$$\vec{E}(\vec{x},t) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} \left[ E_0(\vec{k},\lambda)\,\hat{\varepsilon}_\lambda(\vec{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} + c.c. \right]$$

$\lambda$ runs over two polarization states

$\vec{E}$ is a real field

$$\hat{\varepsilon}_\lambda(-\vec{k}) = \hat{\varepsilon}_\lambda^{*}(\vec{k})$$

$$\vec{\nabla}\cdot\vec{E} = 0 \implies \vec{k}\cdot\hat{\varepsilon}_\lambda(\vec{k}) = 0$$

<u>Examples</u>

(i) linear polarization

e.g. $\hat{k} = (0,0,1) = \hat{\varepsilon}_0$ not a polarization vector

$\hat{\varepsilon}_x = (1,0,0)$

$\hat{\varepsilon}_y = (0,1,0)$

$\vec{k} = \hat{k}\,|\vec{k}|$ wave number of mode

(ii) circular polarization

e.g. $\hat{k} = (0,0,1) = \hat{\varepsilon}_0$

$$\hat{\varepsilon}_\pm = \mp\frac{1}{\sqrt{2}}(\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y)$$

$$\hat{\varepsilon}_m^{*}(k)\cdot\hat{\varepsilon}_{m'}(k) = \delta_{mm'}$$

$$\hat{\varepsilon}_m^{*}(k) = (-1)^m\,\hat{\varepsilon}_{-m}(k)$$

The $\pm$ convention is consistent with phase convention of $Y_{1m}(\theta,\phi)$

$\hat{\varepsilon}_+$ positive helicity (left-circular polarized)

$\hat{\varepsilon}_-$ negative helicity (right-circular polarized)

"optics convention"

Complex field

$$\vec{E}(\vec{x},t) = \mp E_0 (\hat{\varepsilon}_x \pm i \hat{\varepsilon}_y) e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \quad \vec{k} = k\hat{z}$$

$(E_0 \text{ real})$

physical field: $\mathrm{Re}\,\vec{E}$

$$\mathrm{Re}\, E_x(\vec{x},t) = \mp E_0 \cos(kz-\omega t)$$

$$\mathrm{Re}\, E_y(\vec{x},t) = E_0 \sin(kz-\omega t)$$

$$E_0^2 = [\mathrm{Re}\,E_x]^2 + [\mathrm{Re}\,E_y]^2$$

$\Rightarrow$ $\mathrm{Re}\,\vec{E}$ traces out a circle

upper sign: clockwise rotation when viewing an approaching wave (right)

lower sign: anticlockwise rotation (left)

optics convention

(iii) elliptic polarization

$$\vec{E}_0(\vec{k}) = \sum_\lambda E_0(\vec{k},\lambda)\, \hat{\varepsilon}_\lambda(\vec{k})$$

$$= \hat{\varepsilon}_x a_1 e^{i\delta_1} + \hat{\varepsilon}_y a_2 e^{i\delta_2}$$

(where $a_1, a_2$ are real and non-negative)

$\delta_1, \delta_2$ are real

$$\vec{E} = \mathrm{Re}\left(\vec{E}_0\, e^{i(kz-\omega t)}\right) \quad \text{where } \phi = \omega t - kz$$

$$E_x = a_1 \cos(\phi - \delta_1)$$

$$E_y = a_2 \cos(\phi - \delta_2)$$

$$\cos^{-1} A - \cos^{-1} B = \mp \cos^{-1}\left(AB + \sqrt{1-A^2}\sqrt{1-B^2}\right)$$

$$\begin{array}{ll} - & A \geq B \\ + & A < B \end{array}$$

<u>exercise:</u>

$$\frac{E_x^2}{a_1^2} + \frac{E_y^2}{a_2^2} - \frac{2E_xE_y}{a_1a_2}\cos\delta = \sin^2\delta$$

$\vec{E}$ traces out an ellipse. $\qquad \delta \equiv \delta_2 - \delta_1$

$\{a_1, a_2, \delta\}$ are related to the Stokes parameters ($S_i$)

$$S_0 = a_1^2 + a_2^2 \qquad S_2 = 2a_1a_2\cos\delta$$

$$S_1 = a_1^2 - a_2^2 \qquad S_3 = 2a_1a_2\sin\delta$$

note: $S_0^2 = S_1^2 + S_2^2 + S_3^2$.

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complex fields

$$\vec{E}(\vec{x},t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3} E_0(\vec{k},\lambda)\,\hat{\varepsilon}_\lambda(\vec{k})\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

energy density averaged over a cycle (cgs units)

$$\langle u \rangle = \frac{1}{16\pi}\left[\vec{E}\cdot\vec{E}^* + \vec{B}\cdot\vec{B}^*\right] \qquad \text{in vacuum}$$

Poynting vector

$$\langle \vec{S}_c \rangle = \frac{c}{8\pi} \vec{E} \times \vec{B}^* \quad , \quad \langle \vec{S} \rangle = Re \langle \vec{S}_c \rangle$$

complex notation

$$\vec{A}(t) = \vec{A}_0 e^{-i\omega t}$$

$$\vec{B}(t) = \vec{B}_0 e^{-i\omega t}$$

$$Re\, \vec{A}(t) \cdot Re\, \vec{B}(t) =$$

$$\frac{1}{4} \left( \vec{A}_0 e^{-i\omega t} + \vec{A}_0^* e^{i\omega t} \right) \cdot \left( \vec{B}_0 e^{-i\omega t} + \vec{B}_0 e^{i\omega t} \right)$$

$$= \frac{1}{4} \left( \vec{A}_0 \cdot \vec{B}_0^* + \vec{A}_0^* \cdot \vec{B}_0 \right) + \frac{1}{4} \left( \vec{A}_0 \cdot \vec{B}_0 e^{-2i\omega t} + c.c. \right)$$

$$\langle Re\, \vec{A}(t) \cdot Re\, \vec{B}(t) \rangle = \frac{1}{2} Re \left( \vec{A} \cdot \vec{B}^* \right)$$

since $\langle e^{-2i\omega t} \rangle = \langle e^{2i\omega t} \rangle = 0.$

special case: $\vec{A} = \vec{B}$

$$\langle (Re\, \vec{A})^2 \rangle = \frac{1}{2} \vec{A} \cdot \vec{A}^*$$

# Wave propagation in a good conductor

Conductor: homogeneous, isotropic and respects Ohm's law (cgs units)

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

(vacuum would correspond to $\varepsilon = \mu = 1$)

$$\vec{J} = \sigma \vec{E}$$

$\sigma$ = conductivity of conduction electrons

Maxwell's equation in medium

$$\vec{\nabla} \cdot \vec{D} = 4\pi \rho \qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

$\rho$ = "free" charge density

$\vec{J}$ = "free" current density

(continuity equation still holds)

Using Ohm's law

$$\vec{J} = \sigma \vec{E} = \frac{\sigma \vec{D}}{\varepsilon}$$

continuity equation: $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

$$\Longrightarrow \quad \vec{\nabla} \cdot \vec{J} = \frac{\sigma}{\varepsilon} \vec{\nabla} \cdot \vec{D} = \frac{4\pi \rho \sigma}{\varepsilon}$$

$$\Longrightarrow \quad \frac{\partial \rho}{\partial t} = -\frac{4\pi\sigma}{\varepsilon} \rho$$

$$\rho = \rho_0 e^{-\frac{4\pi\sigma}{\varepsilon} t} = \rho_0 e^{-t/\tau}$$

$$\tau \equiv \frac{\varepsilon}{4\pi\sigma} \qquad \text{relaxation time}$$

After transient effects have died out

$$\Longrightarrow \quad \rho = 0 \; !$$

as expected inside a good conductor

(ideal conductor $\sigma \to \infty$)

Now, we must solve:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \frac{1}{\mu} \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \sigma \vec{E} + \frac{\varepsilon}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$
$$\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$\omega$ is real
$\vec{k}$ is complex

(If $\omega$ had an imaginary part, then the wave would attenuate in time.)

I am interested in traveling waves that do not attenuate, so $\omega$ real.

$$\vec{k}\cdot\vec{E}_0 = \vec{k}\cdot\vec{B}_0 = 0$$
$$\vec{B}_0 = \frac{c}{\omega}\vec{k}\times\vec{E}_0$$

and

$$\frac{ic}{\mu\omega}\vec{k}\times(\vec{k}\times\vec{E}_0) + \frac{i\omega\varepsilon}{c}\vec{E}_0 = \frac{4\pi\sigma}{c}\vec{E}_0$$

Use $\vec{k}\times(\vec{k}\times\vec{E}_0) = \vec{k}(\vec{k}\cdot\vec{E}_0) - \vec{E}_0 k^2$
$$= -\vec{E}_0 k^2$$

to get

$$\frac{-ic}{\mu\omega}k^2 + i\omega\frac{\varepsilon}{c} = \frac{4\pi}{c}\sigma$$

$$\boxed{k^2 = \varepsilon\mu\frac{\omega^2}{c^2} + 4\pi i\sigma\frac{\omega\mu}{c^2}}$$

dispersion relation

Note that if $\sigma = 0$

$$k = \sqrt{\varepsilon\mu}\,\frac{\omega}{c}$$

$n = \sqrt{\varepsilon\mu}$ index of refraction

For good conductors, $\sigma \to \infty$
More precisely,

$$\frac{4\pi\sigma}{\omega\varepsilon} \gg 1$$

$$\Rightarrow \quad k^2 \simeq \frac{4\pi i \sigma\omega\mu}{c^2}$$

$$i = e^{i\pi/2}$$
$$i^{1/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

$$k = \frac{1+i}{\sqrt{2}}\sqrt{\frac{4\pi\sigma\omega\mu}{c^2}}$$

Definition: skin depth $\delta$

$$\delta = \frac{c}{\sqrt{2\pi\sigma\omega\mu}} \qquad \text{length scale}$$

[Note: in SI units, $\delta = \sqrt{\frac{2}{\omega\sigma\mu}}$ ]

$$k = \frac{1+i}{\delta}$$

$$e^{i\vec{k}\cdot\vec{x}} = e^{i\hat{k}\cdot\vec{x}/\delta}\, e^{-\hat{k}\cdot\vec{x}/\delta}\,, \quad \vec{k} = k\hat{k} = \left(\frac{1+i}{\delta}\right)\hat{k}$$

(For a good conductor, $\delta \to 0$)

$$\frac{\delta}{\lambda} = \delta\left(\frac{\omega}{2\pi c}\right) = \frac{1}{\pi}\left(\frac{\omega\varepsilon}{4\pi\sigma}\right)^{1/2}\left(\frac{1}{2\varepsilon\mu}\right)^{1/2}$$

But $\dfrac{\omega\varepsilon}{4\pi\sigma} \ll 1$, $\quad \varepsilon\mu \sim O(1)$

$$\Longrightarrow \quad \delta \ll \lambda .$$

Note: in SI units, $\dfrac{\delta}{\lambda} = \dfrac{1}{\pi}\left(\dfrac{\omega\varepsilon}{\sigma}\right)^{1/2}\left(\dfrac{\varepsilon_0\mu_0}{2\varepsilon\mu}\right)^{1/2}$

## Theory of waves in dispersive media

Recall that for static fields (homogeneous, isotropic)

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

For time varying fields,

$$\vec{D}(t) = \varepsilon \vec{E}(t) \qquad ?$$
$$\vec{B}(t) = \mu \vec{H}(t)$$

To maintain causality, you should not have the medium respond instantaneously to the time varying fields.

Keeping the assumption of linearity

$$\vec{D}(t) = \vec{E}(t) + \int_0^\infty d\tau\, f(\tau) E(t-\tau)$$

which builds in causality since $\vec{D}(t)$ depends on $\vec{E}(t')$ for $t' < t$ $\quad (t' = t - \tau)$

Rewrite this as

$$\vec{D}(t) = \vec{E}(t) + \int_{-\infty}^{\infty} d\tau\, f(\tau) E(t-\tau)$$

where $f(\tau) = 0$ for $\tau < 0$.

Next, let us expand $\vec{D}, \vec{E}$ in Fourier modes,

$$\vec{D}(\vec{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{D}(\vec{x},\omega) e^{-i\omega t} d\omega$$

$\boxed{\text{$D(\vec{x},t)$ is a real physical field}}$

$$\Rightarrow \quad \vec{D}(\vec{x},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{D}(\vec{x},t) e^{i\omega t} dt$$

Impose reality: $\vec{D}^*(\vec{x},t) = \vec{D}(\vec{x},t)$

$$\Rightarrow \quad \vec{D}(\vec{x},\omega) = \vec{D}^*(\vec{x},-\omega)$$

I now assert that

$$\boxed{\vec{D}(\vec{x},\omega) = \varepsilon(\omega) \vec{E}(\vec{x},\omega)}$$

$$D(\vec{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varepsilon(\omega) e^{-i\omega t} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{x},t') e^{i\omega t'} dt' \right) d\omega$$

Write $\varepsilon(\omega) = \varepsilon(\omega) - 1 + 1$

Use $\displaystyle \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega = \delta(t-t')$

Hence,

$$\vec{D}(\vec{x},t) = \vec{E}(\vec{x},t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt'$$

$$\times \left[ [\varepsilon(\omega) - 1]\, e^{-i\omega(t-t')}\, \vec{E}(\vec{x},t') \right]$$

Put $t' = t - \tau$

$$\vec{D}(\vec{x},t) = \vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau\, f(\tau)\, \vec{E}(\vec{x}, t-\tau)$$

where

$$f(\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} [\varepsilon(\omega)-1]\, e^{-i\omega\tau}\, d\omega$$

Invert the Fourier transform

$$\boxed{\varepsilon(\omega) = 1 + \int_{-\infty}^{\infty} f(\tau)\, e^{i\omega\tau}\, d\tau}$$

A simple model of dispersive medium

Consider an electron bound by a harmonic force (with damping) acted on by an oscillating electric field $\vec{E}(\vec{x},t) = \vec{E}_0(\vec{x})\, e^{-i\omega t}$