DUE: TUESDAY, JANUARY 23, 2024

1. The energy and the linear momentum of a distribution of electromagnetic fields in vacuum is given (in SI units) by

$$U = \frac{\epsilon_0}{2}\int d^3x\,(\vec{\boldsymbol{E}}^{\,2} + c^2\vec{\boldsymbol{B}}^{\,2})\,, \qquad\qquad \vec{\boldsymbol{P}} = \epsilon_0\int d^3x\,\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\,,$$

where the integration is over all space. Consider an expansion of the electric field in terms of plane waves:

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\left[E_0(\vec{\boldsymbol{k}},\lambda)\,\hat{\boldsymbol{\epsilon}}_\lambda(\vec{\boldsymbol{k}})\,e^{i(\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}}-\omega t)} + \text{c.c.}\right]\,,$$

where $E_0(\vec{\boldsymbol{k}},\lambda)$ is a complex amplitude and c.c. stands for "complex conjugate" of the preceding term. The polarization vector satisfies: $\hat{\boldsymbol{\epsilon}}_\lambda(-\vec{\boldsymbol{k}}) = \hat{\boldsymbol{\epsilon}}^*_\lambda(\vec{\boldsymbol{k}})$.

(a) Show that $\vec{\boldsymbol{P}}$ can be written as

$$\vec{\boldsymbol{P}} = \frac{2\epsilon_0}{c}\sum_\lambda\int\frac{d^3k}{(2\pi)^3}\,|E_0(\vec{\boldsymbol{k}},\lambda)|^2\,\hat{\boldsymbol{k}}\,.$$

Note that all time dependence has canceled out. Explain.

(b) Obtain the corresponding expression for the total energy U. Employing the photon interpretation for each mode $(\vec{\boldsymbol{k}},\lambda)$ of the electromagnetic field, justify the statement that photons are massless.

2. Jackson, problem 7.6.

3. Jackson, problem 7.22. In part (b), assume that $\gamma > 0$ and $\omega_0 > \frac{1}{2}\gamma$.

HINT: In part (a), you should use Jackson eq. (7.120), as Jackson suggests. However, in part (b), the computation is simpler if you use Jackson eq. (7.119). To carry out the principal value prescription, you should make use of eq. (65) of the class handout entitled *Generalized Functions for Physics*. Close the contour in the lower half complex plane with a semicircular arc of radius $R\to\infty$ and employ Cauchy's residue theorem.

EXTRA CREDIT: Show that part (b) can be solved by evaluating the integral obtained from Jackson eq. (7.120).

4. Jackson, problem 7.27.

5. (a) Assume that the vector potential in the Lorenz gauge is given by,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}\,,\,t) = A_0(x,y)(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})e^{i(kz-\omega t)}\,,$$

where $A_0(x,y)$ is a very slowly varying function of position. "Slowly varying" means that the second spatial derivatives of $A_0(x,y)$ can be neglected; however, one must *not* neglect first derivatives of $A_0(x,y)$. After using the Lorenz gauge condition to determine the scalar potential $\Phi(\vec{\boldsymbol{x}}\,,\,t)$, derive the approximate forms for the electric and magnetic fields given in Jackson, problem 7.28.

(b) Using the results of part (a), solve Jackson, problem 7.29.