_{x\to 0}\sin(kx)/x=k$; that is, $\sin(kx)/x$ is regular at $x=0$. Thus,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx=\int_{-\infty}^{\infty}\frac{\sin(kx)}{x}\,dx=\mathrm{sgn}(k)\int_{-\infty}^{\infty}\frac{\sin y}{y}\,dy=\pi\,\mathrm{sgn}(k)\,.\tag{30}$$

Note that the P symbol has no effect on the integral given by eq. (30), since the integrand is regular at $x=0$. The factor of $\mathrm{sgn}(k)$ arises after changing the integration variable, $y=kx$. When $k<0$, the integration limits must be reversed, which then leads to the extra sign. Inserting eqs. (29) and (30) into eq. (28) then yields,

$$\mathrm{P}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x}\,dx=i\pi\,\mathrm{sgn}(k)\,.\tag{31}$$

In light of eqs. (27) and (31), we have verified that the Fourier transform of eq. (20) is satisfied.

Likewise, we can verify that the Fourier transform of eq. (21) is satisfied. The following result is required,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{e^{ikx}}{x\pm i\varepsilon}\,dx=\mp 2\pi i\Theta(\mp k)\,,\tag{32}$$

which was derived in Solution Set 1 in Physics 215. Then, using eq. (31) and employing the two identities, $\mathrm{sgn}(k)=\Theta(k)-\Theta(-k)$ and $1=\Theta(k)+\Theta(-k)$, it follows that the Fourier transform of eq. (21) is

$$\mp 2\pi i\Theta(\mp k)=i\pi\big[\Theta(k)-\Theta(-k)\big]\mp i\pi\big[\Theta(k)+\Theta(-k)\big]\,.\tag{33}$$

It is a simple matter to check that eq. (33) is satisfied for either choice of sign.

Since the Fourier transform of a tempered distribution and its inverse Fourier transform are unique, one can conclude that if the Fourier transforms of eqs. (20) and (21) are satisfied, then eqs. (20) and (21) are valid identities. Thus, the Fourier transform technique exhibited in this Appendix provides a fourth independent derivation of the Sokhotski-Plemelj formula.

[5]When we multiply $\mathrm{Pf}(1/|k|)$ by k, the singularity at $k=0$ is canceled and the prescription indicated by eq. (25) is no longer required. Noting that $k/|k|$ is equal to sign of k for $k\neq 0$, we end up with eq. (26).