Scattering of electromagnetic waves

Two different domains:

(1) scattering at long wavelengths

scattering targets are small in size compared to the wavelengths of the radiation

$kd \ll 1$ (suggests the multipole expansion)

(2) wavelengths are comparable or smaller than the distance scale that characterizes the target

$\lambda \lesssim O(d)$ $\Longrightarrow$ diffraction

Focus on (1).

Thomson scattering

Problem: a plane wave of monochromatic EM wave is incident on a free particle of charge e and mass m.

The oscillating charge radiates

[Quantum mechanically, $\gamma + e^- \to \gamma + e^-$]

Assume non-relativistic motion

Incident plane wave

$$\vec{E}(\vec{x},t) = \hat{\varepsilon}_0 E_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

polarization

$$m\vec{a} = -e\vec{E} \qquad \text{for the electron of charge } -e$$

$(e > 0)$

$$\vec{a} = -\hat{\varepsilon}_0 \frac{e}{m} E_0 e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

[We are ignoring the momentum of the incident photon which is a valid approximation only at low frequencies (long wavelength). If $\frac{\hbar\omega}{c} \gtrsim O(mc)$, then modifications are required. In this region, $\gamma + e^- \to \gamma + e^-$ is called Compton scattering.]

Use Larmor's formula

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\hat{n}\times(\hat{n}\times\vec{a})|^2$$

$$= \frac{e^2}{4\pi c^3} |\hat{n}\times\vec{a}|^2$$

Taking real parts,

$$\vec{a} = -\frac{e}{m}\hat{\varepsilon}_0 E_0 \cos(\vec{k}_0\cdot\vec{x} - \omega t)$$

Time averaged power over one cycle

$$\langle \cos^2(\vec{k}_0 \cdot \vec{x} - \omega t) \rangle = \frac{1}{2}$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4 |E_0|^2}{8\pi m^2 c^3} |\hat{n} \times \hat{\varepsilon}_0|^2$$

Cross-section

$$d\sigma = \frac{\text{energy radiated into } d\Omega \text{ per unit time}}{\text{incident energy per unit area per unit time}}$$

incident (energy) flux

incident flux = time averaged Poynting vector for the incident wave

$$= \frac{c|E_0|^2}{8\pi}$$

Recall:

$$\vec{S} \cdot \hat{n} = \frac{c}{8\pi} (\vec{E} \times \vec{B}^*) \cdot \hat{n}$$
$$= \frac{c}{8\pi} [\vec{E} \times (\hat{n} \times \vec{E}^*)] \cdot \hat{n}$$
$$= \frac{c}{8\pi} |\vec{E}|^2$$

using $\hat{n} \cdot \vec{E} = 0$.

Hence,

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2} \right)^2 |\hat{n} \times \hat{\varepsilon}_0|^2}$$

Thomson cross-section

Define angle θ between $\hat{n}$ and $\hat{\varepsilon}_0$

$\hat{n}$ = direction of scattered wave

$$= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

Then $|\hat{n}\times\hat{\varepsilon}_0|^2 = \sin^2\theta$

$$\int d\Omega \sin^2\theta = \frac{8\pi}{3}$$

$$\boxed{\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = \frac{8\pi}{3} r_c^2}$$

Thomson total cross section

$$r_c \equiv \frac{e^2}{mc^2} = 2.82\times 10^{-13}\ \text{cm}$$ "classical electron radius"

(physically: model the electron as a spherical surface uniformly charged

r_c = radius of sphere

electrostatic energy $\frac{e^2}{r_c}$

$$\frac{e^2}{r_c} = mc^2$$ rest energy of electron)

(Quantum mechanics enters the game at the Compton wavelength $\frac{\hbar}{mc}$)

Note: $r_c = \alpha \frac{\hbar}{mc}$

where $\alpha \equiv \frac{e^2}{\hbar c} \simeq \frac{1}{137}$

Numbers: $\sigma_T = 0.665 \times 10^{-24}\ cm^2$

$= 0.665$ barns

Polarization of the scattered wave $\hat{\varepsilon}$

$\{\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{k} = \hat{n}\}$
orthonormal set
of complex vectors

Identity:

$$\hat{\varepsilon}_1^* \hat{\varepsilon}_1 + \hat{\varepsilon}_2^* \hat{\varepsilon}_2 + \hat{k}\hat{k} = \underbrace{\mathbb{1}}_{\text{"dyad"}}$$

recall: $\sum_n |n\rangle\langle n| = \mathbb{1}$

$$\vec{E} = \hat{\varepsilon}_1(\hat{\varepsilon}_1^* \cdot \vec{E}) + \hat{\varepsilon}_2(\hat{\varepsilon}_2^* \cdot \vec{E}) \qquad \text{since } \hat{n}\cdot\vec{E} = 0$$

$$|\vec{E}|^2 = |\hat{\varepsilon}_1^* \cdot \vec{E}|^2 + |\hat{\varepsilon}_2^* \cdot \vec{E}|^2$$

Then

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} |\hat{\varepsilon}^* \cdot (\hat{n} \times (\hat{n} \times \vec{a}))|^2$$

if the scattered wave has polarization $\hat{\varepsilon}$.

$$\hat{\varepsilon}^* \cdot (\hat{n} \times (\hat{n} \times \vec{a})) = \underbrace{(\hat{\varepsilon}^* \cdot \hat{n})}_{0}(\vec{a}\cdot\hat{n}) - (\hat{\varepsilon}^* \cdot \vec{a})\underbrace{(\hat{n}\cdot\hat{n})}_{1}$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\hat{\varepsilon}^* \cdot \vec{a}|^2$$

Therefore,

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\varepsilon}_0 \cdot \hat{\varepsilon}^*|^2}$$

Unpolarized initial wave means an average over initial polarizations.

Not measuring the polarization of the scattered wave means summing over final state polarizations

"mantra" of scattering theory:

average over initial polarizations
Sum over final polarizations

$$\hat{\varepsilon}^{(1)}\hat{\varepsilon}^{(1)*} + \hat{\varepsilon}^{(2)}\hat{\varepsilon}^{(2)*} = \mathbb{1} - \hat{k}\hat{k}$$

$$\boxed{\sum_{\lambda=1}^{2} \hat{\varepsilon}_i^{(\lambda)} \hat{\varepsilon}_j^{(\lambda)*} = \delta_{ij} - \hat{n}_i \hat{n}_j}$$

$\hat{n} = \hat{k}$

i, j space components of the vector

Polarization sum formula

Example: Thomson scattering, where the polarization of the outgoing radiation is <u>not</u> measured

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \sum_{\lambda=1}^{2} |\hat{\varepsilon}_0 \cdot \hat{\varepsilon}^{(\lambda)*}|^2$$

$$\sum_{\lambda=1}^{2} |\hat{\varepsilon}_0 \cdot \hat{\varepsilon}^{(\lambda)*}|^2 = \sum_{\lambda=1}^{2} \hat{\varepsilon}^*_{0i}\, \hat{\varepsilon}^{(\lambda)}_i\, \hat{\varepsilon}_{0j}\, \varepsilon^{(\lambda)*}_j$$

$\vec{a}\cdot\vec{b} = \sum_{i=1}^{3} a_i b_i$

$= a_i b_i$

using Einstein's summation convention

$$= \hat{\varepsilon}^*_{0i}\, \hat{\varepsilon}_{0j} \sum_{\lambda=1}^{2} \hat{\varepsilon}^{(\lambda)}_i\, \hat{\varepsilon}^{(\lambda)*}_j$$

$$= \hat{\varepsilon}^*_{0i}\, \hat{\varepsilon}_{0j} \left(\delta_{ij} - \hat{n}_i \hat{n}_j\right)$$

$$= \varepsilon^*_0 \cdot \varepsilon_0 - \hat{\varepsilon}_0 \cdot \hat{n}\, \hat{\varepsilon}^*_0 \cdot \hat{n}$$

$$= |\hat{\varepsilon}_0 \times \hat{n}|^2$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\varepsilon}_0 \times \hat{n}|^2$$

Example: Unpolarized Thomson scattering

In this case, we assume that the incoming radiation is unpolarized

$\Longrightarrow$ average over the initial polarizations

and the polarization of the outgoing radiation is not measured

$\Longrightarrow$ sum over the final state polarizations

Hence,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2 \sum_{\lambda}\sum_{\lambda_0} \left|\hat{\varepsilon}_0^{(\lambda_0)} \cdot \hat{\varepsilon}^{(\lambda)*}\right|^2$$

There will be two polarization sums.

For the incoming EM wave, choose $\hat{k}_0 = \hat{z} = (0,0,1)$

Then,

$$\sum_{\lambda_0=1}^{2} \left(\hat{\varepsilon}_0^{(\lambda_0)}\right)_i \left(\hat{\varepsilon}_0^{(\lambda_0)*}\right)_j = \delta_{ij} - \hat{z}_i \hat{z}_j$$

For the outgoing EM wave, $\hat{k} \equiv \hat{n}$

$$\sum_{\lambda=1}^{2} \left(\hat{\varepsilon}^{(\lambda)}\right)_i \left(\hat{\varepsilon}^{(\lambda)*}\right)_j = \delta_{ij} - \hat{n}_i \hat{n}_j .$$

For example

$$\frac{1}{2}\sum_{\lambda_0} \left|\hat{\varepsilon}_0^{(\lambda_0)}\cdot\hat{\varepsilon}^{(\lambda)*}\right|^2 = \frac{1}{2}\hat{\varepsilon}_i^{(\lambda)}\hat{\varepsilon}_j^{(\lambda)*}\sum_{\lambda_0}\left(\hat{\varepsilon}_0^{(\lambda_0)}\right)_j\left(\hat{\varepsilon}_0^{(\lambda_0)*}\right)_i$$

$$= \frac{1}{2}\varepsilon_i^{(\lambda)}\varepsilon_j^{(\lambda)*}\left(\delta_{ij} - \hat{z}_i\hat{z}_j\right)$$

$$= \frac{1}{2}\left(1 - |\hat{z}\cdot\hat{\varepsilon}|^2\right)$$

$$\sum_\lambda 1 = 2$$

$$\sum_\lambda |\hat{z}\cdot\hat{\varepsilon}^{(\lambda)}|^2 = \hat{z}_i\hat{z}_j\sum_\lambda \varepsilon_i^{(\lambda)}\varepsilon_j^{(\lambda)*}$$

$$= \hat{z}_i\hat{z}_j\left(\delta_{ij} - \hat{n}_i\hat{n}_j\right)$$

$$= 1 - (\hat{z}\cdot\hat{n})^2$$

Define $\cos\theta = \hat{z}\cdot\hat{n}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2\left[2 - (1-\cos^2\theta)\right]$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)(1+\cos^2\theta)}$$

$$\sigma_T = \int\frac{d\sigma}{d\Omega}\,d\Omega = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2$$

Electromagnetic scattering in the multipole approximation

$kd \ll 1$ (long wavelength approximation)

Consider incident EM radiation

$$\vec{E}_{inc} = \hat{\varepsilon}_0 E_0 e^{i\vec{k}\cdot\vec{x}}$$

$$\vec{B}_{inc} = \hat{n}_0 \times \vec{E}_{inc}$$

$$\vec{k} \equiv k\,\hat{n}_0 \qquad k_0 = \frac{\omega}{c}$$

e.g., $\vec{E}_{inc}(t) = \mathrm{Re}\, \vec{E}_{inc} e^{-i\omega t}$

Work in vacuum $(\varepsilon = \mu = 1)$

Targets acquire induced dipole moments $\vec{p}$, $\vec{m}$

$$\vec{p}(t) = \mathrm{Re}\, \vec{p} e^{-i\omega t}$$

$$\vec{m}(t) = \mathrm{Re}\, \vec{m} e^{-i\omega t}$$

electric dipole radiated fields

$$\vec{B} = k^2 (\hat{n} \times \vec{p}) \frac{e^{ikr}}{r}$$

$$\vec{E} = k^2 (\hat{n} \times \vec{p}) \times \hat{n} \; \frac{e^{ikr}}{r}$$

magnetic dipole radiated fields

$$\vec{E} = -k^2 (\hat{n} \times \vec{m}) \frac{e^{ikr}}{r}$$

$$\vec{B} = k^2 (\hat{n} \times \vec{m}) \times \hat{n} \frac{e^{ikr}}{r}$$

$$\vec{E}_{sc} = k^2 \frac{e^{ikr}}{r} \left[(\hat{n} \times \vec{p}) \times \hat{n} - \hat{n} \times \vec{m} \right]$$

$$\vec{B}_{sc} = \hat{n} \times \vec{E}_{sc}$$

$$d\sigma = \frac{\text{power radiated into } d\Omega}{\text{incident energy flux}}$$

$$\text{incident energy flux} = \frac{c}{8\pi} |\vec{E}_{inc}|^2$$

$$\text{numerator} = \vec{S} \cdot \hat{n}\, da = \vec{S} \cdot \hat{n}\, r^2 d\Omega$$

If the polarization of the scattered wave is $\hat{\varepsilon}$
" " " " " incident wave is $\hat{\varepsilon}_0$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{r^2 |\hat{\varepsilon}^* \cdot \vec{E}_{sc}|^2}{|\hat{\varepsilon}_0^* \cdot \vec{E}_{inc}|^2}}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{k^4}{|E_0|^2} \left| \hat{\varepsilon}^* \cdot \vec{p} + (\hat{n} \times \hat{\varepsilon}^*) \cdot \vec{m} \right|^2}$$

Since

$$\hat{\varepsilon}^* \cdot \left\{ (\hat{n} \times \vec{p}) \times \hat{n} \right\} = \hat{\varepsilon}^* \cdot \left[\vec{p} - \hat{n}(\hat{n} \cdot \vec{p}) \right] = \hat{\varepsilon}^* \cdot \vec{p}$$

using $\hat{n} \cdot \hat{\varepsilon}^* = 0$.

Note the k^4 dependence (Rayleigh's law)

(connected to the answer to the question "why is the sky blue")

$k = \frac{2\pi}{\lambda}$