1. The theory of electromagnetism in $3+1$ spacetime dimensions can be generalized to $n+1$ spacetime dimensions as follows. The indices of the second-rank totally antisymmetric electromagnetic field strength tensor $F^{\mu\nu}$ now take on values $\mu,\ \nu \in \{0,1,\ldots,n\}$. The dynamical Maxwell equations are given (in gaussian units) by:

$$\partial_\mu F^{\mu\nu} = \frac{S_{n-1}}{c} J^\nu\,, \tag{1}$$

where

$$S_{n-1} \equiv \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{2}$$

is the surface area of an n-dimension ball of unit radius. For example, $S_1 = 2\pi$, $S_2 = 4\pi$, etc. The dual electromagnetic field strength tensor is defined by employing the totally antisymmetric rank $(n+1)$ ϵ-tensor. The latter can be used to express the kinematical Maxwell equations,

$$\epsilon^{\mu\cdots\alpha\beta}\partial_\mu F_{\alpha\beta} = 0\,, \tag{3}$$

where $\cdots$ in eq. (3) represents $n-2$ free indices that are not exhibited explicitly. By convention, we choose $\epsilon^{012\cdots n} = +1$.

(a) In $n+1$ spacetime dimensions, how many independent components are needed to describe $F^{\mu\nu}$? How many of these components represent the electric field and how many of these components represent the magnetic field?

In general, a rank 2 tensor in $n+1$ spacetime dimensions has $(n+1)^2$ components. But, an antisymmetric tensor satisfies $F^{00} = F^{11} = \cdots = 0$ and $F^{\mu\nu} = -F^{\nu\mu}$. Hence, the number of independent components is given by:

$$\tfrac{1}{2}\big[(n+1)^2 - (n+1)\big] = \tfrac{1}{2}n(n+1)\,. \tag{4}$$

As a check, in 4 spacetime dimensions (i.e., with $n=3$), $F^{\mu\nu}$ has 6 components.

The components of the electric field are given by:

$$E^i = F^{i0}\,, \qquad \text{for } i \in \{1,2,\ldots,n\}\,. \tag{5}$$

Since the electric field vector has n components, we can use eq. (4) to conclude that the number of components of the magnetic field is

$$\tfrac{1}{2}n(n+1) - n = \tfrac{1}{2}n(n-1)\,. \tag{6}$$

If you wish to have a more explicit result, recall that for $n=3$, we showed in class that $B^k = -\frac{1}{2}\epsilon^{ijk}F^{ij}$, where there is an implicit sum over the two pairs of repeated indices. In n spatial dimensions, the Levi-Civita tensor has n components. Thus, for $n \geq 4$,

$$B^{k\ell\cdots} = -\tfrac{1}{2}\epsilon^{ijk\ell\cdots}F^{ij}\,, \tag{7}$$

where $\cdots$ represents the remaining $n-4$ space indices. That is, B is a totally antisymmetric rank $n-2$ tensor, which (with a little help from combinatorial mathematics) has

$$\frac{n!}{(n-2)!\,2!} = \tfrac{1}{2}n(n-1)\,, \tag{8}$$

components, in agreement with eq. (6).

(b) Consider the theory of electromagnetism in $2+1$ spacetime dimensions, where $F^{\mu\nu}$ can be constructed by deleting the fourth row and fourth column of the $3+1$ dimensional version of $F^{\mu\nu}$. Note that the electric field vector is now of the form $\vec{\boldsymbol{E}} = \hat{\boldsymbol{x}}E_x + \hat{\boldsymbol{y}}E_y$, as expected, but magnetic field consists of a single "component" which you can denote by B. Define a "dual" electromagnetic field strength tensor and show that it is a Lorentz three-vector of the $2+1$ dimensional spacetime. Determine its components in terms of the electric and magnetic fields. In light of the $2+1$ dimensional version of eq. (3), show that

$$\frac{d}{dt}\int B(\vec{\boldsymbol{x}},t)\,d^2x = 0\,, \tag{9}$$

assuming that the electric and magnetic fields vanish sufficiently fast at spatial infinity.

Starting with $F^{\mu\nu}$ in $3+1$ dimensional spacetime and deleting the four row and column, we obtain:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & -B \\ E_y & B & 0 \end{pmatrix}\,. \tag{10}$$

Note that what was B_z in $3+1$ dimensions is now denoted by B. More explicitly,

$$E^i = F^{i0}\,, \qquad B = -\tfrac{1}{2}\epsilon^{ij}F^{ij}\,, \qquad \text{for } i,j \in \{1,2\}, \tag{11}$$

where ϵ^{ij} is the Levi-Civita tensor in 2 spatial dimensions ($\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 0$), and there is an implicit sum over the pair of repeated indices. In particular, note that

$$E_x \equiv E^1 \quad \text{and} \quad E_y \equiv E^2\,. \tag{12}$$

In $3+1$ dimensional spacetime, the dual of the electromagnetic field strength tensor is an antisymmetric rank 2 tensor that is defined as:

$$\widetilde{F}^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\,, \tag{13}$$

where $F_{\alpha\beta} = g_{\alpha\rho}g_{\beta\sigma}F^{\rho\sigma}$ where $g = \text{diag}(1\,,\,-1\,,\,-1\,,\,-1)$ is the Minkowski metric in $3+1$ spacetime dimensions. In $2+1$ spacetime dimensions, the Levi-Civita tensor has only three indices. Hence, in this case the dual of the electromagnetic field strength tensor is a Lorentz three-vector that is given by

$$\widetilde{F}^{\mu} \equiv -\tfrac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}\,, \tag{14}$$

where the minus sign has been inserted for convenience. In eq. (14), $F_{\alpha\beta} = g_{\alpha\rho}g_{\beta\sigma}F^{\rho\sigma}$, where $g = \text{diag}(1\,,\,-1\,,\,-1)$ is the Minkowski metric in $2+1$ spacetime dimensions. Using eqs. (10) and (14), the components of $\widetilde{F}^\mu$ are:

$$\widetilde{F}^0 = -\tfrac{1}{2}\bigl(\epsilon^{012}F_{12} + \epsilon^{021}F_{21}\bigr) = -F^{12} = B\,, \tag{15}$$

$$\widetilde{F}^1 = -\tfrac{1}{2}\bigl(\epsilon^{102}F_{02} + \epsilon^{120}F_{20}\bigr) = -F^{02} = E^2 = E_y\,, \tag{16}$$

$$\widetilde{F}^2 = -\tfrac{1}{2}\bigl(\epsilon^{201}F_{01} + \epsilon^{210}F_{10}\bigr) = F^{01} = -E^1 = -E_x\,, \tag{17}$$

after making use of eq. (11). Note that eqs. (16) and (17) can be rewritten as

$$\widetilde{F}^i = \epsilon^{ij}E^j\,, \qquad \text{for } i \in \{1,2\}, \tag{18}$$

with an implicit sum over the repeated index j.

In $2+1$ spacetime dimensions, eq. (3) yields

$$\partial_\mu \widetilde{F}^\mu = 0\,, \tag{19}$$

which is equivalent to

$$\frac{1}{c}\frac{\partial F^0}{\partial t} + \partial_i \widetilde{F}^i = 0\,. \tag{20}$$

Using the results of eqs. (15) and (18),

$$\frac{1}{c}\frac{\partial B}{\partial t} + \epsilon^{ij}\partial_i E^j = 0\,. \tag{21}$$

Integrating this result over two-dimensional space,

$$\frac{1}{c}\frac{d}{dt}\int B(\vec{\boldsymbol{x}},t)\,d^2x + \epsilon^{ij}\int \partial_i E^j(\vec{\boldsymbol{x}},t)\,d^2x = 0\,. \tag{22}$$

The second integral in eq. (22) can be converted into a line integral at infinity by using the two-dimensional version of Stoke's theorem. Under the assumption that the electric fields vanish at spatial infinity, the end result is

$$\frac{d}{dt}\int B(\vec{\boldsymbol{x}},t)\,d^2x = 0\,. \tag{23}$$

(c) Consider a reference frame K' that moves at a constant velocity $c\beta\hat{\boldsymbol{x}}$ with respect to reference frame K. Using the behavior of a Lorentz three-vector under a boost, obtain expressions for the electric and magnetic fields, E_x, E_y, and B, in reference frame K' in terms of the corresponding fields in reference frame K. Check that your results coincide with the expected result in $3+1$ dimensional spacetime.

A Lorentz three-vector in $2+1$ spacetime dimensions transforms under a boost similarly to the transformation of a Lorentz four-vector in $3+1$ spacetime dimensions. Writing $w^\mu = (w^0\,;\, w^1\,,\, w^2)$, the corresponding Lorentz transformation is given by

$$w'^{\,0} = \gamma(w^0 - \beta w^1)\,, \tag{24}$$

$$w'^{\,1} = \gamma(w^1 - \beta w^0)\,, \tag{25}$$

$$w'^{\,2} = w^2\,. \tag{26}$$

where $\gamma \equiv (1-\beta^2)^{-1/2}$. The only difference between 2 and 3 spatial dimensions is that in 2 dimensions there is only one spatial direction transverse to $\vec{\boldsymbol{\beta}}$ whereas in 3 dimensions there are two spatial directions transverse to $\vec{\boldsymbol{\beta}}$.

Using eqs. (15)–(17), it follows that

$$B' = \gamma(B - \beta E_y)\,, \tag{27}$$

$$E'_y = \gamma(E_y - \beta B)\,, \tag{28}$$

$$E'_x = E_x\,. \tag{29}$$

To check that eqs. (27)–(29) are correct, note that in $3+1$ spacetime, one can employ eq. (11.148) of Jackson, where the third component of the electric field and the first and second components of the magnetic field are discarded whereas B_3 is identified as B.

(d) In 2 spatial dimensions, there are two different vector differential operators,

$$\vec{\boldsymbol{\nabla}} \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}\,, \tag{30}$$

$$\vec{\boldsymbol{\nabla}}_\perp \equiv \hat{\boldsymbol{x}}\frac{\partial}{\partial y} - \hat{\boldsymbol{y}}\frac{\partial}{\partial x}\,, \tag{31}$$

where $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}_\perp = 0$ (which justifies the notation). Using eqs. (1) and (3), write out Maxwell equations explicitly in terms of the electric field vector, the magnetic field, the charge density and current density vector, and the differential operators defined above. Show that in 2 spatial dimensions, there are only three Maxwell equations (in contrast to the four equations obtained in three spatial dimensions).

In $2+1$ dimensional spacetime, Maxwell's equations are given by eqs. (1) and (19),

$$\partial_\mu F^{\mu\nu} = \frac{2\pi}{c} J^\nu\,, \tag{32}$$

$$\partial_\mu \widetilde{F}^\mu = 0\,. \tag{33}$$

Note that eq. (11) implies that

$$F^{i0} = -F^{oi} = E^i\,, \qquad\qquad F^{ij} = -F^{ji} = -\epsilon^{ij} B\,. \tag{34}$$

Moreover, in terms of components, eqs. (30) and (31) are equivalent to

$$\boldsymbol{\nabla}_i = \partial_i\,, \qquad\qquad \big(\boldsymbol{\nabla}_\perp\big)_i = \epsilon^{ij}\partial_j\,. \tag{35}$$

Using eqs. (10), (15), and (18) and $J^\mu = (c\rho\,;\,\vec{\boldsymbol{J}})$, eq. (32) yields

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{E}} = 2\pi\rho\,, \tag{36}$$

$$\vec{\boldsymbol{\nabla}}_\perp B - \frac{1}{c}\frac{\partial\vec{\boldsymbol{E}}}{\partial t} = \frac{2\pi}{c}\vec{\boldsymbol{J}}\,. \tag{37}$$

Finally, we have already shown that eq. (33) yields eq. (21) which can be rewritten as

$$\frac{1}{c}\frac{\partial B}{\partial t} = \vec{\boldsymbol{\nabla}}_\perp\boldsymbol{\cdot}\vec{\boldsymbol{E}}\,. \tag{38}$$

It is noteworthy that there is no 2 dimensional analog of the 3 dimensional Maxwell equation, $\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{B}} = 0$. A further exploration of electrodynamics in 2 spatial dimensions can be found in a paper by Kirk T. McDonald, *Electrodynamics in 1 and 2 Spatial Dimensions*, which is available at `http://kirkmcd.princeton.edu/examples/2dem.pdf`.

2. Consider a particle of charge e and mass m, which in the presence of a constant uniform magnetic field $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$, performs a circular motion of radius a and angular velocity ω. Assume that the motion is nonrelativistic, i.e., $v = \omega a \ll c$. Choose a coordinate system in which the origin O is the center of the circle.

(a) Determine the electric field $\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t)$ generated by the particle in the far (radiation) zone that is measured by a distant observer located at polar angle θ and azimuthal angle φ with respect to the origin O. You may discard any terms that fall off faster than $\mathcal{O}(1/r)$ where $r \equiv |\vec{\boldsymbol{x}}|$.

Using eq. (14.14) of Jackson,

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}}, t) = \frac{e\,\hat{\boldsymbol{n}} \times \big(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{a}}\big)}{c^2 r}\,, \tag{39}$$

where we have dropped terms of $\mathcal{O}(1/r^2)$. In obtaining eq. (39), we have employed the following approximations that are valid in the radiation zone,

$$R = |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'| \simeq r - \hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{x}}\,'\,, \tag{40}$$

$$\hat{\boldsymbol{n}} \equiv \frac{\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'}{R} \simeq \frac{\vec{\boldsymbol{x}}}{r}\,, \tag{41}$$

where $r \equiv |\vec{\boldsymbol{x}}|$ and $\vec{\boldsymbol{x}}\,'(t)$ is the trajectory of the particle. In general, $\vec{\boldsymbol{a}} \equiv d\vec{\boldsymbol{v}}/dt$ is a function of the retarded time, $t_{\rm ret} \equiv t - |\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}\,'|/c$. However, in the nonrelativistic limit, we may approximate $t_{\rm ret} \simeq t$.

Using eqs. (12.38) and (12.39) of Jackson,

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = \omega\big[\vec{\boldsymbol{v}} \times \hat{\boldsymbol{z}}\big]\,, \tag{42}$$

where the nonrelativistic cyclotron frequency ω is given by

$$\omega = \frac{eB}{mc}\,, \tag{43}$$

after employing the nonrelativistic limit, $\gamma \simeq 1$.

The trajectory of the particle is a circle of radius a in the x–y plane. Hence,

$$\vec{\boldsymbol{x}}'(t) = a\big(\hat{\boldsymbol{x}}\sin\omega t + \hat{\boldsymbol{y}}\cos\omega t\big)\,. \tag{44}$$

Successive differentiations yield

$$\vec{\boldsymbol{v}}(t) = \frac{d\vec{\boldsymbol{x}}'}{dt} = \omega a\big(\hat{\boldsymbol{x}}\cos\omega t - \hat{\boldsymbol{y}}\sin\omega t\big)\,, \tag{45}$$

$$\vec{\boldsymbol{a}}(t) = \frac{d\vec{\boldsymbol{v}}}{dt} = -\omega^2 a\big(\hat{\boldsymbol{x}}\sin\omega t + \hat{\boldsymbol{y}}\cos\omega t\big)\,. \tag{46}$$

One can check that eq. (42) is satisfied.

We can now evaluate $\hat{\boldsymbol{n}} \times \big(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{a}}\big) = \hat{\boldsymbol{n}}\big(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{a}}\big) - \vec{\boldsymbol{a}}$. In spherical coordinates,

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta\,. \tag{47}$$

Hence,

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{a}} = -\omega^2 a\big[\sin\theta\cos\varphi\sin\omega t + \sin\theta\sin\varphi\cos\omega t\big] = -\omega^2 a\sin\theta\sin(\varphi+\omega t)\,, \tag{48}$$

and

$$\begin{aligned}\hat{\boldsymbol{n}} \times \big(\hat{\boldsymbol{n}} \times \vec{\boldsymbol{a}}\big) &= \hat{\boldsymbol{n}}\big(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{a}}\big) - \vec{\boldsymbol{a}} \\ &= -\omega^2 a\Big\{\hat{\boldsymbol{x}}\big[\sin^2\theta\cos\varphi\sin(\varphi+\omega t) - \sin\omega t\big] + \hat{\boldsymbol{y}}\big[\sin^2\theta\sin\varphi\sin(\varphi+\omega t) - \cos\omega t\big] \\ &\qquad + \hat{\boldsymbol{z}}\cos\theta\sin\theta\sin(\varphi+\omega t)\Big\}.\end{aligned} \tag{49}$$

Hence, eq. (39) yields

$$\begin{aligned}\vec{\boldsymbol{E}}(\vec{\boldsymbol{x}},t) = -\frac{e\omega^2 a}{c^2 r}\Big\{&\hat{\boldsymbol{x}}\big[\sin^2\theta\cos\varphi\sin(\varphi+\omega t) - \sin\omega t\big] + \hat{\boldsymbol{y}}\big[\sin^2\theta\sin\varphi\sin(\varphi+\omega t) - \cos\omega t\big] \\ &+ \hat{\boldsymbol{z}}\cos\theta\sin\theta\sin(\varphi+\omega t)\Big\}.\end{aligned} \tag{50}$$

(b) Compute the time average of the angular distribution of the radiated power measured by a distant observer located at polar angle θ and azimuthal angle φ with respect to the origin O.

In the nonrelativistic limit, we can simply employ Larmor's formula given by eq. (14.20) of Jackson,

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\big|\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{a}})\big|^2\,. \tag{51}$$

It is convenient to rewrite the square of the triple product as:

$$\big|\hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\vec{\boldsymbol{a}})\big|^2 = \big[\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{a}}) - \vec{\boldsymbol{a}}\big]\boldsymbol{\cdot}\big[\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\boldsymbol{\cdot}\hat{\boldsymbol{a}}) - \vec{\boldsymbol{a}}\big] = |\vec{\boldsymbol{a}}|^2 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2\,. \tag{52}$$

Then,

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\Big\{|\vec{\boldsymbol{a}}|^2 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2\Big\}\,. \tag{53}$$

Using eqs. (46) and (48), it follows that

$$|\vec{\boldsymbol{a}}|^2 - (\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{a}})^2 = \omega^4 a^2\big[1-\sin^2\theta\sin^2(\varphi+\omega t)\big]\,. \tag{54}$$

Hence, we obtain

$$\frac{dP}{d\Omega} = \frac{e^2\omega^4 a^2}{4\pi c^3}\big[1-\sin^2\theta\sin^2(\varphi+\omega t)\big]\,. \tag{55}$$

The time-averaged angular distribution is obtained by noting that

$$\langle\sin^2(\varphi+\omega t)\rangle = \tfrac{1}{2}\,, \tag{56}$$

when averaged over one cycle. Hence,

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2\omega^4 a^2}{4\pi c^3}\big(1-\tfrac{1}{2}\sin^2\theta\big) = \frac{e^2\omega^4 a^2}{8\pi c^3}\big(1+\cos^2\theta\big)\,. \tag{57}$$

3. Evaluate

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\big]\,, \tag{58}$$

where $f(r)$ is an arbitrary function of the radial variable $r\equiv|\vec{\boldsymbol{x}}|$, and $\vec{\boldsymbol{X}}_{\ell m}((\theta,\varphi)$ is the vector spherical harmonic introduced by Jackson in Chapter 9.

In light of eq. (9.119) of Jackson,

$$\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi) = \frac{1}{\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{L}}\,Y_{\ell m}(\theta,\varphi)\,,\quad \text{where } \vec{\boldsymbol{L}} = -i\,\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}. \tag{59}$$

We first employ the vector identity,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}(\psi\vec{\boldsymbol{A}}) = \vec{\boldsymbol{A}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\psi + \psi\,\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}\,, \tag{60}$$

for any vector quantity $\vec{\boldsymbol{A}}$ and scalar quantity ψ. Thus,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\big]=\vec{\boldsymbol{X}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}f(r)+f(r)\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{X}}=\vec{\boldsymbol{X}}\boldsymbol{\cdot}\hat{\boldsymbol{r}}\,\frac{\partial f}{\partial r}+f(r)\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{X}}\,, \tag{61}$$

where $\hat{\boldsymbol{r}}\equiv|\vec{\boldsymbol{x}}|/r$ is the unit vector in the radial direction. Note that[1]

$$\vec{\boldsymbol{X}}\boldsymbol{\cdot}\hat{\boldsymbol{r}}=\frac{-i}{r\sqrt{\ell(\ell+1)}}\,\vec{\boldsymbol{x}}\boldsymbol{\cdot}\big[\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\varphi)\big]=0\,, \tag{62}$$

since $\vec{\boldsymbol{x}}$ is orthogonal to $\vec{\boldsymbol{x}}\times\vec{\boldsymbol{A}}$ for any vector quantity $\vec{\boldsymbol{A}}$. Moreover,

$$\sqrt{\ell(\ell+1)}\,\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\vec{\boldsymbol{X}}=-i\sqrt{\ell(\ell+1)}\,\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\,Y_{\ell m}(\theta,\varphi)\big]=0\,, \tag{63}$$

where we have used the following result,

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big(\vec{\boldsymbol{x}}\times\vec{\boldsymbol{\nabla}}\big)=\epsilon_{ijk}\partial_i(x_j\partial_k)=\epsilon_{ijk}\big[\delta_{ij}\partial_k+x_j\partial_i\partial_k\big]=0\,, \tag{64}$$

after the implicit sum over the three pairs of repeated indices. Eq. (64) is a consequence of $\partial_i x_j=\delta_{ij}$ and the identities $\epsilon_{ijk}\delta_{ij}=0$ and $\epsilon_{ijk}\partial_i\partial_k=0$. These latter two identities are obtained after noting that ϵ_{ijk} is a completely antisymmetric tensor, whereas δ_{ij} and $\partial_i\partial_k$ are both symmetric under the interchange of their two indices.

We therefore conclude that

$$\vec{\boldsymbol{\nabla}}\boldsymbol{\cdot}\big[f(r)\vec{\boldsymbol{X}}_{\ell m}(\theta,\varphi)\big]=0\,. \tag{65}$$

4. A simple model for an electron of charge $q=-e$ and rest energy $mc^2=0.511\times10^6$ eV consists of a uniform distribution of charge on the surface of a sphere of radius R. Suppose that the electron moves with a speed of $v\ll c$.

(a) Determine the $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields separately for $r<R$ and for $r\geq R$ in the reference frame K where the electron moves with velocity $\vec{\boldsymbol{v}}$. Here, r is the radial coordinate in the reference frame K. In your computation, you should keep terms of $\mathcal{O}(v/c)$, but drop terms of $\mathcal{O}(v^2/c^2)$.

Denote the reference frame where the electron moves with velocity $\vec{\boldsymbol{v}}=c\vec{\boldsymbol{\beta}}$ by K, and denote the rest frame of the electron by K'. Using Gauss' law,

$$\vec{\boldsymbol{E}}'=\begin{cases}-\dfrac{e\,\vec{\boldsymbol{x}}'}{r'^3}\,, & \text{for } r'>R,\\[2ex] 0\,, & \text{for } r'<R,\end{cases} \tag{66}$$

$$\vec{\boldsymbol{B}}'=0\,, \tag{67}$$

where $r'\equiv|\vec{\boldsymbol{x}}'|$.

[1]Since $\vec{\boldsymbol{L}}$ is a purely angular operator (with no component in the $\hat{\boldsymbol{r}}$ direction), it follows that $\hat{\boldsymbol{r}}\boldsymbol{\cdot}\vec{\boldsymbol{L}}=0$. Hence $\hat{\boldsymbol{r}}\boldsymbol{\cdot}\vec{\boldsymbol{X}}=0$, which is another way to establish eq. (62).

Under a boost from reference frame K to reference frame K', we can employ eq. (11.149) of Jackson. Dropping terms of $\mathcal{O}(v^2/c^2)$, we obtain

$$\vec{\boldsymbol{E}}' = \vec{\boldsymbol{E}} + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}\,, \tag{68}$$

$$\vec{\boldsymbol{B}}' = \vec{\boldsymbol{B}} - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}\,, \tag{69}$$

where we have put $\gamma = (1-\beta^2)^{-1/2} = 1+\mathcal{O}(v^2/c^2)$. Inverting the transformation equations above yields,

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}' - \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{B}}'\,, \tag{70}$$

$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}' + \vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{E}}'\,. \tag{71}$$

Inserting the expressions given in eqs. (66) and (67), the end result is

$$\vec{\boldsymbol{E}} = -\frac{e\,\vec{\boldsymbol{x}}'}{r'^3}\,, \tag{72}$$

$$\vec{\boldsymbol{B}} = -\frac{e\,\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{x}}'}{r'^3}\,, \tag{73}$$

for $r > R$, and $\vec{\boldsymbol{E}} = \vec{\boldsymbol{B}} = 0$ for $r < R$.

Using eq. (1) of the class handout entitled *The electromagnetic fields of a uniformly moving charge*, it follows that if reference frames K and K' coincide at time $t = 0$ then

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}} + \frac{\gamma - 1}{\beta^2}(\vec{\boldsymbol{\beta}}\boldsymbol{\cdot}\vec{\boldsymbol{x}})\vec{\boldsymbol{\beta}}\,. \tag{74}$$

In particular,

$$\vec{\boldsymbol{x}}' = \vec{\boldsymbol{x}} + \mathcal{O}(v^2/c^2)\,. \tag{75}$$

Hence, using $\vec{\boldsymbol{v}} = c\vec{\boldsymbol{\beta}}$, we conclude that

$$\vec{\boldsymbol{E}} = -\frac{e\,\vec{\boldsymbol{x}}}{r^3}\,, \tag{76}$$

$$\vec{\boldsymbol{B}} = -\frac{e\,\vec{\boldsymbol{v}} \times \vec{\boldsymbol{x}}}{cr^3}\,, \tag{77}$$

for $r > R$, and $\vec{\boldsymbol{E}} = \vec{\boldsymbol{B}} = 0$ for $r < R$.

(b) From the results of part (a), determine the density of electromagnetic momentum due to the electromagnetic fields of the electron. Then, integrate to obtain the total momentum carried by the fields.

The momentum density in gaussian units [cf. eq. (6.118) of Jackson] is given by

$$\vec{\boldsymbol{g}} = \frac{1}{4\pi c}\vec{\boldsymbol{E}} \times \vec{\boldsymbol{B}} = \frac{e^2}{4\pi c^2}\,\frac{\vec{\boldsymbol{x}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{x}})}{r^6} = \frac{e^2}{4\pi c^2 r^6}\left[r^2\vec{\boldsymbol{v}} - \vec{\boldsymbol{x}}(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{v}})\right], \tag{78}$$

for $r > R$, and $\vec{\boldsymbol{g}} = 0$ for $r < R$. Thus the total field momentum is given by:

$$\vec{\boldsymbol{P}}_{\rm field} = \frac{e^2}{4\pi c^2}\int_R^\infty r^2\,dr \int d\Omega\,\frac{r^2\vec{\boldsymbol{v}} - \vec{\boldsymbol{x}}(\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{v}})}{r^6}\,. \tag{79}$$

Defining $\vec{\boldsymbol{x}}\boldsymbol{\cdot}\vec{\boldsymbol{v}} = rv\cos\theta$, eq. (79) yields

$$\vec{\boldsymbol{P}}_{\rm field} = \frac{e^2}{4\pi c^2}\int_R^\infty \frac{dr}{r^4}\int d\Omega\,\bigl(r^2\vec{\boldsymbol{v}} - rv\cos\theta\,\vec{\boldsymbol{x}}\bigr)\,. \tag{80}$$

Without loss of generality, we may take $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{z}}$. In spherical coordinates, $d\Omega = d\cos\theta\,d\varphi$ and

$$\vec{\boldsymbol{x}} = r\bigl(\hat{\boldsymbol{x}}\sin\theta\cos\varphi + \hat{\boldsymbol{y}}\sin\theta\sin\varphi + \hat{\boldsymbol{z}}\cos\theta\bigr)\,. \tag{81}$$

It immediately follows that $P_x = P_y = 0$ due to the integration over φ. The only nonzero component of $\vec{\boldsymbol{P}}_{\rm field}$ is P_z, which is given by:

$$P_z = \frac{e^2 v}{4\pi c^2}\int_R^\infty \frac{dr}{r^2}\int d\Omega\,(1-\cos^2\theta) = \frac{e^2 v}{4\pi c^2}\,\frac{8\pi}{3R} = \frac{2e^2 v}{3c^2 R}\,. \tag{82}$$

Since this result was obtained under the assumption that $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{z}}$, we can conclude that

$$\vec{\boldsymbol{P}}_{\rm field} = \frac{2e^2}{3c^2 R}\,\vec{\boldsymbol{v}}\,. \tag{83}$$

(c) Find the numerical value of R such that the total momentum carried by the electromagnetic fields is equal to the mechanical momentum $m\vec{\boldsymbol{v}}$ of the electron. How is R related to the classical radius of the electron, $r_c = e^2/(mc^2)$?

In the model of the electron, we identify $\vec{\boldsymbol{P}}_{\rm field} = m\vec{\boldsymbol{v}}$. It then follows that

$$m = \frac{2e^2}{3c^2 R}\,. \tag{84}$$

Solving for R and inserting the relevant numbers,

$$R = \frac{2e^2}{3mc^2} = \frac{2(4.8\times 10^{-10})^2}{3(0.511\times 10^6)(1.6\times 10^{-12})} = 1.88\times 10^{-13}~{\rm cm}\,, \tag{85}$$

where the absolute value of the electron charge in gaussian units is $|e| = 4.8\times 10^{-10}$ esu, the rest energy of the electron is $mc^2 = 0.511\times 10^6$ eV, and the conversion from eV to ergs is 1 eV $= 1.6\times 10^{-12}$ ergs.

The classical radius of the electron (in gaussian units) is defined as

$$r_c \equiv \frac{e^2}{mc^2}\,. \tag{86}$$

Comparing with eq. (85), we conclude that

$$R = \tfrac{2}{3}r_c\,. \tag{87}$$