Radiation from an accelerating charge

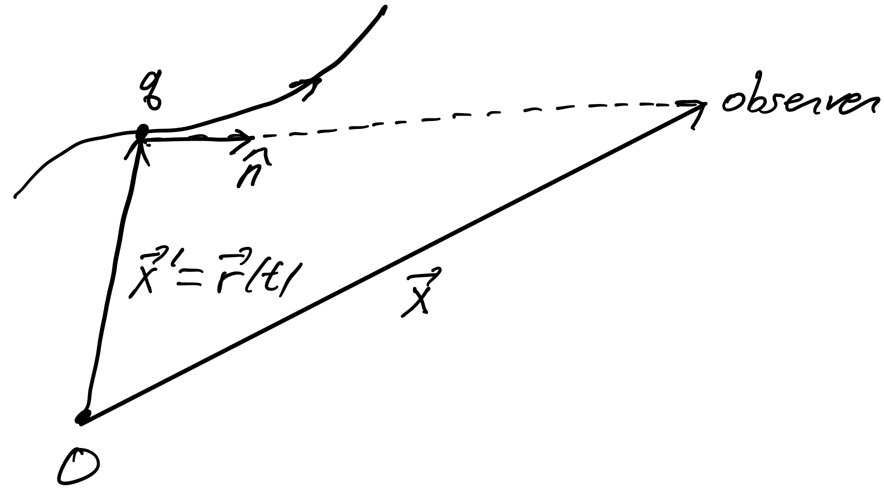
$\vec{R} = \vec{x} - \vec{x}'$

$\hat{n} = \hat{R}$

$R = |\vec{R}|$

$r = |\vec{x}|$

$r' = |\vec{x}'|$



In the radiation zone, $r' \ll r$

$$R = |\vec{x} - \vec{x}'| = (r^2 + r'^2 - 2\vec{x}\cdot\vec{x}')^{1/2}$$
$$= r\left(1 - \frac{2\vec{x}\cdot\vec{x}'}{r^2} + \frac{r'^2}{r^2}\right)^{1/2}$$
$$\simeq r\left(1 - \frac{2\hat{r}\cdot\vec{x}'}{r}\right)^{1/2}$$
$$\simeq r - \hat{r}\cdot\vec{x}'$$

$$\hat{r} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{r}$$

At large $r \gg r'$, we have $\hat{r} \simeq \hat{n}$. Hence,

$$R \simeq r - \hat{n}\cdot\hat{x}'$$

<u>Note:</u>

$$\hat{r} - \hat{n} = \frac{\vec{x}}{r} - \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|} \simeq \frac{\vec{x}}{r} - \frac{\vec{x} - \vec{x}'}{r\left(1 - \frac{\hat{r}\cdot\vec{x}'}{r}\right)}$$
$$\simeq \frac{\vec{x}}{r} - \frac{\vec{x} - \vec{x}'}{r}\left(1 + \frac{\hat{r}\cdot\vec{x}'}{r}\right)$$
$$\simeq \frac{\vec{x}'}{r} - \frac{\vec{x}(\hat{r}\cdot\vec{x}')}{r^2} \simeq \frac{\vec{x}' - \hat{r}(\hat{r}\cdot\vec{x}')}{r} = O\left(\frac{r'}{r}\right)$$

Spectral decomposition of radiation

$$\vec{E}(\vec{x},t) = \int_{-\infty}^{\infty} d\omega\, \vec{E}_\omega(\vec{x}) e^{-i\omega t}$$

Remark: Jackson includes $\frac{1}{\sqrt{2\pi}}$

$\vec{E}(\vec{x},t)$ is a real (physical) field

$$\vec{E}(\vec{x},t) = \vec{E}^*(\vec{x},t)$$

$$\Rightarrow \vec{E}_\omega(\vec{x}) = \vec{E}^*_{-\omega}(\vec{x})$$

$$\vec{E}_\omega(\vec{x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \vec{E}(\vec{x},t) e^{i\omega t}$$

$$P = \int dP = \int \vec{S}(\vec{x},t)\cdot\hat{r}\, da \qquad da = r^2 d\Omega$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

$$\vec{E} = \frac{q}{c}\left.\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}]}{R(1-\hat{n}\cdot\vec{\beta})^3}\right|_{\vec{x}'=\vec{r}(t_{ret})}$$

$$t_{ret} = t - \frac{1}{c}|\vec{x}' - \vec{r}(t_{ret})|$$

$$\vec{B} = \hat{n}(t_{ret})\times\vec{E}$$

$$\vec{E}\times\vec{B} = \vec{E}\times(\hat{n}\times\vec{E}) = \hat{n}|\vec{E}|^2 \qquad \text{since } \hat{n}\cdot\vec{E} = 0$$

$$\vec{S} = \frac{c}{4\pi} |\vec{E}|^2 \hat{n}$$

$$\hat{r}\cdot\hat{n} = \hat{r}\cdot(\hat{r}+\hat{n}-\hat{r}) = 1 + \underbrace{\hat{r}\cdot(\hat{n}-\hat{r})}_{O\left(\frac{r'}{r}\right)} \simeq 1$$

$$P = \frac{c}{4\pi}\int r^2 d\Omega\, |\vec{E}|^2$$

$$= \frac{cr^2}{4\pi}\int d\Omega \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, \vec{E}_\omega^* \cdot \vec{E}_{\omega'}\, e^{i(\omega-\omega')t}$$

as $r \to \infty$

Total energy radiated through the sphere at large r

$$\lim_{r\to\infty} \frac{cr^2}{4\pi}\int d\Omega \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, \vec{E}_\omega^* \cdot \vec{E}_{\omega'}\, e^{i(\omega-\omega')t}$$

Use $$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(\omega-\omega')t} = \delta(\omega-\omega')$$

<u>Remark</u>: This is an idealized situation where the particle accelerates from $t=-\infty$ to $t=+\infty$.

Total energy radiated

$$\lim_{r\to\infty} \frac{cr^2}{2}\int d\Omega \int_{-\infty}^{\infty} d\omega \, |\vec{E}_\omega(\vec{x})|^2$$

$$= \lim_{r\to\infty} cr^2 \int d\Omega \int_{0}^{\infty} d\omega \, |\vec{E}_\omega(\vec{x})|^2$$

since $\vec{E}^*_\omega = \vec{E}_{-\omega}$

$$= \int d\Omega \int_0^\infty d\omega \, \frac{d^2 I}{d\Omega \, d\omega}$$

I = intensity of the radiation

Finally, noting that $R \simeq r - \hat{n}\cdot\vec{x}' \simeq r$

$$|\vec{E}|^2 \sim \frac{1}{R^2}$$

$$\boxed{\frac{d^2 I}{d\Omega \, d\omega} = cR^2 |\vec{E}_\omega|^2}$$

<u>Remark</u>: by a change of variables $t \to t_{ret}$

$$\int_0^\infty d\omega \, \frac{d^2 I}{d\omega \, d\Omega} = \int_{-\infty}^{\infty} dt_{ret} \, \frac{dP(t_{ret})}{d\Omega}$$

$$R\vec{E} = \frac{q}{c} \frac{\hat{n} \times [(\hat{n}-\vec{\beta}) \times \vec{\alpha}]}{(1-\hat{n}\cdot\vec{\beta})^3} \Big|_{\vec{x}'=\vec{r}(t_{ret})}$$

$$\vec{E}_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, \vec{E}(\vec{x},t) e^{i\omega t}$$

$$t_{ret} = t - \frac{1}{c} |\vec{x}' - \vec{r}(t_{ret})|$$

$$t = t_{ret} + \frac{1}{c} |\vec{x}' - \vec{r}(t_{ret})|$$

$$\simeq t_{ret} + \frac{1}{c}(r - \hat{r}\cdot\vec{x}')$$

$$\simeq t_{ret} + \frac{1}{c}(r - \hat{n}\cdot\vec{x}')$$

$$\frac{\partial t}{\partial t_{ret}} = 1 - \hat{n}\cdot\vec{\beta}$$

$$\vec{E}_\omega = \frac{e^{i\omega r/c}}{2\pi} \int_{-\infty}^{\infty} dt_{ret}\, \vec{E}(\vec{x}, t_{ret})\, e^{i\omega(t_{ret} - \vec{x}'\cdot\hat{n}/c)} \times (1 - \hat{n}\cdot\vec{\beta}),$$

where $\vec{x}' = \vec{r}(t_{ret})$

Notation going forward

$t_{ret} \rightarrow t$

$$R|\vec{E}_\omega| = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, R\vec{E}(\vec{x},t)\, e^{i\omega(t - \frac{\hat{n}\cdot\vec{r}(t)}{c})} (1 - \hat{n}\cdot\vec{\beta})$$

Hence,

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt\, e^{i\omega\left(t-\frac{\hat{n}\cdot\vec{r}(t)}{c}\right)} \cdot \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}]}{(1-n\cdot\vec{\beta})^2}\right|^2}$$

One can integrate by parts

$$\frac{d}{dt}\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}} = \frac{d}{dt}\frac{\hat{n}(\hat{n}\cdot\vec{\beta})-\vec{\beta}}{1-\vec{\beta}\cdot\hat{n}}$$

$$= \frac{(1-\vec{\beta}\cdot\hat{n})[\hat{n}(\hat{n}\cdot\vec{\alpha})-\vec{\alpha}] + \vec{\alpha}\cdot\hat{n}[\hat{n}(\hat{n}\cdot\vec{\beta})-\vec{\beta}]}{(1-\vec{\beta}\cdot\hat{n})^2}$$

$$= \frac{(\hat{n}-\vec{\beta})\vec{\alpha}\cdot\hat{n} - \vec{\alpha}[\hat{n}\cdot(\hat{n}-\vec{\beta})]}{(1-\hat{\beta}\cdot\hat{n})^2}$$

$$= \frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\vec{\alpha}]}{(1-\vec{\beta}\cdot\hat{n})^2}$$

$$R\vec{E}_\omega = \frac{q}{2\pi c}\int_{-\infty}^{\infty} dt\, e^{i\omega\left(t-\frac{\hat{n}\cdot\vec{r}(t)}{c}\right)} \frac{d}{dt}\left(\frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right)$$

$$= \frac{q}{2\pi c}\left\{ e^{i\omega\left(t-\frac{\hat{n}\cdot\vec{r}(t)}{c}\right)} \frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\Bigg|_{-\infty}^{\infty}\right.$$

$\lim_{\omega\to\infty} e^{i\omega t} = 0$
in the sense of distributions
[Riemann-Lesbegue]

$$\left. - \int_{-\infty}^{\infty} dt\; i\omega(1-\vec{\beta}\cdot\hat{n})\, e^{i\omega\left(t-\frac{\hat{n}\cdot\vec{r}(t)}{c}\right)} \frac{\hat{n}\times(\hat{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\hat{n}}\right\}$$

Drop term that oscillates infinitely fast, as it effectively averages out to zero.

$$R\vec{E}_\omega = \frac{-iq\omega}{2\pi c}\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\vec{\beta})\, e^{i\omega\left(t-\frac{\hat{n}\cdot\vec{r}(t)}{c}\right)} dt$$

$$\boxed{\frac{d^2 I}{d\omega\, d\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\vec{\beta})\, e^{i\omega\left(t-\frac{\hat{n}\cdot\vec{r}(t)}{c}\right)} dt\right|^2}$$

If $\vec{\beta}$ is constant then $\vec{r}(t) = \vec{v}t$. Integrating over t yields $\delta\left(1-\frac{\hat{n}\cdot\vec{v}}{c}\right) = 0$ since $v < c$.

Application to synchrotron radiation — see Jackson section 14.6

Cherenkov radiation

Radiation of a charge moving at constant velocity in matter.

Consider isotropic, homogeneous medium

$$\vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H} \qquad \text{(cgs units)}$$

ε, μ constants

Maxwell's eqs.

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi\rho}{\varepsilon}, \quad \vec{\nabla} \cdot \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\mu}{c} \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi \vec{J}}{c} + \frac{\varepsilon}{c} \frac{\partial \vec{E}}{\partial t}$$

$\vec{J}, \rho$ are the "free" currents and charges

Note: $\vec{H} \to \vec{H}'/\mu \qquad c \to c\sqrt{\mu\varepsilon}$

$\vec{E} \to \vec{E}'/\sqrt{\mu\varepsilon} \qquad \{e, \rho, \vec{J}\} \to \{e', \rho', \vec{J}'\}\sqrt{\frac{\varepsilon}{\mu}}$

$$\vec{\nabla}\cdot\vec{E}' = 4\pi\rho', \quad \vec{\nabla}\cdot\vec{H}' = 0$$

$$\vec{\nabla}\times\vec{E}' = -\frac{1}{c}\frac{\partial \vec{H}'}{\partial t}$$

$$\vec{\nabla}\times\vec{H}' = \frac{4\pi}{c'}\vec{J}' + \frac{1}{c'}\frac{\partial \vec{E}'}{\partial t}$$

Simplify further by considering medium with $\mu = 1$

$$\vec{E} \to \sqrt{\varepsilon}\,\vec{E}$$

$$\vec{B} \to \vec{B}$$

$$c \to \frac{c}{\sqrt{\varepsilon}}$$

$$e \to \frac{e}{\sqrt{\varepsilon}}$$

For an electron, choose

$$q = -e$$

$$\vec{v} = c\vec{\beta}$$

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2\omega^2\sqrt{\varepsilon}}{4\pi^2 c^3}\left|\int_{-\infty}^{\infty} \hat{n}\times(\hat{n}\times\vec{v})\, e^{i\omega\left(t - \frac{\sqrt{\varepsilon}\,\hat{n}\cdot\vec{r}(t)}{c}\right)} dt\right|^2$$

$n_r = \sqrt{\varepsilon}$ index of refraction.

Suppose $\vec{v}$ is constant. Then $\vec{r}(t) = \vec{v}t$

$$t - \frac{\sqrt{\varepsilon}\,\hat{n}\cdot\vec{r}(t)}{c} = t\left(1 - n_r\frac{\hat{n}\cdot\vec{v}}{c}\right) = t\left(1 - \frac{\vec{k}\cdot\vec{v}}{\omega}\right)$$

since $\omega = kc/n_r$ $\qquad \hat{n} = \vec{k}/|\vec{k}|$.

$$\frac{d^2 I}{d\omega\, d\Omega} = \frac{e^2 \omega^2 n_r}{4\pi^2 c^3} |\hat{n}\times\vec{v}|^2 \left| \int_{-\infty}^{\infty} dt\, e^{i\omega t(1-\vec{k}\cdot\vec{v}/\omega)} \right|^2$$

Note:

$$|\hat{n}\times(\hat{n}\times\vec{v})|^2$$
$$= |\hat{n}(\hat{n}\cdot\vec{v}) - \vec{v}|^2$$
$$= |\vec{v}|^2 - (\hat{n}\cdot\vec{v})^2$$
$$= |\hat{n}\times\vec{v}|^2$$

Here, $\delta\left(1 - \frac{n_r \hat{n}\cdot\vec{v}}{c}\right) \neq 0$

if $v > \frac{c}{n_r}$

Thus, radiation is possible if

$$\frac{c}{n_r} < v < c$$

In problem set 5, you will derive

$$\vec{E}_\omega(\vec{x}) = \frac{-ie\omega}{2\pi c^2 r} e^{i\omega r/c} \int_{-\infty}^{\infty} dt'\, e^{i\omega\left[t' - \frac{\hat{n}\cdot\vec{r}(t')}{c}\right]} \hat{n}\times(\hat{n}\times\vec{v})$$
$$+ O\left(\frac{1}{r^2}\right)$$

For $\vec{r}(t') = \vec{v}t'$

$$\vec{E}_\omega(\vec{x}) = \frac{-ie}{c^2 r} e^{i\omega r/c}\, \hat{n}\times(\hat{n}\times\vec{v})\, \delta\left(1 - \frac{\hat{n}\cdot\vec{v}}{c}\right)$$

after using $\delta\left(\omega\left(1 - \frac{\hat{n}\cdot\vec{v}}{c}\right)\right) = \frac{1}{\omega}\delta\left(1 - \frac{\hat{n}\cdot\vec{v}}{c}\right)$

In a medium,

$$\vec{E}_\omega(\vec{x}) = \frac{-ie}{c^2 r} e^{in_r\omega r/c} \hat{n}\times(\hat{n}\times\vec{v})\, \delta\left(1 - \frac{n_r \hat{n}\cdot\vec{v}}{c}\right)$$

$$\frac{d^2 I}{d\omega\, d\Omega} = n_r c r^2 |E_\omega(\vec{x})|^2$$

$$= \frac{n_r e^2}{c^3} |\hat{n}\times(\hat{n}\times\vec{v})|^2\, \delta^2\left(1 - \frac{n_r \hat{n}\cdot\vec{v}}{c}\right)$$

$$= \frac{n_r e^2 \omega^2}{c^3} |\hat{n}\times\vec{v}|^2\, \delta^2(\omega - \vec{k}\cdot\vec{v})$$

$$\delta^2(\omega - \vec{k}\cdot\vec{v}) = \delta(\omega - \vec{k}\cdot\vec{v})\,\delta(\omega - \vec{k}\cdot\vec{v})$$

$$= \delta(\omega - \vec{k}\cdot\vec{v})\,\delta(0)$$

[used $f(x)\delta(x) = f(0)\delta(x)$]

Interpret $\delta(0) = \frac{1}{2\pi}\int_{-T}^{T} e^{i\omega t}\, dt \Big|_{\omega=0} = \frac{T}{\pi}$

T is the observation time over entire radiation process

$$\frac{d^2 I}{d\omega\, d\Omega} = \int_{-\infty}^{\infty} dT\, \frac{dP}{d\omega\, d\Omega}$$

$$\frac{d^2 I}{d\omega\, d\Omega} = \int_{-\infty}^{\infty} dT\, \frac{dP}{d\omega\, d\Omega}$$

Then, identifying $P = I/(2T)$, we end up with:

$$\boxed{\frac{d^2 P}{d\omega\, d\Omega} = \frac{n_r e^2 \omega^2}{4\pi^2 c^3} |\hat{n}\times\vec{v}|^2 \int_{-\infty}^{\infty} dt\, e^{i\omega(1-n_r \hat{n}\cdot\vec{v}/c)}}$$

the largest contribution to the integral is for $t \sim O\left(\frac{1}{\omega}\right)$

$$\frac{d^2 P}{d\omega\, d\Omega} = \frac{n_r e^2 \omega^2}{2\pi c^3} |\hat{n}\times\vec{v}|^2 \delta\left(\omega\left(1 - \frac{n_r \vec{n}\cdot\vec{v}}{c}\right)\right)$$

Introduce $\vec{k} = \frac{n_r \omega}{c} \hat{n}$

Using $\delta(ax) = \frac{1}{|a|}\delta(x)$,

$$\frac{d^2 P}{d\omega\, d\Omega} = \frac{n_r e^2 \omega^2}{2\pi c^3} |\hat{n}\times\vec{v}|^2 \delta(\omega - \vec{k}\cdot\vec{v})$$

Integrate over $d\Omega = 2\pi\, d\cos\psi$ $\qquad$ $\cos\psi = \hat{n}\cdot\hat{v}$ $\quad$ $(\vec{v} = v\hat{v})$

$$\boxed{\frac{dP}{d\omega} = \frac{e^2 \omega v}{c^2}\left(1 - \frac{c^2}{n_r^2 v^2}\right) \Theta(n_r v - c)}$$

Tamm-Frank

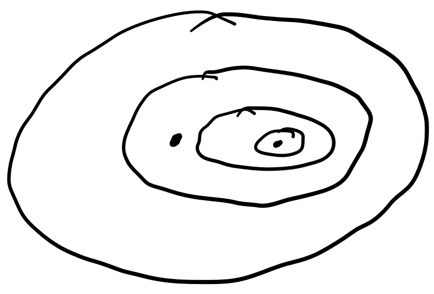
$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \qquad \frac{d\Theta}{dx} = \delta(x)$$

<u>Remark:</u> $n_r = n_r(\omega)$

$$n_r(\omega) \to 1 \text{ as } \omega \to \infty$$

Cherenkov radiation operates only over a narrow (finite) frequency band where $n_r(\omega)\, v > c$.

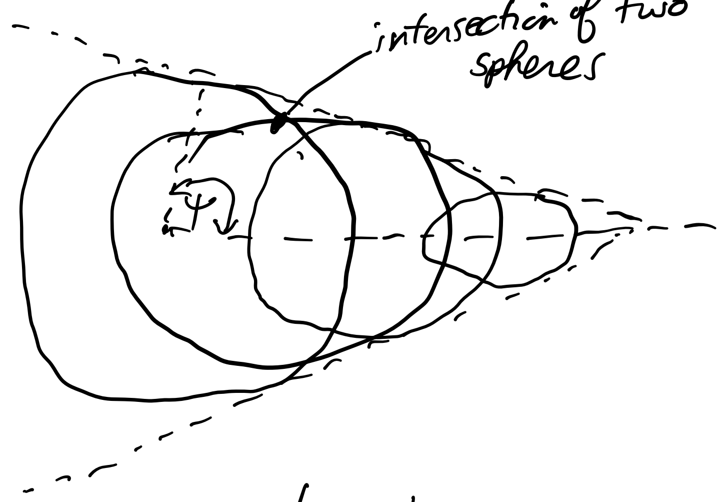
<u>Mach cone</u>



$|\leftarrow vt \rightarrow|$

$|\leftarrow \frac{ct}{n_r} \rightarrow|$

$v < \frac{c}{n_r}$



$|\leftarrow \frac{ct}{n_r} \rightarrow|$

$|\leftarrow vt \rightarrow|$

$v > \frac{c}{n_r}$

two "retarded times"

$$t_{ret} = t - \frac{|\vec{x} - \vec{r}(t_{ret})|}{c}$$

Also $(1 - \hat{n}\cdot\vec{\beta})^{-1}$ has a singularity when $v > \frac{c}{n_r}$.