The Addition Theorem of Spherical Harmonics

The addition theorem for spherical harmonics states that

\[ P_\ell(\cos \theta) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}') Y_{\ell m}^*(\hat{n}'') , \quad \text{where} \ \cos \theta \equiv \hat{n}' \cdot \hat{n}'' . \]  

(1)


An (actively) rotated state vector, corresponding to a state with no internal spin degrees of freedom, is denoted by

\[ |\psi\rangle_R \equiv U[R]|\psi\rangle , \]  

(2)

where \( R \equiv R(\chi \hat{u}) \) is a counterclockwise rotation by an angle \( \chi \) about a fixed axis along the unit vector \( \hat{u} \), and \( U[R] = e^{-i\hat{u} \cdot \mathbf{L}/\hbar} \) is the corresponding unitary operator that acts on quantum states of the Hilbert space. Likewise, the coordinate basis ket \( |\vec{x}\rangle \) can also be rotated,

\[ |\vec{x}'\rangle = U[R]|\vec{x}\rangle , \quad \text{where} \ \vec{x}' = R\vec{x} . \]  

(3)

We also define unit vectors, \( \hat{n} \equiv \vec{x}/|\vec{x}| \) and \( \hat{n}' \equiv \vec{x}'/|\vec{x}'| \), where \( \hat{n}' = R\hat{n} \). With respect to a fixed z-axis, \( \hat{n} \) points in a direction with polar angle \( \theta \) and azimuthal angle \( \phi \), and \( \hat{n}' \) points in a direction with polar angle \( \theta' \) and azimuthal angle \( \phi' \).

Since \( U[R] \) is a unitary operator, we can write,

\[ \psi(\vec{x}) = \langle \vec{x} | \psi \rangle = \langle \vec{x} | U^\dagger[R] U[R] | \psi \rangle = \langle \vec{x}' | \psi \rangle_R = \psi_R(\vec{x}') , \]

after employing eqs. (2) and (3). That is,

\[ \psi_R(\vec{x}') = \psi(\vec{x}) = \psi(R^{-1}\vec{x}') . \]  

(4)

Consider the state vector \( |\psi\rangle = |\ell \ m\rangle \), which is a simultaneous eigenstate of \( \mathbf{L}^2 \) and \( L_z \) with corresponding eigenvalues \( \hbar^2 \ell(\ell + 1) \) and \( \hbar m \), respectively. Then, using eq. (2),

\[ \psi_R(\vec{x}') = \langle \vec{x}' | \psi \rangle_R = \langle \vec{x}' | U[R] | \ell \ m\rangle = \sum_{\ell=0}^{\infty} \sum_{m'=-\ell}^{\ell} \langle \vec{x}' | \ell \ m'\rangle \langle \ell \ m' | U[R] | \ell \ m \rangle . \]  

(5)

Note that \( D_{m'm}^{(\ell)}(R) \delta_{\ell\ell'} \equiv \langle \ell \ m' | U[R] | \ell \ m \rangle \), since \( \{ \mathbf{L}, L_i \} = 0 \) implies that the matrix elements of \( \mathbf{L} \) (as well as any function of \( \mathbf{L} \)) are diagonal in \( \ell \). In addition, \( Y_{\ell m'}(\hat{n}') = \langle \vec{x}' | \ell \ m' \rangle \). Hence, eq. (5) yields

\[ \psi_R(\vec{x}') = \sum_{m'=-\ell}^{\ell} Y_{\ell m'}(\hat{n}') D_{m'm}^{(\ell)}[R] . \]  

(6)

\footnote{Since eq. (4) is true for any \( \vec{x}' \) (which can be treated as a dummy variable), we are free to drop the prime superscript and write \( \psi_R(\vec{x}) = \psi(R^{-1}\vec{x}) \).}

\footnote{Since \( \theta' \) and \( \phi' \) are the polar and azimuthal angles of the unit vector \( \hat{n}' \), we denote \( Y_{\ell m'}(\hat{n}') \equiv Y_{\ell m'}(\theta', \phi') \).}
In light of eq. (4), \( \psi_R(\bar{x}') = \psi(\bar{x}) = \langle \bar{x} | \ell m \rangle = Y_{\ell m}(\hat{n}). \) Hence, eq. (6) yields,

\[
Y_{\ell m}(\hat{n}) = \sum_{m'=-\ell}^{\ell} Y_{\ell m'}(\hat{n}') D_{m'm}^{(\ell)}[R], \quad \text{where } \hat{n}' = R\hat{n}.
\]  

(7)

Note that an equivalent method for deriving eq. (7) starts from the observation that if \( \hat{n}' = R\hat{n} \), then \( |\hat{n}'\rangle = U[R]|\hat{n}\rangle \). It then follows that

\[
|\hat{n}'\rangle = |\hat{n}\rangle U^\dagger[R].
\]  

(8)

Since \( U[R] \) is a unitary operator, we can write,

\[
|\hat{n} \ell m\rangle = |\hat{n}\rangle U^\dagger[R]U[R] |\ell m\rangle = \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} |\hat{n}' \ell' m'\rangle \langle \ell' m'|U[R]|\ell m\rangle
\]

\[
= \sum_{\ell=0}^{\infty} \sum_{m'=-\ell}^{\ell'} \langle \hat{n}' | \ell' m' \rangle D_{m'm}^{(\ell)}[R] \delta_{\ell\ell'} = \sum_{m'=-\ell}^{\ell} \langle \hat{n}' | \ell m' \rangle D_{m'm}^{(\ell)}[R],
\]

which is equivalent to eq. (7).

The \( z \) axis points along the unit vector \( \hat{z} \). Given the rotation \( R \) that appears in eq. (7), we define a unit vector \( \hat{n}'' \equiv R\hat{z} \). If the rotation \( R = R(\alpha, \beta, \gamma) \) is parameterized by its Euler angles, \( \alpha, \beta, \gamma \), then \( R(\alpha, \beta, \gamma) = R(\alpha\hat{z})R(\beta\hat{y})R(\gamma\hat{z}) \), where the three rotations are applied from right to left. Note that the direction of \( \hat{n}'' \) does not depend on \( \gamma \), since \( R(\gamma\hat{z})\hat{z} = \hat{z} \). Hence, \( \hat{n}'' = R(\alpha, \beta, \gamma)\hat{z} = R(\alpha\hat{z})R(\beta\hat{y})\hat{z} \). That is, the unit vector \( \hat{n}'' \) has polar angle \( \beta \) and azimuthal angle \( \alpha \) with respect to the \( z \)-axis.

To obtain the addition theorem for spherical harmonics, we set \( m = 0 \) in eq. (7) and use the identities,\(^3\)

\[
Y_{\ell 0}(\hat{n}) = \sqrt{\frac{2\ell + 1}{4\pi}} P_\ell(\cos \theta),
\]  

(9)

\[
D_{m'0}^{(\ell)}(\alpha, \beta, \gamma) = \sqrt{\frac{4\pi}{2\ell + 1}} Y_{m'}^{*}(\hat{n}''),
\]  

(10)

where \( \theta \) is the polar angle of \( \hat{n} \) with respect to the \( z \)-axis. It then immediately follows that

\[
P_\ell(\cos \theta) = \frac{4\pi}{2\ell + 1} \sum_{m'=-\ell}^{\ell} Y_{\ell m'}(\hat{n}') Y_{\ell m}^{*}(\hat{n}'').
\]  

(11)

after relabeling \( m' \) with \( m \). Finally, we use \( \hat{n}' = R\hat{n} \) and \( \hat{n}'' = R\hat{z} \) to obtain,\(^4\)

\[
\hat{n}' \cdot \hat{n}'' = (R_{ij}n_j)(R_{ik}z_k) = \delta_{jk}n_jz_k = \hat{n} \cdot \hat{z} = \cos \theta.
\]  

(12)

Hence, we have reproduced the addition theorem of spherical harmonics given in eq. (1).

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\(^3\)A proof of eq. (10) appears in Appendix C of the class handout entitled, *Clebsch-Gordan coefficients and the tensor spherical harmonics*.

\(^4\)In eq. (12), there is an implicit sum over the repeated indices. Since the rotation matrix \( R \) is orthogonal, it follows that \( R_{ij}R_{ik} = R_{ij}^TR_{ik} = (R^T)_{jk} = \delta_{jk} \).