Chapter 8

Distributions and Fourier Transforms

8.1 Introduction

In this chapter we develop some more mathematical tools. Again these tools are not important for computational purposes, but they are important as a justification for the calculations performed in practice. Although physicists need not, as a rule, practice mathematics with the same rigour as a mathematicians they need to know mathematics sufficiently well to know when it is safe to be sloppy. Furthermore in areas such as quantum field theory where it is not known whether the difficulties encountered are due to bad mathematics, or bad physics, or both it is important to ensure that the mathematics at least, is correct. To this end, we give here a brief introduction to some of the results of modern analysis. The presentation, although still at a submathematical level is intended for the more mathematically inclined student. We give definitions and theorems, but the proofs for the theorems are only sketched, or omitted altogether. To compensate for this we list several relevant references at the end of the chapter.

8.2 Functionals

A function may be considered as a mapping from a certain well-defined set of numbers called the domain into another set of numbers called the range. Thus, if f denotes a function then $f(x)$ denotes the value of the function f at the point x. This distinction is not always made but clearly there is such a distinction. We shall now consider a mapping whose domain is a set of functions called test functions and whose range is the set of real numbers. Such a mapping is called a functional. If T is a functional then $T(f)$ is the *value* of the functional T at the function f. Thus the arguments of functionals are functions. From the class of all possible functionals we pick out a particularly simple class, namely the linear functionals. A functional T is linear if for f and g belonging to the domain of T and a, b two numbers

$$T(af + bg) = aT(f) + bT(g) . \tag{8.2.1}$$

An example of such a functional is

$$T(f) = \int_{-\infty}^{\infty} t(x)\, f(x)\, dx \tag{8.2.2}$$

where $t(x)$ is a fixed function and $f(x)$ is in the domain of T if the right hand side is convergent. Furthermore a functional T is bounded if for all f in a given space $|T(f)| \leq c \parallel f \parallel$ where c is a positive constant. There is a remarkable theorem for bounded linear functionals on a Hilbert space.

Riesz Representation Theorem

Let $\mathcal{H}$ be a Hilbert space and T a bounded linear functional on $\mathcal{H}$. Then there exists a uniquely determined vector f_T of $\mathcal{H}$ such that

$$T(g) = (f_T, g) \tag{8.2.3}$$

for all $g \in \mathcal{H}$. Conversely, of course, any vector $f \in \mathcal{H}$ defines a bounded linear functional T_f by

$$T_f(g) = (f, g) . \tag{8.2.4}$$

Proof

The proof is rather straightforward and is a proof by construction. Uniqueness is obvious. For suppose f' is another vector besides f_T satisfying (8.2.3), then

$$(f' - f_T, g) = 0 \tag{8.2.5}$$

for all $g \in \mathcal{H}$. Thus $f' - f_T = 0$ as desired. To prove that f_T exists consider the null space N_T of T where

$$N_T = \{g \in \mathcal{H} \mid T(g) = 0\} . \tag{8.2.6}$$

If $N_T = \mathcal{H}$ take $f_T = 0$. This is the trivial case. Now assume $N_T \neq \mathcal{H}$. Then there exists at least one vector $f_0 \neq 0$ belonging to $N_T^{\perp}$, the orthogonal complement of N_T . In this case define

$$f_T = \frac{T(f_0)^*}{\parallel f_0 \parallel^2}\, f_0 . \tag{8.2.7}$$

This is the desired f_T as we now prove. Suppose $g \in N_T$. Then

$$T(g) = 0 = (f_T, g) . \tag{8.2.8}$$

Next, if g is of the form

$$g = \alpha f_0 \tag{8.2.9}$$

then we have

$$(f_T, g) = (f_T, \alpha f_0) = \alpha T(f_0) = T(g) \tag{8.2.10}$$

as required. We now show that any $g \in \mathcal{H}$ can be written

$$g = \alpha f_0 + \beta f_1 \tag{8.2.11}$$

where $f_1 \in N_T$. To prove this recall that

$$T(f_T) \neq 0\,. \tag{8.2.12}$$

Then we have the identity

$$g = \left(g - \frac{T(g)}{T(f_T)} f_T\right) + \frac{T(g)}{T(f_T)} f_T \tag{8.2.13}$$

which is of the form (8.2.11). Thus, since T is linear we have completed our proof and shown that

$$T(g) = (f_T, g) \tag{8.2.14}$$

for all $g \in \mathcal{H}$. This shows that on a Hilbert space the only linear functionals are those given by inner products. We want to extend this notion somewhat. Therefore, it is natural that we must go beyond the concept of Hilbert space.

In general to define a space we must have a criterion for deciding when two points of the space are "close". This criterion defines the *topology* of the space. For example, in the finite dimensional vector spaces $\mathcal{E}_n$ we use the Euclidean norm $(x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ to measure closeness. In Hilbert space we use the norm

$$\| f \| = (f, f)^{1/2} \tag{8.2.15}$$

to measure closeness. For functions one also frequently uses point-wise estimates of the form $|f(x) - g(x)|$. All of these criteria are useful and define different topologies. For functionals one also has an estimate which is derived by analogy with (8.2.2). Thus, if T and S are bounded linear functionals, meaning that there are positive constants c, c' such that

$$T(f) \leq c \| f \| \tag{8.2.16}$$

and

$$S(f) \leq c' \| f \| \tag{8.2.17}$$

for all f in a given space $\mathcal{X}$, then T and S are "close" if $|T(f) - S(f)|$ is small. Here $\| f \|$ denotes the appropriate norm in $\mathcal{X}$. Thus, the notion of "close" (or topology) of the linear functionals on $\mathcal{X}$ is derived from the topology of $\mathcal{X}$ itself.

Dual Space

Let $\mathcal{X}$ be a space of functions with a given topology. Now consider the set $\mathcal{X}'$ of all bounded linear functionals on $\mathcal{X}$. Then $\mathcal{X}'$ is itself a linear vector space with the topology of $\mathcal{X}'$ determined by the topology of $\mathcal{X}$. We call $\mathcal{X}'$ the dual space of $\mathcal{X}$.

An example of these concepts is Hilbert space itself. In this case the dual of $\mathcal{H}$ is $\mathcal{H}$ itself. In fact it is logically correct to consider the inner product on a Hilbert space as being formed by elements from two spaces, the Hilbert space $\mathcal{H}$ and its dual, which is of course a copy of $\mathcal{H}$.

The point of all this is that one can take linear functionals that are as singular or pathological as one wishes if it is possible to find a space of functions sufficiently nice to compensate for these pathologies. The space of nice functions is called the space of test-functions. There are many test-function spaces. One of the most useful of these is the Schwarz space $\mathcal{S}$. Its dual space is called $\mathcal{S}'$, the space of *tempered distributions* and is sufficiently general to encompass almost any kind of "function" we shall encounter. To describe these spaces we need some more terminology.

A function with continuous derivatives up to and including the nth is called C^n. Thus, continuous functions are called C^0. If a function is C^n for all n it is called C^∞. Using this terminology we can define $\mathcal{S}$ as the space of all C^∞ functions which together with their derivatives vanish at infinity faster than the inverse of any polynomial. To make this more explicit we define the sequence of semi-norms [1]

$$\| f \|_{r,n} = \sup_x \left| x^r \frac{d^n f}{dx^n} \right| \tag{8.2.18}$$

where "sup" means "least upper bound". In that case f belongs to $\mathcal{S}$ iff

$$\| f \|_{r,n} < \infty \tag{8.2.19}$$

for all integers r, n. This specifies the topology or notion of closeness in $\mathcal{S}$ So, for example, a sequence $\{f_j\}$ of functions in $\mathcal{S}$ converges to f if for each r and n

$$\lim_{j\to\infty} \| f_j - f \|_{r,n} = 0\ . \tag{8.2.20}$$

In terms of this the *tempered distributions* also have a topology whose definition can be made very similar to the ϵ, δ definition for ordinary functions. Thus, T is continuous at f_0 if given an $\epsilon > 0$ there exist integers r, n and a $\delta > 0$ such that for

$$\| f - f_0 \|_{r,n} < \delta \tag{8.2.21}$$

we have

$$|T(f) - T(f_0)| < \epsilon\ . \tag{8.2.22}$$

One way to ensure that (8.2.22) follows from (8.2.21) is to insist that for all r, n there exists a positive constant c such that

$$|T(f)| \le c \ \| f \|_{r,n} \tag{8.2.23}$$

[1] The difference between a semi-norm and a norm is that a semi-norm may vanish for a given element even though that element is different from zero.

since then

$$|T(f) - T(f_0)| = |T(f - f_0)| \leq c \ \| f - f_0 \|_{r,n} \quad . \tag{8.2.24}$$

In fact, there is a theorem that states that every *continuous linear functional* T on $\mathcal{S}$ satisfies (8.2.23). This means that one can use (8.2.23) to define the topology on $\mathcal{S}'$.

To prepare us for future applications we introduce two more notations for distributions. To specify the value of T at f we have used $T(f)$. We can also write this (T, f). This does not mean we have an inner product, it is simply another way of writing $T(f)$. As a matter of fact as physicists we carry this even a step further and write this as

$$T(f) \equiv \int_{-\infty}^{\infty} T(x)\, f(x)\, dx \quad . \tag{8.2.25}$$

Again this is a purely symbolic way of writing $T(f)$ and does not imply that any integral such as (8.2.25) exists in any of the usual senses of integral. Nevertheless, the notation (8.2.25) is extremely suggestive and thus if applied with due caution one may treat this expression as an integral.

The most common of the distributions so treated is the δ function. It is defined by

$$\delta(f) = f(0) \quad . \tag{8.2.26}$$

On the other hand we frequently write this as

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)\,. \tag{8.2.27}$$

It is an easy matter to prove that no function δ with the property (8.2.27) can exist. [2] However, if we realize that (8.2.27) does not imply a genuine integral of a function δ and is just another, but very suggestive, way of writing (8.2.26) then all objections to writing (8.2.27) are removed. That δ is not a function can also be seen from the fact that although functions may be multiplied by functions to give functions, distributions cannot generally be multiplied by distributions or functions. For example, if we consider the product of $1/x$ and $\delta(x)$ this is not defined in general. Nevertheless there is a smaller domain for which this product makes sense. An even more acute example is the product

$$\delta(x)\delta'(x) \tag{8.2.28}$$

where $\delta'(x)$ is the derivative of $\delta(x)$.

We now define differentiation of distributions. In fact the definition is given by analogy with integration by parts using (8.2.25). Thus, we define

$$\frac{d^n T}{dx^n}(f) = (-1)^n\, T\left(\frac{d^n f}{dx^n}\right) \quad . \tag{8.2.29}$$

[2] See von Neumann's book [8.4], pages 23-25.

This expression is obviously well-defined for all $T \in \mathcal{S}'$ since if $f \in \mathcal{S}$ so is $\frac{d^n f}{dx^n} \in \mathcal{S}$. In the notation (8.2.25) the definition of the derivative reads

$$\int_{-\infty}^{\infty} \frac{d^n T}{dx^n} f(x)\, dx = (-1)^n \int_{-\infty}^{\infty} T(x) \frac{d^n f}{dx^n}\, dx \,. \tag{8.2.30}$$

It is a simple matter to generalize these results to test functions of several variables and distributions over these variables. Thus, if $f(x_1, x_2, \ldots, x_k)$ is an element of $\mathcal{S}^{(k)}$ in each variable, then we may have a distribution T in the dual space $\mathcal{S}'^{(k)}$ such that $T(f)$ is well defined. Again another possible symbolic notation for $T(f)$ would be

$$T(f) = \int T(x_1, x_2, \ldots, x_k)\, f(x_1, x_2, \ldots, x_k)\, dx_1 dx_2 \ldots dx_k \quad . \tag{8.2.31}$$

We emphasize once more that although (8.2.31) looks like an integral it is *not*. This is simply a symbolic way of writing $T(f)$. Nevertheless we shall use this way of writing almost all the time since it is the standard notation for physicists.

8.3 Fourier Transforms

Consider the linear transformations $\mathcal{F}$ and $\bar{\mathcal{F}}$ defined on $\mathcal{S}$ according to

$$(\mathcal{F}f)(p) = \int_{-\infty}^{\infty} e^{-ipx}\, f(x)\, dx \equiv F(p) \tag{8.3.32}$$

$$(\bar{\mathcal{F}}F)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx}\, F(p)\, dp \ . \tag{8.3.33}$$

Clearly (8.3.32) defines a uniformly and absolutely convergent integral since e^{-ipx} can only improve the convergence of an already splendidly convergent integral. We shall now prove that $\mathcal{F}$ and $\bar{\mathcal{F}}$ map $\mathcal{S}$ onto $\mathcal{S}$ in a continuous one to one manner. The proof will give us as a side benefit the formal result

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip(x-y)}\, dp = \delta(x-y) \,. \tag{8.3.34}$$

Consider the expression

$$\lim_{\epsilon \to 0+} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx}\, dp\, e^{-\epsilon p^2} \int_{-\infty}^{\infty} e^{-ipy}\, f(y)\, dy \ . \tag{8.3.35}$$

Now, for $\epsilon > 0$ both integrals exist and we may interchange their order. Furthermore,

$$\int_{-\infty}^{\infty} e^{ipx - \epsilon p^2}\, dp = \sqrt{\frac{\pi}{\epsilon}} \exp\{-x^2/4\epsilon\} \ . \tag{8.3.36}$$

Thus, we get for (8.3.35)

$$\lim_{\epsilon \to 0+} \frac{1}{\sqrt{4\pi\epsilon}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4\epsilon}\, f(y)\, dy \ . \tag{8.3.37}$$

Now consider a circle $(x-y)^2 = R^2$. Clearly, due to the factor $e^{-(x-y)^2/4\epsilon}$ any contribution to the integral (8.3.37) from points outside the circle vanishes in the limit as $\epsilon \to 0$. Thus, we can estimate the difference between $f(x)$ and (8.3.37) by

$$\left|\frac{1}{\sqrt{4\pi\epsilon}} \int_{|x-y\leq R} e^{-(x-y)^2/4\epsilon} [f(y) - f(x)]\, dy\right| \\ \leq \sup_{|x-y|\leq R} |f(x) - f(y)| \to 0 \text{ as } R \to 0 \ . \tag{8.3.38}$$

This justifies (8.3.34) and shows that

$$\bar{\mathcal{F}}\mathcal{F} = 1 \, . \tag{8.3.39}$$

Using (8.3.34) we now also get

$$\mathcal{F}\bar{\mathcal{F}} = 1 \, . \tag{8.3.40}$$

Also we have that $\bar{\mathcal{F}}$, $\mathcal{F}$ map $\mathcal{S}$ onto $\mathcal{S}$ as stated.

Now, in mapping $\mathcal{S}$ onto a copy of itself using $\mathcal{F}$ what happens to $\mathcal{S}'$? In order to keep things well-defined, $\mathcal{S}'$ must be mapped onto $\mathcal{S}'$. Using the symbolic notation of (8.2.25) this is trivial to see. Since $\mathcal{F}$ is a unitary operator on Hilbert space we have for $f, g \in \mathcal{H}$ that

$$(\mathcal{F}f, \mathcal{F}g) = (f, g) \, . \tag{8.3.41}$$

This is known as *Parseval's theorem* and written out reads

$$\int_{-\infty}^{\infty} F^*(p)\, G(p)\, dp = \int_{-\infty}^{\infty} f^*(x)\, g(x)\, dx \tag{8.3.42}$$

or

$$\int_{-\infty}^{\infty} F^*(p)\, dp \int_{-\infty}^{\infty} e^{-ipx}\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx \int_{-\infty}^{\infty} e^{-ipx}\, F^*(p)\, dp \tag{8.3.43}$$

where we have used the formulae defining F and G in terms of f and g and vice-versa. (8.3.43) is already in the desired form to define the Fourier transform of distributions. Thus suppose $g \in \mathcal{S}'$ then (8.3.43) reads

$$(\mathcal{F}g)\,(F^*) = g\left((\mathcal{F}F)^*\right) \, . \tag{8.3.44}$$

Thus we *define* the Fourier transform of distributions in $\mathcal{S}'$ using (8.3.44). In other words if $T \in \mathcal{S}'$ then the Fourier transform $\mathcal{F}T$ is defined by

$$(\mathcal{F}T)\,(f) = T\left((\mathcal{F}f)\right) \tag{8.3.45}$$

where $T \in \mathcal{S}$.

It is now a simple matter to use (8.3.45) to show that the Fourier transform maps $\mathcal{S}'$ onto $\mathcal{S}'$.

8.4 Rigged Hilbert Spaces

To motivate the use and definition of *rigged Hilbert spaces* (also known as *Gel'fand Triples*), we begin by considering the following eigenvalue problem on $\mathcal{H} = \mathcal{L}_2(-\infty, \infty)$

$$p\, u_k = \hbar k\, u_k \; . \tag{8.4.46}$$

Since

$$p = \frac{\hbar}{i} \frac{d}{dx} \tag{8.4.47}$$

we get

$$u_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \; . \tag{8.4.48}$$

Now the operator p corresponds to a physical observable, the momentum and hence the eigenvalue problem (8.4.46) has a definite physical meaning. It tells us what the possible results of measurements of p are and is also supposed to give the probability amplitude for obtaining a given measurement. Nevertheless the "eigenfunctions" $u_k(x)$ are not square-integrable and hence do not belong to our Hilbert space. This is an undesirable situation. It can of course be obviated by forming wave packets. However the plane waves (8.4.48) are particularly convenient for practical calculations and we would be reluctant to have to give up using them. Thus, we are tempted to enlarge our state vector space beyond Hilbert space. Actually this also provides many simplifications in the analysis of operators. However we shall not study that aspect.

To show one possible extension we first note that the functions $u_k(x)$ belong to $\mathcal{S}'$ if we define them as distributions in the following manner

$$u_k(f) = \int_{-\infty}^{\infty} u_k(x)\, f(x)\, dx \; . \tag{8.4.49}$$

Why did we choose $\mathcal{S}'$? The reasons are mainly technical. Thus $\mathcal{F}\mathcal{S}'$ is again $\mathcal{S}'$ and this is desirable. Actually other spaces of distributions may be used, but for the sake of concreteness we concentrate only on $\mathcal{S}'$. Now how does considering u_k as an element of $\mathcal{S}'$ help? To answer this we start with a definition.

Let A be a linear operator in $\mathcal{S}$. This means that A is also a linear operator in $\mathcal{H}$. In fact if A has a dense domain in $\mathcal{S}$ it has a dense domain in $\mathcal{H}$ since $\mathcal{S}$ is dense in $\mathcal{H}$. To see this consider the hermite functions $H_n(x)e^{-x^2/2}$. All of these are in $\mathcal{S}$ and any element in $\mathcal{L}_2(-\infty, \infty)$ can be approximated by linear combinations of these functions. Thus $\mathcal{S}$ is dense in $\mathcal{H}$.

Now given such an operator A then $T \in \mathcal{S}'$ is called a *generalized eigenvector* of A corresponding to the eigenvalue λ if

$$T(Af) = \lambda T(f) \tag{8.4.50}$$

for all $f \in \mathcal{S}$. Notice that by definition $u_k(x)$ is a generalized eigenvector of the momentum operator $p = \hbar/i \, d/dx$ since in the notation (8.2.25) we have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{ikx} \frac{\hbar}{i} \frac{df}{dx} \, dx = -\hbar k \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{ikx} f(x) \, dx \tag{8.4.51}$$

for $f \in \mathcal{S}$. We have simply "integrated" by parts. Thus, we can now legitimately consider functions such as $u_k(x)$ as generalized eigenvectors. We still have to tie this together with the concept of Hilbert space. One more example is in order first.

Formally, the eigenvalue problem

$$x \, g_a(x) = a \, g_a(x) \tag{8.4.52}$$

has the solution

$$g_a(x) = \delta(x - a) \ . \tag{8.4.53}$$

Clearly $\delta(x-a)$ is not square integrable and hence is not in our Hilbert space $\mathcal{H}$. But for $f \in \mathcal{S}$ we have

$$\int_{-\infty}^{\infty} x \, \delta(x-a) \, f(x) \, dx = a \int_{-\infty}^{\infty} \delta(x-a) \, f(x) \, dx \ . \tag{8.4.54}$$

Thus $\delta(x-a)$ is a generalized eigenvector of the position operator x. We now define our *rigged Hilbert space.*

We begin with the space $\mathcal{S}$. On $\mathcal{S}$ are defined a countable sequence of norms $\| f \|_{n.r}$. We now also define on $\mathcal{S}$ an inner product which coincides with the $\mathcal{L}_2$ inner product. Now as stated $\mathcal{S}$ is dense in $\mathcal{L}_2$ and we identify $\mathcal{H}$ with $\mathcal{L}_2$. Thus $\mathcal{S}$ is identified as a subset of $\mathcal{H}$. Together with $\mathcal{S}$ and $\mathcal{H}$ we consider the space $\mathcal{S}'$. The triplet of spaces

$$\mathcal{S} \ , \ \mathcal{H} \ , \ \mathcal{S}'$$

form a rigged Hilbert space. It is usually denoted by

$$\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}' \ .$$

The advantage of the symbolic notation (8.2.11) is now obvious. Thus, "inner products" exist between elements of $\mathcal{H}$ and $\mathcal{H}$ and elements of $\mathcal{S}$ and $\mathcal{S}'$. We do not form inner products between elements of $\mathcal{S}'$ and $\mathcal{S}'$. This is all about rigged Hilbert spaces that we shall need. It is sufficient to provide a justification of most of the manipulations that we shall carry out. Further details are readily available in the references. From now on we shall proceed as if "functions" like $\delta(x)$ and e^{ikx} were elements of $\mathcal{H}$. To justify our manipulations we can always fall back on the concept of rigged Hilbert spaces, but we shall not explicitly do so. As stated at the beginning, this chapter was simply to show that our formal manipulations can be fully justified.

8.5 Problems

8.1 Show that the appropriate normalization for the positive parity solution for scattering from a square well equation (5.7.94) to yield δ-function normalization is $1/\sqrt{\pi}$. You will have to use the continuity of the wavefunction at $x = \pm a$ as well as equation (5.7.101).

8.2 Show that T is a tempered distribution if T is defined by

$$T(f) = \sum_{k=0}^{m} \int_{-\infty}^{\infty} F_k(x) \frac{d^k f(x)}{dx^k} \, dx$$

where F_k are continuous functions bounded by

$$|F_k(x)| \leq C_k(1 + |x|^j)$$

for some C_k and j depending on k. As a matter of fact every tempered distribution can be written in this form. Symbolically one then writes

$$T = \sum_{k=0}^{m} (-1)^k \frac{d^k F_k(x)}{dx^k} \, .$$

This formula cannot be taken literally however since the $F_k(x)$ need not be differentiable. It arises from a formal integration by parts of the first equation above.

8.3 The test function space $\mathcal{D}$ consists of the space of $C^{(\infty)}$ functions of bounded support. The support of a function f, (supp f) is the complement of the largest open set on which the function vanishes. Show that if $\tilde{f} \in \mathcal{FD}$ then $\tilde{f}$ is an entire function.

8.4 Prove the Theorem: The Fourier transform of a *tempered distribution of fast decrease* is a $C^{(\infty)}$ function bounded by a polynomial. A tempered distribution of fast decrease F is of the form

$$F = fT$$

where $f \in \mathcal{S}$ and T is also a tempered distribution.
Hint: To prove that the Fourier transform of F is bounded by a polynomial use the result of problem 8.2.

8.5 Let $f(z)$ be an entire function vanishing rapidly at large $|\Re(z)|$. Show that

$$\lim_{\epsilon \to 0+} \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{1}{x - a + i\epsilon} + \frac{1}{x - a - i\epsilon} \right] f(x) \, dx = P \int_{-\infty}^{\infty} \frac{f(x)}{x - a} \, dx$$

where the principle value integral is defined by

$$P\int_{-\infty}^{\infty} \frac{f(x)}{x-a}\,dx = \lim_{\epsilon\to 0+}\left[\int_{-\infty}^{a-\epsilon} \frac{f(x)}{x-a}\,dx + \int_{a+\epsilon}^{\infty} \frac{f(x)}{x-a}\,dx\right] .$$

Furthermore, show that

$$\lim_{\epsilon\to 0+} \frac{\epsilon/\pi}{x^2+\epsilon^2} = \delta(x) .$$

Hence conclude that considered as distributions

$$\lim_{\epsilon\to 0+} \frac{1}{x-a\pm i\epsilon} = P\frac{1}{x-a} \mp i\pi\delta(x-a)$$

that is,

$$\lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-a\pm i\epsilon} = P\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-a} \mp i\pi f(a) .$$

8.6 Using the result of problem 8.4 and defining

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk = \lim_{\epsilon\to 0+}\frac{1}{2\pi}\left[\int_{0}^{\infty} e^{ik(x+i\epsilon)}\,dk + \int_{-\infty}^{0} e^{ik(x-i\epsilon)}\,dk\right]$$

prove that

$$\int_{-\infty}^{\infty} e^{ikx}\,dk = \delta(x) .$$

8.7 Let $f(k)$ be a $C^{(\infty)}$ function bounded by a polynomial. Show that

$$F(z) = \int_{-\infty}^{\infty} f(k)\,e^{ikx}\,dk$$

is an entire function for $\Im(z) > 0$. Using this and the result of the Theorem proved in problem 8.3 show that every tempered distribution is the boundary value of an analytic function.

8.8 Calculate the Fourier transform of $\delta^{(n)}(x)$.

8.9 Show that

$$x^m\delta^n(x) = \begin{cases} 0 & \text{if } n < m \\ (-1)^m\, m!\,\delta(x) & \text{if } n = m \\ (-1)^m \frac{n!}{(n-m)!}\,\delta^{(n-m)}(x) & \text{if } n > m \end{cases} .$$

Bibliography

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[8.2] There are numerous other books on distribution theory of which we list several below. I.M. Gel'fand and G. E. Shilov, *Generalized Functions Vol.1* - Academic Press, N.Y. (1964).
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[8.3] The original reference for the subject is: L. Schwarz, *Théorie des Distributions*, Hermann, Paris, Part I (1957), Part II (1959).

[8.4] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* - Princeton University Press (1955).