Theory of waves in dispersive media

Recall that for static fields (homogeneous, isotropic)

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

For time varying fields,

$$\vec{D}(t) = \varepsilon \vec{E}(t) \qquad ?$$
$$\vec{B}(t) = \mu \vec{H}(t)$$

To maintain causality, you should not have the medium respond instantaneously to the time varying fields.

Keeping the assumption of linearity

$$\vec{D}(t) = \vec{E}(t) + \int_0^\infty d\tau\, f(\tau) E(t-\tau)$$

which builds in causality since $\vec{D}(t)$ depends on $\vec{E}(t')$ for $t' < t$ $\quad (t' = t - \tau)$

Rewrite this as

$$\vec{D}(t) = \vec{E}(t) + \int_{-\infty}^\infty d\tau\, f(\tau) E(t-\tau)$$

where $f(\tau) = 0$ for $\tau < 0$.

Next, let us expand $\vec{D}, \vec{E}$ in Fourier modes,

$$\vec{D}(\vec{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{D}(\vec{x},\omega) e^{-i\omega t} d\omega$$

$D(\vec{x},t)$ is a real physical field

$$\Longrightarrow \quad \vec{D}(\vec{x},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{D}(\vec{x},t) e^{i\omega t} dt$$

Impose reality: $\vec{D}^*(\vec{x},t) = \vec{D}(\vec{x},t)$

$$\Longrightarrow \quad \vec{D}(\vec{x},\omega) = \vec{D}^*(\vec{x},-\omega)$$

I now assert that

$$\boxed{\vec{D}(\vec{x},\omega) = \varepsilon(\omega) \vec{E}(\vec{x},\omega)}$$

$$D(\vec{x},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varepsilon(\omega) e^{-i\omega t} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{x},t') e^{i\omega t'} dt' \right) d\omega$$

Write $\varepsilon(\omega) = \varepsilon(\omega) - 1 + 1$

Use $$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} d\omega = \delta(t-t')$$

Hence,

$$\vec{D}(\vec{x},t) = \vec{E}(\vec{x},t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt'$$

$$\times \left[[\varepsilon(\omega) - 1] e^{-i\omega(t-t')} \vec{E}(\vec{x},t') \right]$$

Put $t' = t - \tau$

$$\vec{D}(\vec{x},t) = \vec{E}(\vec{x},t) + \int_{-\infty}^{\infty} d\tau\, f(\tau)\, \vec{E}(\vec{x}, t-\tau)$$

where

$$f(\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} [\varepsilon(\omega) - 1] e^{-i\omega\tau} d\omega$$

Invert the Fourier transform

$$\boxed{\varepsilon(\omega) = 1 + \int_{-\infty}^{\infty} f(\tau) e^{i\omega\tau} d\tau}$$

A simple model of dispersive medium

Consider an electron bound by a harmonic force (with damping) acted on by an oscillating electric field $\vec{E}(\vec{x},t) = \vec{E}_0(\vec{x}) e^{-i\omega t}$

We must solve

$$m[\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x}] = e\vec{E}_0 e^{-i\omega t}$$

$\gamma > 0$ (damping)

$$\vec{x} = \vec{x}_0 e^{-i\omega t}$$

Then, $$m\vec{x}_0[-\omega^2 - i\gamma\omega + \omega_0^2] = e\vec{E}_0$$

$$\vec{x}_0 = \frac{e\vec{E}_0}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

electric dipole moment

$$\vec{p} = e\vec{x} = \frac{e^2\vec{E}}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

Let N = number of molecules per unit volume

$$\vec{P} = \frac{Ne^2\vec{E}}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \text{in cgs units}$$

$$\vec{D} = \left(1 + \frac{4\pi Ne^2}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}\right)\vec{E}$$

In this simple model,

$$\boxed{\varepsilon(\omega) = 1 + \frac{4\pi N e^2}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}}$$

Assume that $\mu = 1$. Then the index of refraction is

$$n(\omega) = \sqrt{\varepsilon(\omega)}$$

complex number

Dispersion relation

$$k = \sqrt{\varepsilon\mu}\,\frac{\omega}{c} = \frac{n\omega}{c}$$

Example: Consider a sheet of material made up of N charges per unit volume. Subject the sheet to $\vec{E}(\vec{x},t) = \vec{E}_0(\vec{x})e^{-i\omega t}$

You induce a current density

$$\vec{J} = \rho\vec{v} = Ne\frac{d\vec{x}}{dt} = -i\omega Ne\vec{x}_0 e^{-i\omega t}$$

$$= \frac{-i\omega Ne^2\vec{E}_0 e^{-i\omega t}}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

$$= \sigma\vec{E}$$

where

$$\sigma = \frac{-i\omega N e^2}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}$$

We have recovered Ohm's law $(\vec{J} = \sigma \vec{E})$
with a effective conductivity σ

$$k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi i \sigma}{\omega}\right) \quad (\text{for } \mu = 1)$$

$$\Rightarrow \quad n(\omega) = \left[1 + \frac{4\pi N e^2}{m(\omega_0^2 - \omega^2 - i\gamma\omega)}\right]^{1/2}$$

Note that for $\omega \to \infty$,

$$\varepsilon(\omega) \simeq 1 - \frac{4\pi N e^2}{m\omega^2} \to 1$$

Note that for $\omega \to 0$, $n(\omega) > 1$

Plasma frequency ω_p

$$\omega_p \equiv \frac{4\pi N e^2}{m}$$

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

We can now compute $f(\tau)$

$$f(\tau) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\gamma\omega} \, d\omega$$

$$\omega_0^2 - \omega^2 - i\gamma\omega = -(\omega-\omega_1)(\omega-\omega_2)$$

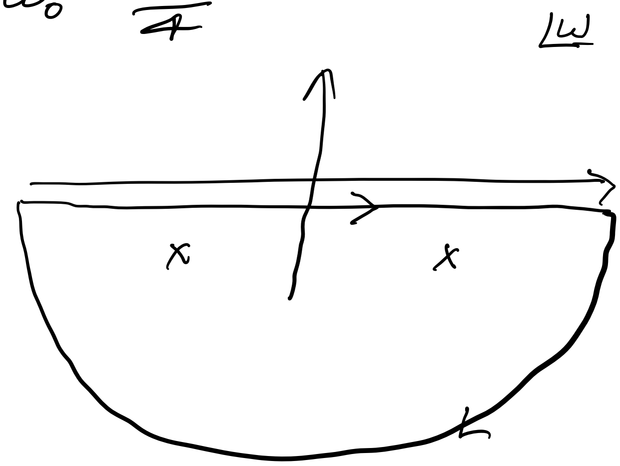
$$\omega_{1,2} = \frac{-i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

Note that

$\gamma > 0$

$\Longrightarrow$

$\mathrm{Im}\, \omega_{1,2} < 0$



if $\tau > 0$

$\Longrightarrow$

$$f(\tau) = -2\pi i \, \Theta(\tau) \frac{\omega_p^2}{2\pi} \left[\frac{e^{-i\omega_2\tau} - e^{-i\omega_1\tau}}{\omega_1 - \omega_2} \right]$$

$$\Theta(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$$

$$\boxed{f(\tau) = \Theta(\tau) e^{-\gamma\tau/2} \omega_p^2 \frac{\sin \nu_0 \tau}{\nu_0}, \quad \nu_0 \equiv \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}$$

$$\varepsilon(\omega) = 1 + \int_{-\infty}^{\infty} d\tau\, f(\tau)\, e^{i\omega\tau}$$

Causality $\Longrightarrow$ $f(\tau) = 0$ for $\tau < 0$.

Properties of $\varepsilon(\omega)$

1. $\varepsilon(-\omega) = \varepsilon^*(\omega^*)$

extended $\varepsilon(\omega)$ to a function in the complex ω plane (using the fact that $f(\tau)$ is a real function)

2. $\varepsilon(\omega)$ is analytic in the upper half complex ω-plane.

To see this, write $\omega = \omega_R + i\omega_I$

$$\varepsilon(\omega) = 1 + \int_0^{\infty} d\tau\, f(\tau)\, e^{i\omega_R\tau} e^{-\omega_I\tau}$$

If $\omega_I \equiv \mathrm{Im}\,\omega > 0$, then $e^{-\omega_I\tau}$ yields a convergent integral.

$$\frac{d\varepsilon}{d\omega} = i\int_0^{\infty} \tau\, d\tau\, f(\tau)\, e^{i\omega_R\tau} e^{-\omega_I\tau}$$

is also convergent

$\Longrightarrow$ analyticity if $\omega_I > 0$

Recall the simple model

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

has poles at $\omega_{1,2} = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

where $\gamma > 0$.

poles are in the lower half complex ω plane.

3. $\varepsilon(\omega) \longrightarrow 1$ as $\omega \to \infty$.

due to the Riemann-Lebesgue lemma

$$\lim_{\omega \to \infty} \int e^{i\omega\tau} f(\tau)\, d\tau = 0$$

often $O\left(\frac{1}{\omega}\right)$

with suitable conditions

e.g. $\int |f(\tau)|\, d\tau < \infty$.

4. $\varepsilon(\omega)$ is analytic on the real ω axis if $f(\tau)$ is a finite function.

true for dielectric medium

not (quite) true for conductors

For conductors ($\sigma \to \infty$)

$$\varepsilon = 1 + \frac{4 i \pi \sigma}{\omega}$$

not analytic at $\omega = 0$.

5. $\text{Im}\, \varepsilon(\omega)$ is related to absorption.

Recall that (energy conservation)

$$\frac{\partial u_{em}}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J}_{total} \cdot \vec{E}$$

$$\vec{J}_{total} \cdot \vec{E} = \frac{\partial u_{mech}}{\partial t}$$

In a medium

$$\vec{J}_{total} = \vec{J} + \vec{J}_p$$

$$\vec{J}_p = c \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \qquad \text{(cgs units)}$$

For simplicity, take $\vec{M} = 0$

In SI units $\vec{P} = \vec{D} - \varepsilon_0 \vec{E}$

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t} \qquad \vec{P} = \frac{\vec{D} - \vec{E}}{4\pi}$$

$$\int d^3x\, \vec{E} \cdot \frac{\partial \vec{P}}{\partial t}$$ = energy per unit time transferred from the EM fields to the medium.

complex fields

$$\vec{E} = \vec{E}_0 e^{-i\omega t}$$

$$\vec{D} = \varepsilon \vec{E} = (\varepsilon_R + i\varepsilon_I)\vec{E}$$

$$\vec{P} = \frac{\varepsilon - 1}{4\pi}\vec{E} = \frac{\varepsilon_R - 1 + i\varepsilon_I}{4\pi}(\vec{E}_0 \cos\omega t - i\vec{E}_0 \sin\omega t)$$

For real physical fields, $\mathrm{Re}\,\vec{E}$, $\mathrm{Re}\,\vec{P}$

$$\vec{E}_R \cdot \frac{\partial \vec{P}_R}{\partial t} = \frac{E_0^2 \omega}{4\pi}\left[(1-\varepsilon_R)\sin\omega t \cos\omega t + \varepsilon_I \cos^2\omega t\right]$$

average over a cycle

$$\langle \sin\omega t \cos\omega t \rangle = 0$$

$$\langle \cos^2\omega t \rangle = \langle \sin^2\omega t \rangle = \frac{1}{2}$$

$$\left\langle \vec{E}_R \cdot \frac{\partial \vec{P}_R}{\partial t} \right\rangle = \frac{E_0^2 \omega\, \varepsilon_I}{8\pi}$$

If $\varepsilon_I \neq 0$, then the medium absorbs energy steadily from the EM fields.

6. You might think that ε_R and ε_I are independent quantities. They are not!

Kramers-Kronig relations

Consider a function $f(z)$ that is analytic in the upper half of the complex z plane (including real axis), such that $f(z) \to 0$ as $|z| \to \infty$ in the upper half plane.

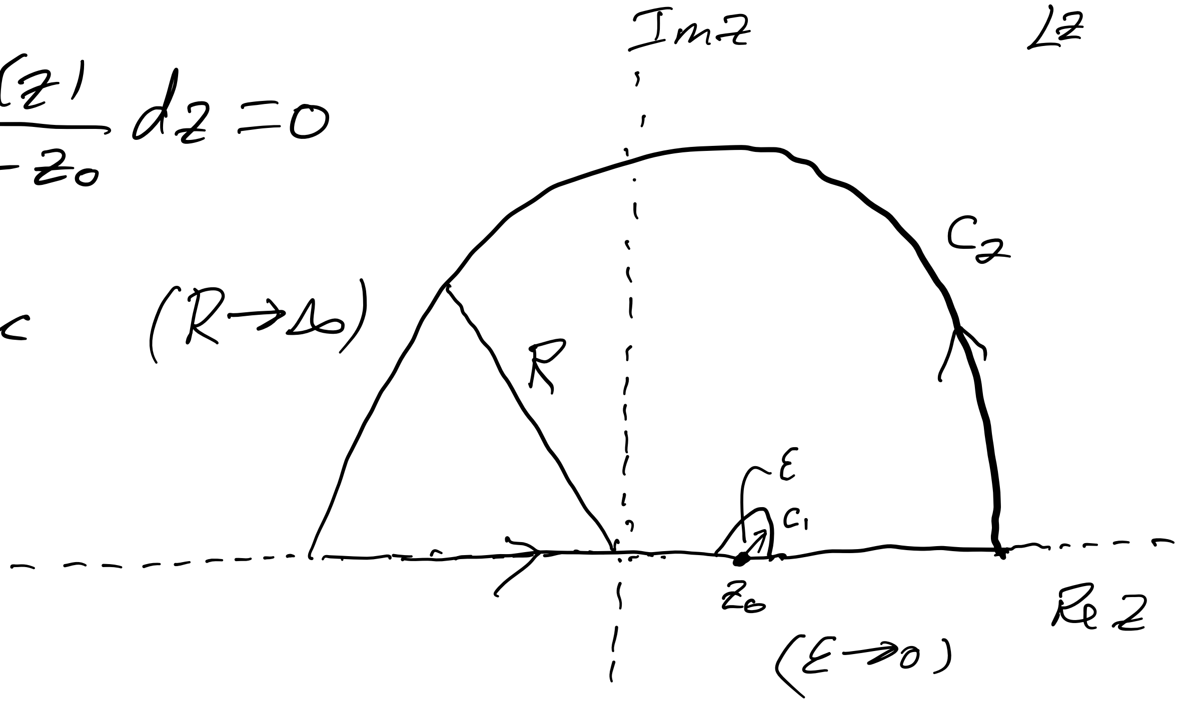
Cauchy's theorem states that for $z_0 \in \mathbb{R}$,

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 0$$

C_1 little arc

C_2 big arc

$(R \to \infty)$



$$\int_{-R}^{z_0-\varepsilon} + \int_{C_1} + \int_{z_0+\varepsilon}^{R} + \int_{C_2} \frac{f(z)}{z-z_0}\, dz = 0$$

Define principal value

$$P\int_{-R}^{R} \frac{f(z)}{z - z_0}\, dz = \lim_{\varepsilon \to 0} \int_{-R}^{z_0-\varepsilon} + \int_{z_0+\varepsilon}^{R} \frac{f(z)}{z-z_0}\, dz$$

example $P\int_{-a}^{a}\frac{dx}{x} = 0$

$$= \lim_{\varepsilon\to 0}\int_{-a}^{-\varepsilon}\frac{dx}{x} + \int_{\varepsilon}^{a}\frac{dx}{x}$$

$$= \ln\left|\frac{\varepsilon}{a}\right| + \ln\left|\frac{a}{\varepsilon}\right|$$

$$= \ln 1 = 0.$$

Next,

$$\lim_{R\to\infty}\int_{C_2}\frac{f(z)}{z-z_0}\,dz = i\int_0^{\pi}\frac{f(Re^{i\theta})}{Re^{i\theta}-z_0}Re^{i\theta}d\theta$$

$z = Re^{i\theta}$

$$\simeq i\int_0^{\pi}f(Re^{i\theta})\,d\theta \to 0.$$

Finally,

$$\lim_{\varepsilon\to 0}\int_{C_1}\frac{f(z)}{z-z_0}\,dz \simeq f(z_0)\int_{\pi}^{0}\frac{i\varepsilon e^{i\theta}d\theta}{\varepsilon e^{i\theta}}$$

$z - z_0 = \varepsilon e^{i\theta}$

$$= -i\pi f(z_0)$$

Hence,

$$P\int_{-\infty}^{\infty}\frac{f(z)}{z-z_0}\,dz - i\pi f(z_0) = 0$$

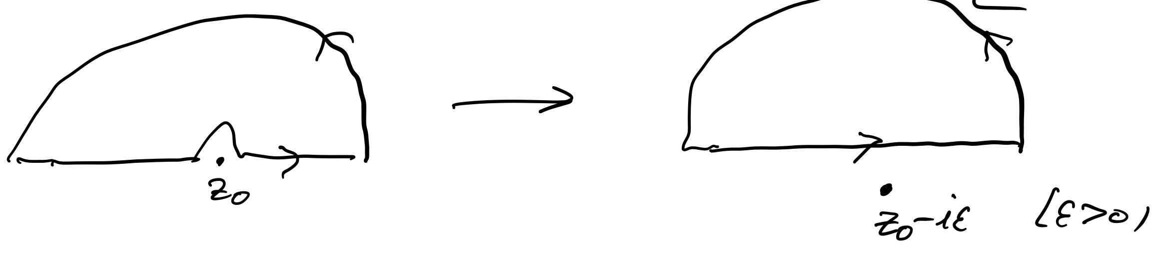
Writing $f(z) = \mathrm{Re}\, f(z) + i\, \mathrm{Im}\, f(z)$

$$\boxed{\begin{aligned} \mathrm{Re}\, f(z) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, f(z')}{z'-z}\, dz' \\ \mathrm{Im}\, f(z) &= \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, f(z')}{z'-z}\, dz' \end{aligned}}$$

called a Hilbert transform pair.

$\mathrm{Re}\, f(z)$, $\mathrm{Im}\, f(z)$ are not independent if $f(z)$ is analytic in the upper half complex z plane and $|f(z)| \to 0$ as $|z| \to \infty$ in the upper half complex z-plane.

<u>Aside:</u> Alternatively, if I replace



$$\int_C \frac{f(z)}{z - z_0 + i\epsilon}\, dz = 0 = P \int_{-\infty}^{\infty} \frac{f(z)}{z - z_0}\, dz - i\pi f(z_0)$$

If I write

$$i\pi f(z_0) = i\pi \int_{-\infty}^{\infty} f(z)\,\delta(z-z_0)\,dz$$

Then, I can interpret:

$$\boxed{\frac{1}{z-z_0+i\varepsilon} = P\,\frac{1}{z-z_0} - i\pi\,\delta(z-z_0) \qquad \varepsilon > 0}$$

Sokhotski–Plemelj formula

Application: $\varepsilon(\omega) - 1$

Recall that $\varepsilon(\omega) \to 1$ as $\omega \to \infty$ and $\varepsilon(\omega)$ is analytic in the upper complex ω-plane. For dielectrics, also analytic on the real ω-axis.

Hence

$$\mathrm{Re}\,\varepsilon(\omega) = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\varepsilon(\omega')}{\omega'-\omega}\,d\omega'$$

$$\mathrm{Im}\,\varepsilon(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\varepsilon(\omega') - 1}{\omega'-\omega}\,d\omega'$$

Recall that $\varepsilon(-\omega) = \varepsilon^*(\omega^*)$.

That is, if $\omega \in \mathbb{R}$, $\varepsilon(-\omega) = \varepsilon^*(\omega)$

$\mathrm{Re}\, \varepsilon(-\omega) = \mathrm{Re}\, \varepsilon(\omega)$ even function

$\mathrm{Im}\, \varepsilon(-\omega) = -\mathrm{Im}\, \varepsilon(\omega)$ odd function

$$\boxed{\begin{aligned} \mathrm{Re}\, \varepsilon(\omega) &= 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \,\mathrm{Im}\, \varepsilon(\omega')}{\omega'^2 - \omega^2}\, d\omega' \\ \mathrm{Im}\, \varepsilon(\omega) &= -\frac{2\omega}{\pi} P \int_0^\infty \frac{\mathrm{Re}\, \varepsilon(\omega') - 1}{\omega'^2 - \omega^2}\, d\omega' \end{aligned}}$$

Kramers-Kronig relations

relates dispersive of the medium to the absorptive properties.

<u>Remark</u>: to apply to conductor ($\sigma \gg 1$), then consider

$$\varepsilon(\omega) \longrightarrow \varepsilon(\omega) - \frac{4\pi i \sigma}{\omega}$$

Check out Jackson for "sum rules".