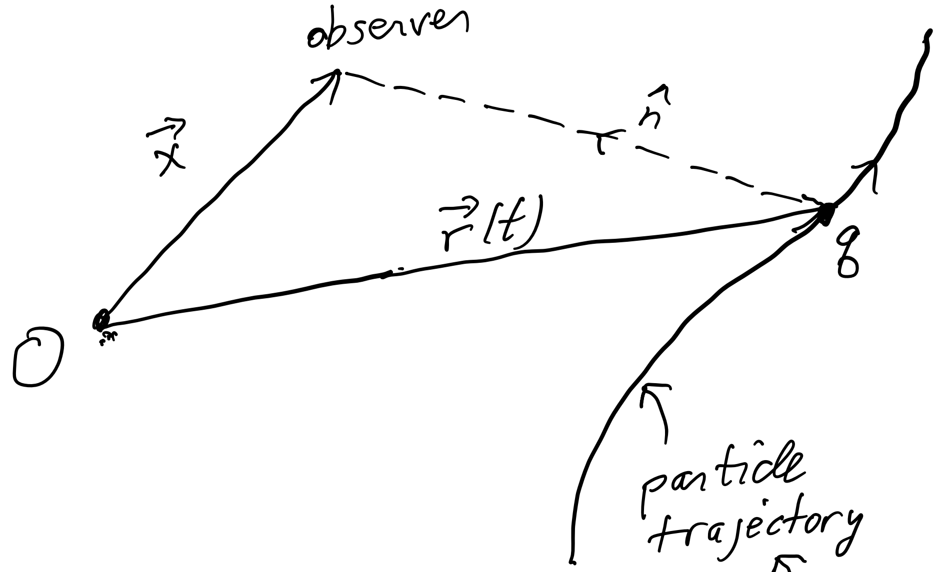
Radiation by moving charges



$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$t' = t - \frac{|\vec{x} - \vec{x}'|}{c} \qquad \vec{x}' = \vec{r}(t)$$

$$\rho(\vec{x}', t) = q\,\delta(\vec{x}' - \vec{r}(t))$$

$$\vec{J}(\vec{x}', t) = \rho(\vec{x}', t)\,\vec{v}(t)$$

$$\vec{A}(\vec{x}, t) = \frac{1}{c}\int d^3x' \, \frac{[\vec{J}(\vec{x}', t')]_{ret}}{|\vec{x} - \vec{x}'|}$$

$$\Phi(\vec{x}, t) = \int d^3x' \, \frac{[\rho(\vec{x}', t')]_{ret}}{|\vec{x} - \vec{x}'|}$$

$$\Phi(\vec{x},t) = q\int d^3x' \, \frac{\delta\left(\vec{x}' - \vec{r}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right)}{|\vec{x}-\vec{x}'|}$$

$$\vec{y} = \vec{x}' - \vec{r}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$d^3x' = \left|\det\left(\frac{\partial x'}{\partial y}\right)\right| d^3y \equiv |J|\, d^3y$$

Inverse matrix

$$\frac{\partial y_i}{\partial x'_j} = \delta_{ij} - \frac{\partial}{\partial x'_j} r_i\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$= \delta_{ij} + v_i\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\frac{\partial}{\partial x'_j}\frac{|\vec{x}-\vec{x}'|}{c}$$

Note that

$$\frac{\partial}{\partial x_i}|\vec{x}| = \frac{\partial}{\partial x_i}(\vec{x}\cdot\vec{x})^{1/2} = \frac{1}{2}(\vec{x}\cdot\vec{x})^{-1/2}\frac{\partial}{\partial x_i}(\vec{x}\cdot\vec{x})$$

$$= (\vec{x}\cdot\vec{x})^{-1/2}\,\vec{x}\cdot\frac{\partial\vec{x}}{\partial x_i}$$

$$= (\vec{x}\cdot\vec{x})^{-1/2}\, x_j \underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}}$$

$$= (\vec{x}\cdot\vec{x})^{-1/2}\, x_i$$

Thus

$$\frac{\partial}{\partial x_i} |\vec{x}| = \frac{x_i}{|\vec{x}|}$$

$$\frac{\partial}{\partial x'_j} |\vec{x}-\vec{x}'| = -\frac{\partial}{\partial (x-x')_j} |\vec{x}-\vec{x}'|$$

$$= \frac{(x'-x)_j}{|\vec{x}-\vec{x}'|}$$

Hence,

$$\frac{\partial y_i}{\partial x'_j} = \delta_{ij} - \frac{1}{c} \frac{(\vec{x}-\vec{x}')_j}{|\vec{x}-\vec{x}'|} v_i\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$\det(\delta_{ij} - a_i b_j) = 1 + \vec{a}\cdot\vec{b}$$

$$J^{-1} \equiv \det\left(\frac{\partial y_i}{\partial x'_j}\right) = 1 - \frac{1}{c|\vec{x}-\vec{x}'|} (\vec{x}-\vec{x}')\cdot\vec{v}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$\vec{\beta} = \frac{\vec{v}}{c}$$

$$\Phi(\vec{x},t) = q \int |J|\, d^3y\, \frac{\delta(\vec{y})}{|x-\vec{x}'|}$$

Set $\vec{y}=0$

$$\Rightarrow \vec{x}' = \vec{r}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)$$

Introduce $R = |\vec{x}-\vec{x}'|$ $\qquad \vec{\beta} = \dfrac{\vec{v}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}{c}$

$$\hat{n} = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|}$$

$$\boxed{\Phi(\vec{x},t) = \left.\frac{q}{R(1-\vec{\beta}\cdot\hat{n})}\right|_{\vec{x}'=\vec{r}\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}}$$

$$\boxed{\vec{A}(\vec{x},t) = \left.\frac{q\vec{\beta}}{R(1-\vec{\beta}\cdot\hat{n})}\right|_{\vec{x}'=\vec{r}\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}}$$

Liénard-Wiechert potentials for a point charge.

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}$$

$$t_{ret} = t - \frac{1}{c}\left|\vec{x} - \vec{r}(t_{ret})\right|$$

$$\frac{\partial t_{ret}}{\partial t} = 1 - \frac{1}{c}\frac{\vec{x}-\vec{r}(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|}\cdot\frac{\partial}{\partial t}\left|\vec{x}-\vec{r}(t_{ret})\right|$$

$$\frac{1}{c}\frac{\partial\vec{r}(t_{ret})}{\partial t} = \vec{\beta}\,\frac{\partial t_{ret}}{\partial t}$$

$$\frac{\partial t_{ret}}{\partial t} = 1 - \vec{\beta}\cdot\hat{n}\,\frac{\partial t_{ret}}{\partial t} \qquad \hat{n} = \frac{\vec{x}-\vec{r}(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|}$$

$$\Rightarrow \boxed{\frac{\partial t_{ret}}{\partial t} = \left.\frac{1}{1-\vec{\beta}\cdot\hat{n}}\right|_{\vec{x}'=\vec{r}\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}}$$

$$dt_{ret} = -\frac{1}{c}\frac{\vec{x}-\vec{r}(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|}\,d(\vec{x}-\vec{r}(t_{ret}))$$

(at fixed t)

$$= -\frac{1}{c}\,\hat{n}\cdot\left[d\vec{x} - c\vec{\beta}(t_{ret})\,dt_{ret}\right]$$

$$dt_{ret}\,(1-\vec{\beta}\cdot\hat{n}) = -\frac{1}{c}\,\hat{n}\cdot d\vec{x}$$

$$\boxed{\vec{\nabla} t_{ret} = \frac{dt_{ret}}{d\vec{x}} = -\frac{1}{c}\,\frac{\hat{n}}{1-\vec{\beta}\cdot\hat{n}}\Bigg|_{\vec{x}'=\vec{r}\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}}$$

Final answer:

$$\vec{E}(\vec{x},t) = \left.\frac{q(1-\beta^2)(\hat{n}-\vec{\beta})}{R^2(1-\vec{\beta}\cdot\hat{n})^3} + \frac{q\,\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\frac{d\vec{\beta}}{dt}\right]}{cR(1-\vec{\beta}\cdot\hat{n})^3}\right|_{\vec{x}'=\vec{r}(t_{ret})}$$

$$\vec{B}(\vec{x},t) = \left.\frac{q(1-\beta^2)\,\vec{\beta}\times\hat{n}}{R^2(1-\vec{\beta}\cdot\hat{n})^3} + \frac{q\,\hat{n}\times\left\{\hat{n}\times\left[(\hat{n}-\vec{\beta})\times\frac{d\vec{\beta}}{dt}\right]\right\}}{CR(1-\vec{\beta}\cdot\hat{n})^3}\right|_{\vec{x}'=\vec{r}(t_{ret})}$$

$$= \hat{n}(t_{ret})\times\vec{E}(\vec{x},t)$$

$$1-\beta^2 = \frac{1}{\gamma^2}$$

If $\frac{d\vec{\beta}}{dt}=0$, then I recover the result previously obtained for a charge moving at constant velocity.

Radiation fields are "proportional" to $\dfrac{d\vec{\beta}}{dt}$.

Power liberated by the accelerating charge at t_{ret}.

$$dP = \vec{S}(\vec{x},t)\cdot\hat{r}\,da$$

$\hat{r} = \dfrac{\vec{x}}{|\vec{x}|}$

$da \simeq R^2 d\Omega$ as $|\vec{x}| \to \infty$

$$P(t_{ret}) = \frac{d\overset{\text{energy}}{E}}{dt_{ret}} = \frac{dE}{dt}\frac{dt}{dt_{ret}}$$

$$= (1-\vec{\beta}\cdot\hat{n})P$$

$\vec{S} = \dfrac{c}{4\pi}(\vec{E}\times\vec{B})$

$$\frac{dP(t_{ret})}{d\Omega} = R^2(1-\vec{\beta}\cdot\hat{n})\,\vec{S}\cdot\hat{n}$$

$\hat{r} \simeq \hat{n}$ as $|\vec{x}| \to \infty$

$$= \frac{cR^2}{4\pi}(1-\vec{\beta}\cdot\hat{n})\left[\vec{E}\times(\hat{n}\times\vec{E})\right]\cdot\hat{n}$$

$$= \frac{cR^2}{4\pi}(1-\vec{\beta}\cdot\hat{n})\left[|\vec{E}|^2 - \underbrace{(\vec{E}\cdot\hat{n})^2}_{0}\right]$$

But $\vec{E}_{rad}\cdot\hat{n} = 0$.

$$\boxed{\frac{dP(t_{ret})}{d\Omega} = \frac{q^2}{4\pi c}\,\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}\,]|^2}{(1-\vec{\beta}\cdot\hat{n})^5}}$$

$$\vec{\alpha} \equiv \frac{d\vec{\beta}}{dt} = \frac{\vec{a}}{c}$$

$$\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}\,] = (\hat{n}-\vec{\beta})\,\hat{n}\cdot\vec{\alpha} - \vec{\alpha}(1-\vec{\beta}\cdot\hat{n})$$

$$|\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}\,]|^2 = (\beta^2-1)(\hat{n}\cdot\vec{\alpha})^2 + \alpha^2(1-\vec{\beta}\cdot\hat{n})^2$$
$$+\,2(\vec{\alpha}\cdot\vec{\beta})(\vec{\alpha}\cdot\hat{n})(1-\vec{\beta}\cdot\hat{n})$$

$$I_1 = \int \frac{(\hat{n}\cdot\vec{\alpha})^2}{(1-\vec{\beta}\cdot\hat{n})^5}\,d\Omega$$

$$I_2 = \int \frac{d\Omega}{(1-\vec{\beta}\cdot\hat{n})^3}$$

see class handout for the evaluation of these integrals

$$I_3 = \int \frac{\hat{n}\cdot\vec{\alpha}}{(1-\vec{\beta}\cdot\hat{n})^4}\,d\Omega$$

$$\boxed{P = \frac{2q^2}{3c}\left[\frac{\alpha^2 - |\vec{\beta}\times\vec{\alpha}|^2}{(1-\beta^2)^3}\right]}$$

$\alpha \equiv |\vec{\alpha}|$ $\qquad \dfrac{1}{(1-\beta^2)^3} = \gamma^6$

Non-relativistic limit

$$\beta = \frac{v}{c} \ll 1$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}|\hat{n}\times(\hat{n}\times\vec{\alpha})|^2$$

$$\hat{n}\times(\hat{n}\times\vec{\alpha}) = \hat{n}(\hat{n}\cdot\vec{\alpha}) - \vec{\alpha}$$

$$|\hat{n}\times(\hat{n}\times\vec{\alpha})|^2 = |\vec{\alpha}|^2 - (\hat{n}\cdot\vec{\alpha})^2$$

$$= \alpha^2 \sin^2\theta$$

θ = angle between $\hat{n}$ and $\vec{\alpha}$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3}\sin^2\theta \qquad P = \frac{2}{3}\frac{q^2 a^2}{c^3}}$$

$\alpha = \dfrac{a}{c}$

Larmor's formula

$$\int d\Omega \sin^2\theta = 2\pi \int_{-1}^{1} d\cos\theta\,(1-\cos^2\theta) = \frac{8\pi}{3}$$

Note: In the general case (relativistic), boost to the instantaneous rest frame:

$$P = \frac{2}{3}\frac{q^2 a_{co}^2}{c^3} \qquad \text{(in the comoving frame)}$$

Eq. (58) of "Examples of four-vectors"

$$a_{co}^2 = \gamma^6\left[|\vec{a}|^2 - \frac{|\vec{v}\times\vec{a}|^2}{c^2}\right]$$

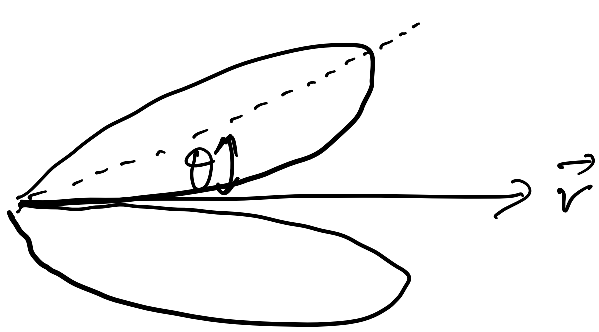
$$\Longrightarrow \quad P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[\alpha^2 - |\vec{\beta}\times\vec{\alpha}|^2\right]$$

Examples: ① relativistic linear motion $\vec{\beta} \parallel \vec{\alpha}$

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

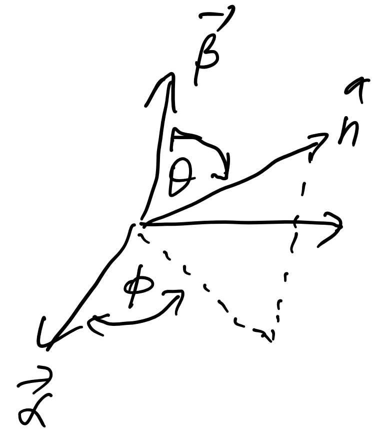
θ = angle between $\hat{n}$ and $\vec{\beta}$

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}\gamma^6$$



$$\theta_{max} \simeq \frac{1}{2\gamma}$$

② relativistic circular motion $\vec{\beta} \perp \vec{\alpha}$ $\quad (\vec{\alpha}\cdot\vec{\beta}=0)$



$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \frac{1}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$

non-relativistic limit

$$\frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3}\left[1-(\hat{n}\cdot\vec{\alpha})^2\right]$$

$$= \frac{q^2 a^2}{4\pi c^3} \sin^2\theta_a$$

$$P = \frac{2}{3}\frac{q^2 a^2}{c^3}\gamma^4$$

Let's compare transverse and longitudinal acceleration at fixed magnitude of the applied force.

$$P = \frac{2q^2}{3c}\gamma^6\left[\alpha^2 - |\vec{\beta}\times\vec{\alpha}|^2\right]$$

$$= -\frac{2}{3}\frac{q^2}{m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}$$

$$m\alpha^\mu = \frac{dp^\mu}{d\tau}$$

acceleration 4-vector

$$\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = \gamma^2\frac{dp_\mu}{dt}\frac{dp^\mu}{dt}$$

$$d\tau = \gamma^{-1}dt$$

$$= \gamma^2\left[\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2 - \left(\frac{d\vec{p}}{dt}\right)^2\right]$$

$$\frac{d\vec{p}}{dt} = \gamma mc\left[\vec{\alpha} + \gamma^2\vec{\beta}(\vec{\beta}\cdot\vec{\alpha})\right]$$

$$\frac{dE}{dt} = \frac{d}{dt}(\gamma mc^2) = mc^2\frac{d\gamma}{dt} = m\gamma^3c^2\vec{\beta}\cdot\vec{\alpha}$$

$$\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = \gamma^4m^2c^2\left[\gamma^4(\vec{\beta}\cdot\vec{\alpha})^2(1-\beta^2) - \alpha^2 - 2\gamma^2(\beta\cdot\alpha)^2\right]$$

Use $1-\beta^2 = \frac{1}{\gamma^2}$

$$\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = -\gamma^6 m^2 c^2\left[\alpha^2 - |\vec{\beta}\times\vec{\alpha}|^2\right]$$

Compare transverse and longitudinal acceleration

1. Transverse $\vec{\beta}\cdot\vec{\alpha} = 0$

$$\frac{dp_\mu}{dt}\frac{dp^\mu}{d\tau} = -\gamma^2\left(\frac{d\vec{p}}{dt}\right)^2$$

after noting that $\frac{dE}{dt} = m\gamma^3 c^2 \vec{\beta}\cdot\vec{\alpha} = 0$

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\left(\frac{dp}{dt}\right)^2$$

2. Longitudinal $\vec{\beta} \parallel \vec{\alpha}$

$$\frac{d\vec{p}}{dt} = \gamma^3 mc\vec{\alpha}$$

$$\frac{dE}{dt} = \gamma^3 mc^2\beta\alpha = c\beta\frac{dp}{d\tau}$$

$$\gamma^2\left[\frac{1}{c^2}\left(\frac{dE}{dt}\right)^2 - \left(\frac{d\vec{p}}{dt}\right)^2\right] = \frac{\gamma^2}{\beta^2-1}\left(\frac{dp}{dt}\right)^2$$

$$= -\left(\frac{dp}{dt}\right)^2$$

$$P = \frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

Importance consequence

Consider a charge accelerating and moving at a extreme relativistic speed, the radiation is a coherent superposition of contributions from longitudinal and transverse acceleration. But the latter dominates!

Hence, at any instant you can view the charge as if it were radiating while traveling along the arc of a circular path with a radius of curvature

$$R = \frac{v^2}{\dot{v}_\perp} \simeq \frac{c^2}{\dot{v}_\perp}$$

(remember that $a = \frac{v^2}{r}$)