Thomson scattering regained

$$\vec{E}_{inc}(\vec{x},t) = \hat{\varepsilon}_0 E_0 e^{ik_0\cdot\vec{x} - i\omega t}$$

$$m\vec{a} = -e\vec{E}_{inc}$$

(for the electron $q=-e$)

$$\vec{a} = \frac{d^2\vec{x}}{dt^2}$$

$$\vec{x}(t) = \frac{\hat{\varepsilon}_0 eE_0}{\omega^2 m} e^{ik\cdot\vec{x} - i\omega t}$$

$$\vec{p}(t) = -e\vec{x}(t) = -\frac{\hat{\varepsilon}_0 e^2}{\omega^2 m} e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\vec{p}(t) = \vec{p}e^{-i\omega t}$$

Hence,

$$\frac{d\sigma}{d\Omega} = \frac{k^4|\hat{\varepsilon}^*\cdot\vec{p}|^2}{|E_0|^2} = \frac{k^4 e^4}{m^2\omega^4}|\hat{\varepsilon}^*\cdot\hat{\varepsilon}_0|^2$$

$$\omega = kc$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\varepsilon}^*\cdot\hat{\varepsilon}_0|^2$$

reproducing Thomson's cross section.

Applications

① Scattering by a small dielectric sphere of radius a

$\uparrow a \ll \lambda$ (small) ; with dielectric constant ε (sphere)

Recall that

$$\phi(r,\theta,\phi) = \begin{cases} \dfrac{-3}{2+\varepsilon} E_0 r \cos\theta , & r<a \\ -E_0 r\cos\theta + \left(\dfrac{\varepsilon-1}{\varepsilon+2}\right) E_0 \dfrac{a^3}{r^2}\cos\theta, & r>a \end{cases}$$

Outside the sphere, there is an induced electric dipole moment

$$\vec{p} = \left(\frac{\varepsilon-1}{\varepsilon+2}\right) a^3 E_0$$

For the scattering problem,

$$\vec{p} = \left(\frac{\varepsilon-1}{\varepsilon+2}\right) a^3 \vec{E}_{inc}$$

$$\vec{m} = 0$$

$$\boxed{\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 |\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2}$$

Unpolarized cross section

average over initial polarizations

sum over final polarizations

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2} k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 (1+\cos^2\theta)$$

$$\sigma = \frac{8\pi}{3} k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2$$

If you measure the final state polarizations, then

$$\frac{d\sigma_\lambda}{d\Omega} = \frac{1}{2} k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 \sum_{\lambda_0} |\hat{\varepsilon}^{(\lambda)*} \cdot \hat{\varepsilon}_0^{(\lambda_0)}|^2$$

$\lambda = 1, 2$ $\qquad$ $\lambda = 1$ in the plane containing $\hat{n}_0, \hat{n}$

$\lambda = 2$ in the plane $\perp$ to the above plane

Polarization asymmetry

$$P \equiv \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta}$$

not power

② Scattering by a perfectly conducting sphere of radius a $(a \ll \lambda)$

$$\phi(r) = \begin{cases} -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta, & r > a \\ 0, & r < a \end{cases}$$

$$\vec{p} = a^3 \vec{E}_{inc}$$

$$\vec{m} = -\frac{a^3}{2}\vec{B}_{inc}$$

<u>Proof:</u> Magnetostatics problem. No $\vec{J}$. $\vec{\nabla}\times\vec{B} = 0$

$$\vec{B} = -\vec{\nabla}\Phi_m$$

$$\vec{\nabla}\cdot\vec{B} = 0 \implies \vec{\nabla}^2\Phi_m = 0$$

boundary conditions

at $r=\infty$ $\quad \vec{B} = B_0 \hat{z}$

$$\Longrightarrow \Phi_m = -B_0 r\cos\theta$$

at $r=a$ (surface), $\vec{B}\cdot\hat{n}_0 = 0$

$$\Longrightarrow \frac{\partial \Phi_m}{\partial r} = 0 \text{ at } r=a$$

$$\phi_m(r) = -B_0 r\cos\theta + \frac{A}{r^2}\cos\theta$$

Applying boundary condition at $r=a$: $\left(\frac{\partial \Phi_m}{\partial r}\right)_{r=a} = 0.$

$$\Longrightarrow A = -\frac{B_0 a^3}{2}.$$

$$\vec{B}_{inc} = \hat{n}_0 \times \vec{E}_{inc}$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}\underbrace{(\hat{n}\times\hat{\varepsilon}^*)\cdot(\hat{n}_0\times\hat{\varepsilon}_0)}_{(\hat{n}\cdot\hat{n}_0)(\hat{\varepsilon}^*\cdot\varepsilon_0) - (\hat{n}\cdot\hat{\varepsilon}_0)(\hat{n}_0\cdot\hat{\varepsilon}^*)}\right|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right]$$

Polarization asymmetry

$$P = \frac{3\sin^2\theta}{5(1+\cos^2\theta) - 8\cos\theta}$$

$\frac{d\sigma}{d\Omega}$ exhibits a strong peak at $\theta = \pi$
(backwards direction)
due to the interference of the electric and magnetic dipole moment contributions.

$$\sigma = \frac{10\pi k^4 a^6}{3}$$

Some calculations

$$\frac{1}{2} \sum_{\lambda, \lambda_0} |\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2 = \frac{1}{2}(1 + \cos^2\theta)$$

$$\frac{1}{2} \sum_{\lambda, \lambda_0} (\hat{n} \cdot \hat{\varepsilon}_0)(\hat{n} \cdot \hat{\varepsilon}_0^*)(\hat{n}_0 \cdot \hat{\varepsilon}^*)(\hat{n}_0 \cdot \hat{\varepsilon})$$
$$= \frac{1}{2}\left[1 - (\hat{n}_0 \cdot \hat{n})^2\right]^2 = \frac{1}{2}(1 - \cos^2\theta)^2$$

$$\sum_{\lambda, \lambda_0} \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 \; \hat{n} \cdot \hat{\varepsilon}_0^* \; \hat{n}_0 \cdot \hat{\varepsilon} = [\delta_{ij} - n_i n_j][\delta_{kl} - (n_0)_k (n_0)_l]$$
$$\times \; \delta_{ik} (n_0)_j \, n_l$$
$$= \hat{n}_0 \cdot \hat{n} - \hat{n}_0 \cdot \hat{n} - \hat{n}_0 \cdot \hat{n} + (\hat{n}_0 \cdot \hat{n})^3$$
$$= \cos\theta(\cos^2\theta - 1) = -\sin^2\theta \cos\theta$$