Thomson scattering regained

$$\vec{E}_{inc}(\vec{x},t) = \hat{\varepsilon}_0 E_0 e^{ik_0\cdot\vec{x} - i\omega t}$$

$$m\vec{a} = -e\vec{E}_{inc}$$

$$\vec{a} = \frac{d^2\vec{x}}{dt^2}$$

(for the electron $q=-e$)

$$\vec{x}(t) = \frac{\hat{\varepsilon}_0 e E_0}{\omega^2 m} e^{ik\cdot\vec{x} - i\omega t}$$

$$\vec{p}(t) = -e\vec{x}(t) = -\frac{\hat{\varepsilon}_0 e^2}{\omega^2 m} e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$\vec{p}(t) = \vec{p}e^{-i\omega t}$$

Hence,

$$\frac{d\sigma}{d\Omega} = \frac{k^4 |\hat{\varepsilon}^*\cdot\vec{p}|^2}{|E_0|^2} = \frac{k^4 e^4}{m^2\omega^4} |\hat{\varepsilon}^*\cdot\hat{\varepsilon}_0|^2$$

$$\omega = kc$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2}{mc^2}\right)^2 |\hat{\varepsilon}^*\cdot\hat{\varepsilon}_0|^2$$

reproducing Thomson's cross section.

Applications

① Scattering by a small dielectric sphere of radius a

$a \ll \lambda$

with dielectric constant ε

Recall that

$$\phi(r,\theta,\phi) = \begin{cases} \dfrac{-3}{2+\varepsilon} E_0 r \cos\theta , & r<a \\ -E_0 r\cos\theta + \left(\dfrac{\varepsilon-1}{\varepsilon+2}\right) E_0 \dfrac{a^3}{r^2}\cos\theta , & r>a \end{cases}$$

Outside the sphere, there is an induced electric dipole moment

$$\vec{p} = \left(\frac{\varepsilon-1}{\varepsilon+2}\right) a^3 E_0$$

For the scattering problem,

$$\vec{p} = \left(\frac{\varepsilon-1}{\varepsilon+2}\right) a^3 \vec{E}_{inc}$$

$$\vec{m} = 0$$

$$\boxed{\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 |\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2}$$

Unpolarized cross section

average over initial polarizations

sum over final polarizations

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{1}{2} k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 (1+\cos^2\theta)$$

$$\sigma = \frac{8\pi}{3} k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2$$

If you measure the final state polarizations, then

$$\frac{d\sigma_\lambda}{d\Omega} = \frac{1}{2} k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2 \sum_{\lambda_0} |\hat{\varepsilon}^{(\lambda)*} \cdot \hat{\varepsilon}_0^{(\lambda_0)}|^2$$

$\lambda = 1, 2$

$\lambda = 1$ in the plane containing $\hat{n}_0, \hat{n}$

$\lambda = 2$ in the plane $\perp$ to the above plane

Polarization asymmetry

$$P \equiv \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}} = \frac{\sin^2\theta}{1+\cos^2\theta}$$

not power

② Scattering by a perfectly conducting sphere of radius a $(a \ll \lambda)$

$$\phi(r) = \begin{cases} -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta, & r > a \\ 0, & r < a \end{cases}$$

$$\vec{p} = a^3 \vec{E}_{inc}$$

$$\vec{m} = -\frac{a^3}{2}\vec{B}_{inc}$$

<u>Proof:</u> Magnetostatics problem. No $\vec{J}$. $\vec{\nabla}\times\vec{B} = 0$

$$\vec{B} = -\vec{\nabla}\Phi_m$$

$$\vec{\nabla}\cdot\vec{B} = 0 \implies \vec{\nabla}^2\Phi_m = 0$$

boundary conditions

at $r=\infty$ $\quad \vec{B} = B_0 \hat{z}$

$$\Longrightarrow \Phi_m = -B_0 r\cos\theta$$

at $r=a$ (surface), $\vec{B}\cdot\hat{n}_0 = 0$

$$\Longrightarrow \frac{\partial \Phi_m}{\partial r} = 0 \text{ at } r=a$$

$$\phi_m(r) = -B_0 r\cos\theta + \frac{A}{r^2}\cos\theta$$

Applying boundary condition at $r=a$: $\left(\frac{\partial \Phi_m}{\partial r}\right)_{r=a} = 0.$

$$\Longrightarrow A = -\frac{B_0 a^3}{2}.$$

$$\vec{B}_{inc} = \hat{n}_0 \times \vec{E}_{inc}$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2} \underbrace{(\hat{n}\times\hat{\varepsilon}^*)\cdot(\hat{n}_0\times\hat{\varepsilon}_0)}_{(\hat{n}\cdot\hat{n}_0)(\hat{\varepsilon}^*\cdot\varepsilon_0) - (\hat{n}\cdot\hat{\varepsilon}_0)(\hat{n}_0\cdot\hat{\varepsilon}^*)} \right|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta \right]$$

Polarization asymmetry

$$P = \frac{3\sin^2\theta}{5(1+\cos^2\theta) - 8\cos\theta}$$

$\frac{d\sigma}{d\Omega}$ exhibits a strong peak at $\theta = \pi$
(backwards direction)
due to the interference of the electric and magnetic dipole moment contributions.

$$\sigma = \frac{10\pi k^4 a^6}{3}$$

Some calculations

$$\frac{1}{2}\sum_{\lambda,\lambda_0} |\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2 = \frac{1}{2}(1+\cos^2\theta)$$

$$\frac{1}{2}\sum_{\lambda,\lambda_0} (\hat{n}\cdot\hat{\varepsilon}_0)(\hat{n}\cdot\hat{\varepsilon}_0^*)(\hat{n}_0\cdot\hat{\varepsilon}^*)(\hat{n}_0\cdot\hat{\varepsilon})$$
$$= \frac{1}{2}\left[1-(\hat{n}_0\cdot\hat{n})^2\right]^2 = \frac{1}{2}(1-\cos^2\theta)^2$$

$$\sum_{\lambda,\lambda_0} \hat{\varepsilon}^*\cdot\hat{\varepsilon}_0\ \hat{n}\cdot\hat{\varepsilon}_0^*\ \hat{n}_0\cdot\hat{\varepsilon} = \left[\delta_{ij} - n_i n_j\right]\left[\delta_{kl} - (n_0)_k (n_0)_l\right]$$
$$\times\ \delta_{ik}(n_0)_j\, n_l$$
$$= \hat{n}_0\cdot\hat{n} - \hat{n}_0\cdot\hat{n} - \hat{n}_0\cdot\hat{n} + (\hat{n}_0\cdot\hat{n})^3$$
$$= \cos\theta(\cos^2\theta - 1) = -\sin^2\theta\cos\theta$$

Spherical wave expansion of a vector plane wave

We have already studied the solutions to

$$(\vec{\nabla}^2 + k^2)\psi(\vec{r}) = 0$$

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_{\ell m}(r) Y_{\ell m}(\theta,\phi)$$

$$f_{\ell m}(r) = a_{\ell m} j_\ell(kr) + b_{\ell m} n_\ell(kr)$$

Consider the fact that $\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}$ is trivially a solution to this equation. Since $e^{i\vec{k}\cdot\vec{r}}$ is non-singular at the origin,

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m} j_\ell(kr) Y_{\ell m}(\theta,\phi)$$

Without loss of generality, we choose $\vec{k} = k\hat{z}$. Then,

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} a_\ell j_\ell(kr) P_\ell(\cos\theta)$$

Projecting out $a_\ell j_\ell(kr)$ using $\int_{-1}^{1} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d\cos\theta = \frac{2}{2\ell+1}\delta_{\ell\ell'}$

we have

$$a_\ell j_\ell(kr) = \frac{2\ell+1}{2}\int_{-1}^{1} e^{ikrw} P_\ell(w)\, dw \qquad w = \cos\theta$$

One can prove that

$$j_\ell(z) = \frac{i^{-\ell}}{2}\int_{-1}^{1} e^{izs} P_\ell(s)\, ds$$

Comparing these results implies that $a_\ell = i^\ell(2\ell+1)$

But, we need not even prove this. Since the unknown is a_ℓ, examine

$$a_\ell j_\ell(kr) = \frac{2\ell+1}{2}\int_{-1}^{1} e^{ikrw} P_\ell(w)\,dw$$

This must be true for all kr. Thus it is true for $kr \to \infty$. In this limit we can use:

$$j_\ell(kr) \sim \frac{\sin\left(kr - \frac{\pi\ell}{2}\right)}{kr}$$

Meanwhile, integrating by parts

$$\int_{-1}^{1} e^{ikrw} P_\ell(w)\,dw = \frac{1}{ikr} e^{ikrw} P_\ell(w)\Big|_{-1}^{1} - \frac{1}{ikr}\int_{-1}^{1} e^{ikrw} P_\ell'(w)\,dw$$

The second term is $O\left(\frac{1}{r^2}\right)$ as can be seen from further partial integration. Using $P_\ell(1) = 1$ and $P_\ell(-1) = e^{i\pi\ell}$,

$$\int_{-1}^{1} e^{ikrw} P_\ell(w)\,dw \sim \frac{1}{ikr}\left[e^{ikr} - e^{i\pi\ell} e^{-ikr}\right]$$

$$= \frac{e^{i\pi\ell/2}}{ikr}\left[e^{i(kr - \frac{\ell\pi}{2})} - e^{-i(kr - \frac{\ell\pi}{2})}\right]$$

$$= \frac{2i^\ell}{kr} \sin\left(kr - \frac{\ell\pi}{2}\right)$$

Thus,

$$a_\ell = (2\ell+1)\, i^\ell$$

Thus,

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr) P_l(\cos\theta)$$

Mor generally,

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr) P_l(\hat{k}\cdot\hat{r})$$

Using the addition theorem,

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr)\, Y_{lm}^*(\theta,\phi)\, Y_{lm}(\theta',\phi')$$

whe $\vec{k}$ makes angles θ', ϕ' with respect to a fixed z-axis.
$\vec{r}$ makes angles θ, ϕ " " " " " " .

Consider next the expansion of a circularly polarized plane wave:

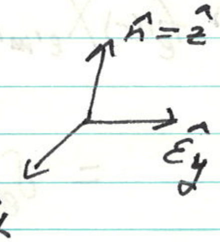
$$\vec{E}(\vec{r}) = \frac{E_0}{\sqrt{2}}(\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y)\, e^{ikz}$$

$$\vec{B}(\vec{r}) = \hat{n}\times\vec{E} = \mp i\vec{E}$$

$\hat{\varepsilon}_x \equiv \hat{x}, \quad \hat{\varepsilon}_y \equiv \hat{y}$

$$\hat{z}\times(\hat{x}\pm i\hat{y}) = \mp i(\hat{x}\pm i\hat{y}) = \mp i\hat{x} + \hat{y}$$

whe $\hat{\varepsilon}_1, \hat{\varepsilon}_2, \hat{n}$ form an orthonormal triad.



$$\hat{n}\times\hat{\varepsilon}_x = \hat{\varepsilon}_y$$
$$\hat{n}\times\hat{\varepsilon}_y = -\hat{\varepsilon}_x$$

We have already seen that the most general form for $\vec{E}, \vec{B}$ which solve

$$(\vec{\nabla}^2 + k^2)\vec{E} = 0$$
$$(\vec{\nabla}^2 + k^2)\vec{B} = 0$$

in a source-free region, are given by:

$$\vec{E}(\vec{r}) = \frac{E_0}{\sqrt{2}} \sum_{\ell m} \left[a_\pm(\ell,m)\, j_\ell(kr)\, \vec{X}_{\ell m} + \frac{i}{k} b_\pm(\ell,m)\, \vec{\nabla} \times j_\ell(kr) \vec{X}_{\ell m} \right]$$

$$\vec{B}(\vec{r}) = \frac{E_0}{\sqrt{2}} \sum_{\ell m} \left[-\frac{i}{k} a_\pm(\ell,m)\, \vec{\nabla} \times j_\ell(kr) \vec{X}_{\ell m} + b_\pm(\ell,m)\, j_\ell(kr)\, \vec{X}_{\ell m} \right]$$

where $$\vec{X}_{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L}\, Y_{\ell m}$$

Since the plane wave is finite everywhere, only $j_\ell(kr)$ appears since $n_\ell(kr)$ diverges as $r \to 0$.

We need to project out the $a_\pm$ and $b_\pm$. The relevant formulae are:

$$\int \left(f_{\ell'}(r) \vec{X}_{\ell' m'}\right)^* \cdot \left(g_\ell(r) \vec{X}_{\ell m}\right) d\Omega = f_\ell^* g_\ell\, \delta_{\ell\ell'} \delta_{mm'}$$

$$\int \left(f_{\ell'}(r) \vec{X}_{\ell' m'}\right)^* \cdot \left[\vec{\nabla} \times g_\ell(r) \vec{X}_{\ell m}\right] d\Omega = 0$$

Note: Use $\vec{\nabla} = \hat{n}\frac{\partial}{\partial r} - \frac{i}{r}\hat{n} \times \vec{L}$ to prove the last result, where $\vec{L} = -i r \hat{n} \times \vec{\nabla}$, and $\hat{n} = \frac{\vec{x}}{r}$.

$$\frac{1}{k^2} \int \left[\vec{\nabla} \times f_{\ell'}(r) \vec{X}_{\ell' m'}\right]^* \cdot \left[\vec{\nabla} \times g_\ell(r) \vec{X}_{\ell m}\right] d\Omega$$

$$= \delta_{\ell\ell'} \delta_{mm'} \left[f_\ell^* g_\ell + \frac{1}{k^2 r^2} \frac{\partial}{\partial r}\left[r f_\ell^* \frac{\partial}{\partial r}(r g_\ell)\right]\right]$$

where f_ℓ and g_ℓ are any linear combination of spherical Bessel functions.

In addition, it is useful to note that:

$$\vec{\nabla} \times f_\ell(r) \vec{X}_{\ell m} = \frac{i\hat{n}\sqrt{\ell(\ell+1)}}{r} f_\ell(r) Y_{\ell m} + \frac{1}{r}\frac{\partial}{\partial r}\left[r f_\ell(r)\right] \hat{n} \times \vec{X}_{\ell m}$$

where $\vec{x} = r\hat{n}$.

The result is

$$\frac{E_0}{\sqrt{2}} a_\pm(l,m)\, j_l(kr) = \int \vec{X}_{lm}^{\,*} \cdot \vec{E}(\vec{r})\, d\Omega$$

$$\frac{E_0}{\sqrt{2}} b_\pm(l,m)\, j_l(kr) = \int \vec{X}_{lm}^{\,*} \cdot \vec{B}(\vec{r})\, d\Omega$$

Inserting $\vec{E}(\vec{r}) = \frac{E_0}{\sqrt{2}}(\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y)e^{ikz}$,

$$\vec{L} \cdot (\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y) = L_\pm = L_\mp^*$$

So,

$$a_\pm(l,m)\, j_l(kr) = \frac{1}{\sqrt{l(l+1)}} \int \left[L_\mp Y_{lm}(\Omega)\right]^* e^{ikz}\, d\Omega$$

$$= \frac{\sqrt{(l\pm m)(l\mp m+1)}}{\sqrt{l(l+1)}} \int Y_{l,m\mp1}^*(\Omega)\, e^{ikz}\, d\Omega$$

Insert $\displaystyle e^{ikz} = \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)}\, j_l(kr)\, Y_{l0}(\theta,0)$

and use the orthogonality of the Y_{lm}. The result:

$$a_\pm(l,m) = i^l \sqrt{4\pi(2l+1)}\, \delta_{m,\pm1}$$

Likewise

$$b_\pm(l,m) = \mp i\, a_\pm(l,m)$$

We end up with

$$\vec{E}(\vec{r}) = \frac{E_0}{\sqrt{2}} \sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \left[j_\ell(kr) \vec{X}_{\ell,\pm 1} \pm \frac{1}{k} \vec{\nabla} \times j_\ell(kr) \vec{X}_{\ell,\pm 1} \right]$$

$$\vec{B}(\vec{r}) = \frac{E_0}{\sqrt{2}} \sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \left[-\frac{i}{k} \vec{\nabla} \times j_\ell(kr) \vec{X}_{\ell,\pm 1} \mp i j_\ell(kr) \vec{X}_{\ell,\pm 1} \right]$$

Note there is no $\ell=0$ term, since $L_\mp Y_{00} = 0$.

<u>interpretation</u>: for a circularly polarized wave, the m values of ± 1 correspond to ± 1 unit of angular momentum per photon parallel to the direction of propagation of the wave.

In particular,

$$(\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y) e^{ikz} = \sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \left[j_\ell(kr) \vec{X}_{\ell,\pm 1} \pm \frac{1}{k} \vec{\nabla} \times j_\ell(kr) \vec{X}_{\ell,\pm 1} \right]$$

which is expansion of a vector circularly polarized plane wave in terms of vector spherical harmonics.

<u>Note</u>: $e^{ikz} = e^{ikr\cos\theta}$. But there is ϕ dependence in $(\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y)$ when expressed in terms of $\hat{r}, \hat{\theta}, \hat{\phi}$. Likewise $\vec{X}_{\ell \pm 1}$ depends on θ, ϕ.

$$\vec{\nabla} \times h_\ell^{(1)}(kr) \vec{X}_{\ell,\pm 1} = \frac{i\hat{n}\sqrt{\ell(\ell+1)}}{r} h_\ell^{(1)}(kr) Y_{\ell m} + \frac{1}{r}\frac{\partial}{\partial r}\left[r h_\ell^{(1)}(kr) \right] \hat{n} \times \vec{X}_{\ell,\pm 1}$$

$$= \left(\frac{\partial}{\partial r} h_\ell^{(1)}(kr) \right) \hat{n} \times \vec{X}_{\ell,\pm 1} + O\left(\frac{1}{r^2}\right)$$

$$= \frac{i(-1)^{\ell+1} e^{ikr}}{r} \hat{n} \times \vec{X}_{\ell,\pm 1} + O\left(\frac{1}{r^2}\right)$$

Scattering of EM waves by a sphere

We have an EM wave incident along $\hat{z}$. The scattered wave consists of outgoing spherical waves. Thus, we write

$$\vec{E}(\vec{r}) = \vec{E}_{inc} + \vec{E}_{sc}$$

$$\vec{B}(\vec{r}) = \vec{B}_{inc} + \vec{B}_{sc}$$

where $\vec{E}_{inc}$ and $\vec{B}_{inc}$ have already been given. The scattered waves must be of the form:

$$\vec{E}_{sc} = \frac{1}{2}\frac{E_0}{\sqrt{2}}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[\alpha_\pm(l)\, h_l^{(1)}(kr)\,\vec{X}_{l,\pm1} \pm \frac{\beta_\pm(l)}{k}\vec{\nabla}\times h_l^{(1)}(kr)\,\vec{X}_{l,\pm1}\right]$$

$$\vec{B}_{sc} = \frac{1}{2}\frac{E_0}{\sqrt{2}}\sum_{l=1}^{\infty} i^l\sqrt{4\pi(2l+1)}\left[\frac{-i\alpha_\pm(l)}{k}\vec{\nabla}\times h_l^{(1)}(kr)\,\vec{X}_{l,\pm1} \mp i\beta_\pm(l)\, h_l^{(1)}(kr)\,\vec{X}_{l,\pm1}\right]$$

under the assumption that the scattering center is spherically symmetric (so no sum over m and only $m=\pm1$ occurs*). The factor of $\frac{1}{2}$ is for convenience.

* Spherical symmetry $\Rightarrow$ angular momentum conservation $\Rightarrow$ only $m=\pm1$ occurs.

In the asymptotic limit,

$$\vec{E}(\vec{r}) = \left[\frac{(\hat{\varepsilon}_x \pm i\hat{\varepsilon}_y)}{\sqrt{2}}e^{ikz} + \frac{e^{ikr}}{r}\vec{f}(\theta,\phi)\right]E_0$$

where $\vec{f}(\theta)$ is called the vector scattering amplitude.

Better: use formula for $\vec{\nabla}\times f_l(r)\vec{X}_l$ given in Jackson 3rd ed. eq (10.60)

Since $h_l^{(1)}(kr) \simeq (-i)^{l+1}\dfrac{e^{ikr}}{kr}$

$$\frac{i}{k}\vec{\nabla}\times\left(\frac{e^{ikr}}{kr}\vec{X}\right) = \frac{i}{k}\vec{\nabla}\left(\frac{e^{ikr}}{kr}\right)\times\vec{X} + \frac{i}{k}\frac{e^{ikr}}{kr}\vec{\nabla}\times\vec{X}$$

$$= -\frac{e^{ikr}}{kr}\hat{n}\times\vec{X} + O\left(\frac{1}{r^2}\right).$$

$$\hat{n} \equiv \frac{\vec{r}}{r}$$

So, as $r \to \infty$

$$\vec{E}_{sc} = \frac{e^{ikr}}{2ikr} \sum_{l=1}^{\infty} \sqrt{4\pi(2l+1)} \left[\vec{X}_{l,\pm 1}\, \alpha_{\pm}(l) \pm i\hat{n} \times \vec{X}_{l,\pm 1}\, \beta_{\pm}(l) \right] \left(\frac{E_0}{\sqrt{2}} \right)$$

Thus

$$\vec{f}(\theta,\phi) = \frac{1}{2ik} \sum_{l=1}^{\infty} \sqrt{2\pi(2l+1)} \left[\vec{X}_{l,\pm 1}\, \alpha_{\pm}(l) \pm i\hat{n} \times \vec{X}_{l,\pm 1}\, \beta_{\pm}(l) \right]$$

As usual, the power scattered is

$$P_{sc} = \frac{c}{8\pi} \mathrm{Re} \int (\vec{E}_{sc} \times \vec{B}_{sc}^{*}) \cdot \hat{n}\, r^2 \, d\Omega$$

As $r \to \infty$, $\vec{B}_{sc} = \hat{n} \times \vec{E}_{sc}$

Recall: $\hat{n} \cdot \vec{L} = 0$ so that $\hat{n} \cdot \vec{X}_{l \pm 1} = 0$.

<u>check</u>:

$$\vec{B}_{sc} = \frac{e^{ikr}}{2ikr} \sum_{l=1}^{\infty} \sqrt{4\pi(2l+1)} \left[\alpha_{\pm}(l)\, \hat{n} \times \vec{X}_{l,\pm 1} \mp i\beta_{\pm}(l)\, \vec{X}_{l,\pm 1} \right] \left(\frac{E_0}{\sqrt{2}} \right)$$

$$(\vec{E}_{sc} \times \vec{B}_{sc}^{*}) \cdot \hat{n} = \left[\vec{E}_{sc} \times (\hat{n} \times \vec{E}_{sc}^{*}) \right] \cdot \hat{n}$$

$$= |\vec{E}_{sc}|^2 \qquad \text{since } \hat{n} \cdot \vec{E}_{sc} = 0$$

$$P_{sc} = \frac{c}{8\pi} |\vec{E}_{sc}|^2 \, r^2 \, d\Omega$$

$$\vec{E}_{sc} = E_0 \frac{e^{ikr}}{r} \vec{f}(\theta,\phi)$$

Incident flux:

$$\frac{c}{8\pi}|\vec{E}_{inc}|^2 = \frac{c}{8\pi}|E_0|^2 \qquad \text{if} \quad \vec{E}_{inc} = (\hat{e}_x \pm i\hat{e}_y)e^{ikz}\frac{E_0}{\sqrt{2}}$$

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{8\pi}{|E_0|^2 c}\frac{dP}{d\Omega} = \frac{1}{|E_0|^2}|\vec{E}_{sc}|^2 r^2 = |\vec{f}(\theta,\phi)|^2$$

$$= \frac{\pi}{2k^2}\left|\sum_{l=1}^{\infty}(2l+1)^{1/2}\left[\alpha_\pm(l)\vec{X}_{l,\pm1} \pm i\beta_\pm(l)\,\hat{n}\times\vec{X}_{l,\pm1}\right]\right|^2$$

Integrating,

$$\sigma_{sc} = \frac{\pi}{2k^2}\sum_{l,l'}(2l+1)^{1/2}(2l'+1)^{1/2}\int d\Omega\left[\alpha_\pm(l)\alpha_\pm^*(l)\,\vec{X}_{l,\pm1}\vec{X}^*_{l',\pm1} + \dots\right]$$

Use:

$$\int \vec{X}^*_{l'm'}(\Omega)\cdot\vec{X}_{lm}(\Omega)\,d\Omega = \delta_{ll'}\delta_{mm'}$$

$$\int X^*_{l'm'}(\Omega)\cdot\left[\hat{n}\times\vec{X}_{lm}\right]d\Omega = 0$$

$$(\hat{n}\times\vec{X}_{lm})\cdot(\hat{n}\times\vec{X}^*_{l'm'}) = \vec{X}_{lm}\cdot\vec{X}^*_{l'm'} \qquad \text{since} \quad \hat{n}\cdot\vec{X}_{lm} = 0.$$

So we get

$$\boxed{\sigma_{sc} = \frac{\pi}{2k^2}\sum_{l=1}^{\infty}(2l+1)\left[|\alpha_\pm(l)|^2 + |\beta_\pm(l)|^2\right]}$$