CHAPTER 3

Dynamics of Classical Fields

§ 3.1 Illustrative Example: The Lagrangian Formalism for a String

We have already discussed the Lagrangian formulation of the dynamics of a system with a finite number of degrees of freedom, say a finite number of particles moving in an external field. Now we want to include the dynamics of fields. We will start by considering a one-dimensional string. It has an important property: If the string is disturbed at one place, then this disturbance may propagate along the string. We can understand this in an intuitive way. The string consists of "atoms." Each atom interacts with its nearest neighbors. Hence, if one atom is disturbed, this disturbance has influence on its neighbor. But this disturbance of a neighbor has influence on the neighbor of the neighbor, etc.! In this way a traveling wave is created that propagates along the string!

In our model the string is composed of "atoms" that in the equilibrium state are evenly spaced throughout the x-axis. (See Figure 3.1.) The important assumption is that the "atoms" are coupled to each other through forces proportional to their relative displacements (Hooke forces).

We will enumerate the "atoms" with an integer n, so that the equilibrium position of the nth "atom" is $x_n = a_n$. If we set the "atoms" in motion, then the nth "atom" will be displaced an amount q_n from its equilibrium position.

When the string is put into vibration, its dynamical evolution is described by the functions

$$q_n = q_n(t), \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots .$$

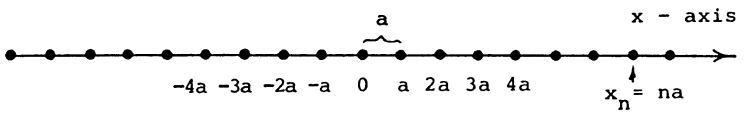

Figure 3.1.

The kinetic energy of the nth "atom" is

$$\frac{1}{2}\, m\dot{q}_n^2\,,$$

and the potential energy associated with the separation of atom n and $n+1$ is:

$$\frac{1}{2}\, mv^2[q_{n+1} - q_n]^2.$$

Hence the total *Lagrangian* for the system is

$$L = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\, m\dot{q}_n^2 - \frac{1}{2}\, mv^2[q_{n+1} - q_n]^2 \right). \tag{3.1}$$

Let us determine the equations of motion. Using (2.6) we get

$$m\ddot{q}_i = mv^2(q_{i+1} - 2q_i + q_{i-1}). \tag{3.2}$$

It is easy to check that the equations of motion actually allow wave solutions. If we put

$$q_n(t) = A\cos(kx_n - \omega t), \tag{3.3}$$

then this will represent a traveling wave. Inserting this into the equation of motion, we get

$$-m\omega^2 A\cos(kx_n - \omega t) = mv^2\cos(kx_n - \omega t)(2\cos(k \cdot a) - 1),$$

or

$$\omega^2 = 2v^2(1 - \cos(k \cdot a)). \tag{3.4}$$

Thus (3.3) is a solution to the equation of motion, provided that (3.4) is satisfied. Relation (3.4) is called a *dispersion relation*. Observe that for small k, i.e., in the long wave-length limit, we may expand the cosine, getting

$$\omega \approx \pm v\sqrt{2\left(1 - 1 + \frac{1}{2}\, k^2a^2\right)} = \pm va \cdot k, \tag{3.5}$$

which is a linear dispersion relation.

Now we want to investigate a *continuous* string, where there are "atoms" everywhere. We can do this by letting $a \to 0$ in our discrete model. We say that we pass to the *continuum limit*.

Now, instead of describing the displacements by the infinite set of numbers $q_j(t)$, we will represent them by a smooth function $q(x, t)$ giving the displacement of the "atom" with equilibrium position x. (See Figure 3.2.)

In the discrete model, the mass density is m/a. When we pass to the continuum limit, we suppose that it approaches a constant ρ, the mass density of the continuous string; i.e.,

$$\frac{m}{a} \to \rho \qquad \text{as } a \to 0.$$

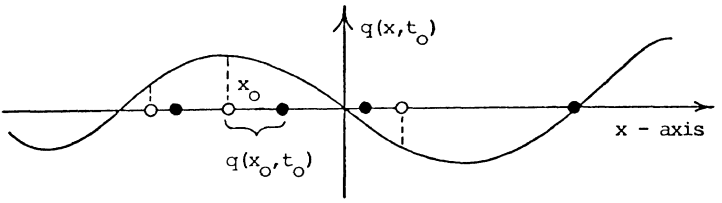

Figure 3.2.

Finally, it will be necessary to make an assumption about v. In the discrete model, the velocity of a wave in the long wave-length limit is (compare (3.3) and (3.5))

$$\frac{\omega}{k} = va.$$

We assume that it approaches a constant c, the velocity of a traveling wave in the continuous string:

$$va \to c \qquad \text{as } a \to 0.$$

With these preliminaries we can investigate what happens to the Lagrangian in the continuum limit. Let us take a look at the kinetic energy:

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \frac{1}{2} m\dot{q}_n^2(t) &= \sum_{x_n=-\infty}^{x_n=+\infty} \frac{1}{2} \frac{m}{a} [\dot{q}(x_n, t)]^2 a \\ &= \sum_{x_n=-\infty}^{x_n=+\infty} \frac{1}{2} \frac{m}{a} [\dot{q}(x_n, t)]^2 \Delta x_n \\ &\to \int_{x=-\infty}^{x=+\infty} \frac{1}{2} \rho[\dot{q}(x, t)]^2 dx.\end{aligned}$$

We can treat the potential energy in a similar way. From the observation

$$q_{n+1}(t) - q_n(t) = q(x_{n+1}, t) - q(x_n, t) \approx a \frac{\partial q}{\partial x} (x_n, t)$$

we get

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} \frac{1}{2} mv^2[q_{n+1}(t) - q_n(t)]^2 &\approx \sum_{x_n=-\infty}^{x_n=+\infty} \frac{1}{2} \frac{m}{a} (va)^2 \left[\frac{\partial q}{\partial x} (x_n, t)\right]^2 a \\ &= \sum_{x_n=-\infty}^{x_n=+\infty} \frac{1}{2} \frac{m}{a} (va)^2 \left[\frac{\partial q}{\partial x} (x_n, t)\right]^2 \Delta x_n \\ &\to \int_{x=-\infty}^{x=+\infty} \frac{1}{2} \rho c^2 \left[\frac{\partial q}{\partial x} (x, t)\right]^2 dx.\end{aligned}$$

Thus for the total Lagrangian we get

$$\sum_{n=-\infty}^{+\infty} \frac{1}{2} m\dot{q}_n^2 - \frac{1}{2} mv^2[q_{n+1} - q_n]^2 \to \int_{x=-\infty}^{+\infty} \left[\frac{1}{2} \rho \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2} \rho c^2 \left(\frac{\partial q}{\partial x} \right)^2 \right] dx, \tag{3.6}$$

showing that in the continuum limit *the Lagrangian is expressed as an integral over space*. The integrand is called the *Lagrangian density*,

$$L\left(q, \frac{\partial q}{\partial t}, \frac{\partial q}{\partial x}\right) = \frac{1}{2} \rho \left(\frac{\partial q}{\partial t} \right)^2 - \frac{1}{2} \rho c^2 \left(\frac{\partial q}{\partial x} \right)^2. \tag{3.7}$$

Observe that it contains not only $\frac{\partial q}{\partial t}$ but also $\frac{\partial q}{\partial x}$! Where did the space derivative $\frac{\partial q}{\partial x}$ come from? It came from the term $(q_{n+1} - q_n)^2$ in the discrete model. Hence, it reflects the *property of local interactions*. Each point in space interacts with its nearest neighbors.

In a similar way we may analyze the equations of motion. In the discrete model we have

$$\ddot{q}_n(t) = v^2[q_{n+1} - 2q_n + q_{n-1}].$$

We may rearrange the term on the right side:

$$\begin{aligned} q_{n+1} - 2q_n + q_{n-1} &= [q(x_{n+1}, t) - q(x_n, t)] - [q(x_n, t) - q(x_n, t)] \\ &\approx a \left[\frac{\partial q}{\partial x} \left(x_n + \frac{q}{2}, t \right) - \frac{\partial q}{\partial x} \left(x_n - \frac{q}{2}, t \right) \right] \\ &\approx a^2 \left[\frac{\partial^2 q}{\partial x^2} (x_n, t) \right]. \end{aligned}$$

Thus we obtain

$$\ddot{q}(x_n, t) \approx v^2 a^2 \frac{\partial^2 q}{\partial x^2} (x_n, t) \to c^2 \frac{\partial^2 q}{\partial x^2}; \qquad \text{i.e.,} \quad \frac{1}{c^2} \frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2}. \tag{3.8}$$

As before, we may look for a solution representing a traveling wave:

$$q(x, t) = A \cos(kx - \omega t). \tag{3.9}$$

If we insert this into the equation of motion (3.8), we get

$$-\omega^2 A \cos(kx - \omega t) = -c^2 k^2 A \cos(kx - \omega t);$$

i.e.,

$$\omega = \pm ck. \tag{3.10}$$

Hence, (3.9) is a solution to the equation of motion, provided that ω satisfies the linear dispersion relation (3.10).

§ 3.2 The Lagrangian Formalism for Relativistic Fields

We should now be motivated for the abstract field theory. We start with a field $\phi(t, \mathbf{x})$ defined throughout space–time. The value of the field at a particular point $\mathbf{x}_0$, $\phi(t, \mathbf{x}_0)$, corresponds to the stretching $q(t, \mathbf{x}_0)$ in the preceding example. The dynamics of the field are governed by a Lagrangian $\mathcal{L}$, which by analogy with the preceding example we write as

$$\mathcal{L} = \int L d^3\mathbf{x}. \tag{3.11}$$

The Lagrangian density L depends not only on the time derivative, but on the space derivatives as well

$$L = L(\phi, \partial_\mu \phi).$$

The presence of space derivatives $\partial_i \phi$ reflects the principle of local interactions.

If we choose two times t_1 and t_2, we may specify the field at these times. Any smooth function $\phi(t, \mathbf{x})$ that satisfies the boundary conditions

$$\phi(t_1, \mathbf{x}) = \phi_1(\mathbf{x}) \quad \text{and} \quad \phi(t_2, \mathbf{x}) = \phi_2(\mathbf{x})$$

represents a possible history of the field. To each such history we associate the action

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt = \int_{t_1}^{t_2} \int L d^3\mathbf{x}\, dt;$$

i.e.,

$$S = \int_\Omega L d^4x, \tag{3.12}$$

where Ω is the four-dimensional region between the hyperplanes $t = t_1$ and $t = t_2$. (See Figure 3.3.) As usual, we want to determine a history $\phi(t, \mathbf{x})$ that extremizes the action. This, of course, leads to the equation of motion for the field. Now suppose that $\phi_0(t, \mathbf{x})$ really extremizes the action. Consider another history,

$$\phi(t, \mathbf{x}) = \phi_0(t, \mathbf{x}) + \epsilon\eta(t, \mathbf{x}),$$

where $\eta(t, \mathbf{x})$ satisfies the boundary conditions $\eta(t_1, \mathbf{x}) = \eta(t_2, \mathbf{x}) = 0$.

Then the action

$$S(\epsilon) = \int_\Omega L(\phi_0 + \epsilon\eta, \partial_\mu\phi_0 + \epsilon\partial_\mu\eta) d^4x$$

has an extremal value when $\epsilon = 0$. Consequently, we get

$$\begin{aligned} 0 = \frac{dS}{d\epsilon}\Big|_{\epsilon=0} &= \int_\Omega \left[\frac{\partial L}{\partial \phi}\Big|_{\epsilon=0} \cdot \eta + \frac{\partial L}{\partial(\partial_\mu\phi)}\Big|_{\epsilon=0} \cdot \partial_\mu\eta \right] d^4x \\ &= \int_\Omega \left[\frac{\partial L}{\partial \phi}\Big|_{\phi_0} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\phi)}\Big|_{\phi_0} \right] \eta d^4x, \end{aligned}$$

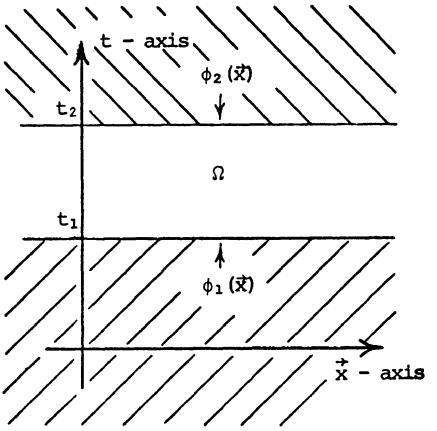

Figure 3.3.

where we have neglected the surface terms due to the boundary conditions on η. But $\eta(t, \mathbf{x})$ was arbitrarily chosen. Therefore, the above result is consistent only if ϕ_0 satisfies the differential equation

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)} = 0. \tag{3.13}$$

This generalizes the Euler–Lagrange equation for a system with a finite number of degrees of freedom. Observe that all the derivatives occur! This has an important consequence: *The equation of motion is Lorentz-invariant, provided that L is a Lorentz scalar.*

We would also like to discuss fields with several components $\phi_a, a = 1, \ldots, n$ (like the Maxwell field A_α). We leave the deduction of the equations of motion as an exercise.

Exercise 3.2.1
Problem: Suppose the field has several components: ϕ_a, $a = 1, \ldots, n$. Show that each of the components must satisfy the appropriate Euler–Lagrange equation

$$\frac{\partial L}{\partial \phi_a} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi_a)} = 0, \qquad a = 1, \ldots, n. \tag{3.14}$$

We may also discuss the energy–momentum corresponding to our field ϕ_a. A direct generalization of the Hamiltonian method suggests that the energy density is given by the *Hamiltonian density*

$$H = \frac{\partial L}{\partial(\partial_0 \phi_a)} \partial_0 \phi_a - L \tag{3.15}$$