Field equations

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu$$

Using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\mathcal{L} = -\frac{1}{8\pi} g_{\mu\alpha} g_{\nu\beta} \left[(\partial^\alpha A^\beta)(\partial^\mu A^\nu) - (\partial^\alpha A^\beta)(\partial^\nu A^\mu) \right]$$
$$- \frac{1}{c} g_{\mu\alpha} J^\alpha A^\mu$$

Lagrange field equations:

$$\frac{\partial \mathcal{L}}{\partial A^\rho} = \partial^\tau \left(\frac{\partial \mathcal{L}}{\partial(\partial^\tau A^\rho)} \right)$$

field degrees of freedom: A^ρ

$$\Longrightarrow \quad \boxed{\partial^\tau F_{\tau\rho} = \frac{4\pi}{c} J_\rho}$$

$$\delta S = 0 \qquad S = \int d^4x\, \mathcal{L}$$

$$\delta(F_{\mu\nu} F^{\mu\nu}) = 2 F_{\mu\nu}\, \delta F^{\mu\nu}$$
$$= 2 F_{\mu\nu}\, \delta(\partial^\mu A^\nu - \partial^\nu A^\mu)$$
$$= 4 F_{\mu\nu}\, \delta(\partial^\mu A^\nu)$$

since $F_{\mu\nu} = -F_{\nu\mu}$

$$= 4 F_{\mu\nu}\, \partial^\mu (\delta A^\nu)$$

$$= 4\, \partial^\mu (F_{\mu\nu}\, \delta A^\nu) - 4 (\partial^\mu F_{\mu\nu})\, \delta A^\nu$$

$$\delta \mathcal{L} = -\frac{1}{4\pi} \partial^\mu (F_{\mu\nu}\, \partial A^\nu) + \frac{1}{4\pi} \delta A^\nu \left[\partial^\mu F_{\mu\nu} - \frac{4\pi}{c} J_\nu \right]$$

$$\delta S = \frac{1}{4\pi} \int \delta A^\nu \left[\partial^\mu F_{\mu\nu} - \frac{4\pi}{c} J_\nu \right] = 0$$

$$\Longrightarrow \quad \partial^\mu F_{\mu\nu} = \frac{4\pi}{c} J_\nu$$

two dynamical Maxwell equations

Two other Maxwell equations

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

two kinematical Maxwell equations

automatically satisfied

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$0 = \varepsilon^{\mu\nu\alpha\beta} \partial_\mu F_{\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta} \partial_\mu (\partial_\alpha A_\beta - \partial_\beta A_\alpha)$$

since $\varepsilon^{\mu\nu\alpha\beta} \partial_\mu \partial_\alpha = 0 = \varepsilon^{\mu\nu\alpha\beta} \partial_\mu \partial_\beta$

Lagrangian of point particle dynamics (relativistic version).

Only possible action that is relativistically invariant is

$$S = -mc^2 \int d\tau$$

τ = proper time

$-mc^2$: constant

$$= -mc^2 \int \frac{d\tau}{dt}\, dt$$

$$S = \int L\, dt$$

$$= -mc^2 \int \gamma^{-1}\, dt$$

$$= -mc^2 \int \sqrt{1 - \frac{v^2}{c^2}}\, dt$$

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$$

(remark: γL is Lorentz invariant)

check nonrelativistic limit

$$L = -mc^2 + \frac{1}{2} m v^2$$

Momentum

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$H = \vec{p}\cdot\vec{v} - L = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{\vec{p}^2c^2 + m^2c^4}$$

Recall $L = L(\vec{x}, \vec{v})$ $\quad \vec{v} = \dot{\vec{x}}$

$H = H(\vec{x}, \vec{p})$

$$\boxed{\begin{array}{l} \mathcal{L}_{int} = -\frac{1}{c} J_\mu A^\mu \\ L_{int} = \int d^3y\, \mathcal{L}_{int} \end{array}}$$

$$\gamma L_{int} = -\frac{1}{c}\int d^3y\, \gamma J_\mu A^\mu$$

For point particles with charge q.

$$(\vec{J} = \rho\vec{v})$$

$$\gamma J_\mu(\vec{x}) = q u_\mu \delta^3(\vec{x}-\vec{y})$$

$$L_{int} = -\frac{q}{\gamma c} u^\mu A_\mu$$

$u^\mu = (\gamma c; \gamma\vec{v})$

$A^\mu = (\Phi, A^\mu)$

$$= -q\Phi + \frac{q}{c}\vec{v}\cdot\vec{A}$$

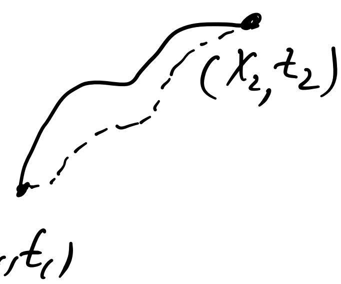
Aside:

$$L_{int} = -\frac{q}{\gamma c}\frac{dx^\mu}{d\tau} A_\mu \qquad d\tau = \frac{dt}{\gamma}$$

$$= -\frac{q}{c}\frac{dx^\mu}{dt} A_\mu$$

$$S_{int} = \int L_{int}\, dt = -\frac{q}{c}\int A_\mu dx^\mu$$

Under a gauge transformation



$$A_\mu \rightarrow A_\mu - \partial_\mu \Lambda(x)$$

$$S_{int} \rightarrow S_{int} + \frac{q}{c}\int_{x_1}^{x_2} \partial_\mu \Lambda(x)\, dx^\mu$$

$$= S_{int} + \frac{q}{c}\left[\Lambda(x_2) - \Lambda(x_1)\right]$$

independent of the extremum path

determines the equations of motion.

Conclusion:

Equations of motion are gauge-independent!

The Lagrangian for a point particle in an electromagnetic field (charge $= q$)

$$\boxed{L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - q\Phi + \frac{q}{c}\vec{v}\cdot\vec{A}}$$

Step I: determine the "canonical momentum" $\vec{P}$

$$\vec{P} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{q}{c}\vec{A}$$

$$= \vec{p} + \frac{q}{c}\vec{A}$$

$\vec{p} =$ "mechanical" momentum

$$H = \vec{P}\cdot\vec{v} - L$$

$$\boxed{H = c\sqrt{\left(P - \frac{q\vec{A}}{c}\right)^2 + m^2c^2} + q\Phi(\vec{x})}$$

Principle of minimal substitution

$$p^\mu \longrightarrow P^\mu - \frac{q}{c}A^\mu$$

In the non-relativistic limit:

$$H = \frac{\left(\vec{P} - \frac{q\vec{A}}{c}\right)^2}{2m} + q\Phi(\vec{x})$$

<u>The equations of motion</u>

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = \frac{\partial L}{\partial \vec{x}}$$

$$\frac{\partial L}{\partial \vec{v}} = \vec{P} = \vec{p} + \frac{q}{c}\vec{A}$$

$$\frac{\partial L}{\partial \vec{x}} = -q\vec{\nabla}\Phi + \frac{q}{c}\vec{\nabla}(\vec{v}\cdot\vec{A})$$

at fixed $\vec{v}$

$$\vec{\nabla}(\vec{v}\cdot\vec{A}) = \vec{v}\cdot\vec{\nabla}\vec{A} + \vec{v}\times(\vec{\nabla}\times\vec{A})$$

$$= \frac{d\vec{A}}{dt} - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times\vec{B}$$

since the chain rule states that

$$\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \sum_i \frac{\partial\vec{A}}{\partial x_i}\frac{d\vec{x}_i}{dt}$$

$$= \frac{\partial\vec{A}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{A}$$

Hence the equations of motion are:

$$\frac{d\vec{p}}{dt} + \frac{q}{c}\frac{d\vec{A}}{dt} = q\left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) + \frac{q}{c}\left[\frac{d\vec{A}}{dt} - \frac{\partial \vec{A}}{\partial t} + \vec{v}\times\vec{B}\right]$$

$$\boxed{\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)}$$

From $E^2 = c^2\vec{p}^2 + m^2c^4$

$$E\frac{dE}{dt} = c^2\vec{p}\cdot\frac{d\vec{p}}{dt}$$

Use $\vec{v} = \dfrac{c^2\vec{p}}{E}$

$$\Rightarrow \frac{dE}{dt} = \vec{v}\cdot\frac{d\vec{p}}{dt} \qquad \vec{F} = \frac{d\vec{p}}{dt}$$

$$= q\vec{v}\cdot\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

$$\boxed{\frac{dE}{dt} = q\vec{v}\cdot\vec{E}}$$

That is, $\dfrac{dp^\mu}{d\tau} = \dfrac{q}{c}F^{\mu\nu}u_\nu$

Dynamics of charge particles in an electromagnetic field

1. Uniform static magnetic field (relativistic)

Equations of motion:

$$\frac{d\vec{p}}{dt} = e\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

e=charge

$$\frac{dE}{dt} = e\vec{v} \cdot \vec{E}$$

For $\vec{E} = 0 \implies \frac{dE}{dt} = 0$ (energy is constant in time)

$E = \gamma m c^2 \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \implies |\vec{v}|$ is constant

$v \equiv |\vec{v}|$

Define $\boxed{\vec{\omega}_B = \frac{e\vec{B}}{\gamma m c} = \frac{ec\vec{B}}{E}}$

$\vec{p} = \gamma m \vec{v}$

$$\Longrightarrow \boxed{\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B}$$

$\vec{v} \times \vec{\omega}_B$ is perpendicular to $\vec{v}$

Take $\vec{B} = B\hat{z} \Longrightarrow \vec{\omega}_B = \omega_B \hat{z}$

$$\vec{v} \times \vec{\omega}_B = (\hat{x} v_y - \hat{y} v_x)\,\omega_B$$

$$\dot{v}_x = \omega_B v_y \qquad \dot{v} \equiv \frac{dv}{dt}$$

$$\dot{v}_y = -\omega_B v_x$$

$$\dot{v}_z = 0 \qquad v_\parallel \equiv v_z$$

Solve using the "complex" velocity
(physical velocity is the real part)

$$\zeta \equiv v_x + i v_y$$

$$\Longrightarrow \dot{\zeta} = -i\omega_B \zeta$$

$$\zeta(t) = \omega_B a\, e^{-i(\omega_B t + \alpha)}$$

Hence,

$$\vec{v}(t) = v_\parallel \hat{z} + \omega_B a (\hat{x} - i\hat{y})\, e^{-i(\omega_B t + \alpha)}$$

Integrate to get

$$\vec{x}(t) = \vec{r}_0 + v_\parallel t\, \hat{z} + i a (\hat{x} - i\hat{y})\, e^{-i(\omega_B t + \alpha)}$$

Take real parts

$$x = x_0 + a\sin(\omega_B t + \alpha)$$

$$y = y_0 + a\cos(\omega_B t + \alpha)$$

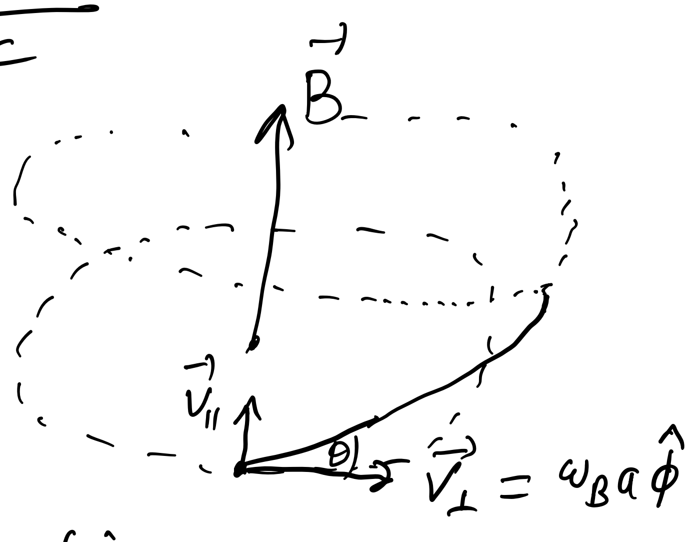
$$z = z_0 + v_{\parallel} t$$

Note: $(x-x_0)^2 + (y-y_0)^2 = a^2$

$$v_\perp = \sqrt{v_x^2 + v_y^2} = \omega_B a = \frac{eBa}{\gamma mc}$$

$$p_\perp = \gamma m v_\perp = \frac{eBa}{c}$$

$$\tan\theta = \frac{v_\parallel}{\omega_B a}$$



helical motion

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2. Crossed $\vec{E}$ and $\vec{B}$ fields (static uniform)

$\hookleftarrow$ $\vec{E}\cdot\vec{B} = 0$

$\frac{dE}{dt} \neq 0$

Method: transform to a new reference frame K' such that either $\vec{E}' = 0$ or $\vec{B}' = 0$.

The frame K' moves with velocity $c\vec{\beta}$ with respect to the frame K

Now, we solve

$$\frac{d\vec{p}'}{dt'} = e\left(\vec{E}' + \frac{\vec{v}'}{c} \times \vec{B}'\right)$$

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{1+\gamma} \vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{1+\gamma} \vec{\beta}(\vec{\beta} \cdot \vec{B})$$

<u>Case 1:</u> $|\vec{E}| < |\vec{B}|$

Then I will choose $\vec{\beta} = \frac{\vec{E} \times \vec{B}}{|\vec{B}|^2}$

Note $\beta \equiv |\vec{\beta}| < 1$

$$\Longrightarrow \vec{E}' = 0$$

$$\vec{B}' = \gamma^{-1} \vec{B}$$

When you return to frame K, there is an additional uniform drift (called the $\vec{E}\times\vec{B}$ drift) in the direction of $\vec{\beta}$ (which is $\perp$ to $\vec{E},\vec{B}$ fields). The drift is independent of the electric charge.

Case 2: $|\vec{E}| > |\vec{B}|$

Choose $\vec{\beta} = \dfrac{\vec{E}\times\vec{B}}{|\vec{E}|^2}$ $\quad 0 \le \beta < 1$

$$\Longrightarrow \quad \vec{B}' = 0$$
$$\vec{E}' = \gamma^{-1}\vec{E}$$

Case 3: $(|\vec{E}| = |\vec{B}|)$

Solve either Case 1 or Case 2 for $\vec{v}(t)$ and $\vec{x}(t)$, and then take limit of $|\vec{E}| \to |\vec{B}|$.

Dynamics of a charged particle with intrinsic spin

Charged particles such as the electron or proton are also point magnetic dipole.

Classically,

$$\vec{m} = \frac{1}{2c} \int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x'$$

For a point particle of charge q

$$\vec{J} = q\vec{v}\,\delta^3(\vec{x}-\vec{x}')$$

$$\Longrightarrow \quad \vec{m} = \frac{q}{2c}\,\vec{x}\times\vec{v} = \frac{q}{2mc}\vec{L}$$

(non-relativistic)

For particles with spin,

$$\vec{m} = \frac{q}{2mc}(\vec{L} + g\vec{S})$$

g-factor

In the rest frame ($\vec{v}=0$), for $q=e$

$$\boxed{\vec{m} = \frac{ge}{2mc}\vec{S}}$$

In an external magnetic field $\vec{B}$,

torque $\vec{N}$

$$\vec{N} = \frac{d\vec{S}}{dt} = \vec{m} \times \vec{B}$$

$$\Longrightarrow$$

$$\boxed{\frac{d\vec{S}}{dt} = \frac{ge}{2mc} \vec{S} \times \vec{B} \quad \text{in the rest frame}}$$

How does this generalize to an arbitrary reference frame?

We shall introduce a spin four vector S^μ

$$S^\mu = (S^0; \vec{S})$$

such that

$$S \cdot p = 0$$

[so in the rest frame

$$S^\mu = (0; \vec{S})]$$

$$p^\mu = (mc; \vec{0})$$