1. [Jackson, problem 11.10]

(a) For the Lorentz boost and rotation matrices $\boldsymbol{K}$ and $\boldsymbol{S}$ show that

$$(\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S})^3 = -\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S}\,, \tag{1}$$

$$(\hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\boldsymbol{K})^3 = \hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\boldsymbol{K}\,, \tag{2}$$

where $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\epsilon}}'$ are any real unit 3-vectors.

We are given

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

To prove eq. (1), we evaluate the matrix $\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S}$ explicitly,

$$\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_3 & \epsilon_2 \\ 0 & \epsilon_3 & 0 & -\epsilon_1 \\ 0 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix},$$

and then compute $(\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S})^3$ via matrix multiplication. Indeed,

$$(\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\epsilon_2^2-\epsilon_3^2 & \epsilon_1\epsilon_2 & \epsilon_1\epsilon_3 \\ 0 & \epsilon_1\epsilon_2 & -\epsilon_1^2-\epsilon_3^2 & \epsilon_2\epsilon_3 \\ 0 & \epsilon_1\epsilon_3 & \epsilon_2\epsilon_3 & -\epsilon_1^2-\epsilon_2^2 \end{pmatrix},$$

and

$$(\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S})^3 = (\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S})^2\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S} = -(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon_3 & \epsilon_2 \\ 0 & \epsilon_3 & 0 & -\epsilon_1 \\ 0 & -\epsilon_2 & \epsilon_1 & 0 \end{pmatrix} = -\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S}\,,$$

after using the fact that $\hat{\boldsymbol{\epsilon}}$ is a real unit 3-vector, which implies that $\epsilon_1^2+\epsilon_2^2+\epsilon_3^2=1$.

To prove eq. (2), we evaluate the matrix $\hat{\boldsymbol\epsilon}\boldsymbol{\cdot}\boldsymbol{K}$ explicitly,

$$\hat{\boldsymbol\epsilon}'\boldsymbol{\cdot}\boldsymbol{K} = \begin{pmatrix} 0 & \epsilon'_1 & \epsilon'_2 & \epsilon'_3 \\ \epsilon'_1 & 0 & 0 & 0 \\ \epsilon'_2 & 0 & 0 & 0 \\ \epsilon'_3 & 0 & 0 & 0 \end{pmatrix}\,, \tag{3}$$

and then compute $(\hat{\boldsymbol\epsilon}'\boldsymbol{\cdot}\boldsymbol{K})^3$ via matrix multiplication. Indeed,

$$(\hat{\boldsymbol\epsilon}'\boldsymbol{\cdot}\boldsymbol{K})^2 = \begin{pmatrix} \epsilon_1'^2+\epsilon_2'^2+\epsilon_3'^2 & 0 & 0 & 0 \\ 0 & \epsilon_1'^2 & \epsilon'_1\epsilon'_2 & \epsilon'_1\epsilon'_3 \\ 0 & \epsilon'_1\epsilon'_2 & \epsilon_2'^2 & \epsilon'_2\epsilon'_3 \\ 0 & \epsilon'_1\epsilon'_3 & \epsilon'_2\epsilon'_3 & \epsilon_3'^2 \end{pmatrix}\,,$$

and

$$(\hat{\boldsymbol\epsilon}'\boldsymbol{\cdot}\boldsymbol{K})^3 = (\hat{\boldsymbol\epsilon}'\boldsymbol{\cdot}\boldsymbol{K})^2\,\hat{\boldsymbol\epsilon}'\boldsymbol{\cdot}\boldsymbol{K} = (\epsilon_1'^2+\epsilon_2'^2+\epsilon_3'^2)\begin{pmatrix} 0 & \epsilon'_1 & \epsilon'_2 & \epsilon'_3 \\ \epsilon'_1 & 0 & 0 & 0 \\ \epsilon'_2 & 0 & 0 & 0 \\ \epsilon'_3 & 0 & 0 & 0 \end{pmatrix} = \hat{\boldsymbol\epsilon}'\boldsymbol{\cdot}\boldsymbol{K}\,,$$

after using the fact that $\hat{\boldsymbol\epsilon}'$ is a real unit 3-vector.

ALTERNATIVE SOLUTION:

The following alternative solution to part (a) is noteworthy. First, observe that the first row and column of S_1, S_2 and S_3 are all zeros. Hence we can simply focus on the remaining 3×3 block. That is, we write the S_i in block matrix form,

$$(S_i)_{jk} = \left(\begin{array}{c|c} 0 & \mathbf{0}^{\mathsf T}_k \\ \hline \mathbf{0}_j & -\epsilon_{ijk} \end{array}\right)\,, \tag{4}$$

where $\mathbf{0}^{\mathsf T}$ is a row vector of three zeros, $\mathbf{0}$ is a column vector of three zeros, and

$$\epsilon_{ijk} = \begin{cases} +1\,, & \text{if } (ijk) \text{ is an even permutation of } (123)\,, \\ -1\,, & \text{if } (ijk) \text{ is an odd permutation of } (123)\,, \\ \ \ 0\,, & \text{otherwise}\,, \end{cases}$$

is the three-dimensional Levi-Civita tensor. After excluding the first row and column, jk labels the three remaining rows and columns of the S_i.

Thus, we can compute $(\hat{\boldsymbol\epsilon}\boldsymbol{\cdot}\boldsymbol{S})^3$ by pretending that the first row and column do not exist. More explicitly,[1]

$$\begin{aligned}(\hat{\boldsymbol\epsilon}\boldsymbol{\cdot}\boldsymbol{S})^3_{jk} &= (\hat{\boldsymbol\epsilon}\boldsymbol{\cdot}\boldsymbol{S})_{j\ell}(\hat{\boldsymbol\epsilon}\boldsymbol{\cdot}\boldsymbol{S})_{\ell m}(\hat{\boldsymbol\epsilon}\boldsymbol{\cdot}\boldsymbol{S})_{mk} = \epsilon_i(S_i)_{j\ell}\,\epsilon_p(S_p)_{\ell m}\,\epsilon_q(S_q)_{mk} \\ &= -\epsilon_i\epsilon_p\epsilon_q\epsilon_{ij\ell}\,\epsilon_{p\ell m}\,\epsilon_{qmk} = \epsilon_i\epsilon_p\epsilon_q\epsilon_{ij\ell}\,\epsilon_{pm\ell}\,\epsilon_{qmk} \\ &= \epsilon_i\epsilon_p\epsilon_q(\delta_{ip}\delta_{jm}-\delta_{im}\delta_{jp})\epsilon_{qmk} = \epsilon_q\epsilon_{qjk} - \epsilon_m\epsilon_j\epsilon_q\epsilon_{qmk}\,, \end{aligned} \tag{5}$$

[1]In eq. (5), we employ the Einstein summation convention. In this derivation, we make use of the antisymmetry properties of the Levi-Civita tensor and employ the identity $\epsilon_{ij\ell}\epsilon_{pm\ell} = \delta_{ip}\delta_{jm} - \delta_{im}\delta_{jp}$.

after noting that $\epsilon_i\epsilon_i = \hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}} = 1$ since $\hat{\boldsymbol{\epsilon}}$ is an arbitrary real unit vector. We now observe that $\epsilon_m\epsilon_j\epsilon_q\epsilon_{qmk} = 0$ since $\epsilon_m\epsilon_q$ is symmetric under the interchange of m and q whereas ϵ_{qmk} is antisymmetric under the same interchange of indices. Thus, eq. (5) yields

$$(\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S})^3_{jk} = \epsilon_q\epsilon_{qjk} = -\epsilon_q(S_q)_{jk} = -(\hat{\boldsymbol{\epsilon}}\boldsymbol{\cdot}\boldsymbol{S})_{jk}\,,$$

which establishes eq. (1).

To establish eq. (2), we rewrite $\hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\boldsymbol{K}$ given by eq. (3) in block matrix form [analogous to the form of the S_i in eq. (4)],

$$(\hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\boldsymbol{K})_{jk} = \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_j & \mathbf{0}_{jk}\end{array}\right)\,, \tag{6}$$

where $\mathbf{0}_{jk}$ stands for the matrix elements of the 3×3 zero matrix. In particular, j labels the row and k labels the column. Then,

$$\begin{aligned}(\hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\boldsymbol{K})^3_{jk} &= \left(\begin{array}{c|c} 0 & \epsilon'_\ell \\ \hline \epsilon'_j & \mathbf{0}_{j\ell}\end{array}\right)\left(\begin{array}{c|c} 0 & \epsilon'_i \\ \hline \epsilon'_\ell & \mathbf{0}_{\ell i}\end{array}\right)\left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_i & \mathbf{0}_{ik}\end{array}\right) = \left(\begin{array}{c|c} \hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}' & \mathbf{0}_i \\ \hline \mathbf{0}^{\mathsf{T}}_j & \epsilon'_j\epsilon'_i\end{array}\right)\left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_i & \mathbf{0}_{ik}\end{array}\right) \\ &= \left(\begin{array}{c|c} 1 & \mathbf{0}_i \\ \hline \mathbf{0}^{\mathsf{T}}_j & \epsilon'_j\epsilon'_i\end{array}\right)\left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_i & \mathbf{0}_{ik}\end{array}\right) = \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_j\,\hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}' & \mathbf{0}_{jk}\end{array}\right) = \left(\begin{array}{c|c} 0 & \epsilon'_k \\ \hline \epsilon'_j & \mathbf{0}_{jk}\end{array}\right) = (\hat{\boldsymbol{\epsilon}}'\boldsymbol{\cdot}\boldsymbol{K})_{jk}\,,\end{aligned}$$

after using the fact that $\hat{\boldsymbol{\epsilon}}'$ is is real unit vector. Once again, eq. (2) is established.

(b) Use the result of part (a) to show that:

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\bigr) = I - \hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\,\sinh\zeta + (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^2\,[\cosh\zeta - 1]\,,$$

where I is the 4×4 identity matrix.

We employ the series expansion for the exponential (which *defines* the matrix exponential),

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\bigr) = \sum_{n=0}^{\infty}\frac{(-\zeta)^n}{n!}\,(\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^n\,. \tag{7}$$

In part (a), we established the following result: $(\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^3 = \hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}$. Hence, it follows that

$$(\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^{2n} = (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^2\,,\qquad (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^{2n+1} = \hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\,,\quad \text{for } n = 1,2,3,\ldots.$$

Thus, we can rewrite the series given in eq. (7) as

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\bigr) = I - \hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\sum_{\substack{n\ \text{odd}\\ n\geq 1}}\frac{\zeta^n}{n!} + (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^2\sum_{\substack{n\ \text{even}\\ n\geq 2}}\frac{\zeta^n}{n!}\,, \tag{8}$$

after using the fact that $(\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^0 = I$ is the 4×4 identity matrix. Using,

$$\sum_{n=0}^{\infty}\frac{\zeta^{2n+1}}{(2n+1)!}=\sinh\zeta\,,\qquad\qquad \sum_{n=0}^{\infty}\frac{\zeta^{2n}}{(2n)!}=\cosh\zeta\,,$$

and noting that the last summation in eq. (8) starts at $n = 2$, we end up with

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\bigr) = I - \hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\,\sinh\zeta + (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^2\,[\cosh\zeta - 1]\,, \tag{9}$$

which is the desired result.

<u>*REMARKS*</u>:

To understand the significance of eq. (9), let us write it explicitly in matrix form. It is convenient to use the block matrix form of eq. (6), where j labels the row and k labels the column,

$$I=\left(\begin{array}{c|c} 1 & \mathbf{0}_k^{\mathsf{T}} \\ \hline \mathbf{0}_j & \delta_{jk}\end{array}\right),\qquad (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})_{jk}=\left(\begin{array}{c|c} 0 & \hat{\boldsymbol{\beta}}_k \\ \hline \hat{\boldsymbol{\beta}}_j & \mathbf{0}_{jk}\end{array}\right),\qquad (\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K})^2_{jk}=\left(\begin{array}{c|c} 1 & \mathbf{0}_k^{\mathsf{T}} \\ \hline \mathbf{0}_j & \hat{\boldsymbol{\beta}}_j\hat{\boldsymbol{\beta}}_k\end{array}\right). \tag{10}$$

Then, eq. (9) yields

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\bigr)=\left(\begin{array}{c|c} \cosh\zeta & -\hat{\boldsymbol{\beta}}_k\sinh\zeta \\ \hline -\hat{\boldsymbol{\beta}}_j\sinh\zeta & \delta_{jk}+\hat{\boldsymbol{\beta}}_j\hat{\boldsymbol{\beta}}_k(\cosh\zeta-1)\end{array}\right).$$

In class, we identified $\zeta = \tanh^{-1}\beta$ as the *rapidity*, which satisfies

$$\gamma=\frac{1}{\sqrt{1-\beta^2}}=\cosh\zeta\,,\qquad\qquad \beta\gamma=\sinh\zeta\,.$$

Hence, after writing $\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{\beta}} = (\beta_1\,,\,\beta_2\,,\,\beta_3)$, it follows that

$$\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\bigr)=\left(\begin{array}{c|c} \gamma & -\gamma\beta_k \\ \hline -\gamma\beta_j & \delta_{jk}+(\gamma-1)\dfrac{\beta_j\beta_k}{\beta^2}\end{array}\right), \tag{11}$$

which we recognize as the boost matrix defined in eq. (11.98) of Jackson.

<u>*AN ALTERNATIVE METHOD FOR COMPUTING* $\exp\bigl(-\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}\bigr)$*:*</u>

Using eq. (3),

$$M\equiv -\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}=\begin{pmatrix} 0 & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & -\zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & 0 & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & 0 & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & 0\end{pmatrix}, \tag{12}$$

In order to compute $f(M) = \exp M$, we shall employ the following formula of matrix algebra. Denote the distinct eigenvalues of the $n \times n$ matrix M by λ_i and define the following polynomial,[2]

$$p(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_m)\,. \tag{13}$$

Then, M is diagonalizable if and only if $p(M) = \mathbf{0}_n$, where $\mathbf{0}_n$ is the $n \times n$ zero matrix. In this case, any function of M is given by[3]

$$f(M) = \sum_{i=1}^{m} f(\lambda_i)\left(\prod_{\substack{j=1\\ j\neq i}}^{m} \frac{M - \lambda_j \boldsymbol{I}_n}{\lambda_i - \lambda_j}\right)\,, \tag{14}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identify matrix and m is the number of distinct eigenvalues.[4]

We first compute the eigenvalues of M, which are roots of the characteristic polynomial,

$$\det(M - \lambda \boldsymbol{I}_4) = \lambda^4 + \frac{\zeta\beta_1}{\beta}\det\begin{pmatrix} -\zeta\beta_1/\beta & 0 & 0 \\ -\zeta\beta_2/\beta & -\lambda & 0 \\ -\zeta\beta_3/\beta & 0 & -\lambda \end{pmatrix} - \frac{\zeta\beta_2}{\beta}\det\begin{pmatrix} -\zeta\beta_1/\beta & -\lambda & 0 \\ -\zeta\beta_2/\beta & 0 & 0 \\ -\zeta\beta_3/\beta & 0 & -\lambda \end{pmatrix}$$

$$- \frac{\zeta\beta_3}{\beta}\det\begin{pmatrix} -\zeta\beta_1/\beta & -\lambda & 0 \\ -\zeta\beta_2/\beta & 0 & -\lambda \\ -\zeta\beta_3/\beta & 0 & 0 \end{pmatrix} = \lambda^2(\lambda^2 - \zeta^2)\,, \tag{15}$$

after using $\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$. Thus, the three distinct eigenvalues of M are $\lambda_i = 0,\ \zeta,\ -\zeta$.

We can check that M is diagonalizable by evaluating:

$$p(M) = M(M - \zeta \boldsymbol{I}_4)(M + \zeta \boldsymbol{I}_4)$$

$$= \begin{pmatrix} 0 & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & -\zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & 0 & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & 0 & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\zeta & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & -\zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & -\zeta & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & -\zeta & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & -\zeta \end{pmatrix}$$

$$\times\begin{pmatrix} \zeta & -\zeta\beta_1/\beta & -\zeta\beta_2/\beta & \zeta\beta_3/\beta \\ -\zeta\beta_1/\beta & \zeta & 0 & 0 \\ -\zeta\beta_2/\beta & 0 & \zeta & 0 \\ -\zeta\beta_3/\beta & 0 & 0 & \zeta \end{pmatrix}$$

$$= \zeta^3\begin{pmatrix} 1 & \beta_1/\beta & \beta_2/\beta & \beta_3/\beta \\ \beta_1/\beta & \beta_1^2/\beta^2 & \beta_1\beta_2/\beta^2 & \beta_1\beta_3/\beta^2 \\ \beta_2/\beta & \beta_1\beta_2/\beta^2 & \beta_2^2/\beta^2 & \beta_2\beta_3/\beta^2 \\ \beta_3/\beta & \beta_1\beta_3/\beta^2 & \beta_2\beta_3/\beta^2 & \beta_3^2/\beta^2 \end{pmatrix}\begin{pmatrix} 1 & -\beta_1/\beta & -\beta_2/\beta & \beta_3/\beta \\ -\beta_1/\beta & 1 & 0 & 0 \\ -\beta_2/\beta & 0 & 1 & 0 \\ -\beta_3/\beta & 0 & 0 & 1 \end{pmatrix} = \mathbf{0}_4\,. \tag{16}$$

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[2]Note that $m \leq n$ since it is possible that the characteristic polynomial possesses one or more multiple roots.

[3]For example, see eqs. (7.36) and (7.3.11) of Carl D. Meyer, *Matrix Analysis and Applied Linear Algebra* (SIAM, Philadelphia, PA, 2000) or Chapter V, Section 2.2 of F.R. Gantmacher, *Theory of Matrices*–Volume I (Chelsea Publishing Company, New York, NY, 1959).

[4]If the $n \times n$ matrix M is not diagonalizable then $p(M) \neq \mathbf{0}_n$, in which case the formula for $f(M)$ is more complicated than the one given in eq. (14).

We now apply eq. (14) to $f(M) = \exp M$. It then follows that

$$\exp M = -\frac{1}{\zeta^2}(M - \zeta \boldsymbol{I}_4)(M + \zeta \boldsymbol{I}_4) + e^{\zeta}\frac{1}{2\zeta^2}M(M + \zeta \boldsymbol{I}_4) + e^{-\zeta}\frac{1}{2\zeta^2}M(M - \zeta \boldsymbol{I}_4)$$
$$= \boldsymbol{I}_4 + \left(\frac{\sinh\zeta}{\zeta}\right) M + \left(\frac{\cosh\zeta - 1}{\zeta^2}\right) M^2 \,. \tag{17}$$

Inserting $M = -\zeta\,\hat{\boldsymbol{\beta}}\boldsymbol{\cdot}\boldsymbol{K}$, we recover the result of eq. (9).

2. [Jackson, problem 11.13] An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' (and the wire) move with velocity $\vec{\boldsymbol{v}}$ parallel to the direction of the wire with respect to the laboratory frame K.

(a) Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

In the rest frame of the wire (i.e. frame K'), choose the z-axis to point along the wire. Then, to compute the electric field, we draw a cylinder of length L and radius r', whose symmetry axis coincides with the z-axis. Applying Gauss' law in gaussian units,

$$\oint_S \vec{\boldsymbol{E}}\,'\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,da = 4\pi Q\,, \tag{18}$$

where Q is the total charge enclosed inside the cylinder. In cylindrical coordinates (r', ϕ', z'),[5] the symmetry of the problem implies that $\vec{\boldsymbol{E}}\,'(r', \phi', z') = E'(r')\hat{\boldsymbol{r}}'$, where $E'(r')$ depends only on the radial distance from the symmetry axis. Choosing the surface S to be the surface of the cylinder, we have $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}'$, and so eq. (18) reduces to

$$2\pi r' L E'(r') = 4\pi Q\,.$$

Defining the linear charge density (i.e. charge per unit length) by $q_0 = Q/L$, we conclude that[6]

$$\vec{\boldsymbol{E}}\,'(r') = \frac{2q_0}{r'}\,\hat{\boldsymbol{r}}'\,. \tag{19}$$

Since there are no moving charges in the rest frame of the wire, it follows that $\vec{\boldsymbol{B}}\,' = 0$.

The transformation laws for the electric and magnetic field between reference frames K and

[5] We denote the radial coordinate of cylindrical coordinates in frame K' to be r' rather than the more traditional ρ', since we reserve the letter ρ for charge density.

[6] The direction of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{z}}$ are the same in frames K and K', so no extra primed-superscript is required on these quantities.

K' are given by[7]

$$\vec{\boldsymbol{E}} = \gamma\left[\vec{\boldsymbol{E}}' - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}'\right] - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}')\,,$$

$$\vec{\boldsymbol{B}} = \gamma\left[\vec{\boldsymbol{B}}' + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'\right] - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}')\,.$$

For this problem, $\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{z}}$. Using the results of part (a), and noting that $r = r'$ (since the radial direction is perpendicular to the direction of the velocity of frame K' with respect to K), it follows that

$$\vec{\boldsymbol{E}} = \frac{2\gamma q_0}{r}\hat{\boldsymbol{r}}\,, \qquad\qquad \vec{\boldsymbol{B}} = \frac{2\gamma\beta q_0}{r}\,\hat{\boldsymbol{\phi}}\,, \tag{20}$$

where we have used $\hat{\boldsymbol{z}}\boldsymbol{\cdot}\hat{\boldsymbol{r}} = 0$ and $\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\phi}}$.

(b) What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

In reference frame K' there are no moving charges, so that $\vec{\boldsymbol{J}}' = 0$. The corresponding charge density is

$$\rho'(r') = \frac{q_0}{2\pi r'}\delta(r')\,. \tag{21}$$

To check this, let us integrate over a cylinder of length L and arbitrary nonzero radius, whose symmetry axis coincides with the z-axis. Then,

$$\int \rho'(r')\,dV = \int \rho'(r')\,r'\,dr'\,d\phi\,dz' = q_0\int dr'\delta(r')dz' = q_0 L = Q\,.$$

Since $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$ is a four-vector, the relevant transformation law between frames K and K' are:

$$c\rho = \gamma(c\rho' + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}')\,, \tag{22}$$

$$\vec{\boldsymbol{J}} = \vec{\boldsymbol{J}}' + \frac{\gamma-1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}c\rho'\,. \tag{23}$$

Plugging in $\vec{\boldsymbol{J}}' = 0$ and the result of eq. (21), and noting that $\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{z}}$ and $r' = r$, it follows that[8]

$$\rho(r) = \frac{\gamma q_0}{2\pi r}\,\delta(r)\,, \qquad\qquad \vec{\boldsymbol{J}} = \frac{\gamma\beta c\,q_0}{2\pi r}\,\hat{\boldsymbol{z}}\,\delta(r) = \rho(r)v\hat{\boldsymbol{z}} = \rho(r)\vec{\boldsymbol{v}}\,, \tag{24}$$

after using $v \equiv \beta\,c$.

[7]Eq. (11.149) of Jackson provides the equations to transform the fields from reference frame K to reference frame K'. To transform the fields from K' to K, simply change the sign of $\vec{\boldsymbol{\beta}}$.

[8]We can interpret $q \equiv \gamma q_0$ as the linear charge density as observed in reference frame K. This is not unexpected due to the phenomenon of length contraction.

(c) From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part (a).

This is an electrostatics and magnetostatics problem, so we can use Gauss' law to compute $\vec{\boldsymbol{E}}$ and Ampère's law to compute $\vec{\boldsymbol{B}}$. The computation of $\vec{\boldsymbol{E}}$ is identical to the one given in part (a) with q_0 replaced by γq_0. Hence, it immediately follows from eq. (19) that

$$\vec{\boldsymbol{E}}(r) = \frac{2\gamma q_0}{r}\,\hat{\boldsymbol{r}}\,,$$

in agreement with eq. (20). Ampère's law in gaussian units is

$$\oint_C \vec{\boldsymbol{B}}\boldsymbol{\cdot} d\vec{\boldsymbol{\ell}} = \frac{4\pi I}{c}\,,$$

where I is the current enclosed in the loop C. With $\vec{\boldsymbol{J}}$ given by eq. (24),

$$I = \int_A \vec{\boldsymbol{J}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,da = \int \rho(r) v\, r\, dr\, d\phi = \gamma q_0 v\,,$$

after noting that $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ points along the direction of the current flow and $da = rdrd\phi$ is the infinitesimal area element perpendicular to the current flow. Using the symmetry of the problem, $\vec{\boldsymbol{B}} = B(r)\hat{\boldsymbol{\phi}}$. Thus, evaluating Ampère's law with a contour C given by a circle centered at $r=0$ that lies in a plane perpendicular to the current flow, $d\vec{\boldsymbol{\ell}} = rd\phi\hat{\boldsymbol{\phi}}$ and

$$2\pi r B(r) = \frac{4\pi I}{c} = \frac{4\pi\gamma q_0 v}{c}\,,$$

which yields

$$\vec{\boldsymbol{B}}(r) = \frac{2\gamma\beta q_0 v}{r}\,\hat{\boldsymbol{\phi}}\,,$$

after using $v = \beta c$, in agreement with eq. (20).

3. [Jackson, problem 11.15] In a certain reference frame, a static uniform electric field E_0 is parallel to the x-axis, and a static uniform magnetic field $B_0 = 2E_0$ lies in the x–y plane, making an angle θ with respect to the x-axis. Determine the relative velocity of a reference frame in which the electric and magnetic fields are parallel. What are the fields in this frame for $\theta \ll 1$ and $\theta \to \frac{1}{2}\pi$?

In frame K, we have

$$\vec{\boldsymbol{E}} = E_0\,\hat{\boldsymbol{x}}\,, \qquad \vec{\boldsymbol{B}} = B_x\,\hat{\boldsymbol{x}} + B_y\,\hat{\boldsymbol{y}}\,, \tag{25}$$

with

$$\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = |\vec{\boldsymbol{E}}|\,|\vec{\boldsymbol{B}}|\cos\theta = E_0 B_0\cos\theta = 2E_0^2\cos\theta\,, \tag{26}$$

after writing $|\vec{\boldsymbol{E}}| = E_0$ and $|\vec{\boldsymbol{B}}| = B_0 = 2E_0$. It follows that

$$B_x = 2E_0\cos\theta\,, \qquad B_y = 2E_0\sin\theta\,. \tag{27}$$

The electric and magnetic fields are parallel in a reference frame K' which is moving at a velocity $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ with respect to reference frame K. That is, the fields in K' satisfy,

$$\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{B}}\,' = 0\,. \tag{28}$$

The electric and magnetic fields in frame K' are related to the corresponding fields in frame K by eq. (11.149) of Jackson,

$$\vec{\boldsymbol{E}}\,' = \gamma(\vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}) - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\,, \tag{29}$$

$$\vec{\boldsymbol{B}}\,' = \gamma(\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}) - \frac{\gamma^2}{\gamma+1}\,\vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,. \tag{30}$$

These relations can be rewritten in the following form,

$$\vec{\boldsymbol{E}}'_{\parallel} = \vec{\boldsymbol{E}}_{\parallel}\,, \qquad\qquad \vec{\boldsymbol{B}}'_{\parallel} = \vec{\boldsymbol{B}}_{\parallel}\,, \tag{31}$$

$$\vec{\boldsymbol{E}}'_{\perp} = \gamma\left(\vec{\boldsymbol{E}}_{\perp} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}_{\perp}\right)\,, \qquad\qquad \vec{\boldsymbol{B}}'_{\perp} = \gamma\left(\vec{\boldsymbol{B}}_{\perp} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}_{\perp}\right)\,. \tag{32}$$

In eqs. (31) and (32), fields with a $\parallel$ subscript are parallel to $\vec{\boldsymbol{\beta}}$ and fields with a $\perp$ subscript are perpendicular to $\vec{\boldsymbol{\beta}}$. For example,

$$\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}_{\parallel} = 0 \quad \text{and} \quad \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\perp} = 0\,,$$

which implies that

$$\vec{\boldsymbol{E}}_{\parallel} = \frac{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}}{\beta^2} \quad \text{and} \quad \vec{\boldsymbol{E}}_{\perp} = \vec{\boldsymbol{E}} - \frac{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}}{\beta^2} = \frac{\vec{\boldsymbol{\beta}} \times (\vec{\boldsymbol{E}} \times \vec{\boldsymbol{\beta}})}{\beta^2}\,.$$

The form of eqs. (31) and (32) suggests that the relative velocity $\vec{\boldsymbol{v}}$ should point in the z-direction. That is,

$$\vec{\boldsymbol{\beta}} = \beta\,\hat{\boldsymbol{z}}\,,$$

in which case $\vec{\boldsymbol{E}}_{\parallel} = E_z\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{B}}_{\parallel} = B_z\hat{\boldsymbol{z}}$. Since $E_z = B_z = 0$, it follows from eq. (31) that $E'_z = B'_z = 0$. Using eq. (32), the transverse fields are given by

$$E'_x = \gamma(E_x - \beta B_y) = \gamma E_0(1 - 2\beta\sin\theta)\,, \qquad E'_y = \gamma(E_y + \beta B_x) = 2\beta\gamma E_0\cos\theta\,, \tag{33}$$

$$B'_x = \gamma(B_x + \beta E_y) = 2\gamma E_0\cos\theta\,, \qquad B'_y = \gamma(E_y + \beta B_x) = \gamma E_0(2\sin\theta - \beta)\,, \tag{34}$$

after using eqs. (25)–(27). Moreover, eq. (28) implies that $E'_xB'_y - E'_yB'_x = (\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{B}}\,')_z = 0$. Inserting the results for the primed fields in this last equation, it then follows that

$$\gamma^2E_0^2(1 - 2\beta\sin\theta)(2\sin\theta - \beta) - 4\beta\gamma^2E_0^2\cos^2\theta = 0\,.$$

Multiplying out the factors above and writing $\cos^2\theta = 1 - \sin^2\theta$, the above equation simplifies to

$$2\beta^2\sin\theta - 5\beta + 2\sin\theta = 0\,. \tag{35}$$

This is a quadratic equation in β which is easily solved. The larger of the two roots is greater than 1, which we reject since $0 \leq \beta \leq 1$ (i.e., $0 \leq v \leq c$). The smaller of the two roots is non-negative and less than 1. Thus, we conclude that

$$\beta = \frac{v}{c} = \frac{5 - \sqrt{25 - 16\sin^2\theta}}{4\sin\theta}\,. \tag{36}$$

The two limiting cases are easily analyzed. In the case of $\theta \ll 1$, we can work to first order in θ. From eq. (36) we find that $\beta \simeq \frac{2}{5}\theta$. Since $\theta \ll 1$ it follows that $\beta \ll 1$, in which case

$$\gamma = (1-\beta^2)^{-1/2} \simeq 1 + \mathcal{O}(\beta^2)\,.$$

Since we are working to first order in θ, we also must work to first order in β. In particular we can neglect terms such as $\beta\theta$. Hence, in this limiting case, eqs. (33) and (34) yield

$$\vec{\boldsymbol{E}}\,' = \tfrac{1}{2}\vec{\boldsymbol{B}}\,' = E_0(\hat{\boldsymbol{x}} + 2\beta\hat{\boldsymbol{y}})\,, \qquad \text{for } \beta \simeq \tfrac{2}{5}\theta \ll 1\,, \tag{37}$$

where we have neglected terms that are second order (or higher) in β. Finally, in the limit of $\theta \to \frac{1}{2}\pi$, eq. (36) yields $\beta = \frac{1}{2}$. Then $\gamma = 2/\sqrt{3}$, and eqs. (33) and (34) yield

$$\vec{\boldsymbol{E}}\,' = 0\,, \qquad \vec{\boldsymbol{B}}\,' = \sqrt{3}E_0\hat{\boldsymbol{y}}\,, \qquad \text{for } \theta = \tfrac{1}{2}\pi\,. \tag{38}$$

<u>*REMARK 1*</u>:

Recall that in class, we showed that the quantity $F^{\mu\nu}\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} = -4\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}$ is a Lorentz invariant. This means that if $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are perpendicular in one frame, then they must be perpendicular in all frames. Thus, if $\theta = \frac{1}{2}\pi$ in frame K and $\theta = 0$ in frame K', then it must be true that either the electric field or the magnetic field (or both) vanish in frame K', since the only vector that is both perpendicular and parallel to a given fixed nonzero vector is the zero vector. This is indeed the case here, as can be seen in eq. (38).

<u>*REMARK 2*</u>:

It is easy to show that eq. (36) implies that $0 \leq \beta \leq \frac{1}{2}$. If we multiply the numerator and denominator of eq. (36) by $5 + \sqrt{25 - 16\sin^2\theta}$, we obtain,

$$\beta = \frac{4\sin\theta}{5 + \sqrt{25 - 16\sin^2\theta}}\,.$$

Since the polar angle lies in the range $0 \leq \theta \leq \pi$ or equivalently $0 \leq \sin\theta \leq 1$, it follows immediately that $\beta \geq 0$ (where $\beta = 0$ corresponds to $\sin\theta = 0$ as expected). Finally, it is easy to verify that

$$\frac{4\sin\theta}{5 + \sqrt{25 - 16\sin^2\theta}} \leq \frac{1}{2}\,. \tag{39}$$

Since the denominator on the left hand side above is positive, we can rewrite eq. (39) as

$$4\sin\theta \leq \tfrac{1}{2}\left(5 + \sqrt{25 - 16\sin^2\theta}\right)\,. \tag{40}$$

This inequality is manifestly true for $\sin\theta = 0$. For $\sin\theta > 0$, eq. (40) can be rearranged into the following form

$$8\sin\theta - 5 \leq \sqrt{25 - 16\sin^2\theta}\,. \tag{41}$$

Squaring both sides and simplifying the resulting expression then yields $\sin\theta\,(\sin\theta - 1) \leq 0$. Dividing both sides of the equation by $\sin\theta$ (which is assumed positive) yields $0 \leq \sin\theta \leq 1$, which is valid for all polar angles θ. Hence, eq. (39) is established. The inequality becomes an equality if $\sin\theta = 1$, in which case $\beta = \frac{1}{2}$.

REMARK 3: Non-uniqueness of the solution

In our analysis above, we found one solution to the problem. However, it is easy to see that there are an infinite number of solutions. That is, there are an infinite number of Lorentz boost matrices such that

$$F'^{\mu\nu} = \Lambda(\vec{\boldsymbol{\beta}})^\mu{}_\alpha \Lambda(\vec{\boldsymbol{\beta}})^\nu{}_\beta F^{\alpha\beta}\,, \tag{42}$$

where $F^{\alpha\beta}$ is the electromagnetic field strength tensor made up of the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields given in eqs. (25) and (27), $F'^{\mu\nu}$ is the electromagnetic field strength tensor made up of the $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ fields such that $\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{B}}\,' = 0$, and $\Lambda(\vec{\boldsymbol{\beta}})$ is the Lorentz boost matrix in the direction of $\vec{\boldsymbol{\beta}}$ given in eq. (11). We have already found one such boost matrix, namely $\Lambda(\beta\hat{\boldsymbol{z}})$, where β is given by eq. (36). This boost matrix produces the $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ fields given in eqs. (33) and (34). Since $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ are parallel in the primed reference frame, we can write

$$\vec{\boldsymbol{E}}\,' = E'\hat{\boldsymbol{n}}\,, \qquad\qquad \vec{\boldsymbol{B}}\,' = B'\hat{\boldsymbol{n}}\,, \tag{43}$$

where $\hat{\boldsymbol{n}}$ is the common direction of $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$. Using eq. (33), one obtains an explicit form for $\hat{\boldsymbol{n}}$ that is given by,

$$\hat{\boldsymbol{n}} = \frac{(1-2\beta\sin\theta)\hat{\boldsymbol{x}} + 2\beta\cos\theta\,\hat{\boldsymbol{y}}}{\sqrt{1-4\beta\sin\theta+4\beta^2}} = \frac{(\sin\theta - 2\beta)\vec{\boldsymbol{E}} + \beta\cos\theta\,\vec{\boldsymbol{B}}}{E_0\sin\theta\sqrt{1-4\beta\sin\theta+4\beta^2}}\,, \tag{44}$$

where β is given by eq. (36). We used eqs. (25)–(27) to obtain the final expression above.

If one applies the following Lorentz transformation to reference frame K,

$$\Lambda = \Lambda(\beta'\hat{\boldsymbol{n}})\Lambda(\beta\hat{\boldsymbol{z}})\,, \tag{45}$$

then in the resulting reference frame K'' the $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ fields are also parallel, for *any* choice of β'. This result follows from eq. (31), which states that the components of the electric and magnetic field that are parallel to the boost direction are unaffected by the Lorentz transformation. Having found the reference frame K' after applying $\Lambda(\beta\hat{\boldsymbol{z}})$ where $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ are parallel and point in the $\hat{\boldsymbol{n}}$ direction, one can perform an arbitrary boost in the direction parallel to $\hat{\boldsymbol{n}}$ without modifying $\vec{\boldsymbol{E}}\,'$ and $\vec{\boldsymbol{B}}\,'$ further.

One can evaluate the right hand side of eq. (45) explicitly. Here, I will make use of Paweł Klimas, *Lecture Notes on Classical Electrodynamics*, which has been posted to the Physics 214 webpage. Using eqs. (1.73) and (1.78) of Klimas' notes,

$$\Lambda(\beta'\hat{\boldsymbol{n}})\Lambda(\beta\hat{\boldsymbol{z}}) = \mathcal{O}\Lambda(\vec{\boldsymbol{\beta}}\,'')\,, \tag{46}$$

where $\mathcal{O}$ is a Lorentz transformation corresponding to a pure rotation[9] and

$$\vec{\boldsymbol{\beta}}'' = \frac{1}{1+\beta\beta'\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}}\left[\frac{\beta'\hat{\boldsymbol{n}}}{\gamma} + \left(1+\frac{\gamma\beta\beta'}{\gamma+1}\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}}\right)\beta\hat{\boldsymbol{z}}\right], \tag{47}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$. In light of eq. (44), it follows that $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{z}} = 0$, and eq. (47) simplifies to[10]

$$\vec{\boldsymbol{\beta}}'' = \beta'(1-\beta^2)^{1/2}\,\hat{\boldsymbol{n}} + \beta\hat{\boldsymbol{z}}\,. \tag{48}$$

Note that the parallel electric and magnetic field remain parallel if one transforms the reference frame by a pure rotation. Thus, we can neglect the pure rotation $\mathcal{O}$ in eq. (45) to conclude that starting from reference frame K, the application of the boost $\Lambda(\vec{\boldsymbol{\beta}}'')$ to produce reference frame K'' yields $\vec{\boldsymbol{E}}''$ and $\vec{\boldsymbol{B}}''$ fields that are parallel.

To summarize, the complete answer to the problem posed by Jackson (although probably not what Jackson meant to ask) is that any boost of the form $\Lambda\big(\beta'(1-\beta^2)^{1/2}\,\hat{\boldsymbol{n}} + \beta\hat{\boldsymbol{z}}\big)$, where β and $\hat{\boldsymbol{n}}$ are fixed by eqs. (36) and (44), respectively, will yield a reference frame K'' such that the $\vec{\boldsymbol{E}}''$ and $\vec{\boldsymbol{B}}''$ fields are parallel, for *any choice of the parameter* β', where $0 \leq \beta' \leq 1$.

ALTERNATIVE SOLUTION:

Plugging the results for the electric and magnetic fields in reference frame K' given by eqs. (29) and (30) into eq. (28), one can work out the following expressions. First,

$$(\vec{\boldsymbol{E}}+\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\times(\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}) = \vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}}\big[E^2+B^2-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})\big] + \vec{\boldsymbol{E}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}) + \vec{\boldsymbol{B}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,, \tag{49}$$

where $E \equiv |\vec{\boldsymbol{E}}|$ and $B \equiv |\vec{\boldsymbol{B}}|$. Second,

$$(\vec{\boldsymbol{E}}+\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}})\times\vec{\boldsymbol{\beta}} = -\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + \beta^2\vec{\boldsymbol{B}}\,, \tag{50}$$

$$\vec{\boldsymbol{\beta}}\times(\vec{\boldsymbol{B}}-\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}) = \vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}) + \beta^2\vec{\boldsymbol{E}}\,. \tag{51}$$

Hence, we obtain,

$$\begin{aligned}\vec{\boldsymbol{E}}'\times\vec{\boldsymbol{B}}' = \gamma^2\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}}\left\{\gamma^2[E^2+B^2-\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})] - \frac{\gamma^3}{\gamma+1}\big[(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})^2 + (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\right\} \\ +\gamma^2\big[\vec{\boldsymbol{E}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}) + \vec{\boldsymbol{B}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\big]\left[1-\frac{\gamma\beta^2}{\gamma+1}\right] - \frac{\gamma^3}{\gamma+1}\left\{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}} - (\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\right\}.\end{aligned} \tag{52}$$

We can simplify the above expression by using $\gamma^2 = (1-\beta^2)^{-1}$, which yields $\beta^2 = (\gamma^2-1)/\gamma^2$. Hence,

$$1-\frac{\gamma\beta^2}{\gamma+1} = 1-\frac{\gamma-1}{\gamma} = \frac{1}{\gamma}\,. \tag{53}$$

[9]The rotation $\mathcal{O}$ is called the Wigner rotation. As explained below eq. (48), the parallel electric and magnetic fields remain parallel under a pure rotation, and thus we will not require an explicit expression for the Wigner rotation in this problem.

[10]If we define $\beta'' \equiv |\vec{\boldsymbol{\beta}}''|$, then $\beta''^{\,2} = \beta'^{\,2}(1-\beta^2)+\beta^2$. One can then check that $0 \leq \beta^2, \beta'^{\,2} \leq 1$ implies that $0 \leq \beta''^{\,2} \leq 1$, as required by special relativity.

We then end up with,

$$\vec{\boldsymbol{E}}'\times\vec{\boldsymbol{B}}' = \gamma^2\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}+\gamma\big[\vec{\boldsymbol{E}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})+\vec{\boldsymbol{B}}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\big]-\gamma^2 h\vec{\boldsymbol{\beta}}-\frac{\gamma^3}{\gamma+1}\Big\{(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}})\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{B}}-(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{E}}\Big\}\,, \tag{54}$$

where

$$h \equiv E^2+B^2-k\big[E^2B^2-(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]-\frac{\gamma}{\gamma+1}\Big\{\big[k_1E^2+k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\big]^2+\big[k_1(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})+k_2B^2\big]^2\Big\}\,. \tag{55}$$

The only way to satisfy $\vec{\boldsymbol{E}}'\times\vec{\boldsymbol{B}}'=0$ is if the right hand side of eq. (54) is proportional to $\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}$.[11]

One way of ensuring that the right hand side of eq. (54) is proportional to $\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}$ is to take $\vec{\boldsymbol{\beta}}$ to be parallel to $\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}$. That is, there exists a nonzero constant k such that

$$\vec{\boldsymbol{\beta}} = k\,\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}}\,. \tag{56}$$

Note that eq. (56) implies that $\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}=\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}=0$. Hence, eq. (54) simplifies to,

$$\vec{\boldsymbol{E}}'\times\vec{\boldsymbol{B}}' = \gamma^2\vec{\boldsymbol{\beta}}\left[\frac{1+\beta^2}{k}-E^2-B^2\right]\,. \tag{57}$$

It then follows that

$$\vec{\boldsymbol{E}}'\times\vec{\boldsymbol{B}}'=0 \quad\Longrightarrow\quad \frac{1+\beta^2}{k}=E^2+B^2\,. \tag{58}$$

Using eqs. (25) and (27), $E^2+B^2=5E_0^2$ and

$$\vec{\boldsymbol{\beta}} = k\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} = \frac{2E_0^2 k\sin\theta}{\beta}\vec{\boldsymbol{\beta}}\,. \tag{59}$$

Thus, one can identify,

$$k=\frac{\beta}{2E_0^2\sin\theta}\,. \tag{60}$$

Plugging this result into eq. (58) yields,

$$2\sin\theta(1+\beta^2)=5\beta\,, \tag{61}$$

which reproduces the result previously obtained in eq. (35).

As a check of our calculation, let us verify explicitly that $\vec{\boldsymbol{E}}'$ is parallel to $\vec{\boldsymbol{B}}'$. Inserting eq. (56) into eqs. (29) and (30) yields,

$$\vec{\boldsymbol{E}}' = \gamma\vec{\boldsymbol{E}}\left(1-kB^2\right)+\gamma k\vec{\boldsymbol{B}}(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,, \tag{62}$$

$$\vec{\boldsymbol{B}}' = \gamma\vec{\boldsymbol{B}}\left(1-kE^2\right)+\gamma k\vec{\boldsymbol{E}}(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\,. \tag{63}$$

[11] Recall that if $\{\vec{\boldsymbol{v}}_i\}$ is a set of linearly independent vectors, then the only solution to $\sum_i c_i\vec{\boldsymbol{v}}_i=0$ is $c_i=0$ for all i.

In light of eqs. (25)–(27) and eq. (60),

$$\vec{\boldsymbol{E}}' = \gamma E_0\big[(1-2\beta\sin\theta)\hat{\boldsymbol{x}} + 2\beta\cos\theta\,\hat{\boldsymbol{y}}\big]\,, \tag{64}$$

$$\vec{\boldsymbol{B}}' = \gamma E_0\big[2\cos\theta\,\hat{\boldsymbol{x}} + (2\sin\theta-\beta)\hat{\boldsymbol{y}}\big]\,. \tag{65}$$

We can now check that

$$\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = [2\sin\theta(1+\beta^2) - 5\beta]\hat{\boldsymbol{z}} = 0\,, \tag{66}$$

after employing eq. (61), which completes the check of the calculation.

The two limiting cases are now easily analyzed. In the case of $\theta \ll 1$, we can work to first order in θ. As noted below eq. (36), $\beta \simeq \frac{2}{5}\theta$ and $\gamma = (1-\beta^2)^{-1/2} \simeq 1 + \mathcal{O}(\beta^2)$. Since we are working to first order in θ, we also must work to first order in β. In particular we can neglect terms such as $\beta\theta$. Hence, in this limiting case, eqs. (64) and (65) yield

$$\vec{\boldsymbol{E}}' = \tfrac{1}{2}\vec{\boldsymbol{B}}' = E_0(\hat{\boldsymbol{x}} + 2\beta\hat{\boldsymbol{y}})\,, \qquad \text{for } \beta \simeq \tfrac{2}{5}\theta \ll 1\,, \tag{67}$$

where we have neglected terms that are second order (or higher) in β. Finally, in the limit of $\theta \to \frac{1}{2}\pi$, eq. (36) yields $\beta = \frac{1}{2}$. Then $\gamma = 2/\sqrt{3}$, and eqs. (64) and (65) yield

$$\vec{\boldsymbol{E}}' = 0\,, \qquad \vec{\boldsymbol{B}}' = \sqrt{3}E_0\hat{\boldsymbol{y}}\,, \qquad \text{for } \theta = \tfrac{1}{2}\pi\,. \tag{68}$$

Thus, we have reproduced the results of eqs. (37) and (38).

<u>*REMARK 4*</u>:

Another strategy to find all possible boosts that result in parallel electric and magnetic fields is to start with eqs. (29) and (30) and impose the condition $\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = 0$ to determine the most general form for the boost. We again denote the boost parameter by $\vec{\boldsymbol{\beta}}$.

Since $\vec{\boldsymbol{E}}$, $\vec{\boldsymbol{B}}$ and $\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}$ are three linearly independent vectors, $\vec{\boldsymbol{\beta}}$ can be written in the following form,

$$\vec{\boldsymbol{\beta}} = k_1\vec{\boldsymbol{E}} + k_2\vec{\boldsymbol{B}} + k\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\,, \tag{69}$$

where the constants k_1, k_2 and k are to be determined. It then follows that,

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = E^2k_1 + (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})k_2\,, \qquad \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})k_1 + B^2k_2\,,$$

$$\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}} = -k_2\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} - k\big[(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{E}} - E^2\vec{\boldsymbol{B}}\big]\,, \qquad \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}} = k_1\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} + k\big[(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\vec{\boldsymbol{B}} - B^2\vec{\boldsymbol{E}}\big]\,.$$

$$\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}(\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}) = k|\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}|^2 = k\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,,$$

and $\beta \equiv |\vec{\boldsymbol{\beta}}|$, where

$$\beta = k_1^2E^2 + 2k_1k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + k_2^2B^2 + k^2\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,. \tag{70}$$

This last equation is needed to obtain an expression for $\gamma \equiv (1-\beta^2)^{-1/2}$. Plugging the above results into eq. (54) yields,

$$\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = c_1\vec{\boldsymbol{E}} + c_2\vec{\boldsymbol{B}} + c_3\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}}\,, \tag{71}$$

in reference frame K'', where

$$c_1 = -\gamma^2 k_1 h + \gamma\big[k_1 E^2 + k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\big] + \frac{\gamma^3 k k_1}{\gamma+1}\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,, \tag{72}$$

$$c_2 = -\gamma^2 k_2 h + \gamma\big[k_1(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + k_2 B^2\big] + \frac{\gamma^3 k k_2}{\gamma+1}\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big]\,, \tag{73}$$

$$c_3 = \gamma^2(1-kh) - \frac{\gamma^3}{\gamma+1}\big[k_1^2E^2 + 2k_1k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + k_2^2B^2\big]\,. \tag{74}$$

and h is given by eq. (55), which we rewrite below for the reader's convenience,

$$h \equiv E^2 + B^2 - k\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big] - \frac{\gamma}{\gamma+1}\bigg\{\big[k_1E^2 + k_2(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})\big]^2 + \big[k_1(\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}}) + k_2B^2\big]^2\bigg\}\,. \tag{75}$$

To find solutions $\{k_1, k_2, k\}$ to the equation $\vec{\boldsymbol{E}}' \times \vec{\boldsymbol{B}}' = 0$, we set $c_1 = c_2 = c_3 = 0$. This yields three nonlinear equations for the three unknowns, k_1, k_2 and k. The one solution obtained previously with $\vec{\boldsymbol{\beta}} = \beta_0\hat{\boldsymbol{z}}$ corresponds to $k_1 = k_2 = 0$ and $kh = 1$, where k is given by eq. (60). Here, we write β_0 to distinguish this special case from the general case under consideration. In this special case, $c_1 = c_2 = 0$ automatically and $c_3 = 0$ yields $kh = 1$ which implies that

$$k(E^2 + B^2) - k^2\big[E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2\big] = 1\,. \tag{76}$$

Using eqs. (25)–(27), $E^2 + B^2 = 5E_0^2$ and $E^2B^2 - (\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}})^2 = 4E_0^4\sin^2\theta$. Hence,

$$4E_0^4k^2\sin^2\theta - 5E_0^2k + 1 = 0\,. \tag{77}$$

Using eq. (60) to eliminate k (replacing β with β_0 as noted above), eq. (56) is equivalent to

$$\beta_0^2 - \frac{5\beta_0}{2\sin\theta} + 1 = 0\,, \tag{78}$$

which yields eq. (61) for the special case of $\vec{\boldsymbol{\beta}} = \beta_0\hat{\boldsymbol{z}}$, as expected.

More generally, one can verify that eq. (48) provides a family of solutions to eqs. (72)–(74) with $c_1 = c_2 = c_3 = 0$. In light of eqs. (44) and (48), we can identify,

$$k_1 = \frac{\beta'(\sin\theta - 2\beta_0)}{\gamma_0E_0\sin\theta\sqrt{1 - 4\beta_0\sin\theta + 4\beta_0^2}}\,, \tag{79}$$

$$k_2 = \frac{\beta'\beta_0\cos\theta}{\gamma_0E_0\sin\theta\sqrt{1 - 4\beta_0\sin\theta + 4\beta_0^2}}\,, \tag{80}$$

$$k = \frac{\beta_0}{2E_0^2\sin\theta}\,, \tag{81}$$

where β_0 is given by eq. (78), $\gamma_0 \equiv (1-\beta_0^2)^{-1/2}$ and β' is an arbitrary number such that $0 \leq \beta' \leq 1$. I have checked using Mathematica that after plugging in eqs. (79)–(81) into eqs. (72)–(75) along with $E^2 = E_0^2$, $B^2 = 4E_0^2$, and $\vec{\boldsymbol{E}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = 2E_0^2\cos\theta$, the end result is,

$$c_1 = -\frac{2\gamma C E_0\bigl[2(\beta_0^2+1)\sin\theta - 5\beta_0\bigr]}{(1+\gamma)\sin\theta}\left[\gamma + \frac{1-\frac{1}{2}\beta_0\sin\theta - C^2\cos^2\theta}{1-\beta_0^2 - C^2(1-4\beta_0\sin\theta+4\beta_0^2)}\right], \tag{82}$$

$$c_2 = \frac{\gamma C E_0\cos\theta\bigl[2(\beta_0^2+1)\sin\theta - 5\beta_0\bigr]}{(1+\gamma)\sin\theta}\left[\gamma + \frac{1-C^2(1-2\beta_0\sin\theta)}{1-\beta_0^2 - C^2(1-4\beta_0\sin\theta+4\beta_0^2}\right], \tag{83}$$

$$c_3 = \frac{\gamma^2\bigl[2(\beta_0^2+1)\sin\theta - 5\beta_0\bigr]\bigl[1+\gamma-\gamma C^2(1-2\beta_0\sin\theta)\bigr]}{2(1+\gamma)\sin\theta}, \tag{84}$$

where

$$C \equiv \frac{\beta'}{\gamma_0\sqrt{1-4\beta_0\sin\theta+4\beta_0^2}}\,. \tag{85}$$

Indeed, if β_0 satisfies eq. (78) then we find that $c_1 = c_2 = c_3 = 0$. Thus, I have verified that a boost to the frame with boost parameter given by eq. (48) yields $\vec{\boldsymbol{E}}\,' \times \vec{\boldsymbol{B}}\,' = 0$. I believe that $\{k_1, k_2, k\}$ given by eqs. (79)–(81) provides all possible solutions, but I do not have a proof of this statement.

4. [Jackson, problem 11.18] The electric and magnetic fields of a particle of charge q moving in a straight line with speed $v = \beta c$, given by eq. (11.52) of Jackson, become more and more concentrated as $\beta \to 1$, as indicated in Fig. 11.9 on p. 561 of Jackson. Choose axes so that the charge moves along the z axis in the positive direction, passing the origin at $t = 0$. Let the spatial coordinates of the observation point be (x, y, z) and define the transverse vector $\vec{\boldsymbol{r}}_\perp$, with components x and y. Consider the fields and the source in the limit of $\beta = 1$.

(a) Show that the fields can be written as

$$\vec{\boldsymbol{E}} = 2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,, \qquad\qquad \vec{\boldsymbol{B}} = 2q\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,, \tag{86}$$

where $\hat{\boldsymbol{v}}$ is a unit vector in the direction of the particle's velocity.

We begin with eq. (11.154) on p. 560 of Jackson,

$$\vec{\boldsymbol{E}} = \frac{q\vec{\boldsymbol{R}}}{R^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}\,, \tag{87}$$

where ψ is the angle between the vectors $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{R}}$. I have modified Jackson's notation by employing the symbol $\vec{\boldsymbol{R}}$ for the vector that points from the charge q to the observation point $\vec{\boldsymbol{r}} = (x, y, z)$ in reference frame K.[12] Eq. (87) was also derived in class along with the corresponding result for the magnetic field,

$$\vec{\boldsymbol{B}} = \frac{q(\vec{\boldsymbol{v}}\times\vec{\boldsymbol{R}})}{cR^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}\,. \tag{88}$$

[12] Jackson denotes the vector that points from the charge q to the observation point (x, y, z) by $\vec{\boldsymbol{r}}$. However, I prefer to employ $\vec{\boldsymbol{r}}$ to represent the vector that points from the origin of reference frame K to the observation point, as shown in Fig. 1.

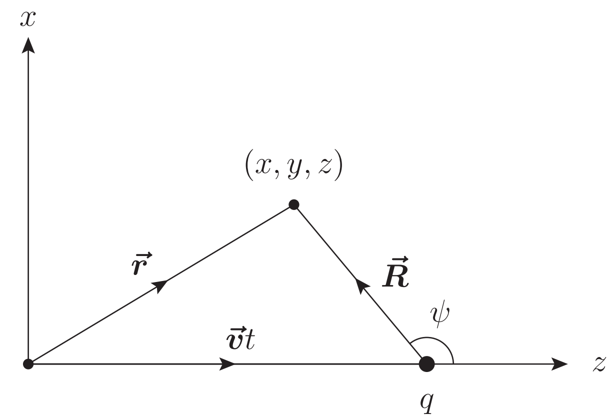

Figure 1: A charge q moving at constant velocity $\vec{\boldsymbol{v}}$ in the z-direction as seen from reference frame K. The angle ψ is defined so that $\hat{\boldsymbol{v}}\boldsymbol{\cdot}\hat{\boldsymbol{R}} = \cos\psi$.

The reference frame K is exhibited in Fig. 1. It is evident from this figure that

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{r}} - \vec{\boldsymbol{v}}t\,. \tag{89}$$

The velocity vector is taken to lie along the z-direction. That is, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{z}}$.

It is convenient to introduce the notation where

$$\vec{\boldsymbol{r}}_\perp = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}\,, \qquad\qquad \vec{\boldsymbol{r}}_\parallel = z\vec{\boldsymbol{z}}\,, \tag{90}$$

so that $\vec{\boldsymbol{r}}_\perp\boldsymbol{\cdot}\vec{\boldsymbol{v}} = 0$ and $\vec{\boldsymbol{r}}_\parallel \times \vec{\boldsymbol{v}} = 0$. Likewise, we can resolve the vector $\vec{\boldsymbol{R}}$ into components parallel and perpendicular to the velocity vector,

$$\vec{\boldsymbol{R}} = \vec{\boldsymbol{R}}_\parallel + \vec{\boldsymbol{R}}_\perp\,,$$

where

$$\vec{\boldsymbol{R}}_\parallel \equiv R_\parallel\hat{\boldsymbol{z}} = (z-vt)\hat{\boldsymbol{z}}\,, \qquad\qquad \vec{\boldsymbol{R}}_\perp = \vec{\boldsymbol{r}}_\perp\,. \tag{91}$$

after making use of eq. (89). In particular, note that $|\vec{\boldsymbol{R}}_\perp| \equiv R_\perp = R\sin\psi$. It follows that

$$\begin{aligned} R^3(1-\beta^2\sin^2\psi)^{3/2} &= (R^2-\beta^2R^2\sin^2\psi)^{3/2} = (R_\perp^2 + R_\parallel^2 - \beta^2R_\perp^2)^{3/2} \\ &= [R_\parallel^2 + R_\perp^2(1-\beta^2)]^{3/2} = (R_\parallel^2 + R_\perp^2/\gamma^2)^{3/2}\,. \end{aligned} \tag{92}$$

Note that in obtaining eq. (92) we used $R^2 = R_\perp^2 + R_\parallel^2$ and $\gamma \equiv (1-\beta^2)^{-1/2}$. Moreover, since $\vec{\boldsymbol{R}}_\perp = \vec{\boldsymbol{r}}_\perp$ [cf. eq. (91)], we may replace $R_\perp$ with $r_\perp \equiv |\vec{\boldsymbol{r}}_\perp| = (x^2+y^2)^{1/2}$ in the above formulae. Eqs. (87), (91) and (92) then yield

$$\vec{\boldsymbol{E}} = \frac{\gamma q\big[\vec{\boldsymbol{r}}_\perp + (z-vt)\hat{\boldsymbol{z}}\big]}{(\gamma^2R_\parallel^2 + r_\perp^2)^{3/2}}\,. \tag{93}$$

Likewise, eqs.(88), (91) and (92) yield

$$\vec{\boldsymbol{B}} = \frac{\gamma q(\vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp)}{c(\gamma^2R_\parallel^2 + r_\perp^2)^{3/2}}\,. \tag{94}$$

Consider the limit of $\beta \to 1$. In this limit, $\gamma \to \infty$, and we see that

$$\lim_{\gamma\to\infty} \frac{\gamma}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = \begin{cases} 0\,, & \text{if } R_\parallel \neq 0\,, \\ \infty\,, & \text{if } R_\parallel = 0\,. \end{cases}$$

This implies that

$$\lim_{\gamma\to\infty} \frac{\gamma}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = k\delta(R_\parallel)\,, \tag{95}$$

for some constant k. Note that in light of eq. (91),

$$\lim_{\gamma\to\infty} R_\parallel = z - ct\,,$$

since $\gamma \to \infty$ in the limit of $v \to c$. To determine k, we integrate eq. (95) from $-\infty$ to ∞, since $R_\parallel$ can be any real number (either positive, negative or zero) depending on the value of the time t. Thus, employing the substitution $u = \gamma R_\parallel$,

$$k = \int_{-\infty}^{\infty} \frac{\gamma\, dR_\parallel}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = \int_{-\infty}^{\infty} \frac{du}{(u^2 + r_\perp^2)^{3/2}} = \frac{u}{r_\perp^2 (u^2 + r_\perp^2)^{1/2}}\bigg|_{-\infty}^{\infty} = \frac{2}{r_\perp^2}\,.$$

Hence, we conclude that

$$\lim_{\gamma\to\infty} \frac{\gamma}{(\gamma^2 R_\parallel^2 + r_\perp^2)^{3/2}} = \frac{2}{r_\perp^2}\delta(z - ct)\,.$$

Inserting this result back into eqs. (114) and (94) yields

$$\vec{\boldsymbol{E}} = 2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z - ct)\,, \qquad \vec{\boldsymbol{B}} = 2q\frac{\hat{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z - ct)\,. \tag{96}$$

In obtaining $\vec{\boldsymbol{E}}$ above, we noted that in the limit of $v \to c$, the z-component of the electric field is proportional to $(z - ct)\delta(z - ct) = 0$ due to a property of the delta function. In obtaining $\vec{\boldsymbol{B}}$ above, we noted that $\lim_{v\to c} \vec{\boldsymbol{v}}/c = \hat{\boldsymbol{v}}$. Finally, since the delta function is an even function of its argument, we can write $\delta(z - ct) = \delta(ct - z)$, and eq. (86) is verified.

(b) Show by substitution into the Maxwell equations that these fields are consistent with the 4-vector source density

$$J^\alpha = qcv^\alpha \delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(ct - z)\,,$$

where the 4-vector $v^\alpha = (1\,;\,\hat{\boldsymbol{v}})$.

The four-vector current is given by $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$. Hence, using the Maxwell equations in gaussian units,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 4\pi\rho = \frac{4\pi J^0}{c}\,.$$

Hence, using eq. (96) and noting that $E_z = 0$, it follows that

$$J^0 = \frac{c}{4\pi}\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = \frac{c}{4\pi}\left(\vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\vec{\boldsymbol{E}} + \frac{\partial E_z}{\partial z}\right) = \frac{qc}{2\pi}\delta(z - ct)\vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right)\,, \tag{97}$$

where

$$\vec{\boldsymbol{\nabla}}_\perp \equiv \hat{\boldsymbol{x}}\,\partial/\partial x + \hat{\boldsymbol{y}}\,\partial/\partial y\,. \tag{98}$$

For $\vec{\boldsymbol{r}}_\perp \equiv x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} \neq 0$, an elementary computation yields

$$\vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0\,. \tag{99}$$

To determine the behavior at $\vec{\boldsymbol{r}}_\perp = 0$, we consider the two-dimensional analogue of the divergence theorem,

$$\int_A dxdy\,\vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right) = \oint_C r_\perp d\phi\,\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\boldsymbol{\cdot}\hat{\boldsymbol{r}}_\perp = \int_0^{2\pi} d\phi = 2\pi\,, \tag{100}$$

where A is a circular disk and C is the circular boundary of the disk. Note that $\hat{\boldsymbol{r}}_\perp = \vec{\boldsymbol{r}}_\perp/r_\perp$ is the outward normal to the circular boundary.

Eqs. (99) and (100) imply that

$$\vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right) = 2\pi\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\,, \tag{101}$$

where $\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)$ is a two-dimensional delta function. Inserting this result into eq. (97), we end up with

$$J^0 = qc\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)\,. \tag{102}$$

Next, we employ the Maxwell equation,

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{B}} - \frac{1}{c}\frac{\partial\vec{\boldsymbol{E}}}{\partial t} = \frac{4\pi}{c}\vec{\boldsymbol{J}}, \tag{103}$$

to evaluate $\vec{\boldsymbol{J}}$. First, we compute

$$\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp = \hat{\boldsymbol{z}}\times(x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}) = x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}}\,, \tag{104}$$

where we have used the fact that $\vec{\boldsymbol{v}}$ points in the z direction. It then follows that

$$\vec{\boldsymbol{\nabla}}\times\left[\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right] = \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2}\,\delta(z-ct) & \dfrac{x}{x^2+y^2}\,\delta(z-ct) & 0 \end{pmatrix}. \tag{105}$$

Evaluating the determinant and making use of eqs. (90), (98) and (101) yields,

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\times\left[\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right] &= -\frac{x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}}{x^2+y^2}\delta'(z-ct) + \left\{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\}\delta(z-ct) \\ &= -\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta'(z-ct) + \hat{\boldsymbol{z}}\,\vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\right)\delta(z-ct) \\ &= -\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta'(z-ct) + 2\pi\hat{\boldsymbol{z}}\,\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)\,. \end{aligned} \tag{106}$$

The prime refers to differentiation with respect to z. Finally, we compute

$$\frac{\partial}{\partial t}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right)=-c\frac{\partial}{\partial z}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right)=-\frac{c\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta'(z-ct)\,. \tag{107}$$

Inserting eq. (96) into eq. (103) and using eqs. (106) and (107), we obtain

$$\begin{aligned}\vec{\boldsymbol{J}}&=\frac{qc}{2\pi}\vec{\boldsymbol{\nabla}}\times\left[\frac{\hat{\boldsymbol{v}}\times\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right]-\frac{q}{2\pi}\frac{\partial}{\partial t}\left(\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(z-ct)\right)\\&=\frac{qc}{2\pi}\left\{-\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta'(z-ct)+2\pi\hat{\boldsymbol{z}}\,\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)+\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta'(z-ct)\right\}\\&=qc\hat{\boldsymbol{v}}\,\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)\,,\end{aligned} \tag{108}$$

after using the fact that $\hat{\boldsymbol{v}}=\hat{\boldsymbol{z}}$. Combining eqs. (102) and (108), we can write

$$J^\alpha=qcv^\alpha\delta^{(2)}(\vec{\boldsymbol{r}}_\perp)\delta(z-ct)\,,$$

where the four-vector $v^\alpha=(1\,;\,\hat{\boldsymbol{v}})$.

(c) Show that the fields of part (a) are derivable from either of the following 4-vector potentials:

$$A^0=A^z=-2q\delta(ct-z)\ln(\lambda r_\perp)\,,\qquad\qquad \vec{\boldsymbol{A}}_\perp=0\,, \tag{109}$$

or

$$A^0=A^z=0\,,\qquad\qquad \vec{\boldsymbol{A}}_\perp=-2q\Theta(ct-z)\vec{\boldsymbol{\nabla}}_\perp\ln(\lambda r_\perp)\,, \tag{110}$$

where λ is an irrelevant parameter setting the scale of the logarithm. Show that the two potentials differ by a gauge transformation and find the corresponding gauge function χ.

The four-vector potential is $A^\mu=(\Phi\,;\,\vec{\boldsymbol{A}})$. Given the four-vector potential, the electromagnetic fields are determined by

$$\vec{\boldsymbol{E}}=-\vec{\boldsymbol{\nabla}}A^0-\frac{1}{c}\frac{\partial\vec{\boldsymbol{A}}}{\partial t}\,,\qquad\qquad \vec{\boldsymbol{B}}=\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{A}}\,.$$

Inserting the scalar and vector potentials given in eq. (109),

$$\begin{aligned}\vec{\boldsymbol{E}}&=2q\vec{\boldsymbol{\nabla}}\Big[\delta(ct-z)\ln(\lambda r_\perp)\Big]+\frac{2q}{c}\hat{\boldsymbol{z}}\ln(\lambda r_\perp)\frac{\partial}{\partial t}\delta(ct-z)\\&=2q\delta(z-ct)\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\left[\tfrac{1}{2}\ln(x^2+y^2)+\ln\lambda\right]+2q\hat{\boldsymbol{z}}\ln(\lambda r_\perp)\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\delta(ct-z)\\&=2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\delta(z-ct)\,,\end{aligned}$$

after using $\vec{\boldsymbol{r}}_\perp = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ and $r_\perp^2 = x^2 + y^2$. In particular, note that

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) f(ct - z) = 0\,,$$

for any function of $ct - z$. Using eq. (109) to compute the magnetic field,

$$\begin{aligned}
\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} &= \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & -2q\ln(\lambda r_\perp)\delta(ct-z) \end{pmatrix} \\
&= -2q\delta(ct-z)\left\{\hat{\boldsymbol{x}}\frac{\partial}{\partial y}\ln(\lambda r_\perp) - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\ln(\lambda r_\perp)\right\} \\
&= -2q\delta(ct-z)\left\{\hat{\boldsymbol{x}}\frac{\partial}{\partial y}\left[\tfrac{1}{2}\ln(x^2+y^2) + \ln\lambda\right] - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\left[\tfrac{1}{2}\ln(x^2+y^2) + \ln\lambda\right]\right\} \\
&= -\frac{2q}{r_\perp^2}(y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}})\delta(ct-z) = 2q\frac{\hat{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,,
\end{aligned}$$

after employing eq. (104).

Repeating these calculations using eq. (110),

$$\vec{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial\vec{\boldsymbol{A}}_\perp}{\partial t} = 2q\delta(ct-z)\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\left[\tfrac{1}{2}\ln(x^2+y^2) + \ln\lambda\right] = 2q\frac{\vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,,$$

after using the relation between the delta function and the step function, $\delta(x) = \dfrac{d}{dx}\Theta(x)$. In the computation of the magnetic field, we require the following result:

$$\vec{\boldsymbol{\nabla}}_\perp \ln(\lambda r_\perp) = \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\left[\tfrac{1}{2}\ln(x^2+y^2) + \ln\lambda\right] = \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}}{x^2+y^2}\,.$$

Hence, it follows that

$$\begin{aligned}
\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}} &= -2q\det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \Theta(ct-z)\dfrac{x}{x^2+y^2} & \Theta(ct-z)\dfrac{y}{x^2+y^2} & 0 \end{pmatrix} \\
&= \frac{y\hat{\boldsymbol{x}} - x\hat{\boldsymbol{y}}}{x^2+y^2}\delta(ct-z) + \hat{\boldsymbol{z}}\,\Theta(ct-z)\left\{\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\} \\
&= 2q\frac{\hat{\boldsymbol{v}} \times \vec{\boldsymbol{r}}_\perp}{r_\perp^2}\,\delta(ct-z)\,,
\end{aligned}$$

after employing eq. (104) and noting that

$$\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)=-\frac{2xy}{(x^2+y^2)^2}+\frac{2xy}{(x^2+y^2)^2}=0\,.$$

Finally, we demonstrate that eqs. (109) and (110) differ by a gauge transformation. Under a gauge transformation (using gaussian units),

$$\vec{\boldsymbol{A}}\to\vec{\boldsymbol{A}}'=\vec{\boldsymbol{A}}+\vec{\boldsymbol{\nabla}}\chi\,,\qquad\qquad A^0\to A'^0=A^0-\frac{1}{c}\frac{\partial\chi}{\partial t}\,.$$

Denoting A^μ by eq. (109) and A'^μ by eq. (110), it follows that

$$\begin{aligned}\frac{\partial\chi}{\partial t}&=-2qc\delta(ct-z)\ln(\lambda r_\perp)\,,\\ \vec{\boldsymbol{\nabla}}_\perp\chi&=-2q\Theta(ct-z)\vec{\boldsymbol{\nabla}}_\perp\ln(\lambda r_\perp)\,,\\ \frac{\partial\chi}{\partial z}&=2q\delta(ct-z)\ln(\lambda r_\perp)\,.\end{aligned}$$

The solution to these equations can be determined by inspection,

$$\chi(\vec{\boldsymbol{x}},t)=-2q\Theta(ct-z)\ln(\lambda r_\perp)\,,$$

up to an overall additive constant.

5. [Jackson, problem 11.19] A particle of mass M and 4-momentum P decays into two particles of mass m_1 and m_2.

(a) Use the conservation of energy and momentum in the form $p_2 = P - p_1$, and use the invariance of scalar products of 4-vectors to show that the total energy of the first particle in the rest frame of the decaying particle is[13]

$$E_1=\frac{M^2+m_1^2-m_2^2}{2M}\,,$$

and that E_2 is obtained by interchanging m_1 and m_2.

In the rest frame of the decaying particle (with 3-momentum $\vec{\boldsymbol{P}}$, relativistic energy E and mass M), we can explicitly write out the momentum four-vectors,[7]

$$P=(M\,;\,\vec{\boldsymbol{0}})\,,\qquad p_1=(E_1\,;\,\vec{\boldsymbol{p}}_1)\,,\qquad p_1=(E_2\,;\,\vec{\boldsymbol{p}}_2)\,.\tag{111}$$

In particular, $\vec{\boldsymbol{P}}=\vec{\boldsymbol{0}}$ (the zero-vector), since by definition the decaying particle in its rest frame is at rest, i.e. its velocity is zero so that $\vec{\boldsymbol{P}}=\gamma M\vec{\boldsymbol{v}}=\vec{\boldsymbol{0}}$. Likewise, $E=\gamma M$, but in the rest frame $\gamma=1$ so that $E=M$ as indicated in eq. (111).

[13]Note that on p. 566, Jackson indicates that in particle kinematics, the symbols p and m stand for cp and mc^2, respectively. We follow this convention in this problem.

To obtain an expression for E_1, we start from the conservation of the 4-momentum,

$$p_2 = P - p_1 \,.$$

Squaring this equation, and using the fact that squares of the corresponding 4-momenta are Lorentz-invariant, $P^2 = M^2$ and $p_i^2 = m_i$ (for $i = 1, 2$), it follows that

$$m_2^2 = M^2 + m_1^2 - 2P \cdot p_1 \,. \tag{112}$$

The dot product of the two 4-momenta is a Lorentz invariant and can be evaluated in any reference frame. In particular, using the rest-frame forms for the 4-momenta given in eq. (111), it immediately follows that $P \cdot p_1 = ME_1$. Inserting this result in eq. (112) yields

$$m_2 = M^2 + m_1^2 - 2ME_1 \,.$$

Solving for E_1 then yields

$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} \,. \tag{113}$$

To obtain E_2 one can repeat the derivation above starting from $p_1 = P - p_2$. Squaring this equation and using $P \cdot p_2 = ME_2$, we see that the resulting derivation is identical to the previous one with m_1 and m_2 interchanged. Thus,

$$E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M} \,. \tag{114}$$

Alternatively, we can obtain E_2 from energy conservation, $P^0 = p_1^0 + p_2^0$. That is,

$$E_1 + E_2 = M \,. \tag{115}$$

Inserting eq. (113) into eq. (115) and solving for E_2, we again obtain eq. (114).

<u>*REMARK*</u>:

Although Jackson does not ask you to compute the magnitude of the 3-momentum of the decaying particles, this can be easily done. By momentum conservation, $\vec{\boldsymbol{P}} = \vec{\boldsymbol{p}}_1 + \vec{\boldsymbol{p}}_2 = \vec{\boldsymbol{0}}$, from which it follows that $\vec{\boldsymbol{p}}_1 = -\vec{\boldsymbol{p}}_2$. Hence, we can re-express the rest-frame four vectors p_1 and p_2 given in eq. (111) as $p_1 = (E_1\,;\,\vec{\boldsymbol{p}})$ and $p_2 = (E_2\,;\,-\vec{\boldsymbol{p}})$, where

$$\vec{\boldsymbol{p}} \equiv \vec{\boldsymbol{p}}_1 = -\vec{\boldsymbol{p}}_2 \,. \tag{116}$$

Using the formula for relativistic energy, $E_1^2 = |\vec{\boldsymbol{p}}\,|^2 + m_1^2$, and the formula for E_1 given in eq. (113), we can solve for $|\vec{\boldsymbol{p}}\,|$,

$$|\vec{\boldsymbol{p}}\,| = \sqrt{E_1^2 - m_1^2} = \frac{\sqrt{(M^2 + m_1^2 - m_2^2)^2 - 4M^2 m_1^2}}{2M} \,.$$

In the literature, this formula is often written as

$$|\vec{\boldsymbol{p}}\,| = \frac{\lambda^{1/2}(M^2, m_1^2, m_2^2)}{2M} \,,$$

where the function $\lambda(x, y, z)$ is defined by

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz\,.$$

This function is ubiquitous in relativistic kinematics and is known as the triangle function.[14]

(b) Show that the *kinetic energy* T_i of the ith particle in the same frame is:

$$T_i = \Delta M \left(1 - \frac{m_i}{M} - \frac{\Delta M}{2M}\right)\,, \tag{117}$$

where $\Delta M \equiv M - m_1 - m_2$ is the mass excess or Q-value of the process.

The kinetic energy T_i of the ith particle is defined as (cf. footnote 7):

$$T_i = E_i - m_i\,,$$

That is, we subtract the rest energy from the relativistic energy of the particle. Using eq. (113), it immediately follows that:

$$T_1 = \frac{M^2 + m_1^2 - m_2^2}{2M} - m_1 = \frac{(M - m_1)^2 - m_2^2}{2M} = \frac{(M - m_1 - m_2)(M - m_1 + m_2)}{2M}\,.$$

If we define $\Delta M \equiv M - m_1 - m_2$, then simple algebra yields

$$\frac{M - m_1 + m_2}{2M} = 1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\,.$$

Hence, it follows that

$$T_1 = \Delta M \left(1 - \frac{m_1}{M} - \frac{\Delta M}{2M}\right)\,. \tag{118}$$

The calculation of T_2 proceeds similarly by interchanging m_1 and m_2. Since ΔM is invariant under this interchange, it follows that:

$$T_2 = \Delta M \left(1 - \frac{m_2}{M} - \frac{\Delta M}{2M}\right)\,. \tag{119}$$

Thus, we have confirmed eq. (117). Note that

$$T_1 + T_2 = \Delta M \left(2 - \frac{\Delta M + m_1 + m_2}{M}\right) = \Delta M\,, \tag{120}$$

after using the definition of ΔM given below eq. (117).

[14]The function λ is called the triangle function since $\frac{1}{4}\left[-\lambda(x, y, z)\right]^{1/2}$ is the area of a triangle with sides of lengths $\sqrt{x}$, $\sqrt{y}$ and $\sqrt{z}$. However, in the application to the decay of one particle into two particles, we have $\lambda(M^2, m_1^2, m_2^2) \geq 0$ if the decay is kinematically allowed (i.e. consistent with energy conservation), in which case $M \geq m_1 + m_2$ and the corresponding triangle of lengths M, m_1 and m_2 does not exist. The triangle function can be expressed in many equivalent ways. For further details, see E. Byckling and K. Kajantie, *Particle Kinematics* (John Wiley & Sons Ltd., London, UK, 1973). This book is considered to be the bible for relativistic kinematics, although it is sadly long out of print.

(c) The charged pi-meson ($M = 139.6$ MeV) decays into a mu-meson[15] ($m_1 = 105.7$ MeV) and a neutrino ($m_2 = 0$). Calculate the kinetic energies of the mu-meson and the neutrino in the pi-meson's rest frame. The unique kinetic energy of the muon is the signature of a two-body decay. It entered importantly in the discovery of the pi-meson in photographic emulsions by Powell and coworkers in 1947.

For charged pi-meson decay into a muon and a neutrino, the sum of the muon and neutrino kinetic energies [cf. eq. (120)] is $T_1 + T_2 = \Delta M = 139.6 - 105.7 = 33.9$ MeV. Using eqs. (118) and (119),

$$T_1 = \left(1 - \frac{105.7}{139.6} - \frac{33.9}{2(139.6)}\right) 33.9\text{ MeV} = (0.1214)(33.9\text{ MeV}) = 4.1\text{ MeV}\,,$$
$$T_2 = \left(1 - \frac{33.9}{2(139.6)}\right) 33.9\text{ MeV} = (0.8786)(33.9\text{ MeV}) = 29.8\text{ MeV}\,.$$

6. [Jackson, problem 11.27, part (a)] A charge density ρ' of zero total charge, but with a dipole moment $\vec{\boldsymbol{p}}'$, exists in reference frame K'. There is no current density in K'. The frame K' moves with velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ in the frame K. Find the charge and current densities ρ and $\vec{\boldsymbol{J}}$ in the frame K and show that there is a magnetic dipole moment $\vec{\boldsymbol{m}} = \frac{1}{2}(\vec{\boldsymbol{p}}' \times \vec{\boldsymbol{\beta}})$, correct to first order in β. What is the electric dipole moment $\vec{\boldsymbol{p}}$ in K to the same order in β?[16]

In frame K', we have $\vec{\boldsymbol{J}}' = 0$ and the charge density $\rho'(\vec{\boldsymbol{x}}')$ is assumed to be time-independent. Furthermore, the total charge in frame K' is zero, i.e.

$$\int \rho'(\vec{\boldsymbol{x}}')\, d^3x' = 0\,, \tag{121}$$

whereas the electric dipole moment in frame K', denoted by $\vec{\boldsymbol{p}}'$, is assumed to be nonzero. By definition, the electric dipole moment is given by

$$\vec{\boldsymbol{p}}' = \int \vec{\boldsymbol{x}}'\, \rho'(\vec{\boldsymbol{x}}')\, d^3x'\,. \tag{122}$$

Frame K' moves with velocity $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\beta}}c$ with respect to frame K. We shall employ eqs. (22) and (23), which we repeat here:

$$c\rho = \gamma(c\rho' + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}')\,, \tag{123}$$
$$\vec{\boldsymbol{J}} = \vec{\boldsymbol{J}}' + \frac{\gamma - 1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}c\rho'\,. \tag{124}$$

Setting $\vec{\boldsymbol{J}}' = 0$, it follows that

$$\rho(\vec{\boldsymbol{x}}) = \gamma\rho'(\vec{\boldsymbol{x}}')\,, \qquad \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}) = \gamma\vec{\boldsymbol{\beta}}c\rho'(\vec{\boldsymbol{x}}') = \vec{\boldsymbol{\beta}}c\rho(\vec{\boldsymbol{x}})\,. \tag{125}$$

[15] The nomenclature *mu-meson* is no longer standard. The currently accepted name for this particle is the muon.

[16] Jackson denotes the electric dipole moment vector in frame K' by $\vec{\boldsymbol{p}}$, which is misleading notation. Thus, I have denoted the electric dipole moment in frame K' by $\vec{\boldsymbol{p}}'$ and the electric dipole moment in frame K by $\vec{\boldsymbol{p}}$.

Hence, the electric dipole moment in frame K is given by

$$\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x\,, \tag{126}$$

and the magnetic dipole moment is defined (in gaussian units) in frame K as

$$\vec{\boldsymbol{m}} = \frac{1}{2c}\int \vec{\boldsymbol{x}}\times\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})\,d^3x = \frac{1}{2}\int \vec{\boldsymbol{x}}\times\vec{\boldsymbol{\beta}}\,\rho(\vec{\boldsymbol{x}})\,d^3x = \frac{1}{2}\left\{\int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x\right\}\times\vec{\boldsymbol{\beta}}\,,$$

after employing eq. (125) in the second step above. In light of eq. (126), it follows that

$$\vec{\boldsymbol{m}} = \tfrac{1}{2}\,\vec{\boldsymbol{p}}\times\vec{\boldsymbol{\beta}}\,. \tag{127}$$

In order to relate $\vec{\boldsymbol{p}}$ to $\vec{\boldsymbol{p}}'$, we need to compare $\rho(\vec{\boldsymbol{x}})$ and $\rho'(\vec{\boldsymbol{x}}')$. Eq. (11.19) of Jackson gives the four-vector $x'^{\,\mu}$ in terms of x^μ. The inverse relation is obtained by changing the sign of $\vec{\boldsymbol{\beta}}$, which yields

$$x_0 = \gamma(x_0' + \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}')\,, \qquad\qquad \vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}' + \frac{\gamma-1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}x_0'\,. \tag{128}$$

Working to first order in β, we can set $\gamma = (1-\beta^2)^{-1/2} \simeq 1$, and

$$x_0 \simeq x_0' + \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}'\,, \qquad\qquad \vec{\boldsymbol{x}} \simeq \vec{\boldsymbol{x}}' + \vec{\boldsymbol{\beta}}x_0'\,. \tag{129}$$

In the same approximation, $d^3x' \simeq d^3x$ and $\rho(\vec{\boldsymbol{x}}) \simeq \rho'(\vec{\boldsymbol{x}}')$. Hence, using eq. (121),

$$\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x \simeq \int (\vec{\boldsymbol{x}}' + \vec{\boldsymbol{\beta}}x_0')\rho'(\vec{\boldsymbol{x}}')\,d^3x' = \int \vec{\boldsymbol{x}}'\rho'(\vec{\boldsymbol{x}}')\,d^3x' = \vec{\boldsymbol{p}}'\,. \tag{130}$$

to first order in β. Using eqs. (127) and (130), we conclude that to first order in β,

$$\vec{\boldsymbol{m}} = \tfrac{1}{2}\,\vec{\boldsymbol{p}}'\times\vec{\boldsymbol{\beta}}\,.$$

<u>*ADDENDUM:*</u>

Although Jackson only asks for the result to first order in β, it is not too hard to derive the exact result. By assumption, $\rho'(\vec{\boldsymbol{x}}')$ is a time-independent charge density in frame K' which satisfies eq. (121) [i.e. the total charge vanishes]. We are asked to compute $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{m}}$ in frame K at a fixed time x_0. Using eq. (128), it follows that

$$x_0' = \frac{x_0}{\gamma} - \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}'\,.$$

Inserting this result back into eq. (128) then yields

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}' + \left[\frac{\gamma-1}{\beta^2} - \gamma\right](\vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}')\vec{\boldsymbol{\beta}} + \vec{\boldsymbol{\beta}}x_0\,.$$

We can simplify this expression by noting that $\gamma = (1-\beta^2)^{-1/2}$ implies that $\beta^2\gamma^2 = \gamma^2 - 1$.

Substituting for β^2 above,

$$\frac{\gamma-1}{\beta^2}-\gamma=-\frac{\gamma}{\gamma+1}\,. \tag{131}$$

Hence,

$$\vec{\boldsymbol{x}}=\vec{\boldsymbol{x}}\,'-\frac{\gamma}{\gamma+1}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}}+\vec{\boldsymbol{\beta}}x_0\,. \tag{132}$$

We can evaluate the Jacobian of the transformation, eq. (132),[17]

$$\frac{\partial x_i}{\partial x'_j}=\delta_{ij}-\frac{\gamma}{\gamma+1}\,\beta_i\beta_j\,, \tag{133}$$

which we evaluate at fixed x_0. To compute the determinant of the Jacobian, we use the following general result which is easily proved (see the Appendix following this addendum):

$$\det(\delta_{ij}-a_ia_j)=1-|\vec{\boldsymbol{a}}|^2\,.$$

Hence,

$$\det\left(\frac{\partial x_i}{\partial x'_j}\right)=1-\frac{\gamma\beta^2}{\gamma+1}=\frac{1}{\gamma+1}\left[\gamma+1-\frac{\gamma^2-1}{\gamma}\right]=\frac{1}{\gamma}\,.$$

It follows that

$$d^3x=\det\left(\frac{\partial x_i}{\partial x'_j}\right)\,d^3x'=\frac{d^3x'}{\gamma}\,, \tag{134}$$

which is just the well-known length contraction in special relativity. Using eqs. (125) and (134), it follows that the charge dq located inside an infinitesimal volume element is

$$dq=\rho(\vec{\boldsymbol{x}})\,d^3x=\rho'(\vec{\boldsymbol{x}}\,')\,d^3x'\,, \tag{135}$$

since the factors of γ cancel out. Eq. (135) is expected since the electric charge is a Lorentz scalar, which must be independent of the reference frame used to evaluate it.

We now can compute $\vec{\boldsymbol{p}}$ in frame K. Using eqs. (132) and (135),

$$\vec{\boldsymbol{p}}=\int\vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x=\int\vec{\boldsymbol{x}}\,\rho'(\vec{\boldsymbol{x}}\,')\,d^3x'=\int\left[\vec{\boldsymbol{x}}\,'-\frac{\gamma}{\gamma+1}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}}+\vec{\boldsymbol{\beta}}x_0\right]\,\rho'(\vec{\boldsymbol{x}}\,')\,d^3x'\,. \tag{136}$$

Using eq. (122), it follows that

$$\vec{\boldsymbol{p}}=\vec{\boldsymbol{p}}\,'-\frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,')\vec{\boldsymbol{\beta}} \tag{137}$$

where we have used eq. (121) [i.e., the total charge in frame K' vanishes] to eliminate the last term in eq. (136). Indeed, if we work to first order in β, then the second term on the right hand side of eq. (137) can be dropped and we recover the result of eq. (130).

To obtain $\vec{\boldsymbol{m}}$, we start with the exact result obtained in eq. (127). Inserting eq. (137) for $\vec{\boldsymbol{p}}$ and using $\vec{\boldsymbol{\beta}}\times\vec{\boldsymbol{\beta}}=0$, it follows that[18]

$$\vec{\boldsymbol{m}}=\tfrac{1}{2}\,\vec{\boldsymbol{p}}\,'\times\vec{\boldsymbol{\beta}}\,, \tag{138}$$

which we now see is exact to all orders in β.

[17] In eq. (133), we do not distinguish between lowered and raised indices, as all involved quantities are three-dimensional.

[18] In the literature, one often finds the result of eq. (138) quoted without the factor of $\frac{1}{2}$. For a discussion of this discrepancy, see V. Hnizdo, *Magnetic dipole moment of a moving electric dipole*, American Journal of Physics **80**, 645–647 (2012).

REMARK:

One can derive eq. (137) more directly as follows. Using eqs. (125) and (128),

$$\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x = \gamma\int\left[\vec{\boldsymbol{x}}\,' + \frac{\gamma-1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}x_0'\right]\rho'(\vec{\boldsymbol{x}}\,')\,d^3x$$

$$= \gamma\int\left[\vec{\boldsymbol{x}}\,' + \frac{\gamma-1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}x_0'\right]\rho'(\vec{\boldsymbol{x}}\,')\,\delta(x_0 - ct)d^4x\,.$$

In the last step, I inserted the integral $\int\delta(x_0 - ct)dx_0 = 1$ which does not change the result. This is useful, since $d^4x = d^4x' = d^3x'dx_0'$. Using eq. (128) to express x_0 in terms of x_0',

$$\vec{\boldsymbol{p}} = \int\left[\vec{\boldsymbol{x}}\,' + \frac{\gamma-1}{\beta^2}\,(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}} + \gamma\vec{\boldsymbol{\beta}}x_0'\right]\rho'(\vec{\boldsymbol{x}}\,')\,\delta\left(x_0' + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,' - \frac{ct}{\gamma}\right)d^3x'dx_0'\,,$$

after writing $\gamma\delta(x_0 - ct) = \delta([x_0 - ct]/\gamma) = \delta(x_0' + \vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,' - ct/\gamma)$. Integrating over x_0' yields

$$\vec{\boldsymbol{p}} = \int\left[\vec{\boldsymbol{x}}\,' + \left(\frac{\gamma-1}{\beta^2} - \gamma\right)(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,')\vec{\boldsymbol{\beta}}\right]\rho'(\vec{\boldsymbol{x}}\,')\,d^3x' + ct\vec{\boldsymbol{\beta}}\int\rho'(\vec{\boldsymbol{x}}\,')\,d^3x'\,.$$

Using eqs. (121) and (131) along with $\vec{\boldsymbol{p}}' = \int\vec{\boldsymbol{x}}\,'\rho'(\vec{\boldsymbol{x}}\,')\,d^3x'$, we end up with

$$\vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}' - \frac{\gamma}{\gamma+1}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}')\vec{\boldsymbol{\beta}}\,,$$

which reproduces eq. (137). Of course, the two derivations are equivalent.

APPENDIX: Proof of a determinantal formula

Theorem: Let A be an $n\times n$ matrix, whose matrix elements are given by:

$$A_{ij} = \delta_{ij} - a_i a_j\,. \tag{139}$$

Then,

$$\det A = 1 - |\vec{\boldsymbol{a}}|^2\,, \qquad \text{where } |\vec{\boldsymbol{a}}|^2 = \sum_{i=1}^n a_i^2\,. \tag{140}$$

Proof: The determinant of A is defined as

$$\det A = \epsilon_{i_1 i_2\cdots i_n}\,A_{1i_1}A_{2i_2}\cdots A_{ni_n}\,,$$

where there is an implicit sum over repeated indices. Plugging in eq. (139),

$$\det A = \epsilon_{i_1 i_2\cdots i_N}(\delta_{1i_1} - a_1 a_{i_1})(\delta_{2i_2} - a_2 a_{i_2})\cdots(\delta_{ni_n} - a_n a_{i_n})\,.$$

Expanding the product above, and using the Kronecker deltas to perform the sums, we obtain

$$\begin{aligned}\det A &= \epsilon_{123\cdots n} - a_1 a_{i_1}\epsilon_{i_1 23\cdots n} - a_2 a_{i_2}\epsilon_{1 i_2 3\cdots n} - \cdots - a_n a_{i_n}\epsilon_{123\cdots i_n}\\ &= \epsilon_{123\cdots n}(1 - a_1^2 - a_2^2 - \cdots - a_n^2)\,,\end{aligned} \tag{141}$$

after performing the final set of summations and using the fact that $\epsilon_{i_1 i_2 \cdots i_n}$ vanishes unless all of its indices are distinct. Note the absence of any terms in eq. (141) that are quartic (or higher order) in the a_i, since all such terms will be symmetric under the interchange of two indices that are summed against two corresponding indices of the Levi-Civita tensor. For example,

$$\epsilon_{i_1 i_2 \cdots i_N} a_1 a_{i_1} a_2 a_{i_2} = 0 \,,$$

since $a_{i_1} a_{i_2}$ is symmetric under the interchange of i_1 and i_2 whereas $\epsilon_{i_1 i_2 \cdots i_N}$ is antisymmetric under this interchange of indices.

Since $\epsilon_{123 \cdots n} = 1$, eq. (141) yields

$$\det A = 1 - \sum_{i=1}^{n} a_i^2 \,,$$

which completes the proof.