That is, $g_{\mu\nu}$ is an <u>invariant</u> tensor

In general, objects that satisfy $S' = S$ are called Lorentz scalars.

<u>example</u>: $g_{\mu\nu} x^\mu x^\nu = x^2 = c^2 t^2 - |\vec{x}|^2$.

Note:

$w^\nu = V_\mu A^{\mu\nu}$ transforms as a contravariant four-vector.

[Aside: In three dimensional Euclidean space, the metric tensor is δ_{ij}. Transformation laws are

$$A_{ij} = R_{ik} R_{jl} A_{kl}$$

where $R^T R = I$. One difference is that there is no distinction between covariant and contravariant indices.]

<u>Levi-Civita tensor</u>

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & , \{\mu\nu\alpha\beta\} \text{ is an even permutation of } \{0123\} \\ -1 & , \text{odd permutation} \\ 0 & , \text{otherwise} \end{cases}$$

$\varepsilon^{0123} = +1$

$(\varepsilon_{0123} = -1)$

e.g. $\varepsilon^{0132} = -1$.

Under proper Lorentz transformations,

$$\varepsilon'^{\mu\nu\alpha\beta} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\Lambda^{\alpha}{}_{\tau}\Lambda^{\beta}{}_{\kappa}\,\varepsilon^{\rho\sigma\tau\kappa}$$

$$= \varepsilon^{\mu\nu\alpha\beta}(\det\Lambda) \qquad \text{definition of determinant}$$

$$= \varepsilon^{\mu\nu\alpha\beta} \qquad \det\Lambda = +1$$

for proper Lorentz transformations

$\varepsilon^{\mu\nu\alpha\beta}$ is a <u>pseudo-invariant tensor</u>.

I can construct a pseudo-scalar

$$\varepsilon^{\mu\nu\alpha\beta}a_\mu b_\nu c_\alpha d_\beta .$$

[<u>Beware</u>: Some books define $\varepsilon_{0123} = +1$]

The list of all possible invariant tensors (under proper Lorentz transformations) are

$$g_{\mu\nu}, \; g^{\mu\nu}, \; \varepsilon^{\mu\nu\alpha\beta}, \; \varepsilon_{\mu\nu\alpha\beta}$$

and products thereof.

$g_{\mu\nu}g_{\alpha\beta}$ an invariant 4th rank tensor.

Identities involving $\varepsilon^{\mu\nu\alpha\beta}$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta} = -24$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\sigma} = -6\delta^{\beta}_{\sigma}$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\rho\sigma} = -2(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho})$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\rho\sigma\tau} = -\det\begin{pmatrix} \delta^{\nu}_{\rho} & \delta^{\nu}_{\sigma} & \delta^{\nu}_{\tau} \\ \delta^{\alpha}_{\rho} & \delta^{\alpha}_{\sigma} & \delta^{\alpha}_{\tau} \\ \delta^{\beta}_{\rho} & \delta^{\beta}_{\sigma} & \delta^{\beta}_{\tau} \end{pmatrix}$$

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\gamma\rho\sigma\tau} = -\det\begin{pmatrix} \\ & 4\times 4 & \\ \\ \end{pmatrix}$$

exercise: write out the 4×4 matrix above.

Examples of four-vectors

1. Velocity four-vector

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = (\gamma c\,;\,\gamma\vec{v}) \qquad \vec{v} = \frac{d\vec{x}}{dt}$$

τ = proper time $\quad d\tau^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ (Lorentz scalar)

$$d\tau = \gamma^{-1} dt \qquad \gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$v = |\vec{v}|$$

$$u^2 \equiv g_{\mu\nu} u^\mu u^\nu$$
$$= \gamma^2 (c^2 - |\vec{v}|^2) = c^2 \gamma^2 \left(1 - \frac{v^2}{c^2}\right)$$
$$= c^2$$

Under Lorentz transformations,

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu$$

Suppose in an inertial reference frame K,
the velocity four-vector is u^μ.
The reference frame K' is moving at velocity $\vec{w}$
with respect to K.
What is u'^μ? Use $\Lambda^\mu{}_\nu$ boost matrix, with
boost parameter $\vec{\beta}_w = \frac{\vec{w}}{c}$

$$u^\mu = (\gamma c; \gamma \vec{v}), \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$
$$u'^\mu = (\gamma' c; \gamma' \vec{v}'), \quad \gamma' = \left(1 - \frac{v'^2}{c^2}\right)^{-1/2}$$

Relation between $\vec{v}$ and $\vec{v}'$

$$\boxed{\vec{v}' = \frac{1}{1 - \frac{\vec{v}\cdot\vec{w}}{c^2}} \left[\frac{1}{\gamma_w} \left(\vec{v} - \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2} \vec{w} \right) - \left(1 - \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right) \vec{w} \right]}$$

$$\gamma_w = \frac{1}{\sqrt{1-\beta_w^2}} \qquad \vec{\beta}_w \equiv \frac{\vec{w}}{c}$$

Law of addition of velocities

Special case: $\vec{v} \parallel \vec{w}$

$$\vec{v} = \left(\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\right)\vec{w}$$

$\vec{v} = k\vec{w}$

$\vec{v}\cdot\vec{w} = k|\vec{w}|^2$

$$\Longrightarrow \qquad \vec{v}\,' = \frac{\vec{v}-\vec{w}}{1-\frac{\vec{v}\cdot\vec{w}}{c^2}}$$

In limit of $c \to \infty$, $\vec{v}\,' = \vec{v}-\vec{w}$.

2. Momentum four-vector

$$p^\mu = m u^\mu = \left(\frac{E}{c};\ \vec{p}\right)$$

m = mass of the particle (intrinsic property of the particle)

Lorentz scalar

$$\vec{p} = \gamma m \vec{v}$$

$$E = \gamma m c^2$$

$$\vec{v} = \frac{\vec{p}\,c^2}{E}$$ after dividing the two equations above.

$$p^2 = g_{\mu\nu}\, p^\mu p^\nu = (p^0)^2 - |\vec{p}|^2 = m^2 c^2$$

$$p^0 = \frac{E}{c} \implies \boxed{E^2 = c^2|\vec{p}|^2 + m^2c^4}$$

non-relativistic limit $|\vec{p}| \to 0$

$$E \simeq \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{|\vec{p}|^2}{2m}}_{\text{non-relativistic kinetic energy}}$$

$$\boxed{T = \sqrt{c^2|\vec{p}|^2 + m^2c^4} - mc^2}$$

relativistic generalization of the kinetic energy.

3. Force and acceleration four-vectors.

$$\vec{F} = \frac{d\vec{p}}{dt}$$ valid in relativity

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m \vec{v}) = \gamma m \frac{d\vec{v}}{dt} + m\vec{v}\frac{d\gamma}{dt}$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{\vec{v}\cdot\vec{v}}{c^2}\right)^{-1/2} = \frac{\gamma^3}{c^2}\,\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$\boxed{\vec{F} = \gamma m \left[\frac{d\vec{v}}{dt} + \frac{\gamma^2}{c^2}\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right)\vec{v}\right]}$$

relativistic generalization of Newton's 2ND law.

Two special cases

1. $\vec{v} \parallel \frac{d\vec{v}}{dt}$ (linear motion)

$$\frac{d\vec{v}}{dt} = k\vec{v} \implies k = \frac{1}{v^2}\,\vec{v}\cdot\frac{d\vec{v}}{dt}$$

$$= \frac{\vec{v}}{v^2}\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right)$$

$$\implies \vec{F} = \gamma^3 m \frac{d\vec{v}}{dt}, \quad \text{linear motion}$$

2. $\vec{v} \perp \frac{d\vec{v}}{dt}$ (circular motion)

$$\Rightarrow \quad \vec{F} = \gamma m \frac{d\vec{v}}{dt}, \quad \text{circular motion}$$

relativistic mass (??)

Should it be defined to be γm, as in circular motion?

Should it be defined to be $\gamma^3 m$, as in linear motion?

What about in other cases, where $\vec{F}$ is not even proportional to $\frac{d\vec{v}}{dt}$?

Conclusion: "relativistic mass" is not a useful concept. It is best to regard "mass" as an intrinsic property of a particle, i.e. a Lorentz scalar quantity that does not depend on the reference frame.

See L.B. Okun, Physics Today 42, 31-36 (1989).

3. Force and acceleration 4-vectors

Last time we saw that the relativistic generalization of Newton's second law was

$$\vec{F} = \gamma m \left[\frac{d\vec{v}}{dt} + \frac{\gamma^2}{c^2}\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right)\vec{v} \right]$$

where $\vec{v} \equiv \frac{d\vec{x}}{dt}$. Introducing $\vec{a} \equiv \frac{d\vec{v}}{dt}$, we can rewrite the above equation as

$$\vec{F} = \gamma m \left[\vec{a} + \frac{\gamma^2}{c^2}(\vec{v}\cdot\vec{a})\vec{v} \right]$$

$$\vec{F}\cdot\vec{v} = \gamma m \vec{v}\cdot\vec{a}\left(1 + \frac{\gamma^2 v^2}{c^2}\right) = \gamma^3 m \vec{v}\cdot\vec{a}$$

since $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

$$\frac{dE}{dt} = \frac{d}{dt}(\gamma m c^2) = mc^2 \frac{d\gamma}{dt} = \gamma^3 m \vec{v}\cdot\vec{a}$$

$$\frac{dE}{dt} = \vec{F}\cdot\vec{v} \qquad \text{(power)}$$

<u>Force 4-vector</u> (Minkowski force)

$$K^\mu = \frac{dp^\mu}{d\tau} = \left(\frac{\gamma \vec{F}\cdot\vec{v}}{c} ; \gamma\vec{F} \right) \qquad \vec{K} = \gamma\vec{F}$$

Acceleration 4-vector

$$\alpha^\mu = \frac{du^\mu}{d\tau} = \Big(\frac{\gamma^4}{c}\vec{v}\cdot\vec{a} \, ; \; \underbrace{\gamma^2\vec{a} + \frac{\gamma^4}{c^2}(\vec{v}\cdot\vec{a})\vec{v}}_{\vec{\alpha}} \Big)$$

Since $p^\mu = m u^\mu$

$$K^\mu = m\alpha^\mu$$

Properties of α^μ

note:

$A^\mu B_\mu = A_\mu B^\mu$

$= g_{\mu\nu}A^\mu B^\mu$

(i) $\alpha^\mu u_\mu = 0$

Proof: Recall $u^\mu u_\mu = c^2$

$$0 = \frac{d}{d\tau}(u^\mu u_\mu) = 2u^\mu \frac{du_\mu}{d\tau} = 2u^\mu \alpha_\mu$$

Since u^μ is timelike $\quad (u^\mu u_\mu > 0)$

$\Rightarrow \alpha^\mu$ is spacelike $\quad (\alpha^\mu \alpha_\mu < 0)$

$$\alpha^\mu \alpha_\mu = -\gamma^4\Big[|\vec{a}|^2 + \frac{\gamma^2}{c^2}(\vec{v}\cdot\vec{a})^2\Big]$$

$$= -\gamma^6\Big[|\vec{a}|^2 - \frac{|\vec{v}\times\vec{a}|^2}{c^2}\Big]$$

$$= -\gamma^6|\vec{a}|^2\Big(1 - \frac{v^2}{c^2}\sin^2\theta\Big) < 0$$

since $|v| < c$

θ = angle between $\vec{v}$ and $\vec{a}$

$$g \equiv \sqrt{-\alpha_\mu \alpha^\mu}$$

Constant acceleration in relativity
is not constant $\vec{a}$
Instead it is constant g.

Example: constant linear acceleration $\vec{a} \parallel \vec{v}$

$$g = \gamma^3 |\vec{a}|$$

$$\frac{dv}{dt} = |\vec{a}| = \gamma^{-3} g = \left(1 - \frac{v^2}{c^2}\right)^{3/2} g$$

$$v(t) = \frac{gt}{\left(1 + \frac{g^2 t^2}{c^2}\right)^{1/2}}$$

$$\lim_{t \to \infty} v(t) = c$$

The trajectory of the particle $(\vec{v} = v\hat{x})$

$$ct = \frac{c^2}{g} \sinh\left(\frac{g\tau}{c}\right), \quad x = \frac{c^2}{g} \cosh\left(\frac{g\tau}{c}\right)$$

$$\Longrightarrow \quad x^2 - c^2 t^2 = \frac{c^4}{g}$$ (hyperbola in spacetime)

This is called hyperbolic motion.