1. [Jackson, problem 14.4] Using the Liénard-Wiechert fields, discuss the time-averaged power radiated per unit solid angle in nonrelativistic motion of a point particle with charge e, moving:

(a) along the z axis with instantaneous position $z(t) = a\cos\omega_0(t)\,$,

(b) in a circle of radius R in the x–y plane with constant angular frequency ω_0.

Sketch the angular distribution of the radiation of the radiation and determine the total power radiated in each case.

(a) Case 1: Non-relativistic motion of a point particle with charge e moving along the z-axis with instantaneous position $z(t) = a\cos\omega_0(t)\,$.

We make use of eq. (14.20) of Jackson, which is relevant for non-relativistic motion,

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\left|\boldsymbol{\hat{n}}\times\left(\boldsymbol{\hat{n}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right|^2\,, \tag{1}$$

where

$$\vec{\boldsymbol{\beta}} = \frac{\vec{\boldsymbol{v}}}{c} = \frac{1}{c}\frac{d\vec{\boldsymbol{x}}}{dt}\,.$$

In this case, we have

$$\vec{\boldsymbol{x}}(t) = \boldsymbol{\hat{z}}\,a\cos\omega_0 t\,,$$

which yields

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = -\boldsymbol{\hat{z}}\,\frac{a\omega_0^2}{c}\,\cos\omega_0 t\,.$$

Working out the absolute square of the triple product in eq. (1),

$$\left|\boldsymbol{\hat{n}}\times\left(\boldsymbol{\hat{n}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right|^2 = \left|\boldsymbol{\hat{n}}\left(\boldsymbol{\hat{n}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right) - \frac{d\vec{\boldsymbol{\beta}}}{dt}\right|^2 = \left|\frac{d\vec{\boldsymbol{\beta}}}{dt}\right|^2 - \left(\boldsymbol{\hat{n}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)^2 \tag{2}$$

$$= \frac{a^2\omega_0^4}{c^2}\,\cos^2\omega_0 t\,\left[1-(\boldsymbol{\hat{n}}\boldsymbol{\cdot}\boldsymbol{\hat{z}})^2\right] = \frac{a^2\omega_0^4}{c^2}\,\cos^2\omega_0 t\,\sin^2\theta\,.$$

In obtaining the final result above, we chose to work in a coordinate system in which the origin corresponds to the instantaneous position of the charged particle, and the unit vector $\boldsymbol{\hat{n}}$ has polar angle θ and azimuthal angle ϕ with respect to the z-axis,

$$\boldsymbol{\hat{n}} = \boldsymbol{\hat{x}}\,\sin\theta\,\cos\phi + \boldsymbol{\hat{y}}\,\sin\theta\,\sin\phi + \boldsymbol{\hat{z}}\,\cos\theta\,. \tag{3}$$

The time-averaged power is easily obtained by noting that[1]

$$\langle\cos^2\omega_0 t\rangle = \tfrac{1}{2}\,.$$

[1] To compute the time-average of $\cos^2\omega_0 t$, note that the time averages satisfy $\langle\cos^2\omega_0 t\rangle = \langle\sin^2\omega_0 t\rangle$, and $\cos^2\omega_0 t + \sin^2\omega_0 t = 1$.

Hence, it follows that

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2 a^2 \omega_0^4}{8\pi c^3} \sin^2\theta\,. \tag{4}$$

In Figure 1, the angular distribution of the radiated power is exhibited as a polar plot.

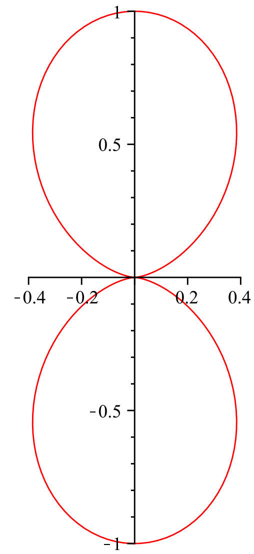


Figure 1: A polar plot of the angular distribution of the power radiated by a charged particle moving non-relativistically along the z axis with instantaneous position $z(t) = a\cos\omega_0(t)$. The angular distribution is given by eq. (4) and is proportional to $\sin^2\theta$. This plot was created with Maple software.

Integrating over the solid angle yields the total radiated power,

$$\langle P \rangle = \frac{e^2 a^2 \omega_0^4}{3c^3}\,.$$

(b) Case 2: Non-relativistic motion of a point particle with charge e moving in a circle of radius R in the x–y plane with constant angular frequency ω_0.

For circular motion in the x–y plane, the trajectory of the particle is given by

$$\vec{\boldsymbol{x}}(t) = R(\hat{\boldsymbol{x}}\cos\omega_0 t + \hat{\boldsymbol{y}}\sin\omega_0 t)\,.$$

Then, we easily compute

$$\frac{d\vec{\boldsymbol{\beta}}}{dt} = \frac{1}{c}\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = -\frac{\omega_0^2}{c}\,\vec{\boldsymbol{x}}(t)\,.$$

We again choose to work in a coordinate system in which the origin corresponds to the instantaneous position of the charged particle, and the unit vector $\hat{\boldsymbol{n}}$ given by eq. (3) has polar angle θ and azimuthal angle ϕ with respect to the z-axis. Consequently,

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\frac{d\vec{\boldsymbol{\beta}}}{dt} = -\frac{\omega_0^2 R}{c}\left(\cos\omega_0 t\,\sin\theta\,\cos\phi + \sin\omega_0 t\,\sin\theta\,\sin\phi\right)\,.$$

Evaluating the absolute square of the triple cross product as in part (a) [cf. eq. (2)], we obtain:

$$\left|\hat{\boldsymbol{n}}\times\left(\hat{\boldsymbol{n}}\times\frac{d\vec{\boldsymbol{\beta}}}{dt}\right)\right|^2 = \frac{\omega_0^4R^2}{c^2}\left[1-\sin^2\theta(\cos\phi\,\cos\omega_0 t+\sin\phi\,\sin\omega_0 t)^2\right]$$
$$= \frac{\omega_0^4R^2}{c^2}\left[1-\sin^2\theta\,\cos^2(\omega_0 t-\phi)\right]\,.$$

Using eq. (1), it follows that

$$\frac{dP}{d\Omega}=\frac{e^2\omega_0^4R^2}{4\pi c^3}\left[1-\sin^2\theta\,\cos^2(\omega_0 t-\phi)\right]\,.$$

The time-averaged power is easily obtained by noting that $\langle\cos^2(\omega_0 t-\phi)\rangle=\frac{1}{2}$. Employing the trigonometric identity, $1-\frac{1}{2}\sin^2\theta=\frac{1}{2}(1+\cos^2\theta)$, it follows that

$$\left\langle\frac{dP}{d\Omega}\right\rangle=\frac{e^2\omega_0^4R^2}{8\pi c^3}\left(1+\cos^2\theta\right)\,. \tag{5}$$

In Figure 2, the angular distribution of the radiated power is exhibited as a polar plot.

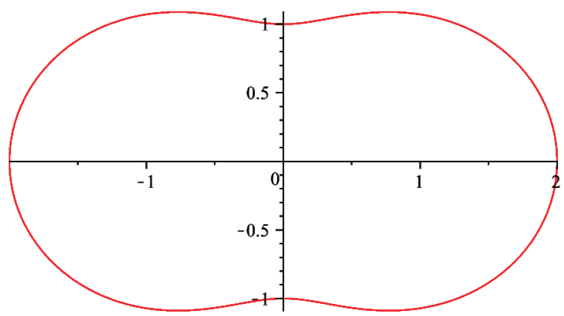


Figure 2: A polar plot of the angular distribution of the power radiated by a charged particle moving non-relativistically in a circle of radius R in the x–y plane with constant angular frequency ω_0. The angular distribution is given by eq. (5) and is proportional to $1+\cos^2\theta$. This plot was created with Maple software.

Integrating over solid angles yields the total radiated power,

$$\langle P\rangle=\frac{2e^2\omega_0^4R^2}{3c^3}\,.$$

2. [Jackson, problem 14.5] A *nonrelativistic* particle of charge ze, mass m, and kinetic energy E makes a *head-on* collision with a fixed central force field of finite range. The interaction is repulsive and described by a potential $V(r)$, which becomes greater than E at close distances.

(a) Show that the total energy radiated is given by

$$\Delta W=\frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\rm min}}^{\infty}\left|\frac{dV}{dr}\right|^2\frac{dr}{\sqrt{V\left(r_{\rm min}\right)-V(r)}}$$

where $r_{\rm min}$ is the closest distance of approach in the collision.

Using Larmor's formula given in eq. (14.22) of Jackson for a nonrelativistic particle of charge ze and mass m,

$$P(t) = \frac{2z^2e^2}{3c^3}|\vec{\boldsymbol{a}}(t)|^2\,, \tag{6}$$

where $\vec{\boldsymbol{a}}$ is the acceleration of the particle (which depends on the time t). Newton's second law yields

$$\vec{\boldsymbol{F}} = -\vec{\boldsymbol{\nabla}}V = m\vec{\boldsymbol{a}}\,, \tag{7}$$

where V is the potential energy, which is assumed to depend on the radial coordinate r alone. Thus, we can identify

$$\vec{\boldsymbol{a}} = -\frac{1}{m}\frac{dV}{dr}\hat{\boldsymbol{r}}\,, \tag{8}$$

where $\hat{\boldsymbol{r}}$ is a unit vector pointing in the radial direction. Inserting this result into eq. (6) yields

$$P(t) = \frac{2z^2e^2}{3m^2c^3}\left|\frac{dV}{dr}\right|^2\,. \tag{9}$$

The total energy radiated by the particle (denoted by ΔW below) is therefore given

$$\Delta W = \int_{-\infty}^{\infty} P(t)\,dt = \frac{2z^2e^2}{3m^2c^3}\int_{-\infty}^{\infty}\left|\frac{dV}{dr}\right|^2\,dt\,, \tag{10}$$

where $r = r(t)$ depends on the time coordinate t.

We can assume that at $t = -\infty$, the particle is infinitely far from the fixed central force field (whose center will define the origin of our coordinate system). It is convenient to choose $t = 0$ to be the time of closest approach, in which the particle is at a distance $r_{\rm min}$ from the origin. The result of the interaction is to turn the particle around. The particle now retraces its original trajectory (in the opposite direction); $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ of the incoming charge simply reverse their signs and the particle ends up infinitely far away at $t = \infty$. In particular, the energy radiated by the particle on its way in is equal to the energy radiated on its way out. Hence,

$$\int_{-\infty}^{\infty}\left|\frac{dV}{dr}\right|^2\,dt = 2\int_0^{\infty}\left|\frac{dV}{dr}\right|^2\,dt\,. \tag{11}$$

It then follows that

$$\Delta W = \frac{4z^2e^2}{3m^2c^3}\int_{r_{\rm min}}^{\infty}\left|\frac{dV}{dr}\right|^2\,\frac{dt}{dr}\,dr = \frac{4z^2e^2}{3m^2c^3}\int_{r_{\rm min}}^{\infty}\left|\frac{dV}{dr}\right|^2\,\frac{dr}{v}\,, \tag{12}$$

after changing variables from t to $r(t)$ and noting that $dt/dr = (dr/dt)^{-1} = v^{-1}$, where v is the velocity of the particle when the particle is located at $r = r(t)$.

One can determine $v(t)$ from the conservation of energy. The total energy of the particle is given by

$$E = \tfrac{1}{2}mv^2 + V(r) = \text{constant}\,, \tag{13}$$

It then follows that

$$v = \sqrt{\frac{2[E - V(r)]}{m}}\,. \tag{14}$$

Moreover, $r_{\min}$ is determined by the condition that $v = 0$ when $r = r_{\min}$. Thus, $E = V(r_{\min})$. Hence,

$$v = \sqrt{\frac{2[V(r_{\min}) - V(r)]}{m}}\,. \tag{15}$$

Inserting this result into eq. (12) yields

$$\Delta W = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\int_{r_{\min}}^{\infty}\left|\frac{dV}{dr}\right|^2 \frac{dr}{\sqrt{V\,(r_{\min}) - V(r)}} \tag{16}$$

(b) If the interaction is a Coulomb potential $V(r) = zZe^2/r$, show that the total energy radiated is

$$\Delta W = \frac{8}{45}\frac{zmv_0^5}{Zc^3}$$

where v_0 is the velocity of the charge at infinity.

We now use eq. (13) by evaluating E at $r = \infty$. Since $V(\infty) = 0$, it follows that $E = \tfrac{1}{2}mv_0^2$. In part (a), we noted that $E = V(r_{\min})$. It then follows that

$$V(r_{\min}) = \frac{zZe^2}{r_{\min}} = \tfrac{1}{2}mv_0^2\,. \tag{17}$$

Plugging the above expressions for $V(r)$ and $V(r_{\min})$ into eq. (16), we obtain

$$\Delta W = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\,(zZe^2)^{3/2}\int_{r_{\min}}^{\infty}\frac{dr}{r^4}\left(\frac{1}{r_{\min}} - \frac{1}{r}\right)^{-1/2}, \tag{18}$$

where

$$r_{\min} = \frac{2zZe^2}{mv_0^2}\,, \tag{19}$$

in light of eq. (17). It is convenient to define $x \equiv r/r_{\min}$, and change the integration variable from r to x. Then,

$$\Delta W = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\,\frac{(zZe^2)^{3/2}}{r_{\min}^{5/2}}\int_1^{\infty}\frac{dx}{x^{7/2}\sqrt{x-1}}\,. \tag{20}$$

One more change of variables is needed to convert the integral into a more familiar form. Defining $y = 1/x$, it follows that

$$\int_1^{\infty}\frac{dx}{x^{7/2}\sqrt{x-1}} = \int_0^1\frac{y^2\,dy}{\sqrt{1-y}}\,. \tag{21}$$

Recall the well-known Beta integral,

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,,\quad \text{for } \operatorname{Re} p > 0 \text{ and } \operatorname{Re} q > 0, \tag{22}$$

where the gamma function satisfies $\Gamma(x+1) = x\Gamma(x)$, with $\Gamma(n) = (n-1)!$ for nonnegative integers n. Hence,

$$\int_0^1 \frac{y^2\,dy}{\sqrt{1-y}} = \frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} = \frac{2\,\Gamma(\frac{1}{2})}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\,\Gamma(\frac{1}{2})} = \frac{16}{15}\,. \tag{23}$$

Inserting the results of eqs. (19), (21), and (23) back into eq. (20), we end up with

$$\Delta W = \frac{4}{3}\frac{z^2e^2}{m^2c^3}\sqrt{\frac{m}{2}}\,(zZe^2)^{3/2}\left(\frac{mv_0^2}{2zZe^2}\right)^{5/2}\frac{16}{15} = \frac{8}{45}\frac{zmv_0^5}{Zc^3}\,. \tag{24}$$

3. In class, we showed that the angular distribution of the power radiated by a point particle of charge e moving along a trajectory $\vec{\boldsymbol{r}}(t)$ at velocity $c\vec{\boldsymbol{\beta}}(t) \equiv d\vec{\boldsymbol{r}}(t)/dt$ is given by:

$$\frac{dP}{d\Omega} = \lim_{r\to\infty}\frac{cr^2}{4\pi}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega''\,\vec{\boldsymbol{E}}^*_{\omega'}(\vec{\boldsymbol{x}})\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\omega''}(\vec{\boldsymbol{x}})\,e^{i(\omega'-\omega'')t}\,,$$

where r is the distance of the observer from the origin and the Fourier coefficient of the electric field vector is given by

$$\boldsymbol{E}_\omega(\vec{\boldsymbol{x}}) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)e^{i\omega t} \tag{25}$$

(a) Derive the following expression for the Fourier coefficient,

$$\boldsymbol{E}_\omega(\vec{\boldsymbol{x}}) = -\frac{ie\,\omega\,e^{i\omega r/c}}{2\pi rc}\int_{-\infty}^{\infty} dt\;\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega(t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c)}\,,$$

where $\hat{\boldsymbol{n}}$ is a unit vector pointing from the charge to the observer.[2]

If a point particle with charge e moves along a trajectory $\vec{\boldsymbol{r}}(t)$ at velocity $\vec{\boldsymbol{v}} \equiv c\vec{\boldsymbol{\beta}}$ with acceleration $\vec{\boldsymbol{a}} = c\,d\vec{\boldsymbol{\beta}}/dt$, then the leading order behavior of the electric and magnetic fields at large distances (in gaussian units) is given by:

$$\vec{\boldsymbol{E}} = \frac{e}{cr}\left(\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^3}\right)_{\vec{\boldsymbol{x}}\,'=\vec{\boldsymbol{r}}(t_{\rm ret})} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{26}$$

$$\vec{\boldsymbol{B}} = \hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{27}$$

where the retarded time is defined as $t_{\rm ret} \equiv t - \left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{r}}(t_{\rm ret})\right|/c$. Note that in eq. (26) the velocity $c\vec{\boldsymbol{\beta}}$ is equal to the derivative of $\vec{\boldsymbol{x}}\,' = \vec{\boldsymbol{r}}(t_{\rm ret})$ with respect to the retarded time,

$$\vec{\boldsymbol{\beta}} \equiv \vec{\boldsymbol{\beta}}(t_{\rm ret}) = \frac{1}{c}\frac{d\vec{\boldsymbol{r}}(t_{\rm ret})}{dt_{\rm ret}}\,. \tag{28}$$

[2] As noted by Jackson below his eq. (14.62), assuming that the observation point $\vec{\boldsymbol{x}}$ is located very far away from the region of space where the acceleration occurs, the unit vector $\hat{\boldsymbol{n}}$ can be very well approximated as being constant in time.

Inserting eq. (26) into eq. (25) yields,

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{e}{2\pi c r}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\left(\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^3}\right)_{\vec{\boldsymbol{x}}\,'=\vec{\boldsymbol{r}}(t_{\rm ret})}. \tag{29}$$

Define $t' \equiv t_{\rm ret}$ and change the variable of integration in eq. (29),

$$t = t' + \frac{1}{c}|\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')| \quad\Longrightarrow\quad dt = \frac{dt}{dt'}\,dt' = \left(1-\frac{\big(\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')\big)\boldsymbol{\cdot} d\vec{\boldsymbol{r}}/dt'}{c|\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')|}\right)dt'\,.$$

Noting that $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}} = d\vec{\boldsymbol{r}}(t')/dt'$ [cf. eq. (28)] and

$$\hat{\boldsymbol{n}} = \frac{\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')|}\,,$$

it follows that $dt = (1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})dt'$. Hence,

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{e}{2\pi c r}\int_{-\infty}^{\infty} dt'\, e^{i\omega[t'+|\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')|/c]}\left(\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^2}\right). \tag{30}$$

For large values of $r\equiv|\vec{\boldsymbol{x}}|$, we can approximate

$$\hat{\boldsymbol{n}} = \frac{\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')}{|\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')|} = \frac{\vec{\boldsymbol{x}}}{r}\left[1+\mathcal{O}\left(\frac{1}{r}\right)\right]\,,$$

so that $\vec{\boldsymbol{x}}\simeq r\hat{\boldsymbol{n}}$ and

$$\begin{aligned} t'+\frac{1}{c}\,|\vec{\boldsymbol{x}}-\vec{\boldsymbol{r}}(t')| &= t'+\frac{1}{c}\sqrt{r^2-2\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')+r'^{\,2}} = t'+\frac{r}{c}\left[1-\frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right] \\ &\simeq t'+\frac{1}{c}\Big(r-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')\Big)\,, \end{aligned}$$

where $r'\equiv|\vec{\boldsymbol{r}}(t')|$. Inserting the above result into eq. (30) yields

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = \frac{e}{2\pi c r}\,e^{i\omega r/c}\int_{-\infty}^{\infty} dt'\, e^{i\omega[t'-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]}\left(\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}})^2}\right). \tag{31}$$

Employing the identity,

$$\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{(1-\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})^2} = \frac{d}{dt'}\left(\frac{\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})}{1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}}\right)\,, \tag{32}$$

we can integrate by parts and drop the surface term.[3] Hence, eqs. (31) and (32) yield,

$$\begin{aligned}
\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) &= -\frac{e}{2\pi cr}\,e^{i\omega r/c}\int_{-\infty}^{\infty}dt'\left(\frac{\hat{\boldsymbol{n}}\times\left[(\hat{\boldsymbol{n}}-\vec{\boldsymbol{\beta}})\times\dfrac{d\vec{\boldsymbol{\beta}}}{dt}\right]}{1-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\beta}}}\right)\frac{d}{dt'}\,e^{i\omega[t'-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]} \\
&= -\frac{ie\omega}{2\pi cr}\,e^{i\omega r/c}\int_{-\infty}^{\infty}dt\;\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,, \qquad (33)
\end{aligned}$$

after dropping the primed superscripts by writing t in place of t' (which after all is just a dummy integration variable), and using $c\,\vec{\boldsymbol{\beta}}(t') = d\vec{\boldsymbol{r}}(t')/dt'$.

(b) [Jackson, problem 14.13] Using the results of part (a) and the Poisson sum formula, show explicitly that if the motion of a radiating particle repeats itself with periodicity T, then the continuous frequency spectrum becomes a discrete spectrum containing frequencies that are integral multiples of the fundamental. Show that a general expression for the time-averaged power radiated per unit solid angle in each multiple m of the fundamental frequency $\omega_0 = 2\pi/T$ is given by

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{T}\int_0^T dt\,\frac{dP}{d\Omega} \equiv \sum_{m=1}^{\infty}\frac{dP_m}{d\Omega}\,,$$

where

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3}\left|\int_0^{2\pi/\omega_0}\vec{\boldsymbol{v}}(t)\times\hat{\boldsymbol{n}}\,\exp\left[im\omega_0\left(t-\frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)}{c}\right)\right]dt\right|^2. \qquad (34)$$

It is convenient to rewrite the integral in eq. (33) as

$$\int_{-\infty}^{\infty}dt\;\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]} = \sum_{m=-\infty}^{\infty}\int_{mT}^{(m+1)T}dt\;\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,. \qquad (35)$$

Since the motion is periodic, we have

$$\vec{\boldsymbol{r}}(t+T) = \vec{\boldsymbol{r}}(t) \quad\text{and}\quad \vec{\boldsymbol{\beta}}(t+T) = \vec{\boldsymbol{\beta}}(t)\,,$$

where $T \equiv 2\pi/\omega_0$ defines the fundamental frequency ω_0. Let us define a new variable, $t' \equiv t - mT$. Then, eq. (35) takes the following form,

$$\int_{-\infty}^{\infty}dt\;\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]} = \sum_{m=-\infty}^{\infty}e^{i\omega mT}\int_0^T dt'\;\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}')\,e^{i\omega[t'-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]}\,, \qquad (36)$$

where $\vec{\boldsymbol{\beta}}' \equiv \vec{\boldsymbol{\beta}}(t')$.

At this point, we can apply the Poisson sum formula,[4]

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{i\omega mT} = \sum_{m=-\infty}^{\infty}\delta(\omega T - 2\pi m)\,.$$

[3]The justification for dropping the surface term is discussed on pp. 675–676 of Jackson.

[4]See Section 5 of the class handout entitled *Generalized Functions for Physics*.

Hence, eq. (36) can be rewritten as

$$\int_{-\infty}^{\infty} dt\ \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]} = \sum_{m=-\infty}^{\infty}\delta(\omega T-2\pi m)\int_0^T dt\ \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,,$$

after again removing the superscript primes from t' from the right hand side above as a notational convenience. Note that the δ-function enforces the condition,

$$\omega = \frac{2\pi m}{T} = m\omega_0\,, \qquad \text{for } m=0,\pm1,\pm2,\ldots,$$

which implies that the frequency spectrum is discrete.

Thus we can rewrite eq. (33), obtained in part (a), as

$$\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}}) = -\frac{ie\omega}{2\pi cr}\,e^{i\omega r/c}\sum_{m=-\infty}^{\infty}\delta(\omega T-2\pi m)\int_0^T dt\ \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}})\,e^{i\omega[t-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c]}\,. \qquad (37)$$

The power radiated per unit solid angle (in gaussian units) is given by

$$\frac{dP}{d\Omega} = \lim_{r\to\infty} r^2\,\vec{\boldsymbol{S}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\,, \qquad \text{where } \vec{\boldsymbol{S}} = \frac{c}{4\pi}(\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}})\,,$$

and $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ are the real physical fields. For large distances r, we have $\vec{\boldsymbol{B}}\simeq\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}$, as noted in eq. (27), in which case

$$\vec{\boldsymbol{E}}\times\vec{\boldsymbol{B}} = \vec{\boldsymbol{E}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{E}}) = \hat{\boldsymbol{n}}|\vec{\boldsymbol{E}}|^2\,,$$

after using $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{E}}=0$ (i.e., the electromagnetic radiation is transverse). Hence, it follows that

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\lim_{r\to\infty} r^2|\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)|^2\,. \qquad (38)$$

Inverting the Fourier transform defined in eq. (25),

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = \int_{-\infty}^{\infty} d\omega\,\vec{\boldsymbol{E}}_\omega(\vec{\boldsymbol{x}})\,e^{-i\omega t}\,,$$

and inserting the result into eq. (38) yields

$$\frac{dP}{d\Omega} = \lim_{r\to\infty}\frac{cr^2}{4\pi}\int_{-\infty}^{\infty}d\omega'\int_{-\infty}^{\infty}d\omega''\,\vec{\boldsymbol{E}}^*_{\omega'}\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{\omega''}\,e^{i(\omega'-\omega'')t}\,.$$

Using eq. (37) in the above expression, we obtain

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega'\int_{-\infty}^{\infty}d\omega''\,\delta(\omega'T-2\pi m)\,\delta(\omega''T-2\pi n)\,e^{i(\omega'-\omega'')(t-r/c)}\,\omega'\omega''$$

$$\times\int_0^T dt'\ \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}')\,e^{i\omega[t'-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]}\int_0^T dt''\ \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{\beta}}'')\,e^{i\omega''[t''-\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t'')/c]}\,.$$

We can now perform the integrals over ω' and ω'' using the δ-functions, which set $\omega' = m\omega_0$ and $\omega'' = n\omega_0$, respectively (where $\omega_0 \equiv 2\pi/T$). Thus,

$$\frac{dP}{d\Omega} = \frac{e^2\omega_0^2}{4\pi c T^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} mn\, e^{i\omega_0(m-n)(t-r/c)} \int_0^T dt' \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}')\, e^{i\omega[t' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t')/c]}$$
$$\times \int_0^T dt''\, \hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}'')\, e^{i\omega''[t'' - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t'')/c]}\,. \qquad (39)$$

Since $dP/d\Omega$ depends on t, we shall integrate over one cycle,

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{T}\int_0^T \frac{dP}{d\Omega}\, dt\,.$$

Taking the time-average of eq. (39), the integration over t is straightforward, as it depends only on the following integral,

$$\frac{1}{T}\int_0^T e^{i\omega_0 t(m-n)}\, dt = \delta_{mn}\,.$$

The sums over m and n in eq. (39) now collapse into a single sum over m. Noting that

$$\left[\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}')\right]\boldsymbol{\cdot}\left[\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}'')\right] = (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}')\boldsymbol{\cdot}(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}}'')\,, \qquad (40)$$

the end result is,

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2\omega_0^2}{4\pi c T^2} \sum_{m=-\infty}^{\infty} m^2 \left| \int_0^T (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})\, e^{im\omega_0(t - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c)} dt \right|^2$$
$$= \frac{e^2\omega_0^2}{2\pi c T^2} \sum_{m=1}^{\infty} m^2 \left| \int_0^T (\hat{\boldsymbol{n}} \times \vec{\boldsymbol{\beta}})\, e^{im\omega_0(t - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)/c)} dt \right|^2\,, \qquad (41)$$

after noting that positive and negative m contribute equally to the sum over m (whereas the $m = 0$ contribution to the sum vanishes). Thus, using $\vec{\boldsymbol{v}}(t) = c\vec{\boldsymbol{\beta}}$ and $T = 2\pi/\omega_0$ in eq. (41), we can write:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \sum_{m=1}^{\infty} \frac{dP_m}{d\Omega}\,, \qquad (42)$$

where

$$\frac{dP_m}{d\Omega} = \frac{e^2\omega_0^4 m^2}{(2\pi c)^3} \left| \int_0^{2\pi/\omega_0} \vec{\boldsymbol{v}}(t) \times \hat{\boldsymbol{n}}\, \exp\left[im\omega_0\left(t - \frac{\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{r}}(t)}{c}\right)\right] dt \right|^2\,. \qquad (43)$$

4. [Jackson, problem 13.9] Assuming that Plexiglas or Lucite has an index of refraction of 1.50 in the visible region, compute the angle of emission of visible Cherenkov radiation for electrons and protons as a function of their kinetic energies in MeV. Determine how many quanta with wavelengths between 4000 and 6000 Å are emitted per centimeter of the path in Lucite by a 1 MeV electron, a 500 MeV proton, and a 5 GeV proton.

Using eq. (13.50) of Jackson, the angle of emission θ_c is obtained from:

$$\cos\theta_c = \frac{1}{\beta\sqrt{\epsilon}}\,, \tag{44}$$

and the index of refraction is $n_r = \sqrt{\epsilon}$. To compute β given the kinetic energy T, we recall that

$$T = E - mc^2 = \gamma mc^2 - mc^2 = mc^2\left[\sqrt{\frac{1}{1-\beta^2}} - 1\right].$$

Solving for β, it follows that

$$\beta^2 = 1 - \frac{1}{\left(1 + \dfrac{T}{mc^2}\right)^2}\,,$$

from which β is easily obtained,

$$\beta = \frac{T\sqrt{1 + \dfrac{2mc^2}{T}}}{T + mc^2}\,.$$

Hence, eq. (44) yields

$$\cos\theta_c = \frac{1}{n_r}\left(1 + \frac{mc^2}{T}\right)\left(1 + \frac{2mc^2}{T}\right)^{-1/2}\,. \tag{45}$$

Note that $mc^2 = 0.511$ MeV for the electron and $mc^2 = 938$ MeV for the proton. Inserting these numbers along with $n_r = 1.5$ in eq. (45), one obtains the angle of emission of visible Cherenkov radiation for electrons and protons as a function of their kinetic energies in MeV.

To determine the number of quanta emitted per path length, we first use eq. (13.48) of Jackson:[5]

$$\left(\frac{dE}{dx}\right)_{\rm rad} = \frac{e^2}{c^2}\int_{n_r > 1/\beta} \omega\left(1 - \frac{1}{\beta^2 n_r^2}\right)d\omega\,.$$

Assuming that n_r is independent of ω in the frequency range of interest, we integrate from $\omega = \omega_1$ to $\omega = \omega_2$ to obtain,

$$\left(\frac{dE}{dx}\right)_{\rm rad} = \frac{e^2}{2c^2}\left(1 - \frac{1}{\beta^2 n_r^2}\right)(\omega_2^2 - \omega_1^2)\,.$$

For the range 4000 Å$\leq \lambda \leq$ 6000Å, where 1 Å$= 10^{-8}$ cm, we have

$$\omega_1 = \frac{2\pi c}{\lambda_1} = \frac{2\pi(3\times 10^{10}~\text{cm}\cdot\text{s}^{-1})}{4\times 10^{-5}~\text{cm}} = 4.71\times 10^{15}~\text{s}^{-1}\,,$$

$$\omega_2 = \frac{2\pi c}{\lambda_2} = \frac{2}{3}\omega_1 = 3.14\times 10^{15}~\text{s}^{-1}\,.$$

[5]Note that the charge of the moving particle is denoted by ze in Jackson. In this problem, $z = \pm 1$ for the proton and electron, respectively, so that $z^2 = 1$.

The energy of one quantum is $\hbar\omega$. Hence, it follows that

$$\frac{dN}{d\omega\, dx} = \frac{1}{\hbar\omega}\frac{dE}{d\omega\, dx}\,,$$

where N is the number of quanta radiated. Thus,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = \frac{e^2}{\hbar c^2}\int_{\omega_1}^{\omega_2}\left(1-\frac{1}{\beta^2 n_r^2}\right)d\omega = \frac{e^2}{\hbar c^2}\left(1-\frac{1}{\beta^2 n_r^2}\right)(\omega_2-\omega_1)\,.$$

We can rewrite the above equation by using $\omega = kc/n_r = 2\pi c/(n_r\lambda)$ [cf. eq. (7.5) of Jackson] and by introducing the fine structure constant,

$$\alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,.$$

It follows that

$$\left(\frac{dN}{dx}\right)_{\rm rad} = \frac{2\pi\alpha}{n_r}\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\left(1-\frac{1}{\beta^2 n_r^2}\right)\,. \tag{46}$$

We now plug in the relevant numbers into eq. (46).

<u>Case 1</u>: For a $T = 1$ MeV electron,

$$\frac{1}{\beta} = \left(1+\frac{mc^2}{T}\right)\left(1+\frac{2mc^2}{T}\right)^{-1/2} = 1.0626\,.$$

Hence,

$$\cos\theta_c = \frac{2}{3\beta} = 0.7084 \quad\Longrightarrow\quad \theta_c = 44.9^\circ\,,$$

and

$$1-\frac{1}{\beta^2 n_r^2} = 1-\frac{(1.0626)^2}{(1.5)^2} = 0.4982\,.$$

Eq. (46) then yields,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = \frac{2\pi}{1.5}\left(\frac{1}{137}\right)\left(\frac{1}{4\times 10^{-5}~{\rm cm}}-\frac{1}{6\times 10^{-5}~{\rm cm}}\right)(0.4982) = 127~{\rm quanta/cm}\,.$$

<u>Case 2</u>: For a $T = 500$ MeV proton,

$$\frac{1}{\beta} = 1.3193\,.$$

Hence,

$$\cos\theta_c = 0.8795 \quad\Longrightarrow\quad \theta = 28.4^\circ\,.$$

Eq. (46) then yields,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = 58~{\rm quanta/cm}\,.$$

Case 3: For a $T = 5$ GeV proton,

$$\frac{1}{\beta} = 1.0127\,.$$

Hence,

$$\cos\theta_c = 0.6751 \qquad \Longrightarrow \qquad \theta = 47.5^\circ\,.$$

Eq. (46) then yields,

$$\left(\frac{dN}{dx}\right)_{\rm rad} = 140 \text{ quanta/cm}\,.$$

5. [Jackson, problem 10.1]

(a) Show that for arbitrary initial polarizations, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{d\sigma}{d\Omega}(\hat{\boldsymbol{\epsilon}}_0, \hat{\boldsymbol{n}}_0, \hat{\boldsymbol{n}}) = k^4 a^6 \Big[\tfrac{5}{4} - |\hat{\boldsymbol{\epsilon}}_0 \boldsymbol{\cdot} \hat{\boldsymbol{n}}|^2 - \tfrac{1}{4}|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 - \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}\Big]\,,$$

where $\hat{\boldsymbol{n}}_0$ and $\hat{\boldsymbol{n}}$ are the directions of the incident and scattered electromagnetic waves, respectively, while $\hat{\boldsymbol{\epsilon}}_0$ is the (perhaps complex) unit polarization vector of the incident radiation ($\hat{\boldsymbol{\epsilon}}_0^*\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 1\,;\ \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 0$.)

Our starting point is eq. (10.14) of Jackson,

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\hat{\boldsymbol{\epsilon}}^*\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 - \tfrac{1}{2}(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^*)\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)\right|^2\,.$$

For arbitrary initial polarization $\hat{\boldsymbol{\epsilon}}_0$, the scattering cross section summed over the final state polarizations is

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \sum_\lambda \left|\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 - \tfrac{1}{2}(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)\right|^2\,. \tag{47}$$

We shall evaluate the polarization sum using the following identity derived in the class handout entitled *Polarization Vectors and Polarization Sums*,

$$\sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)} = \delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j\,, \tag{48}$$

where the $\hat{\boldsymbol{n}}_i$ ($i \in \{1,2,3\}$) are the Cartesian components of the unit vector $\hat{\boldsymbol{n}} \equiv \vec{\boldsymbol{k}}/k$. Expanding out the terms in eq. (47), we first evaluate

$$\sum_\lambda |\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j = (\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j\,[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j] = 1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\,, \tag{49}$$

after using $\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^* = 1$ in the final step.

Similarly,

$$\sum_\lambda |(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_{ijk}\, \hat{\boldsymbol{n}}_j \hat{\boldsymbol{\epsilon}}_k^{(\lambda)\,*} \hat{\boldsymbol{\epsilon}}_{\ell mn}\, \hat{\boldsymbol{n}}_m \hat{\boldsymbol{\epsilon}}_n^{(\lambda)} (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)_i (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*)_\ell\,,$$

where the summation over repeated index pairs is implied by the Einstein summation convention. Using the polarization sum identity given by eq. (48),

$$\sum_\lambda |(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}_0)|^2 = \epsilon_{ijk}\epsilon_{\ell mn}(\delta_{kn} - \hat{\boldsymbol{n}}_k \hat{\boldsymbol{n}}_n)(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)_i (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*)_\ell\,.$$

Since ϵ_{ijk} is a totally antisymmetric tensor, it follows that $\epsilon_{ijk}\,\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_k = 0$. Employing the identity,

$$\epsilon_{ijk}\epsilon_{\ell mn}\delta_{kn} = \epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\,,$$

we end up with

$$\begin{aligned}\sum_\lambda |(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_m (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)_i (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*)_\ell \\ &= |\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0|^2 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2\,,\end{aligned}$$

after noting that $\hat{\boldsymbol{n}}_j\hat{\boldsymbol{n}}_m\delta_{jm} = \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = 1$. Finally, we can expand out the square of the cross product,

$$|\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0|^2 = 1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 = 1\,,$$

after using $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 0$ (which follows from the fact that the polarization vector is transverse to the direction of propagation of the electromagnetic wave). Hence, we conclude that

$$\sum_\lambda |(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = 1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2\,.$$

All that remains is to evaluate the cross-term in eq. (47).

$$\begin{aligned}\sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)})_j(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*)_j &= \sum_\lambda \epsilon_{jk\ell}\,\hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_\ell^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i\hat{\boldsymbol{n}}_k(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*)_j \\ &= \epsilon_{jk\ell}(\delta_{i\ell} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_\ell)(\hat{\boldsymbol{\epsilon}}_0)_i\hat{\boldsymbol{n}}_k(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*)_j \\ &= \epsilon_{jk\ell}\,\hat{\boldsymbol{n}}_k(\hat{\boldsymbol{\epsilon}}_0)_\ell(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*)_j = (\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}_0)\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0^*) \\ &= (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*) - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 \\ &= \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\,.\end{aligned}$$

Collecting all the above results, it follows that

$$\frac{d\sigma}{d\Omega} = k^4a^6\left\{1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 + \tfrac{1}{4}\left[1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2\right] - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\right\},$$

which simplifies to

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\tfrac{5}{4} - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 - \tfrac{1}{4}|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\right], \tag{50}$$

as required.

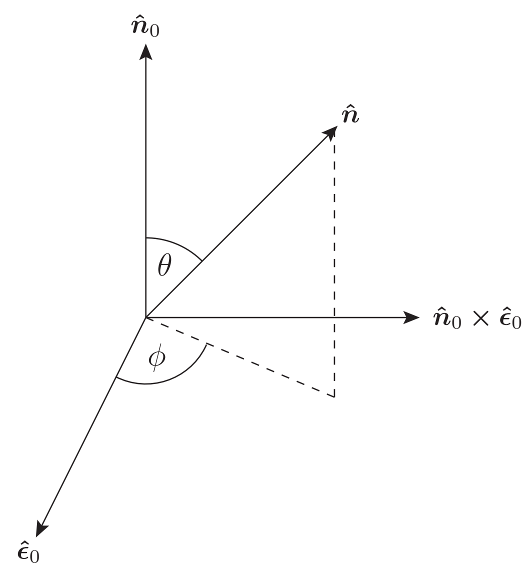
<u>*REMARK:*</u> If you express the square of the vector cross product in eq. (47) as the sum of products of dot products before carrying out the polarization sums, you will arrive at a different form for eq. (50). Nevertheless, it is possible to show that the two forms are equivalent. This alternative method is provided at the end of this Solution Set.

(b) If the incident radiation is linearly polarized, show that cross section is

$$\frac{d\sigma}{d\Omega}(\hat{\boldsymbol{\epsilon}}_0, \hat{\boldsymbol{n}}_0, \hat{\boldsymbol{n}}) = k^4 a^6 \left[\tfrac{5}{8}(1 + \cos^2\theta) - \cos\theta - \tfrac{3}{8}\sin^2\theta\cos 2\phi\right],$$

where $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of the linear polarization.

We set up our coordinate system as follows:



The components of the corresponding unit vectors are:

$$\hat{\boldsymbol{\epsilon}}_0 = (1\,,\,0\,,\,0), \qquad \hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0 = (0\,,\,1\,,\,0)\,, \qquad \hat{\boldsymbol{n}} = (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)\,.$$

It follows that

$$\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \sin\theta\cos\phi\,, \qquad \hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0) = \sin\theta\sin\phi\,, \qquad \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \cos\theta\,.$$

Hence, eq. (50) yields

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\tfrac{5}{4} - \sin^2\theta\cos^2\phi - \tfrac{1}{4}\sin^2\theta\sin^2\phi - \cos\theta\right]. \tag{51}$$

Writing $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$ and $\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$, eq. (51) takes the following form,

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\tfrac{5}{8}(1+\cos^2\theta) - \cos\theta - \tfrac{3}{8}\sin^2\theta\cos 2\phi\right]. \tag{52}$$

(c) What is the ratio of the scattered intensities at $\theta = \frac{1}{2}\pi$, $\phi = 0$ and $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$? Explain physically in terms of the induced multipoles and their radiation patterns.

Using eq. (52), it follows that

$$\frac{\dfrac{d\sigma}{d\Omega}\left(\theta = \tfrac{1}{2}\pi\,,\ \phi = 0\right)}{\dfrac{d\sigma}{d\Omega}\left(\theta = \tfrac{1}{2}\pi\,,\ \phi = \tfrac{1}{2}\pi\right)} = \frac{1}{4}\,.$$

If we trace back the origin of the various contributions, we see that the electric dipole scattering originates from

$$1 - |\boldsymbol{\hat{n}}\boldsymbol{\cdot}\boldsymbol{\hat{\epsilon}}|^2 = 1 - \sin^2\theta\cos^2\phi \quad \xrightarrow[\theta=\frac{1}{2}\pi]{} \quad \sin^2\phi\,,$$

whereas the magnetic dipole scattering originates from

$$\tfrac{1}{4}\left[1 - |\boldsymbol{\hat{n}}\boldsymbol{\cdot}(\boldsymbol{\hat{n}}_0\times\boldsymbol{\hat{\epsilon}}_0)|^2\right] = \tfrac{1}{4}\left(1 - \sin^2\theta\sin^2\phi\right) \quad \xrightarrow[\theta=\frac{1}{2}\pi]{} \quad \tfrac{1}{4}\cos^2\phi\,.$$

Thus, at $\theta = \frac{1}{2}\pi$, $\phi = 0$, we have pure magnetic dipole scattering. In contrast, at $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$, we have pure electric dipole scattering, whose contribution is four times larger than the magnetic dipole scattering contribution at $\theta = \frac{1}{2}\pi$, $\phi = 0$. The factor of four originates from the relative factor of two between the electric dipole moment $\vec{\boldsymbol{p}}$ [cf. eq. (10.12) of Jackson] and the magnetic dipole moment $\vec{\boldsymbol{m}}$ [cf. eq. (10.13) of Jackson] that are induced by the electric and magnetic fields of the incoming plane wave.

At $\theta = \frac{1}{2}\pi$, $\phi = 0$, we see that $\boldsymbol{\hat{n}}$ points in the direction of $\boldsymbol{\hat{\epsilon}}_0$. But $\boldsymbol{\hat{n}}$ points in the direction of the outgoing wave, whereas $\boldsymbol{\hat{\epsilon}}_0$ is parallel to the direction of the electric field of the incoming plane wave. Since the latter is also parallel to the direction of $\vec{\boldsymbol{p}}$, we conclude that in this case $\boldsymbol{\hat{n}}$ is parallel to $\vec{\boldsymbol{p}}$. It follows that $\boldsymbol{\hat{\epsilon}}^*$ must be perpendicular to $\vec{\boldsymbol{p}}$ (since the former is necessarily perpendicular to $\boldsymbol{\hat{n}}$), in which case $\boldsymbol{\hat{\epsilon}}^*\boldsymbol{\cdot}\vec{\boldsymbol{p}} = 0$. Eq. (10.4) of Jackson then implies that the scattering in this case is entirely due to the magnetic dipole term.

Similarly, at $\theta = \frac{1}{2}\pi$, $\phi = \frac{1}{2}\pi$, we see that $\boldsymbol{\hat{n}}$ points in the direction of $\boldsymbol{\hat{n}}_0\times\boldsymbol{\hat{\epsilon}}_0$, which is parallel to the direction of the magnetic field of the incoming plane wave. Since the latter is also parallel to $\vec{\boldsymbol{m}}$, we conclude that in this case $\boldsymbol{\hat{n}}$ is parallel to $\vec{\boldsymbol{m}}$. It follows that $\boldsymbol{\hat{n}}\times\boldsymbol{\hat{\epsilon}}_0^*$ must be perpendicular to $\vec{\boldsymbol{m}}$, in which case $(\boldsymbol{\hat{n}}\times\boldsymbol{\hat{\epsilon}}_0^*)\boldsymbol{\cdot}\vec{\boldsymbol{m}} = 0$. Eq. (10.4) of Jackson then implies that the scattering in this case is entirely due to the electric dipole term.

Alternative evaluation of eq. (47)

In the evaluation of eq. (47), one might be tempted to employ the vector identity,

$$(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0) = (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0) - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\,. \tag{53}$$

Then, eq. (47) takes the following form:

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \sum_\lambda \left|\left(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right)\left[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\right] + \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*})\right]\right|^2\,. \tag{54}$$

Expanding out the squared quantity yields

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \Bigg[\left[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\right]^2 \sum_\lambda \left|\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right|^2 + \tfrac{1}{4}\left|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right|^2 \sum_\lambda \left|\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\right|^2$$

$$+ \operatorname{Re}\left\{\left[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\right](\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*) \sum_\lambda \left(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right)\left(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}\right)\right\}\Bigg]. \tag{55}$$

We have already used eqs. (48) and (49) to obtain,

$$\sum_\lambda |\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j = (\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{\epsilon}}_0^*)_j\,[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j] = 1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\,. \tag{56}$$

Similarly,

$$\sum_\lambda \left|\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\right|^2 = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_j = (\hat{\boldsymbol{n}}_0)_i(\hat{\boldsymbol{n}}_0)_j\,[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j] = 1 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2\,. \tag{57}$$

and

$$\sum_\lambda \left(\hat{\boldsymbol{\epsilon}}^{(\lambda)\,*}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right)\left(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}^{(\lambda)}\right) = \sum_\lambda \hat{\boldsymbol{\epsilon}}_i^{(\lambda)\,*}\hat{\boldsymbol{\epsilon}}_j^{(\lambda)}(\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}_0)_j = (\hat{\boldsymbol{\epsilon}}_0)_i(\hat{\boldsymbol{n}}_0)_j\,[\delta_{ij} - \hat{\boldsymbol{n}}_i\hat{\boldsymbol{n}}_j] = -(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\,, \tag{58}$$

where we have used the fact that $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = 0$ (since the electromagnetic wave is transverse to the direction of propagation). Inserting the polarization sums obtained above into eq. (55) yields,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= k^4 a^6 \Bigg\{\left[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\right]^2\left[1 - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2\right] + \tfrac{1}{4}\left|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right|^2\left[1 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2\right] \\ &\qquad - \left[1 - \tfrac{1}{2}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\right](\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)\left|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right|^2\Bigg\} \\ &= k^4 a^6\left[1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0 + \tfrac{1}{4}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2 - \tfrac{3}{4}\left|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\right|^2\right].\end{aligned} \tag{59}$$

To see that eq. (59) is equivalent to eq. (50), we need to evaluate

$$|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = \epsilon_{ijk}\,\hat{\boldsymbol{n}}_i(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{\epsilon}}_0)_k\,\epsilon_{mpq}\hat{\boldsymbol{n}}_m(\hat{\boldsymbol{n}}_0)_p(\hat{\boldsymbol{\epsilon}}_0^*)_q\,, \tag{60}$$

where the Einstein summation convention is being used to sum over the repeated indices. Eq. (60) can be simplified by employing the following identity,

$$\epsilon_{ijk}\epsilon_{mpq} = \det\begin{pmatrix}\delta_{im} & \delta_{ip} & \delta_{iq}\\ \delta_{jm} & \delta_{jp} & \delta_{jq}\\ \delta_{km} & \delta_{kp} & \delta_{kq}\end{pmatrix}$$

$$= \delta_{im}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) - \delta_{ip}(\delta_{jm}\delta_{kq} - \delta_{jq}\delta_{km}) + \delta_{iq}(\delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km})\,. \tag{61}$$

Thus, it follows that

$$\begin{aligned}
&|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = \epsilon_{ijk}\hat{\boldsymbol{n}}_i(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{\epsilon}}_0)_k\,\epsilon_{mpq}\hat{\boldsymbol{n}}_m(\hat{\boldsymbol{n}}_0)_p(\hat{\boldsymbol{\epsilon}}_0^*)_q\\
&= \big[\delta_{im}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}) - \delta_{ip}(\delta_{jm}\delta_{kq} - \delta_{jq}\delta_{km}) + \delta_{iq}(\delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km})\big]\hat{\boldsymbol{n}}_i(\hat{\boldsymbol{n}}_0)_j(\hat{\boldsymbol{\epsilon}}_0)_k\hat{\boldsymbol{n}}_m(\hat{\boldsymbol{n}}_0)_p(\hat{\boldsymbol{\epsilon}}_0^*)_q\\
&= \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}\big[(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*) - (\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*)\big] - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\big[(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*) - (\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*)(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)\big]\\
&\qquad\qquad + \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^*\big[(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0) - (\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0)\big]\,.
\end{aligned} \tag{62}$$

The above expression can be further simplified by using the fact that $\hat{\boldsymbol{\epsilon}}_0$ is a complex unit vector that satisfies $\hat{\boldsymbol{\epsilon}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^* = 1$ and the real unit vectors $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}_0$ satisfy $\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}} = \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0 = 1$. In addition, due to the transverse nature of the incoming electromagnetic wave, it follows that $\hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0 = \hat{\boldsymbol{n}}_0\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0^* = 0$, as previously noted. Thus, we end up with

$$|\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{\epsilon}}_0)|^2 = 1 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2 - \big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2\,. \tag{63}$$

Plugging the result of eq. (63) into eq. (50) yields,

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= k^4a^6\Big[\tfrac{5}{4} - |\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0|^2 - \tfrac{1}{4}\big[1 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2 - \big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2\big] - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0\Big]\\
&= k^4a^6\big[1 - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0 + \tfrac{1}{4}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{n}}_0)^2 - \tfrac{3}{4}\big|\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{\epsilon}}_0\big|^2\big],
\end{aligned} \tag{64}$$

which coincides with eq. (59).