We shall see that for $2^\ell$-pole radiation ($E\ell$ and $M_\ell$)

$$P \propto c k^{2\ell+2}.$$

---

M1 and E2 radiation due to general time-dependent sources (not necessarily harmonic)

Previously, we wrote down the expression for the radiation field,

$$\vec{B}_{\text{rad}}(\vec{x},t) = -\frac{1}{c^2 r}\,\hat{n}\times\frac{\partial}{\partial t}\int d^3x'\,\vec{J}\left(\vec{x}', t-\frac{|\vec{x}-\vec{x}'|}{c}\right)$$

We then expanded ($kd \ll 1$)

$$|\vec{x}-\vec{x}'| \simeq r - \vec{x}'\cdot\hat{n} + O\left(\frac{1}{r}\right) \qquad (\vec{x}\equiv r\hat{n})$$

$$J\left(\vec{x}', t-\frac{|\vec{x}-x'|}{c}\right) = \vec{J}\left(\vec{x}', t-\frac{r}{c}\right) + \frac{\vec{x}'\cdot\hat{n}}{c}\left.\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right|_{t'=t-\frac{r}{c}} + \ldots$$

yields E1 radiation

yields M1 and E2 radiation

Note that $$\left.\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right|_{t'=t-\frac{r}{c}} = \frac{\partial\vec{J}(\vec{x}',t-\frac{r}{c})}{\partial t}$$

since $r=|\vec{x}|$ is fixed.

So, we can use the same trick as in the last lecture by writing

$$\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\vec{x}'\cdot\hat{n} = \frac{1}{2}\left[\vec{x}'\cdot\hat{n}\,\vec{J} + \hat{n}\cdot\vec{J}\,\vec{x}'\right] - \frac{1}{2}\hat{n}\times(\vec{x}'\times\vec{J})$$

## For M1

$$\vec{B}_{M1} = \frac{1}{c^2 r} \hat{n} \times \left[ \hat{n} \times \frac{\partial^2}{\partial t^2} \vec{\mu}(t - \frac{r}{c}) \right]$$

$$\vec{E}_{M1} = \vec{B}_{M1} \times \hat{n}$$

$E1 \rightarrow M1$

interchanges $\vec{E}_{E1} \rightarrow \vec{B}_{M1}$ $\quad \vec{p} \longrightarrow \vec{\mu}$

$\vec{B}_{E1} \rightarrow -\vec{E}_{M1}$

$$P_{M1} = \frac{2}{3c^3} |\ddot{\vec{\mu}}|^2$$

$$P_{E1} = \frac{2}{3c^3} |\ddot{\vec{p}}|^2$$

## For E2

Use again

$$\vec{\nabla}_i' (x_k' x_\ell' J_i) = x_k' J_\ell + x_\ell' J_k + x_k' x_\ell' \vec{\nabla} \cdot \vec{J}$$

vanish when integrated over $d^3x'$

End result is:

$$\vec{B}_{E2} = -\frac{1}{6c^3 r} \hat{n} \times \frac{\partial^3}{\partial t^3} \vec{Q}(t - \frac{r}{c})$$

$$\vec{E}_{E2} = \vec{B}_{E2} \times \hat{n}$$

$\dot{f} \equiv \frac{\partial f}{\partial t}$

Total radiated power

$$P_{E2} = \frac{1}{180 c^5} \sum_{ij} |\dddot{Q}_{ij}|^2$$

Example: rotating charge $q$ in the x-y plane

$$\rho(\vec{x},t) = \frac{q}{r}\,\delta(r-R)\,\delta(z)\,\delta(\phi-\omega t)$$

$$\vec{p}(t) = \int d^3x\, \vec{x}\, \rho(\vec{x},t) \qquad \vec{x} = (r\cos\phi, r\sin\phi, z)$$

$$p_x(t) = q\int \frac{\delta(r-R)}{r}\,\delta(\phi-\omega t)\,\delta(z)\, r\cos\phi\; r\,dr\,d\phi\,dz$$

$$= qR\int_0^{2\pi} \delta(\phi-\omega t)\cos\phi\, d\phi$$

$$= qR\cos\omega t$$

Similarly,

$$p_y(t) = qR\sin\omega t$$

$$p_z(t) = 0$$

$$\ddot{p}_x = -qR\omega^2\cos\omega t$$

$$\ddot{p}_y = -qR\omega^2\sin\omega t$$

$$|\ddot{\vec{p}}|^2 = (\ddot{p}_x)^2 + (\ddot{p}_y)^2 + (\ddot{p}_z)^2 = q^2R^2\omega^4$$

$$P_{E1} = \frac{2|\ddot{\vec{p}}|^2}{3c^3} = \frac{2q^2R^2\omega^4}{3c^3} = \frac{2cq^2R^2k^4}{3}$$

after using $\omega = kc$

Using complex notation, we define the complex vector $\vec{p}$ such that

$$\vec{p}(t) = \mathrm{Re}\,(\vec{p}\,e^{-i\omega t})$$

It follows that

$$\vec{p} = qR(1, i, 0)$$

since

$$\vec{p}(t) = qR\left(\mathrm{Re}\,e^{-i\omega t}, \mathrm{Re}\,ie^{-i\omega t}, 0\right)$$

$$= qR(\cos\omega t, \sin\omega t, 0)$$

reproduces our previous result.

For harmonic sources, we derived previously

$$P_{E1} = \frac{ck^4}{3}|\vec{p}|^2$$

Inserting $\vec{p} = qR(1, i, 0)$

$$|\vec{p}|^2 = 2q^2R^2 \qquad (\text{since } |\vec{p}|^2 = |p_x|^2 + |p_y|^2 + |p_z|^2)$$

so that

$$P_{E1} = \frac{2cq^2R^2k^4}{3}$$

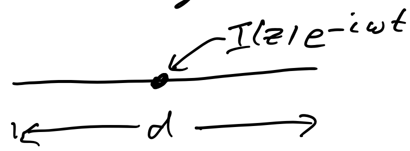
which reproduces our previous result.

# Centerfed linear antenna

in the electric dipole approximation

Antenna lies along the z-axis (length $d$)

$$|z| \leq \frac{d}{2}$$




$$\boxed{I(z)e^{-i\omega t} = I_0\left(1 - \frac{2|z|}{d}\right)e^{-i\omega t}}$$

"real" current is $Re\left[I(z)e^{-i\omega t}\right]$

Note: $I(z = \pm\frac{d}{2}) = 0$ "standing waves"

In the electric dipole approximation $kd \ll 1$

"radiation" zone: $d \ll \lambda \ll r$ ← distance to observer

Given:

$$\vec{J}(\vec{x}) = I_0\left(1 - \frac{2|z|}{d}\right)\delta(x)\delta(y)\hat{z}$$

$$I = \int \vec{J}\cdot d\vec{a} \qquad d\vec{a} = \hat{z}\,dx\,dy$$

For harmonic sources, $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} = i\omega\rho$$

$$\rho(\vec{x}) = \frac{-i}{\omega}\frac{dJ_z}{dz} = \frac{2iI_0}{\omega d}\,\varepsilon(z)\delta(x)\delta(y)$$

$$\varepsilon(z) = \operatorname{sgn} z = \begin{cases} +1, & z > 0 \\ -1, & z < 0 \end{cases} \qquad |z| = z\,\varepsilon(z)$$

$$\vec{p}(t) = \vec{p}\, e^{-i\omega t} \qquad \begin{matrix} x\delta(x) = 0 \\ y\delta(y) = 0 \end{matrix}$$

$$\vec{p} = \int \vec{x}\, \rho(\vec{x})\, d^3x = \int (x\hat{x} + y\hat{y} + z\hat{z})\, \rho(\vec{x})\, d^3x$$

$$= \hat{z} \int_{-\frac{d}{2}}^{\frac{d}{2}} \frac{2i I_0}{\omega d}\, \varepsilon(z)\, z\, dz$$

$$= \frac{4i I_0}{\omega d}\, \hat{z} \int_0^{d/2} z\, dz$$

$$\boxed{\vec{p} = \frac{i d I_0}{2\omega}\, \hat{z}} \qquad \omega = kc$$

$$P = \frac{1}{3} c k^4 |\vec{p}|^2 = \frac{I_0^2 (kd)^2}{12c}$$

<u>Radiation resistance</u> $R_{rad}$

$$P = R_{rad} \langle I^2 \rangle \qquad \langle I^2 \rangle = \langle (\mathrm{Re}\, I)^2 \rangle_{z=0}$$

$$= \frac{1}{2} I_0^2 R_{rad} \qquad = \frac{1}{2} I_0^2$$

$$\Rightarrow R_{rad} = \frac{(kd)^2}{6c} = 5 (kd)^2 \text{ ohms}$$

since $\langle \cos^2 \omega t \rangle = \frac{1}{2}$

# Radiated energy, momentum and angular momentum

Using results from Chapter 6 of Jackson.

The energy decrease in a volume $V$ is equal to the rate of energy flowing out of $V$ across a boundary $S$.

$$-\frac{dU_{total}}{dt} = \oint \vec{S}\cdot d\vec{a} \qquad d\vec{a} = \hat{n}\, r^2 d\Omega$$

$$U_{total} = \int (u_{fields} + u_{mech})\, d^3x$$

$$u_{fields} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \qquad \text{(cgs units)}$$

$$\frac{\partial u_{mech}}{\partial t} = \vec{J}\cdot\vec{E}$$

$$P = -\frac{dU_{total}}{dt}$$

$$\frac{dP}{d\Omega} = (\vec{S}\cdot\hat{n})\, r^2 \qquad \vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$$

Likewise,

$$-\frac{d\vec{P}_{total}}{dt} = -\oint T_{ij}\,\hat{n}_j\,da$$

yields the total radiated linear momentum

$-T_{ij}\,\hat{n}_j$ = $i^{th}$ component of the outward momentum flux through $S$

$T_{ij}$ = Maxwell stress tensor.

$$T_{ij} = \frac{1}{4\pi}\left[E_iE_j + B_iB_j - \frac{1}{2}\delta_{ij}\left(\vec{E}^2+\vec{B}^2\right)\right]$$

Finally

$da = r^2 d\Omega$

$$-\frac{d\vec{L}_{total}}{dt} = \oint M_{ij}\,\hat{n}_j\,da$$

where $M_{ij} = -\varepsilon_{ik\ell}\,x_k\,T_{\ell j}$

$M_{ij}\,\hat{n}_j$ = $i^{th}$ component of the outward angular momentum flux through $S$.

$$\overleftrightarrow{M} = \overleftrightarrow{T}\times\vec{x} = -\vec{x}\times\overleftrightarrow{T}$$

$$-M_{ij} = \varepsilon_{ijk}\, x_k\, T_{\ell j}$$

$$= \frac{1}{4\pi}\, \varepsilon_{i k \ell}\, x_k \left[ E_\ell E_j + B_\ell B_j - \frac{1}{2}\delta_{\ell j}\left[\vec{E}^2 + \vec{B}^2\right]\right]$$

$$M_{ij}\hat{n}_j = -\frac{1}{4\pi}\left\{ (\vec{x}\times\vec{E})\,\hat{n}\cdot\vec{E} + (\vec{x}\times\vec{B})\,\hat{n}\cdot\vec{B}\right\}$$

after using $\vec{x}\times\hat{n} = 0$ $\quad \vec{x} = r\hat{n}$

Hence,

$$\boxed{-\frac{d\vec{L}_{total}}{dt} = -\frac{1}{4\pi}\int r^3 d\Omega \left[ (\hat{n}\times\vec{E})\,\hat{n}\cdot\vec{E} + (\hat{n}\times\vec{B})\,\hat{n}\cdot\vec{B}\right]}$$

rate of angular momentum radiated

For complex fields,

$$\frac{dP}{d\Omega} = \frac{cr^2}{8\pi}\,\mathrm{Re}\,(\vec{E}\times\vec{B}^*)\cdot\hat{n}$$

extra factor of $\frac{1}{2}$ (and complex conjugate one of the two field)

Recall that for the radiation fields,

$$\hat{n}\cdot\vec{B}_{rad} = \hat{n}\cdot\vec{E}_{rad} = 0 \quad \text{at } O\left(\frac{1}{r}\right).$$

Jackson derives the electric dipole radiation fields from

$$\vec{A}(\vec{x},t) = \frac{1}{cr}e^{-i\omega t}\int d^3x'\,\vec{J}(\vec{x}')e^{ik|\vec{x}-\vec{x}'|} + O\left(\frac{1}{r^2}\right)$$

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}$$

<u>E1</u>: $e^{ik|\vec{x}-\vec{x}'|} = 1$

$$\vec{A}_{E1}(\vec{x}) = \frac{-ik}{r}e^{ikr}\vec{p} + O\left(\frac{1}{r^2}\right)$$

$$\vec{B}_{E1}(\vec{x}) = \vec{\nabla}\times A_{E1}(\vec{x})$$

$$= \frac{k^2}{r}e^{ikr}\,\hat{n}\times\vec{p}\left(1-\frac{1}{ikr}\right) + O\left(\frac{1}{r^3}\right)$$

$$\vec{\nabla}\times\vec{B} = -\frac{i\omega}{c}\vec{E} = -ik\vec{E} \quad \text{for harmonic sources}$$

$$\vec{E} = \frac{i}{k}\vec{\nabla}\times\vec{B}$$

Hence,

$$\vec{E}_{E1}(\vec{x}) = k^2 (\hat{n}\times\vec{p})\times\hat{n}\,\frac{e^{ikr}}{r} + \left[3\hat{n}(\hat{n}\cdot\vec{p})-\vec{p}\right]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr} + O\left(\frac{1}{r^4}\right)$$

Note: the formula

$\vec{E} = \vec{B}\times\hat{n}$ valid at $O\left(\frac{1}{r}\right)$ only.

Exercise: Show that to leading order in the $\frac{1}{r}$ expansion, where

$$\vec{\tau} \equiv -\frac{d\vec{L}_{total}}{dt}$$

Show that $\vec{E} = \vec{B}\times\hat{n} + O\left(\frac{1}{r^2}\right)$, $\vec{B} = \hat{n}\times\vec{E} + O\left(\frac{1}{r^2}\right)$, and

$$\frac{d\vec{\tau}}{d\Omega} = \frac{r^2}{4\pi}\,\vec{x}\times(\vec{E}\times\vec{B}) + O\left(\frac{1}{r}\right).$$

$\hat{n} = \vec{x}/r$

$\frac{1}{4\pi c}\,\vec{x}\times(\vec{E}\times\vec{B})$ = angular momentum density

# Multipole Expansion of Radiation Fields

Harmonic fields

$$\vec{E}(\vec{x},t) = \vec{E}(\vec{x})e^{-i\omega t}$$
$$\vec{B}(\vec{x},t) = \vec{B}(\vec{x})e^{-i\omega t}$$

In the radiation zone, $\vec{J} = \rho = 0$.

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = ik\vec{B}$$
$$\vec{\nabla}\times\vec{B} = -ik\vec{E}$$

$k = \frac{\omega}{c}$

Eliminate $\vec{E}$

$$\vec{\nabla}\times\frac{i}{k}(\vec{\nabla}\times\vec{B}) = ik\vec{B}$$

$$\boxed{(\vec{\nabla}^2 + k^2)\vec{B}(\vec{x}) = 0}$$

homogeneous Helmholtz equations

Eliminate $\vec{B}$

$$\boxed{(\vec{\nabla}^2 + k^2)\vec{E}(\vec{x}) = 0}$$

Theorem: Let $\vec{F}$ be any vector field satisfying $\vec{\nabla}\cdot\vec{F}=0$. Then there exist scalar functions $\psi, \chi$ such that

$$\vec{F} = \vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi$$

where

$\psi, \chi$ = Debye potentials

$$\vec{L} = -i\vec{x}\times\vec{\nabla} = i\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

$\psi, \chi$ are unique up to

arbitrary radial function

$$\psi(\vec{x}) \rightarrow \psi(\vec{x}) + f(r) \qquad r \equiv |\vec{x}|$$

$$\chi(\vec{x}) \longrightarrow \chi(\vec{x}) + g(r)$$

Note that

$$\vec{L}^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1)Y_{\ell m}(\theta,\phi)$$

Operator identities

$$\vec{L}\cdot(\vec{\nabla}\times\vec{L}) = 0 \qquad \vec{\nabla}\cdot\vec{L} = 0$$

$$\vec{x}\cdot\vec{L} = 0 \qquad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{L}) = 0$$

$$\vec{x}\cdot(\vec{\nabla}\times\vec{L}) = i\vec{L}^2 \qquad \vec{L}\,\vec{\nabla}^2 = \vec{\nabla}^2\vec{L}$$

These identities imply $\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot[\vec{L}\psi + (\vec{\nabla}\times\vec{L})\chi] = 0.$