The Poisson sum formula

The Poisson sum formula takes on a number of different forms in the literature. Here is one useful version,

\[ \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} = \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m). \]  

(1)

You will use this version of the Poisson sum formula in solving problem 14.13 of Jackson.

To prove this formula, consider the following periodic function, defined by:

\[ f(x) = \frac{1}{2} - \frac{x}{2\pi}, \quad 0 \leq x \leq 2\pi, \]  

(2)

where \( f(x + 2\pi) = f(x) \). Note that \( f(x) \) is discontinuous at \( x = 2\pi m \), where \( m = 0, \pm 1, \pm 2, \ldots \). It follows that one can expand \( f(x) \) in a Fourier series:

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \]

where

\[ c_n = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-inx} f(x) \, dx. \]

Inserting \( f(x) \) as given by eq. (2), one easily obtains:

\[ c_0 = 0, \quad c_n = \frac{-i}{2\pi n}, \quad (n \neq 0). \]

That is,

\[ f(x) = -\frac{i}{2\pi} \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{e^{inx}}{n}. \]  

(3)

Consider the derivative of \( f(x) \), which we denote by \( f'(x) \). From its definition [eq. (1)], \( f'(x) = -1/(2\pi) \) for \( x \neq 2\pi m \) \((m = 0, \pm 1, \pm 2, \ldots)\). At \( x = 2\pi m \), the discontinuity of \( f(x) \) can be described by the step function \( \Theta(x) \). Specifically, in the vicinity of \( x = 2\pi m \),

\[ f(x) = -\frac{1}{2} + \Theta(x - 2\pi m), \quad \text{for } x \simeq 2\pi m. \]  

(4)
That is, \( f(x) = -\frac{1}{2} \) for \( x = 2\pi m - \epsilon \) and \( f(x) = \frac{1}{2} \) for \( x = 2\pi m + \epsilon \), where \( \epsilon > 0 \) is an infinitesimal quantity. Taking the derivative of eq. (4) yields:

\[
f'(x) = \delta(x - 2\pi m), \quad \text{for } x \approx 2\pi m.
\]

We conclude that:

\[
f'(x) = -\frac{1}{2\pi} + \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m).
\] (5)

We can also compute \( f'(x) \) by differentiating the Fourier series of \( f(x) \) term-by-term. Using eq. (3), we obtain:

\[
f'(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} = \frac{1}{2\pi} \left[ -1 + \sum_{n=-\infty}^{\infty} e^{inx} \right].
\] (6)

Equating eqs. (5) and (6) yields the desired result announced in eq. (1).

Actually, the most common form for the Poisson sum formula is as follows. Given a function \( f(t) \) and its Fourier transform,

\[
F(\omega) \equiv \int_{-\infty}^{\infty} e^{i\omega t} f(t) dt,
\] (7)

then the Poisson sum formula is given by:

\[
\sum_{m=-\infty}^{\infty} f(\alpha m) = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right),
\] (8)

where \( \alpha \) is any number. One can derive eq. (8) by inserting the integral expression for \( F \) [eq. (7)] on the right-hand side of eq. (8), and then performing the sum over \( n \) using eq. (1). The resulting integrals are then trivially performed, and the left hand side of eq. (8) is immediately obtained.

For further details, see for example M.J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions*, pp. 67–71.