The power spectrum of Cherenkov radiation

Consider a charge e moving along a trajectory, $\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{r}}(t')$, where t' is the retarded time. We choose the origin of our coordinate system to be in the region of space near the trajectory of the charge. First, suppose that the charge is moving with velocity $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ and acceleration $\vec{\boldsymbol{\alpha}} = d\vec{\boldsymbol{v}}/dt = c\, d\vec{\boldsymbol{\beta}}/dt$. The radiation emitted by the charge is detected by an observer located at the point $\vec{\boldsymbol{x}}$. We define the unit vector $\hat{\boldsymbol{n}}$ by

$$\hat{\boldsymbol{n}} \equiv \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t)}{|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t)|}\,, \tag{1}$$

which points along the direction from the charge to the observer. Let us define $R \equiv |\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t)|$ and $r \equiv |\vec{\boldsymbol{x}}|$. Assuming that the observation point is very far away from the region of space where the trajectory of the charge is located, then

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}} + \mathcal{O}\left(\frac{1}{r}\right)\,. \tag{2}$$

That is, to paraphrase Jackson [see the text on p. 675 below eq. (14.62)], the unit vector $\hat{\boldsymbol{n}}$ is constant in time to a very good approximation.

In class, we showed that the power spectrum for a radiating charge e in vacuum is given by (in gaussian units)

$$\frac{d^2 I}{d\omega d\Omega} = \lim_{r\to\infty} cr^2 |\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}})|^2\,, \tag{3}$$

where[1]

$$\begin{aligned}
\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t') e^{i\omega t'}\, dt' \\
&= \frac{e}{2\pi c^2 r}\, e^{i\omega r/c} \int_{-\infty}^{\infty} dt'\, e^{\,i\omega[t' - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{r}}(t')/c]}\, \frac{\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \vec{\boldsymbol{\alpha}}]}{(1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\beta}})^2} + \mathcal{O}\left(\frac{1}{r^2}\right)\,,
\end{aligned} \tag{4}$$

where $\vec{\boldsymbol{\alpha}} \equiv d\vec{\boldsymbol{\beta}}/dt'$. Noting that:

$$\frac{d}{dt'}\left(\frac{\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})}{1 - \vec{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{n}}}\right) = \frac{\hat{\boldsymbol{n}} \times [(\hat{\boldsymbol{n}} - \vec{\boldsymbol{\beta}}) \times \vec{\boldsymbol{\alpha}}]}{(1 - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\beta}})^2}\,,$$

we may integrate eq. (4) by parts. Assuming that the surface term can be dropped, we find:

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{-ie\omega}{2\pi c^2 r}\, e^{i\omega r/c} \int_{-\infty}^{\infty} dt'\, e^{\,i\omega[t' - \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{r}}(t')/c]}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) + \mathcal{O}\left(\frac{1}{r^2}\right)\,.$$

[1] Note that because the electric field, $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t')$ is real, it follows that $\vec{\boldsymbol{E}}_{-\omega}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{E}}^*_\omega(\vec{\boldsymbol{x}})$. Hence it suffices to consider only positive frequencies.

We now suppose that the charge is moving at constant velocity $\vec{\boldsymbol{v}}$, with no acceleration. In this case, the particle trajectory is given by $\vec{\boldsymbol{r}}(t') = \vec{\boldsymbol{v}}t'$. Using the fact that we can approximate $\hat{\boldsymbol{n}}$ as being time-independent [cf. eq. (2)],

$$\begin{aligned}
\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) &= \frac{-ie\omega}{2\pi c^2 r}\, e^{i\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) \int_{-\infty}^{\infty} dt'\, e^{i\omega t'[1-\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c]} + \mathcal{O}\left(\frac{1}{r^2}\right) \\
&= \frac{-ie}{c^2 r}\, e^{i\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad (5)
\end{aligned}$$

where in the last step, we used $\delta\left(\omega[1-\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c]\right) = \omega^{-1}\delta\left(1-\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c\right)$, since by assumption, ω is non-negative. We may use eq. (5) to determine $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$:

$$\begin{aligned}
\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) &= \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) \\
&= -\frac{2\pi ie}{c^2 r}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) \delta\left(t - \frac{r}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right).
\end{aligned}$$

Of course, since $|\vec{\boldsymbol{v}}| < c$, it follows that $\delta\left(1-\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c\right) = 0$. Hence as expected, there is no radiation from a charge moving at constant velocity.

In an isotropic, homogeneous medium where $\epsilon \neq 1$, the above results apply if we make the following transformations:[2] $\vec{\boldsymbol{E}} \to n_r \vec{\boldsymbol{E}}$, $c \to c/n_r$ and $e \to e/n_r$, where the index of refraction is $n_r \equiv \sqrt{\epsilon}$. In this case,

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{-ie\omega}{2\pi c^2 r}\, e^{in_r\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}) \int_{-\infty}^{\infty} dt'\, e^{i\omega t'[1-n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c]} + \mathcal{O}\left(\frac{1}{r^2}\right) \qquad (6)$$

Evaluating the above integral, we obtain:

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{-ie}{c^2 r}\, e^{in_r\omega r/c}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right). \qquad (7)$$

If $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}} = c/n_r$ (which is possible if the charged particle is moving faster than the speed of light in the medium, c/n_r), then $\delta(1 - n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c) \neq 0$ and the resulting electric field does exhibit an $\mathcal{O}(1/r)$ behavior at large r. Indeed, one can verify that[3]

$$\begin{aligned}
\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) &= \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) \\
&= -\frac{ie}{c^2 r}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(1 - \frac{n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-n_r r/c)} + \mathcal{O}\left(\frac{1}{r^2}\right) \\
&= -\frac{2\pi ie}{c^2 r}\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})\, \delta\left(t - \frac{n_r r}{c}\right) \delta\left(1 - \frac{n_r\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right). \qquad (8)
\end{aligned}$$

[2]For simplicity, we assume that the magnetic permeability $\mu = 1$.

[3]The delta functions in eq. (8) imply that the electric field is singular on the surface of the Mach cone where $\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}} = c/n_r$ and $r = ct/n_r$. These singularities arise due to an idealization of the problem (e.g. the assumption of a point charge); in a more realistic setting these singularities are smoothed out. For example, see Glenn S. Smith, *Cherenkov radiation from a charge of finite size or a bunch of charges*, American Journal of Physics **61**, 147–155 (1993).

Thus, radiation can occur—this is Cherenkov radiation. To compute the power spectrum, one may be tempted to insert eq. (7) into eq. (3). However, this results in a square of a delta-function which requires a careful interpretation.

Here, we provide a calculational method that avoids the square of the delta-function. Insert eq. (6) into eq. (3) [after replacing $\vec{\boldsymbol{E}} \to n_r \vec{\boldsymbol{E}}$ and $c \to c/n_r$ in the latter], and note the identity $|\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}})|^2 = |\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}|^2$. Then,

$$\frac{d^2 I}{d\omega d\Omega} = n_r c r^2 |\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}})|^2 = \frac{n_r e^2 \omega^2}{4\pi^2 c^3} |\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}|^2 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, e^{i\omega(t'-t'')(1-n_r \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c)}\,.$$

It is convenient to introduce two new integration variables:

$$T \equiv \tfrac{1}{2}(t'+t'')\,, \qquad\qquad t \equiv t' - t''\,.$$

Note that the Jacobian of the transformation from (t', t'') to (t, T) is unity. This change of variables is inspired by the treatment of Cherenkov radiation given in *Classical Electrodynamics* by Julian Schwinger et al. I quote from this textbook:

> [The time] t is of order $1/\omega$, thus setting the time scale for the emission of radiation.[4] This microscopic time scale may be much smaller than macroscopic time intervals; for example, for visible light, $t \sim 10^{-15}$ sec. The time T is then interpreted as the average (macroscopic) time of emission...

We therefore define the *power* distribution of the radiation by:

$$\frac{d^2 I}{d\omega d\Omega} = \int_{-\infty}^{\infty} dT\, \frac{dP}{d\omega d\Omega}\,.$$

It then immediately follows that:

$$\begin{aligned} \frac{dP}{d\omega d\Omega} &= \frac{n_r e^2 \omega^2}{4\pi^2 c^3}\, |\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}|^2 \int_{-\infty}^{\infty} dt\, e^{i\omega t(1-n_r \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}/c)} \\ &= \frac{n_r e^2 \omega^2}{2\pi c^3}\, |\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}|^2 \delta\left(\omega\left(1 - \frac{n_r \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{v}}}{c}\right)\right)\,. \end{aligned}$$

Note that $dP/d\omega d\Omega$ is independent of T; that is, the rate of energy emission is constant in (macroscopic) time.

If we introduce the wave number vector in the medium, $\vec{\boldsymbol{k}} \equiv (n_r\omega/c)\hat{\boldsymbol{n}}$, then the above result can be rewritten as:

$$\frac{dP}{d\omega d\Omega} = \frac{n_r e^2 \omega^2}{2\pi c^3}\, |\hat{\boldsymbol{n}} \times \vec{\boldsymbol{v}}|^2 \delta(\omega - \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{v}})\,.$$

Finally, if we integrate over $d\Omega = 2\pi\, d\cos\psi$, where $\cos\psi \equiv \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{v}}$ and $v \equiv |\vec{\boldsymbol{v}}|$, we arrive at the Tamm-Frank formula:

$$\boxed{\frac{dP}{d\omega} = \frac{e^2 \omega\, v}{c^2}\left(1 - \frac{c^2}{n_r^2 v^2}\right)\Theta(n_r v - c)} \tag{9}$$

[4] In Fourier integrals of the form $\int_{-\infty}^{\infty} e^{i\omega t} F(t)dt$, where $F(t)$ is a well behaved function, the important range of t that contributes to the integral is of order $1/\omega$.

Note that I have employed the step function,

$$\Theta(x) \equiv \begin{cases} 1\,, & \text{for } x > 0\,, \\ 0\,, & \text{for } x < 0\,, \end{cases}$$

in eq. (9) to emphasize that radiation only occurs if $v > c/n_r$. One should in mind that $n_r = n_r(\omega)$ depends on the frequency. In general, $n_r(\omega) \to 1$ as $\omega \to \infty$. Thus, Cherenkov radiation operates only over a narrow (finite) band of ω in which $n_r(\omega)v > c$.