This is an open book exam with a time limit of three hours. You may refer to Jackson's text, the class handouts and solution sets, or any other material linked to the course webpage. You may also consult your own personal notes and any reference of integrals or other mathematical facts. However, you may not collaborate with anyone else during the exam.

Note that you do not have to derive all results from scratch. However, if you use a particular result, you should cite its source (e.g. an equation in Jackson, an equation derived in a class lecture, or an equation that appears in a solution set or in a class handout).

This exam consists of four problems consisting of ten individual parts in total, each of which is worth ten points. Use this information to manage your time appropriately during the exam.

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1. The theory of electromagnetism in $3+1$ spacetime dimensions can be generalized to $n+1$ spacetime dimensions as follows. The indices of the second-rank totally antisymmetric electromagnetic field strength tensor $F^{\mu\nu}$ now take on values $\mu,\ \nu \in \{0, 1, \ldots, n\}$. The dynamical Maxwell equations are given (in gaussian units) by:

$$\partial_\mu F^{\mu\nu} = \frac{S_{n-1}}{c} J^\nu\,, \tag{1}$$

where

$$S_{n-1} \equiv \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{2}$$

is the surface area of an n dimension ball of unit radius. For example, $S_1 = 2\pi$, $S_2 = 4\pi$, etc. The dual electromagnetic field strength tensor is defined by employing the totally antisymmetric rank $(n+1)$ ϵ-tensor. The latter can be used to express the kinematical Maxwell equations,

$$\epsilon^{\mu\cdots\alpha\beta}\partial_\mu F_{\alpha\beta} = 0\,, \tag{3}$$

where $\cdots$ in eq. (3) represents $n-2$ free indices that are not exhibited explicitly. By convention, we choose $\epsilon^{012\cdots n} = +1$.

(a) In $n+1$ spacetime dimensions, how many independent components are needed to describe $F^{\mu\nu}$? How many of these components represent the electric field and how many of these components represent the magnetic field?

(b) Consider the theory of electromagnetism in $2+1$ spacetime dimensions, where $F^{\mu\nu}$ can be constructed by deleting the fourth row and fourth column of the $3+1$ dimensional version of $F^{\mu\nu}$. Note that the electric field vector is now of the form $\vec{\boldsymbol{E}} = \hat{\boldsymbol{x}}E_x + \hat{\boldsymbol{y}}E_y$, as expected, but magnetic field consists of a single "component" which you can denote by B. Define a "dual" electromagnetic field strength tensor and show that it is a Lorentz three-vector of the $2+1$ dimensional spacetime. Determine its components in terms of the electric and magnetic fields. In light of the $2+1$ dimensional version of eq. (3), show that

$$\frac{d}{dt}\int B(\vec{\boldsymbol{x}},t)\,d^2x = 0\,, \tag{4}$$

assuming that the electric and magnetic fields vanish sufficiently fast at spatial infinity.

(c) Consider a reference frame K' that moves at a constant velocity $c\beta\hat{\boldsymbol{x}}$ with respect to reference frame K. Using the behavior of a Lorentz three-vector under a boost, obtain expressions for the electric and magnetic fields, E_x, E_y, and B, in reference frame K' in terms of the corresponding fields in reference frame K. Check that your results coincide with the expected result in $3+1$ dimensional spacetime.

(d) In 2 spatial dimensions, there are two different vector differential operators,

$$\vec{\boldsymbol{\nabla}} \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\,, \tag{5}$$

$$\vec{\boldsymbol{\nabla}}_\perp \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial y} - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\,, \tag{6}$$

where $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}_\perp = 0$ (which justifies the notation). Using eqs. (1) and (3), write out Maxwell equations explicitly in terms of the electric field vector, the magnetic field, the charge density and current density vector, and the differential operators defined above. Show that in 2 spatial dimensions, there are only three Maxwell equations (in contrast to the four equations obtained in three spatial dimensions).

2. Consider a particle of charge e and mass m, which in the presence of a constant uniform magnetic field $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$, performs a circular motion of radius a and angular velocity ω. Assume that the motion is nonrelativistic, i.e., $v = \omega a \ll c$. Choose a coordinate system in which the origin O is the center of the circle.

(a) Determine the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t)$ generated by the particle in the far (radiation) zone that is measured by a distant observer located at polar angle θ and azimuthal angle φ with respect to the origin O. You may discard any terms that fall off faster than $\mathcal{O}(1/r)$ where $r \equiv |\vec{\boldsymbol{x}}|$.

(b) Compute the time average of the angular distribution of the radiated power measured by a distant observer located at polar angle θ and azimuthal angle φ with respect to the origin O.

3. Evaluate

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\left[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\right],$$

where $f(r)$ is an arbitrary function of the radial variable $r \equiv |\vec{\boldsymbol{x}}|$, and $\vec{\boldsymbol{X}}_{\ell m}((\theta,\varphi)$ is the vector spherical harmonic introduced by Jackson in Chapter 9.

4. A simple model for an electron of charge $q = -e$ and rest energy $mc^2 = 0.511 \times 10^6$ eV consists of a uniform distribution of charge on the surface of a sphere of radius R. Suppose that the electron moves with a speed of $v \ll c$.

(a) Determine the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields separately for $r < R$ and for $r \geq R$ in the reference frame K where the electron moves with velocity $\vec{\boldsymbol{v}}$. Here, r is the radial coordinate in the reference frame K. In your computation, you should keep terms of $\mathcal{O}(v/c)$, but drop terms of $\mathcal{O}(v^2/c^2)$.

HINT: First compute the fields in the rest frame of the electron.

(b) From the results of part (a), determine the density of electromagnetic momentum due to the electromagnetic fields of the electron. Then, integrate to obtain the total momentum carried by the fields.

HINT: Without loss of generality, you may assume that $\vec{\boldsymbol{v}}$ points in the z-direction. Carry out the integration in spherical coordinates, where $\vec{\boldsymbol{x}} = r\big(\hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta\big)$.

(c) Find the numerical value of R such that the total momentum carried by the electromagnetic fields is equal to the mechanical momentum $m\vec{\boldsymbol{v}}$ of the electron. How is R related to the classical radius of the electron, $r_c = e^2/(mc^2)$?