1. The energy and the linear momentum of a distribution of electromagnetic fields in vacuum is given (in SI units) by

\[ U = \frac{\varepsilon_0}{2} \int d^3r (\vec{E}^2 + c^2 \vec{B}^2), \quad \vec{P} = \varepsilon_0 \int d^3r \vec{E} \times \vec{B}, \]

where the integration is over all space. Consider an expansion of the electric field in terms of plane waves:

\[ \vec{E}(\vec{r}, t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} \left[ E_0(\vec{k}, \lambda) \hat{\epsilon}_\lambda(\vec{k}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} + \text{c.c.} \right], \]

where \( E_0(\vec{k}, \lambda) \) is a complex amplitude and c.c. stands for “complex conjugate” of the preceding term. The polarization vector satisfies: \( \hat{\epsilon}_\lambda(-\vec{k}) = \hat{\epsilon}_\lambda^*(\vec{k}) \).

(a) Show that \( \vec{P} \) can be written as

\[ \vec{P} = \frac{2\varepsilon_0}{c} \sum_\lambda \int \frac{d^3k}{(2\pi)^3} |E_0(\vec{k}, \lambda)|^2 \hat{k}. \]

Note that all time dependence has canceled out. Explain.

(b) Obtain the corresponding expression for the total energy \( U \). Employing the photon interpretation for each mode \((\vec{k}, \lambda)\) of the electromagnetic field, justify the statement that photons are massless.

2. Jackson, problem 7.4

3. Jackson, problem 7.6

4. Jackson, problem 7.27

5. (a) Assume that the vector potential in the Lorenz gauge is given by,

\[ \vec{A}(\vec{x}, t) = A_0(x, y)(\hat{x} \pm i\hat{y})e^{i(kz - \omega t)}, \]

where \( A_0(x, y) \) is a very slowly varying function of position. “Slowly varying” means that the second spatial derivatives of \( A_0(x, y) \) can be neglected; however, one must not neglect first derivatives of \( A_0(x, y) \). After using the Lorenz gauge condition to determine the scalar potential \( \Phi(\vec{x}, t) \), derive the approximate forms for the electric and magnetic fields given in Jackson, problem 7.28.

(b) Jackson, problem 7.29