# Lorentz transformations

change reference frames

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

a matrix $\mu$ rows
$\nu$ columns

$$x'^{\mu} x'_{\mu} = x^{\nu} x_{\nu}$$

$$g_{\mu\lambda} x'^{\mu} x'^{\lambda} = g_{\nu\sigma} x^{\nu} x^{\sigma}$$

$$(\Lambda^{\mu}{}_{\nu}\, g_{\mu\lambda}\, \Lambda^{\lambda}{}_{\sigma} - g_{\nu\sigma})\, x^{\nu} x^{\sigma} = 0$$

must be true for arbitrary $x^{\nu}$

$$\boxed{\Lambda^{\mu}{}_{\nu}\, g_{\mu\lambda}\, \Lambda^{\lambda}{}_{\sigma} = g_{\nu\sigma}}$$

defining equation for a Lorentz transformation

or equivalently

$$G = \mathrm{diag}(1,-1,-1,-1)$$

$$\boxed{\Lambda^{T} G \Lambda = G}$$

recall how matrices multiply

$$(\Lambda^{T})_{\nu}{}^{\mu} = \Lambda^{\mu}{}_{\nu}$$

$$(AB)_{ij} = \sum_{k} A_{ik} B_{kj}$$

(If $G$ were the identity matrix, then $\Lambda^{T} G \Lambda = G \Rightarrow \Lambda^{T}\Lambda = I$)

$\Lambda$ is a pseudo-orthogonal matrix

$O(1,3)$ mostly minus

$O(3,1)$ mostly plus $\quad G \to -G$

$$O(1,3) \cong O(3,1)$$

Take determinant of $\Lambda^T G \Lambda = G$ $\quad$ $\det G = -1$

$\det \Lambda^T = \det \Lambda$

$$\Longrightarrow \quad (\det \Lambda)^2 = 1$$

$$\boxed{\det \Lambda = \pm 1}$$

Return to $\Lambda^\mu{}_\nu \, g_{\mu\lambda} \Lambda^\lambda{}_\sigma = g_{\nu\sigma}$

Put $\nu = \sigma = 0$ (free indices). Then $g_{00} = 1$ and

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1$$

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^1{}_0)^2 + (\Lambda^2{}_0)^2 + (\Lambda^3{}_0)^2$$

$$\Longrightarrow \quad (\Lambda^0{}_0)^2 \geq 1$$

That is $\boxed{\Lambda^0{}_0 \geq 1 \quad \text{or} \quad \Lambda^0{}_0 \leq -1}$

There are four classes of Lorentz transformations:

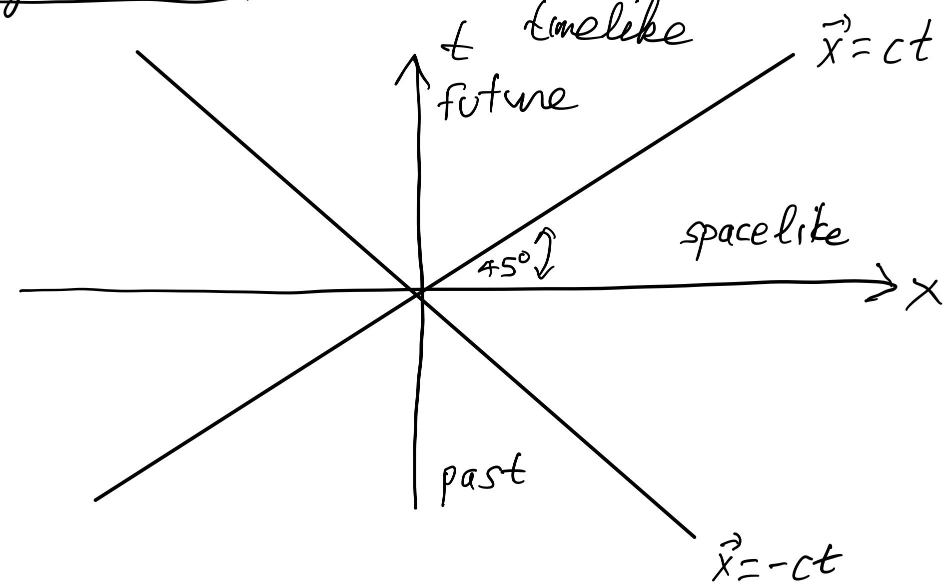
$\det \Lambda = 1, \ \Lambda^0{}_0 \geq 1 \longleftarrow$ proper orthochronous

$\det \Lambda = 1, \ \Lambda^0{}_0 \leq -1$

$\left.\begin{array}{l} \det \Lambda = -1, \ \Lambda^0{}_0 \geq 1 \\ \det \Lambda = -1, \ \Lambda^0{}_0 \leq -1 \end{array}\right\}$ improper

$\Lambda^0{}_0 \geq 1$ preserves the sense of time (orthochronous)
($x^0, x'^0$ have the same sign)

Light cone




timelike $x^2 > 0$
spacelike $x^2 < 0$

orthochronous Lorentz transformations map future light cone into the future light cone

The four classes of Lorentz transformations are disconnected and only the proper orthochronous Lorentz transformations are continuously connected to the identity $I$

Note $I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a Lorentz transformation

$\Lambda^T G \Lambda = G \quad (\Lambda = I)$

$SO(1,3) \cong SO(3,1)$ proper Lorentz transformations

"special", meaning $\det \Lambda = 1$

proper orthochronous: $SO_+(1,3)$

or $SO_\uparrow(1,3)$

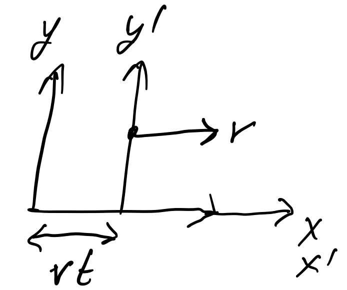
## Examples of Lorentz transformations

1. $$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix} \qquad R^T R = I$$

satisfies $\Lambda^T G \Lambda = G$

2. Boosts

simple case




At $t=0$, origins of two systems coincide

boost in the x-direction

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh \zeta & -\sinh \zeta & 0 & 0 \\ -\sinh \zeta & \cosh \zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Note $\cosh^2 \zeta - \sinh^2 \zeta = 1$

ensures that $(x'^0)^2 - (x'^1)^2 = (x^0)^2 - (x^1)^2$

$$x'^2 = x^2$$
$$x'^3 = x^3 \qquad\qquad (x^0 = ct)$$

$$x'^1 = 0 = -\sinh\zeta\, ct + \cosh\zeta\, x^1$$

$$\boxed{\tanh \zeta = \frac{v}{c} \equiv \beta} \qquad \zeta = \text{rapidity}$$

$$\cosh \zeta \equiv \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\sinh \zeta = \beta\gamma$$

$$x'^0 = \gamma(x^0 - \beta x^1)$$
$$x'^1 = \gamma(x^1 - \beta x^0)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

i.e., $\quad t' = \dfrac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}, \quad x' = \dfrac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$

$$y' = y, \quad z' = z$$

Boost in an arbitrary direction $\vec{v} = c\vec{\beta}$

$$x_0' = \gamma(x_0 - \vec{\beta}\cdot\vec{x}) = x_0 \cosh\zeta - \hat{\beta}\cdot\vec{x}\sinh\zeta$$

$$\vec{x}\,' = \vec{x} + \frac{(\gamma-1)}{\beta^2}(\vec{\beta}\cdot\vec{x})\vec{\beta} - \gamma\beta x_0$$

$$= \vec{x} + \hat{\beta}\left[(\cosh\zeta - 1)\,\vec{x}\cdot\hat{\beta} - x_0\sinh\zeta\right]$$

$$\hat{\beta} \equiv \frac{\vec{\beta}}{\beta}, \quad \text{and} \quad \zeta = \tanh^{-1}\beta$$

where $\beta \equiv |\vec{\beta}|$ $\qquad (x_0 = x^0 = ct)$

Note that

$$x_0' = \gamma(x_0 - \vec{\beta}\cdot\vec{x}_{\parallel})$$

That is $\vec{x} = \vec{x}_{\parallel} + \vec{x}_{\perp}$ where $\vec{x}_{\perp}\cdot\vec{\beta} = 0$, $\vec{x}_{\parallel}\times\vec{\beta} = 0$

$$\vec{\beta} = \frac{\beta\,\vec{x}_{\parallel}}{|\vec{x}_{\parallel}|} \iff \vec{x}_{\parallel} = \left(\frac{\vec{\beta}\cdot\vec{x}}{\beta^2}\right)\vec{\beta}$$

$$= (\hat{\beta}\cdot\vec{x})\,\hat{\beta}$$

$$\boxed{\begin{aligned} \vec{x}_{\parallel}' &= \gamma(\vec{x}_{\parallel} - \vec{\beta}x_0) \\ \vec{x}_{\perp}' &= \vec{x}_{\perp} \end{aligned}}$$

$$\vec{x}_{\perp} = \vec{x} - \vec{x}_{\parallel}$$

Thus, the most general Lorentz boost matrix is

$$\Lambda = \left(\begin{array}{c|c} \gamma & -\gamma\vec{\beta} \\ \hline -\gamma\vec{\beta} & \delta_{ij} + (\gamma - 1)\hat{\beta}_i \hat{\beta}_j \end{array}\right)$$

$$\hat{\beta} = \frac{\vec{\beta}}{\beta}$$

row

$$\Lambda^{\mu}{}_{\bullet\nu}$$

column

$$\left(\begin{array}{c|c} \Lambda^{0}{}_{0} & \Lambda^{0}{}_{j} \\ \hline \Lambda^{i}{}_{0} & \Lambda^{i}{}_{j} \end{array}\right) \qquad i, j = 1, 2, 3$$

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derivatives of four-vectors

$$\partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} = \left( \frac{\partial}{\partial x_0} ; \frac{\partial}{\partial x_i} \right)$$

$$= \left( \frac{1}{c}\frac{\partial}{\partial t} ; -\vec{\nabla} \right)$$

$$x_i = -x^i$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}; \vec{\nabla}\right)$$

"del"

D'Alenbertian

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

is a scalar under Lorentz transformations.

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Proper time $\tau$

$$d\tau^2 = \frac{1}{c^2} dx_\mu dx^\mu$$

(Lorentz scalar quantity)

$$= \frac{1}{c^2}\left(c^2 dt^2 - \sum_i dx^i dx^i\right)$$

$$v^i = \frac{dx^i}{dt}$$

$$= \frac{1}{c^2}(c^2 - v^2) dt^2$$

$$= (1-\beta^2) dt^2 = \gamma^{-2} dt^2$$

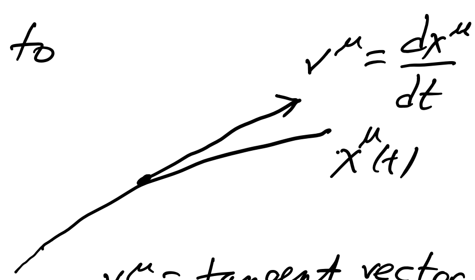
$$\boxed{t_2 - t_1 = \int_{\tau_1}^{\tau_2} \gamma(\tau)\, d\tau}$$

$\gamma \geq 1$

time dilation!

Consider a general coordinate transformation from a system of coordinates $x^\mu$ to

$$x'^\mu = f^\mu(\vec{x}, t)$$




$$dx^\mu \to dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

$v^\mu$ = tangent vector

by the chain rule.

For Lorentz transformations,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$\Lambda^\mu{}_\nu$ is independent of $x$.

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$$

$$dx'^\mu = \Lambda^\mu{}_\nu dx^\nu$$

Any vector $A^\mu$ that transforms like

$$A^\mu \longrightarrow A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$$

is called <u>contravariant</u>.

Suppose I consider a field $\phi(x)$

$$\partial_\mu \phi \equiv \frac{\partial \phi}{\partial x^\mu}$$

$$\frac{\partial \phi}{\partial x^\mu} \longrightarrow \frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi}{\partial x^\nu}$$

by the chain rule

Any vector $A_\mu$ that transforms like

$$A_\mu \longrightarrow A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu$$

is called <u>covariant</u>.

You recognize the Jacobian matrix

$$J = \frac{\partial x'^\mu}{\partial x^\nu}$$

$$J^{-1} = \frac{\partial x^\nu}{\partial x'^\mu}$$

Note that a four-vector can be expanded in a basis

$$V = V^\mu \hat{e}_{(\mu)} \qquad \text{basis } \{ \hat{e}_{(0)}, \hat{e}_{(1)}, \hat{e}_{(2)}, \hat{e}_{(3)} \}$$

not a spacetime index

active transformation: transform $V^\mu \to V'^\mu$
leave basis vectors fixed

passive transformation: fixes $V^\mu$
transforms basis vectors in an opposite sense

Covariant vectors transform in the same sense as $\hat{e}_{(\mu)}$

Generalize to tensors

$$A'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} A^{\alpha\beta}$$

2ND rank contravariant tensor

$$A'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} A_{\alpha\beta}$$

2ND rank covariant tensor

$$A'_{\mu}{}^{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} A_{\alpha}{}^{\beta}$$

2ND rank mixed tensor

For "Lorentz" tensors

$$\frac{\partial x'^{\mu}}{\partial x^{\alpha}} = \Lambda^{\mu}{}_{\alpha}$$

$$A'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} A_{\mu\nu}$$

$$A'_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\mu} (\Lambda^{-1})^{\beta}{}_{\nu} A_{\alpha\beta}$$

How does the metric transform?

$$g'_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\mu} (\Lambda^{-1})^{\beta}{}_{\nu} g_{\alpha\beta}$$

Recall that $\Lambda$ is defined by $g_{\alpha\beta} = \Lambda^{\rho}{}_{\alpha} \Lambda^{\sigma}{}_{\beta} g_{\rho\sigma}$

$$g'_{\mu\nu} = \Lambda^{\rho}{}_{\alpha} (\Lambda^{-1})^{\alpha}{}_{\mu} \Lambda^{\sigma}{}_{\beta} (\Lambda^{-1})^{\beta}{}_{\nu} g_{\rho\sigma}$$

$$= \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} g_{\rho\sigma} = g_{\mu\nu}$$

That is, $g_{\mu\nu}$ is an invariant tensor

In general, objects that satisfy $S' = S$ are called Lorentz scalars.

example: $g_{\mu\nu} x^\mu x^\nu = x^2 = c^2 t^2 - |\vec{x}|^2$.

Note:

$w^\nu = V_\mu A^{\mu\nu}$ transforms as a contravariant four-vector.

[Aside: In three dimensional Euclidean space, the metric tensor is $\delta_{ij}$. Transformation laws are

$$A_{ij} = R_{ik} R_{jl} A_{kl}$$

where $R^T R = I$. One difference is that there is no distinction between covariant and contravariant indices.]

## Levi-Civita tensor

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & , \{\mu\nu\alpha\beta\} \text{ is an even permutation of } \{0123\} \\ -1 & , \text{ odd permutation} \\ 0 & , \text{ otherwise} \end{cases}$$

$\varepsilon^{0123} = +1$

$(\varepsilon_{0123} = -1)$

e.g. $\varepsilon^{0132} = -1$.