1. Consider an oversimplified model of an antenna consisting of a thin wire of length ℓ and negligible cross section, carrying a harmonically varying current density flowing in the z direction. The (complex) current in the wire is given by $Ie^{-i\omega t}$, where I is a constant (independent of position).

(a) Show that the (complex) current density takes the form:

$$\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \hat{\boldsymbol{z}} I e^{-i\omega t}\delta(x)\delta(y)[\Theta(z+\tfrac{1}{2}\ell) - \Theta(z-\tfrac{1}{2}\ell)]\,, \tag{1}$$

by verifying that eq. (1) implies that the corresponding current is given by $Ie^{-i\omega t}$, where the step function $\Theta(x) \equiv 1$ if $x > 0$ and $\Theta(x) \equiv 0$ if $x < 0$. Here, we assume that the point $z = 0$ corresponds to the midpoint of the antenna.

First we note that

$$\Theta(z+\tfrac{1}{2}\ell) - \Theta(z-\tfrac{1}{2}\ell) = \begin{cases} 0\,, & z > \ell/2\,, \\ 1\,, & |z| < \ell/2\,, \\ 0\,, & z < -\ell/2\,. \end{cases}$$

Thus, $\vec{\boldsymbol{J}} = 0$ if $z > \ell/2$ or $z < -\ell/2$. For $|z| < \ell/2$,

$$J_z = Ie^{-i\omega t}\delta(x)\delta(y)\,, \qquad\qquad J_x = J_y = 0\,.$$

The current is obtained by computing

$$\int \vec{\boldsymbol{J}}\cdot d\vec{\boldsymbol{a}} = \int \vec{\boldsymbol{J}}\cdot\hat{\boldsymbol{z}}\,dxdy = \int J_z\,dxdy = Ie^{-i\omega t}\,.$$

(b) Prove that there is an oscillating charge density at $z = \pm\frac{1}{2}\ell$ (*i.e.*, at both ends of the antenna), but the charge density vanishes at any interior point on the antenna.

The continuity equation is

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} + \frac{\partial\rho}{\partial t} = 0\,.$$

For $\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t) = \vec{\boldsymbol{J}}(\vec{\boldsymbol{x}})e^{-i\omega t}$ and $\rho(\vec{\boldsymbol{x}},t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$, the continuity equation then reads:

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = i\omega\rho(\vec{\boldsymbol{x}})\,.$$

Using $\vec{\boldsymbol{J}}$ given in part (a),

$$\begin{aligned}\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{J}} = \frac{\partial J_z}{\partial z} &= I\delta(x)\delta(y)\frac{\partial}{\partial z}\left[\Theta(z+\tfrac{1}{2}\ell) - \Theta(z-\tfrac{1}{2}\ell)\right] \\ &= I\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right]\,.\end{aligned}$$

Setting this result to $i\omega\rho(\vec{\boldsymbol{x}})$, we conclude that:

$$\rho(\vec{\boldsymbol{x}}) = -\frac{iI}{\omega}\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] ,$$

which corresponds to two point charges located at the two ends of the antenna. Moreover, $\rho(\vec{\boldsymbol{x}}, t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ indicates that the point charges have magnitudes that oscillate in time. (As usual, we take the real part of $\rho(\vec{\boldsymbol{x}}, t)$ to find the corresponding physical quantity.)

(c) Show that the antenna acts like an oscillating electric dipole moment, $\vec{\boldsymbol{p}}\,e^{-i\omega t}$. Evaluate $\vec{\boldsymbol{p}}$ in terms of the current I, the antenna length ℓ and the angular frequency ω.

The electric dipole moment is given by

$$\vec{\boldsymbol{p}}(t) = \int \vec{\boldsymbol{x}}\rho(\vec{\boldsymbol{x}}, t)d^3x = e^{-i\omega t}\int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x = \vec{\boldsymbol{p}}\,e^{-i\omega t}\,, \tag{2}$$

after employing $\rho(\vec{\boldsymbol{x}}, t) = \rho(\vec{\boldsymbol{x}})e^{-i\omega t}$ and defining,

$$\vec{\boldsymbol{p}} \equiv \int \vec{\boldsymbol{x}}\,\rho(\vec{\boldsymbol{x}})\,d^3x\,. \tag{3}$$

We therefore compute:

$$\begin{aligned}
\vec{\boldsymbol{p}} &= \int \vec{\boldsymbol{x}}\rho(\vec{\boldsymbol{x}})d^3x \\
&= \int (x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})\left(\frac{-iI}{\omega}\right)\delta(x)\delta(y)\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] dxdydz \\
&= \frac{-iI}{\omega}\hat{\boldsymbol{z}}\int zdz\left[\delta(z+\tfrac{1}{2}\ell) - \delta(z-\tfrac{1}{2}\ell)\right] = \frac{iI\ell}{\omega}\hat{\boldsymbol{z}}\,.
\end{aligned}$$

(d) Calculate the angular distribution of the radiated power, $dP/d\Omega$, assuming that $\lambda \gg \ell$, where λ is the wavelength of the emitted radiation. Express your answer in terms of the current I, the antenna length ℓ and the wavelength λ. Integrate over angles to obtain the total radiated power.

For $\lambda \gg \ell$, the electric dipole approximation is very accurate. Hence, we can neglect all other multipole contributions. Using eq. (9.23) of Jackson (in SI units),

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}k^4|\vec{\boldsymbol{p}}|^2\sin^2\theta = \frac{c^2 Z_0}{32\pi^2}\frac{I^2\ell^2k^4}{\omega^2}\sin^2\theta\,, \tag{4}$$

where $Z_0 \equiv \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space. Recalling that $\omega = kc$ and $k = 2\pi/\lambda$, the above result can be written as:

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8}\left(\frac{\ell}{\lambda}\right)^2\sin^2\theta\,. \tag{5}$$

Integrating over angles by using

$$\int \sin^2\theta\, d\Omega = 2\pi \int_{-1}^{1} (1-\cos^2\theta)\, d\cos\theta = \frac{8\pi}{3}\,,$$

we end up with:

$$P = \frac{\pi Z_0 I^2}{3}\left(\frac{\ell}{\lambda}\right)^2 . \tag{6}$$

Note that one can also derive eq. (6) by using eq. (9.24) of Jackson,

$$P = \frac{c^2 Z_0 k^4}{12\pi}\,|\vec{\boldsymbol{p}}|^2 = \frac{c^2 Z_0}{12\pi}\left(\frac{2\pi}{\lambda}\right)^4\left(\frac{I^2\ell^2}{c^2}\right)\left(\frac{\lambda}{2\pi}\right)^2 = \frac{\pi Z_0 I^2}{3}\left(\frac{\ell}{\lambda}\right)^2 . \tag{7}$$

The results above have been given in SI units. To convert eqs. (4)–(7) to Gaussian units, one can simply replace $Z_0 \to 4\pi/c$. This can be understood by writing the impedance of free space [cf. eq. (9.5) of Jackson] as,

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\epsilon_0 c}\,. \tag{8}$$

Moreover, using Table 3 on p. 782 of Jackson, we must replace $I \to \sqrt{4\pi\epsilon_0}\, I$, when converting a formula expressed in SI units to gaussian units. Thus,

$$Z_0 I^2 \to \frac{1}{\epsilon_0 c}\, 4\pi\epsilon_0 I^2 = \frac{4\pi I^2}{c} \tag{9}$$

which is consistent with replacing Z_0 with $4\pi/c$ as asserted above.

2. An electron of charge e and mass m moves in a plane perpendicular to a uniform magnetic field B. If the energy loss by radiation is neglected, the orbit is a circle of some radius R. Let E be the total relativistic energy of the electron, and assume that $E \gg mc^2$ (corresponding to ultra-relativistic motion).

(a) Express B analytically in terms of the parameters given above. Compute numerically the required magnetic field B, in gauss, for the case of $R = 30$ meters and $E = 2.5$ GeV.

For circular motion,

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = -\frac{v^2}{R}\,\hat{\boldsymbol{r}}\,. \tag{10}$$

Since the circular motion is in a plane that is perpendicular to the magnetic field $\vec{\boldsymbol{B}}$, it follows that $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{v}}$ and $\hat{\boldsymbol{r}}$ are mutually orthogonal vectors. Moreover, eqs. (12.38) and (12.39) on p. 585 of Jackson yield (in gaussian units),

$$\frac{d\vec{\boldsymbol{v}}}{dt} = \frac{e}{\gamma m c}\vec{\boldsymbol{v}} \times \vec{\boldsymbol{B}}\,. \tag{11}$$

Thus, if $\vec{\boldsymbol{B}}$ points in the z-direction, then $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\theta}}$ and the circular motion is clockwise in the x–y plane. Combining eqs. (10) and (11), it follows that

$$B = \frac{\gamma m v c}{eR}\,.$$

In the ultra-relativistic limit, we have $v \simeq c$ so that

$$B \simeq \frac{\gamma m c^2}{eR} = \frac{E}{eR}\,, \qquad (12)$$

where $E = \gamma mc^2$ is the total relativistic energy of the electron. Note that eq. (12) has been derived using gaussian units. Plugging in numbers, and recalling that 1 eV $= 1.6\times 10^{-19}$ J and 1 J $= 10^7$ ergs, it follows that

$$B = \frac{(2.5\times 10^9~\text{eV})(1.6\times 10^{-12}~\text{ergs/eV})}{(4.8\times 10^{-10}~\text{esu})(3\times 10^3~\text{cm})} = 2.78\times 10^3~\text{gauss}\,. \qquad (13)$$

One can convert eq. (12) into SI units by replacing $eB \to ecB$.[1] In this case, eq. (13) would be replaced by

$$B = \frac{(2.5\times 10^9~\text{eV})(1.6\times 10^{-19}~\text{J/eV})}{(1.6\times 10^{-19}~\text{C})(30~\text{m})(3\times 10^8~\text{m/s})} = 0.278~\text{T}\,.$$

Converting tesla to gauss using 1 T $= 10^4$ gauss using Table 4 on p. 783 of Jackson, we recover the result of eq. (13).

(b) In fact, the electron radiates electromagnetic energy. Suppose that the energy loss per revolution, ΔE, is small compared to E. Express the ratio $\Delta E/E$ analytically in terms of the parameters given above.

Using eq. (14.46) on p. 671 of Jackson,

$$P = \frac{2e^2 a^2}{c^3}\gamma^4\,,$$

where $a \equiv |d\vec{\boldsymbol{v}}/dt|$ and t is the charge's own time (which is denoted by t' in section 3 of Chapter 14 of Jackson). For circular motion, the magnitude of the acceleration is $a = v^2/R$. One orbit covers a distance of $2\pi R$ in a time $\Delta t = 2\pi R/v$. Thus, the energy lost per orbit is

$$\Delta E = P\,\Delta t = \frac{2e^2}{c^3}\left(\frac{v^2}{R}\right)^2\left(\frac{2\pi R}{v}\right)\gamma^4 = \frac{4\pi e^2}{3R}\left(\frac{v}{c}\right)^3\gamma^4\,.$$

In the ultra-relativistic limit, $v \simeq c$, so we end up with

$$\Delta E \simeq \frac{4\pi e^2}{3R}\gamma^4\,.$$

[1]The easiest way to see this is to note that the magnetic force on a charge e in SI units is $e\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}$, whereas in gaussian units, the magnetic force is given by $e\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}/c$.

Dividing by the total relativistic energy $E = \gamma mc^2$, and substituting $\gamma \equiv E/(mc^2)$, it follows that

$$\frac{\Delta E}{E} \simeq \frac{4\pi e^2}{3mc^2 R}\left(\frac{E}{mc^2}\right)^3 . \tag{14}$$

(c) Evaluate the ratio obtained in part (b) numerically using the values of R and E given in part (a). Note that the rest mass of the electron is $mc^2 = 511$ keV.

Plugging in the numbers into eq. (14),

$$\frac{\Delta E}{E} \simeq \frac{4\pi(4.8\times 10^{-10}\text{ esu})^2}{3(0.511\times 10^6\text{ eV})(1.6\times 10^{-12}\text{ ergs/eV})(3\times 10^3\text{ cm})}\left(\frac{2.5\times 10^9\text{ eV}}{0.511\times 10^6\text{ eV}}\right)^3 = 4.6\times 10^{-5} . \tag{15}$$

To check the dimensions (since $\Delta E/E$ must be dimensionless), note that in gaussian units,

$$1\text{ esu} = 1\text{ statcoulomb} = 1\text{ dyne}^{1/2}\cdot\text{cm} = 1\ (\text{erg}\cdot\text{cm})^{1/2}. \tag{16}$$

Eq. (14) has been derived using gaussian units. In SI units, e^2 is replaced by $e^2/(4\pi\epsilon_0)$. In this case, using the numerical value of ϵ_0 given at the bottom of Table 3 on p. 782 of Jackson, the modified eq. (14) would yield

$$\begin{aligned}\frac{\Delta E}{E} &\simeq \frac{(1.6\times 10^{-19}\text{ C})^2}{3(0.511\times 10^6\text{ eV})(1.6\times 10^{-19}\text{ J/eV})(30\text{ m})(8.854\times 10^{-12}\text{ F/m})}\left(\frac{2.5\times 10^9\text{ eV}}{0.511\times 10^6\text{ eV}}\right)^3 \\ &= 4.6\times 10^{-5} ,\end{aligned} \tag{17}$$

which again produces the same result obtained in eq. (15). Note that the unit of capacitance (farad) is given by 1 F = 1 C/volt = 1 C^2/J, so that $\Delta E/E$ is dimensionless, as expected.

3. A charged particle of mass m and charge e with relativistic velocity $\vec{\boldsymbol{v}}_0 = v_0\hat{\boldsymbol{z}}$ enters a medium where it is slowed down by a force that is proportional to its velocity. That is, $\vec{\boldsymbol{F}} = d\vec{\boldsymbol{p}}/dt = -\eta\vec{\boldsymbol{v}}$, where η is a positive dimensionful constant. The time t refers to the moving charge and $t = 0$ when the particle enters the medium.

(a) Using relativistic mechanics, determine the acceleration of the charged particle as a function of its velocity, mass and η.

This is a one dimensional problem, since the particle moves in a straight line. The relativistic equation of motion for the particle is given by:

$$\vec{\boldsymbol{F}} = \frac{d\vec{\boldsymbol{p}}}{dt} = \frac{d}{dt}(\gamma m\vec{\boldsymbol{v}}) ,$$

where $\vec{\boldsymbol{p}} = \gamma m\vec{\boldsymbol{v}}$ is the relativistic momentum. Writing $\gamma \equiv (1 - v^2/c^2)^{-1/2}$, where $v \equiv |\vec{\boldsymbol{v}}|$, and remembering that the velocity $\vec{\boldsymbol{v}}$ and γ depend on time,

$$\frac{d}{dt}(\gamma m\vec{\boldsymbol{v}}) = m\vec{\boldsymbol{v}}\frac{d\gamma}{dt} + \gamma m\frac{d\vec{\boldsymbol{v}}}{dt} = \gamma m\left[\frac{d\vec{\boldsymbol{v}}}{dt} + \frac{\gamma^2}{c^2}\vec{\boldsymbol{v}}\left(\vec{\boldsymbol{v}}\cdot\frac{d\vec{\boldsymbol{v}}}{dt}\right)\right] , \tag{18}$$

where we have used

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{\gamma^3}{c^2}\left(\vec{\boldsymbol{v}}\cdot\frac{d\vec{\boldsymbol{v}}}{dt}\right).$$

For linear motion, $\vec{\boldsymbol{v}}$ and $d\vec{\boldsymbol{v}}/dt$ are parallel vectors. Thus,

$$\vec{\boldsymbol{v}}\left(\vec{\boldsymbol{v}}\cdot\frac{d\vec{\boldsymbol{v}}}{dt}\right) = v^2\frac{d\vec{\boldsymbol{v}}}{dt}.$$

Inserting this result back into eq. (18) yields,

$$\vec{\boldsymbol{F}} = \frac{d}{dt}(\gamma m\vec{\boldsymbol{v}}) = \gamma m\frac{d\vec{\boldsymbol{v}}}{dt}\left(1 + \frac{v^2/c^2}{1 - v^2/c^2}\right) = \gamma^3 m\frac{d\vec{\boldsymbol{v}}}{dt}. \tag{19}$$

Here, we have rederived a result previously obtained in eq. (25) of the class handout entitled *Examples of four-vectors.*

Using $\vec{\boldsymbol{F}} = -\eta\vec{\boldsymbol{v}}$, we can solve eq. (19) for the acceleration $\vec{\boldsymbol{a}} \equiv d\vec{\boldsymbol{v}}/dt$,

$$\vec{\boldsymbol{a}} = -\frac{\eta}{\gamma^3 m}\vec{\boldsymbol{v}}.$$

If we denote $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{v}}$ and $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{v}}$ is a unit vector in the direction of the motion, then

$$a = -\frac{\eta v}{\gamma^3 m}, \tag{20}$$

which indicates that the particle is *decelerating.*

(b) Determine the angular distribution of the instantaneous power radiated once the particle has entered the medium and slowed down to a velocity v. The polar and azimuthal angles of the emitted radiation are defined relative to the z-axis which lies along the direction of the particle velocity. In your calculation, you may neglect the effect of the medium on the emitted radiation (*i.e.*, you should treat the radiation as if it were emitted in the vacuum.)

Using the relativistic Larmor formula for an accelerating charge in linear motion given in eq. (14.39) on p. 669 of Jackson,[2]

$$\frac{dP}{d\Omega} = \frac{e^2a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}, \tag{21}$$

where θ is the direction of emitted radiation and $\beta \equiv v/c$. Inserting the result for the acceleration a obtained in eq. (20) into eq. (21), one obtains

$$\frac{dP}{d\Omega} = \frac{e^2\eta^2v^2}{4\pi\gamma^6m^2c^3}\frac{1-\cos^2\theta}{(1-\beta\cos\theta)^5}. \tag{22}$$

[2] In this problem, the time t refers to the charge's own time. This is what Jackson denotes by $t' = t_{\rm ret}$ in section 3 of Chapter 14.

(c) How much energy is emitted in the form of electromagnetic radiation from the time the particle enters the medium until it slows down and reaches zero velocity? Express your answer in terms of the parameters e, m, c, η and v_0.

To obtain the total instantaneous power emitted by the particle after slowing down to a velocity v, we simply integrate eq. (22) over the polar and azimuthal angles,

$$P = \frac{e^2\eta^2 v^2}{4\pi\gamma^6 m^2 c^3}\, 2\pi \int_{-1}^{1} \frac{1-\cos^2\theta}{(1-\beta\cos\theta)^5}\, d\cos\theta \,.$$

Setting $x = \cos\theta$ and using

$$\int_{-1}^{1} \frac{1-x^2}{(1-\beta x)^5}\, dx = \frac{4}{3}\gamma^6 \,,$$

we end up with

$$P = \frac{2e^2\eta^2 v^2}{3m^2c^3} \,. \tag{23}$$

Note that the instantaneous radiated power integrated over all solid angles can be directly obtained from eq. (14.26) on p. 666 of Jackson. Since the velocity and acceleration are parallel, it follows that

$$P = \frac{2e^2a^2}{3c^3}\gamma^6 \,.$$

Using eq. (20) for the acceleration a in the above formula, we immediately recover eq. (23).

The power is defined by $P = dE/dt$. It is convenient to rewrite dt in terms of dv^2. Note that

$$dv^2 = d(\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}) = 2\,\vec{\boldsymbol{v}}\cdot d\vec{\boldsymbol{v}} = 2\vec{\boldsymbol{v}}\cdot\frac{d\vec{\boldsymbol{v}}}{dt}\,dt = 2\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}\,dt \,.$$

Using eq. (20), it follows that

$$dv^2 = -\frac{2\eta v^2}{m\gamma^3}\,dt \,.$$

or equivalently,

$$v^2 dt = -\frac{m\gamma^3}{2\eta} dv^2 \,.$$

Writing $\gamma^3 = (1 - v^2/c^2)^{-3/2}$, it follows that

$$E = \int P\,dt = \frac{2e^2\eta^2}{3m^2c^3}\int v^2 dt = -\frac{e^2\eta}{3mc^3}\int_{v_0^2}^{0} \frac{dv^2}{(1-v^2/c^2)^{3/2}} \,,$$

where $v_0 \equiv v(t=0)$. The last integral is elementary, and the final result is,

$$E = \frac{2e^2\eta}{3mc}(\gamma_0 - 1) \,,$$

where $\gamma_0 \equiv (1 - v_0^2/c^2)^{-1/2}$.

<u>*NOTE*</u>: All the formulae given in Chapter 14 of Jackson are given in gaussian units. To convert the results of Problem 3 to SI units, simply replace $e^2 \to e^2/(4\pi\epsilon_0)$ as previously noted below eq. (16).