This is an open book exam with a time limit of three hours. You may refer to Jackson's text, a second textbook of your choosing, and any printed material that you downloaded (e.g., solution sets and class handouts) from the 2024 Physics 214 course webpage. You may also consult your own personal notes and any reference of integrals or other mathematical facts. However, you may not *use your laptops or the internet. Moreover, you may* not *collaborate with anyone else during the exam.*

Note that you do not have to derive all results from scratch. However, if you use a particular result, you should cite its source (e.g. an equation in Jackson, an equation derived in a class lecture, or an equation that appears in a solution set or in a class handout).

This exam consists of three problems with ten individual parts in total. Each part is worth ten points. Use this information to manage your time appropriately during the exam.

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1. Consider an oversimplified model of an antenna consisting of a thin wire of length ℓ and negligible cross section, carrying a harmonically varying current density flowing in the z direction. The (complex) current in the wire is given by $Ie^{-i\omega t}$, where I is a constant (independent of position).

(a) Show that the (complex) current density takes the form:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}}, t) = \hat{\boldsymbol{z}} I e^{-i\omega t}\delta(x)\delta(y)[\Theta(z + \tfrac{1}{2}\ell) - \Theta(z - \tfrac{1}{2}\ell)]\,, \tag{1}$$

by verifying that eq. (1) implies that the corresponding current is given by $Ie^{-i\omega t}$, where the step function $\Theta(x) \equiv 1$ if $x > 0$ and $\Theta(x) \equiv 0$ if $x < 0$. Here, we have assumed that the point $z = 0$ corresponds to the midpoint of the antenna.

(b) Prove that there is an oscillating charge density at $z = \pm\frac{1}{2}\ell$ (i.e., at both ends of the antenna), but the charge density vanishes at any interior point on the antenna.

(c) Show that the antenna acts like an oscillating electric dipole moment, $\vec{\boldsymbol{p}}\,e^{-i\omega t}$. Evaluate $\vec{\boldsymbol{p}}$ in terms of the current I, the antenna length ℓ and the angular frequency ω.

(d) Calculate the angular distribution of the radiated power, $dP/d\Omega$, assuming that $\lambda \gg \ell$, where λ is the wavelength of the emitted radiation. Express your answer in terms of the current I, the antenna length ℓ and the wavelength λ. Integrate over angles to obtain the total radiated power.

2. An electron of charge e and mass m moves in a plane perpendicular to a uniform magnetic field B. If the energy loss by radiation is neglected, the orbit is a circle of some radius R. Let E be the total relativistic energy of the electron, and assume that $E \gg mc^2$ (corresponding to ultra-relativistic motion).

(a) Express B analytically in terms of the parameters given above. Compute numerically the required magnetic field B, in gauss, for the case of $R = 30$ meters and $E = 2.5$ GeV.

(b) In fact, the electron radiates electromagnetic energy. Suppose that the energy loss per revolution, ΔE, is small compared to E. Express the ratio $\Delta E/E$ analytically in terms of the parameters given above.

HINT: You can assume that the power is constant (independent of time) during the time Δt it takes for the electron to complete one orbit. In the ultra-relativistic limit, $v \simeq c$, so you can easily compute Δt.

(c) Evaluate the ratio obtained in part (b) numerically using the values of R and E given in part (a). Note that the rest mass of the electron is $mc^2 = 511$ keV.

3. A charged particle of mass m and charge e with relativistic velocity $\vec{\boldsymbol{v}}_0 = v_0\hat{\boldsymbol{z}}$ enters a medium where it is slowed down by a force that is proportional to its velocity. That is, $\vec{\boldsymbol{F}} = d\vec{\boldsymbol{p}}/dt = -\eta\vec{\boldsymbol{v}}$, where η is a positive dimensionful constant. The time t refers to the moving charge and $t = 0$ when the particle enters the medium.

(a) Using relativistic mechanics, determine the acceleration of the charged particle as a function of its velocity, mass and η.

HINT: This is a one dimensional problem, since the particle moves in a straight line.

(b) Determine the angular distribution of the instantaneous power radiated once the particle has entered the medium and slowed down to a velocity v. The polar and azimuthal angles of the emitted radiation are defined relative to the z-axis which lies along the direction of the particle velocity. In your calculation, you may neglect the effect of the medium on the emitted radiation (*i.e.*, you should treat the radiation as if it were emitted in the vacuum.)

(c) How much energy is emitted in the form of electromagnetic radiation from the time the particle enters the medium until it slows down and reaches zero velocity? Express your answer in terms of the parameters e, m, c, η and v_0.

HINT: First, compute the total power. The following integral may be useful,

$$\int_{-1}^{1} \frac{1-x^2}{(1-\beta x)^5}\, dx = \frac{4}{3}\gamma^6\,,$$

where $\beta \equiv v/c$ and $\gamma \equiv (1-\beta^2)^{-1/2}$. Then to determine the energy, change the integration variable from time t to velocity v and integrate from $v_0 \equiv v(t=0)$ to zero velocity.