<u>Example</u>: (1) oscillating charge q in the $\hat{z}$ direction

$$\vec{x}(t) = a\hat{z}\cos\omega t = \hat{z}\,\mathrm{Re}\, a e^{-i\omega t}$$

In complex notation

$$\vec{x}(t) = \hat{z}\, a e^{-i\omega t}$$

Recall

$$Q_{\ell m} = \int d^3x\, r^\ell\, Y^*_{\ell m}(\Omega)\, \rho(\vec{x})$$

electric dipole radiation ($\ell = 1$)

$$Q_{1m}(t) = \int d^3x\, r\, Y^*_{1m}(\Omega)\, q\, \delta(\vec{x} - \hat{z} a e^{-i\omega t})$$

$$= \sqrt{\frac{3}{4\pi}}\, q\, a\, e^{-i\omega t}\, \delta_{m0}$$

since $r Y^*_{10}(\Omega) = \sqrt{\frac{3}{4\pi}}\, z$ $\qquad$ $\hat{z} = (0,0,1)$

In the spherical basis, $\hat{z} = (0, \underset{m=0}{\underset{\uparrow}{1}}, 0)$

$$\Longrightarrow \quad Q_{1m} = \sqrt{\frac{3}{4\pi}}\, q a\, \delta_{m0}$$

(2) oscillating charge q rotating counterclockwise in the x-y plane (radius = $\frac{a}{\sqrt{2}}$)

$$\vec{x}(t) = \frac{a}{\sqrt{2}}(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$$

$$= \frac{a}{\sqrt{2}}\,\mathrm{Re}\left[(\hat{x}+i\hat{y})e^{-i\omega t}\right]$$

Complex vector $\vec{x}(t) = \frac{a}{\sqrt{2}}(\hat{x}+i\hat{y})e^{-i\omega t}$

$$= \frac{a}{\sqrt{2}}(e^{-i\omega t})\,(1, i, 0)$$

$$\xrightarrow[\text{basis}]{\text{spherical}} \quad a(0, 0, \underbrace{1}_{m=-1})$$

$$Q_{1m} = -\sqrt{\frac{3}{4\pi}}\, q\, a\, \delta_{m1}$$

Case 1: $$\frac{dP^{E10}}{d\Omega} = \frac{4\pi c k^4}{9}|Q_{10}|^2 \underbrace{|\vec{X}_{10}|^2}_{\frac{3}{8\pi}\sin^2\theta}$$

$$= \frac{ck^4}{8\pi} q^2 a^2 \sin^2\theta$$

Case 2: $$\frac{dP^{E11}}{d\Omega} = \frac{ck^4}{16\pi} q^2 a^2 (1+\cos^2\theta)$$

since $$|\vec{X}_{11}|^2 = \frac{3}{16\pi}(1+\cos^2\theta)$$

$$P^{E10} = P^{E11} = \frac{ck^4}{3} q^2 a^2$$

<u>Beware</u> !!

In the case of the rotating charge

$$\rho(\vec{x},t) = \frac{q}{r^2}\delta(r-a)\,\delta(\cos\theta)\,\delta(\phi-\omega t)$$

$$Q_{1m}(t) = \int d^3x \; r\, Y^*_{1m}(\Omega)\, \rho(\vec{x},t)$$

$$= q a\, Y^*_{1m}\left(\frac{\pi}{2}, \omega t\right)$$

$$Q_{1,\pm 1}(t) = \mp\sqrt{\frac{3}{8\pi}}\, q a\, e^{\mp i\omega t}$$

$$Q_{10}(t) = 0$$

But $\rho(\vec{x},t) \neq \mathrm{Re}\left(\rho(\vec{x})e^{-i\omega t}\right)$

Lagrangian and Hamiltonian formulation of electrodynamics

Relativistic classical field theory

$$L = \int d^3x \, \mathcal{L}$$

$\mathcal{L}$ = Lagrange density (also inaccurately called the Lagrangian)

<u>action</u>

$$S = \int dt\, L = \frac{1}{c}\int d^4x\, \mathcal{L}$$

$$d^4x = d^3x\, dx_0 = c\, d^3x\, dt$$

d^4x is a Lorentz invariant

action S is Lorentz invariant

$\Longrightarrow$ $\mathcal{L}$ is Lorentz invariant

Field equation will be determined by minimizing the action S.

In classical mechanics $L = L(q_i, \dot{q}_i)$ $\quad i = 1, 2, \ldots, N$

In relativistic field theory,

fields $\phi(\vec{x}, t)$

$$\mathcal{L} = \mathcal{L}(\phi(\vec{x},t),\ \partial_\mu \phi(\vec{x},t))$$

$$S = S[\phi, \partial_\mu \phi] \qquad \text{functional}$$

$$\delta S[\phi] = 0$$

$$= \delta \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$= \int d^4x \; \delta \mathcal{L}(\phi, \partial_\mu \phi)$$

$$= \int d^4x \left[\mathcal{L}(\phi + \delta\phi, \partial_\mu \phi + \delta \partial_\mu \phi) - \mathcal{L}(\phi, \partial_\mu \phi) \right]$$

$$= \int d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta \partial_\mu \phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] + \ldots$$

Note: $\delta(\partial_\mu \phi) = \partial_\mu(\phi + \delta\phi) - \partial_\mu \phi$

$$= \partial_\mu(\delta\phi)$$

$$\delta S[\phi] = 0 = \int d^4x \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu(\delta\phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right]$$

$$0 = \int d^4x \, \delta\phi \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right]$$

Hence,

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}$$

Field equations (of motion)

What is $\mathcal{L}$ for electrodynamics?

1. Depend at most quadratically on the fields and its derivatives

   [$\Longrightarrow$ linear field equations (superposition)]

2. It must be gauge invariant [action]

3. It must be Lorentz invariant

[4. Depends at most quadratically on the number of derivatives]

For E&M, we have A_μ

$F_{\mu\nu}$

Only Lorentz invariant, gauge invariant quantities:

$$F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

subtlety

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu K^\mu$$

where $K^\mu = 2\varepsilon^{\mu\nu\alpha\beta}A_\nu \partial_\alpha A_\beta$

$$= \varepsilon^{\mu\nu\alpha\beta}A_\nu F_{\alpha\beta}$$

$$\int d^4x \, \partial_\mu K^\mu = 0$$

Conclusion

$$\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

note: minus sign to ensure that the extremum of the action is a minimum.

note: $1/16\pi$ defines the EM units (gaussian cgs)

[Remark: in quantum field theory (or particle physics) the standard units adopted are called rationalized cgs units (eliminates some factors of 4π).

e.g. $\vec{\nabla}\cdot\vec{E} = \rho$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$]

In terms of $\vec{E}$, $\vec{B}$ fields

$$\mathcal{L} = \frac{1}{8\pi}\left(\vec{E}^2 - \vec{B}^2\right)$$

In electrodynamics,

$$L = L_{particles} + L_{fields} + L_{int}$$

For the interactions

$$\mathcal{L}_{int} = -\frac{1}{c} J_\mu A^\mu$$

$$S_{int} = -\frac{1}{c^2}\int d^4x\, J_\mu A^\mu$$

Under a gauge transformation,

$$A^\mu \longrightarrow A^\mu - \partial^\mu \Lambda$$

$$S_{int} \to S_{int} + \frac{1}{c^2}\int d^4x\, J_\mu \partial^\mu \Lambda$$

$$= S_{int} - \frac{1}{c^2}\int d^4x\, \partial^\mu J_\mu \Lambda$$

$$= S_{int}$$

because $\partial^\mu J_\mu = 0$

Finally, examine

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}[A_\mu] = -\frac{1}{16\pi}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$-\frac{1}{c} J_\mu A^\mu$$

Applying the field equations

$$\boxed{\partial^\alpha F_{\alpha\beta} = \frac{4\pi}{c} J_\beta}$$

The other two Maxwell equations are satisfied by virtue of writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Longrightarrow \quad \partial^\mu \tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \partial^\mu F^{\alpha\beta} = 0$$

<u>Derivation of the field equations</u>

$$\mathcal{L} = -\frac{1}{16}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{c} J_\mu A^\mu$$

$$= -\frac{1}{8\pi}\left[(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)\right] - \frac{1}{c} J_\mu A^\mu$$

$$= -\frac{1}{8\pi} g_{\mu\alpha} g_{\nu\beta}\left[(\partial^\alpha A^\beta)(\partial^\mu A^\nu) - (\partial^\alpha A^\beta)(\partial^\nu A^\mu)\right] - \frac{1}{c} g_{\mu\alpha} J^\alpha A^\mu$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial A^\sigma} = -\frac{1}{c} g_{\mu\alpha} J^\alpha \delta^\mu_\sigma = -\frac{1}{c} g_{\sigma\beta} J^\alpha = -\frac{1}{c} J_\sigma$$

$$-8\pi \frac{\partial \mathcal{L}}{\partial(\partial^\rho A^\sigma)} = g_{\mu\alpha} g_{\nu\beta} \Big[\delta^\alpha_\rho \delta^\beta_\sigma (\partial^\mu A^\nu) + \delta^\mu_\rho \delta^\nu_\sigma (\partial^\alpha A^\beta) - \delta^\alpha_\rho \delta^\beta_\sigma (\partial^\nu A^\mu) - \delta^\nu_\rho \delta^\mu_\sigma (\partial^\alpha A^\beta) \Big]$$

$$= 2(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$= 2F_{\rho\sigma}$$

Thus,

$$\partial^\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial^\rho A^\sigma)} \right) = \frac{\partial \mathcal{L}}{\partial A^\sigma}$$

yields

$$\partial^\rho F_{\rho\sigma} = \frac{4\pi}{c} J_\sigma$$

$$\delta S = 0 \qquad S = \int d^4x\, \mathcal{L}$$

$$\begin{aligned}
\delta(F_{\mu\nu}F^{\mu\nu}) &= 2F_{\mu\nu}\,\delta F^{\mu\nu} \\
&= 2F_{\mu\nu}\,\delta(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= 4F_{\mu\nu}\,\delta(\partial^\mu A^\nu)
\end{aligned}$$

since
$F_{\mu\nu} = -F_{\nu\mu}$

$$= 4F_{\mu\nu}\,\partial^\mu(\delta A^\nu)$$

$$= 4\,\partial^\mu(F_{\mu\nu}\,\delta A^\nu) - 4(\partial^\mu F_{\mu\nu})\,\delta A^\nu$$

$$\delta \mathcal{L} = -\frac{1}{4\pi}\partial^\mu(F_{\mu\nu}\,\partial A^\nu) + \frac{1}{4\pi}\delta A^\nu\left[\partial^\mu F_{\mu\nu} - \frac{4\pi}{c}J_\nu\right]$$

$$\delta S = \frac{1}{4\pi}\int \delta A^\nu\left[\partial^\mu F_{\mu\nu} - \frac{4\pi}{c}J_\nu\right] = 0$$

$$\Longrightarrow \quad \partial^\mu F_{\mu\nu} = \frac{4\pi}{c}J_\nu$$

two dynamical Maxwell equations

Two other Maxwell equations

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

two kinematical Maxwell equations

automatically satisfied

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$0 = \varepsilon^{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}\partial_\mu(\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

since $\varepsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\alpha = 0 = \varepsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\beta$