The Wigner-Eckart Theorem and the Projection Theorem

The Wigner-Eckart theorem states that the matrix element of an irreducible tensor operator, $T_q^{(k)}$ (where $q = -k, -k+1, \ldots, k-1, k$) with respect to the angular momentum basis is given by

$$\langle \alpha'\, j'\, m' | T_q^{(k)} | \alpha\, j\, m \rangle = \langle j\, k\, ;\, m\, q \,|\, j\, k\, ;\, j'\, m' \rangle \, \frac{\langle \alpha'\, j' \| T^{(k)} \| \alpha\, j \rangle}{\sqrt{2j+1}}\,, \tag{1}$$

where $\langle \alpha'\, j' \| T^{(k)} \| \alpha\, j \rangle$ is the *reduced matrix element* of $T^{(k)}$, which is independent of q, m and m', and the factor of of $(2j+1)^{1/2}$ on the right hand side of eq. (1) is conventional. The notation of the Clebsch-Gordan coefficients in these notes follow that of Sakurai and Napolitano.

For example, given any scalar operator, $S \equiv T_0^{(0)}$, eq. (1) yields,

$$\langle \alpha'\, j'\, m' | S | \alpha\, j\, m \rangle = \delta_{jj'} \delta_{mm'} \frac{\langle \alpha'\, j' \| S \| \alpha\, j \rangle}{\sqrt{2j+1}}\,, \tag{2}$$

after noting the relation of the Clebsch-Gordan coefficients, $\langle j\, 0\, ;\, m\, 0 \,|\, j\, 0\, ;\, j'\, m' \rangle = \delta_{jj'} \delta_{mm'}$. The result of eq. (2) is not surprising since acting with a scalar operator on the state $|j\, m\rangle$ imparts no angular momentum.

Likewise, if we consider a rank-one irreducible tensor operator $A^{(1)}$, then the application of eq. (1) implies that

$$\langle \alpha'\, j'\, m' | A_q^{(1)} | \alpha\, j\, m \rangle = \langle j\, 1\, ;\, m\, q \,|\, j\, 1\, ;\, j'\, m' \rangle \, \frac{\langle \alpha'\, j' \| A^{(1)} \| \alpha\, j \rangle}{\sqrt{2j+1}}\,, \tag{3}$$

which vanishes unless the Clebsch-Gordan coefficient, $\langle j\, 1\, ;\, m\, q \,|\, j\, 1\, ;\, j'\, m' \rangle \neq 0$. The latter condition is satisfied only when $m' = m + q$ and $|j-1| \leq j' \leq j+1$. These inequalities imply that if $j = 0$ then only $j' = 1$ is allowed (i.e., the case $j = j' = 0$ is excluded). In the case of $j = j' \neq 0$, one can obtain an explicit expression for the reduced matrix element, $\langle \alpha'\, j \| A^{(1)} \| \alpha\, j \rangle$, by deriving a result known as the Projection Theorem.

The Projection Theorem states that for any vector operator, $\vec{\boldsymbol{A}}$, and $j \neq 0$,

$$\langle \alpha'\, j\, m' | A_q^{(1)} | \alpha\, j\, m \rangle = \frac{\langle \alpha'\, j\, m | \vec{\boldsymbol{J}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} | \alpha\, j\, m \rangle}{\hbar^2 j(j+1)} \, \langle j\, m' | J_q^{(1)} | j\, m \rangle\,, \tag{4}$$

where $A_q^{(1)}$ and $J_q^{(1)}$ (for $q = +1, 0, -1$) are the components of the vectors, $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{J}}$, with respect to the spherical basis. Moreover, in the derivation of eq. (4), one obtains

$$\langle \alpha'\, j \| A^{(1)} \| \alpha\, j \rangle = \frac{1}{\hbar} \sqrt{\frac{2j+1}{j(j+1)}} \, \langle \alpha'\, j\, m | \vec{\boldsymbol{J}} \boldsymbol{\cdot} \vec{\boldsymbol{A}} | \alpha\, j\, m \rangle\,. \tag{5}$$

The proof of the Projection Theorem is given on pp. 254–255 of Sakurai and Napolitano. In these notes, I shall provide slightly different proof that follows the three steps outlined in Exercise 15.3.4 on p. 420 of R. Shankar, *Principles of Quantum Mechanics*, 2nd edition (Springer Science, New York, NY, 1994).

Step 1: First, we shall use the expression[1]

$$\langle j\,1\,;\,j\,0\,|\,j\,1\,;\,j\,j\rangle = \sqrt{\frac{j}{j+1}}\,, \tag{6}$$

and the Wigner-Eckart theorem to show that

$$\langle\alpha'\,j'\|J^{(1)}\|\alpha\,j\rangle = \delta_{\alpha\alpha'}\,\delta_{jj'}\,\hbar[j(j+1)(2j+1)]^{1/2}\,.$$

The vector operator $\vec{\boldsymbol{J}}$ has the following spherical components,

$$J^{(1)}_{\pm 1} = \mp\frac{1}{\sqrt{2}}\,(J_x \pm iJ_y) = \mp\frac{J_\pm}{\sqrt{2}}\,, \qquad\qquad J^{(1)}_0 = J_z\,.$$

Using the Wigner-Eckart theorem,

$$\langle\alpha'\,j'\,m'|J^{(1)}_0|\alpha\,j\,m\rangle = \langle j\,1\,;\,m\,0\,|\,j\,1\,;\,j'\,m'\rangle\,\frac{\langle\alpha'\,j'\|J^{(1)}\|\alpha\,j\rangle}{\sqrt{2j+1}}\,. \tag{7}$$

Since $J^{(1)}_0 = J_z$, the left-hand side of eq. (7) is easily evaluated.

$$\langle\alpha'\,j'\,m'|J^{(1)}_0|\alpha\,j\,m\rangle = \hbar m\,\delta_{\alpha\alpha'}\,\delta_{jj'}\,\delta_{mm'}\,.$$

Inserting this result in eq. (7) and setting $m = m' = j$ yields

$$\langle j\,1\,;\,j\,0\,|\,j\,1\,;\,j'\,j\rangle\,\frac{\langle\alpha'\,j'\|J^{(1)}\|\alpha\,j'\rangle}{\sqrt{2j+1}} = \hbar j\delta_{\alpha\alpha'}\,\delta_{jj'}\,. \tag{8}$$

Note that $\langle j\,1\,;\,j\,0\,|\,j\,1\,;\,j'\,j\rangle = 0$ unless $j' = j\pm 1$ or $j' = j$. Thus, for $j' \neq \pm 1$, 0, eq. (8) holds automatically. It then follows that

$$\langle\alpha'\,j'\|J^{(1)}\|\alpha\,j\rangle = 0\,, \qquad \text{for } j' = j\pm 1\,.$$

If $j' = j$ eq. (8) yields

$$\langle j\,1\,;\,j\,0\,|\,j\,1\,;\,j\,j\rangle\,\frac{\langle\alpha'\,j\|J^{(1)}\|\alpha\,j\rangle}{\sqrt{2j+1}} = \hbar j\delta_{\alpha\alpha'}\,.$$

Using eq. (6), it follows that

$$\langle\alpha'\,j\|J^{(1)}\|\alpha\,j\rangle = \hbar\,\sqrt{j(j+1)(2j+1)}\,\delta_{\alpha\alpha'}\,.$$

This allows us to rewrite eq. (8) as

$$\langle\alpha'\,j'\|J^{(1)}\|\alpha\,j\rangle = \hbar\,\sqrt{j(j+1)(2j+1)}\,\delta_{\alpha\alpha'}\,\delta_{jj'}\,. \tag{9}$$

[1]In the class handout, *Clebsch-Gordan coefficients and the tensor spherical harmonics*, Table 2 provides explicit expressions for $\langle\ell\,1\,;\,m-m_s\,,\,m_s\,|\,\ell\,1\,;\,j\,m\rangle$ for $j=\ell+1,\ell,\ell-1$. If one replaces ℓ with j' and m with m' in this table, the resulting expressions also apply to $\langle j',1\,;\,m'-m_s\,,\,m_s\,|\,j'\,1\,;\,j\,m'\rangle$ for $j = j'+1, j', j'-1$, where j' can be either integral or half-integral. In particular, $\langle j',1\,;\,m'\,,\,0\,|\,j'\,1\,;\,j\,m'\rangle = m'/\sqrt{j'(j'+1)}$. Setting $j' = m' = j$ then yields eq. (6).

Step 2: Second, we shall make use of $\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}} = J_zA_z + \frac{1}{2}(J_-A_+ + J_+A_-)$, where $A_\pm \equiv A_x \pm iA_y$ to argue that

$$\langle\alpha'\, j\, m'|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle = c_{mm'}\frac{\langle\alpha'\, j\|A^{(1)}\|\alpha\, j\rangle}{\sqrt{2j+1}}\,,$$

where $c_{mm'} \equiv c\,\delta_{mm'}$ and $c = \hbar[j(j+1)]^{1/2}$ is a constant that is independent of α, α' and $\vec{\boldsymbol{A}}$.

Since $\vec{\boldsymbol{J}}$ and $\vec{\boldsymbol{A}}$ are vector operators, it follows that $\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}$ is a scalar operator. Then eq. (2) implies that $c_{mm'} \equiv c\,\delta_{mm'}$ for some constant c. Next, if we define $A_\pm \equiv A_x \pm iA_y$, then

$$\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}} = \tfrac{1}{2}(J_-A_+ + J_+A_-) + J_zA_z\,.$$

It is more convenient to employ the spherical components of $\vec{\boldsymbol{A}}$,

$$A^{(1)}_{\pm 1} = \mp\frac{1}{\sqrt{2}}\,(A_x \pm iA_y) = \mp\frac{1}{\sqrt{2}}A_\pm\,, \qquad\qquad A^{(1)}_0 = A_z\,.$$

Hence,

$$\begin{aligned}\langle\alpha'\, j\, m|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle &= \langle\alpha'\, j\, m|\tfrac{1}{2}(-\sqrt{2}\,J_-A^{(1)}_{+1} + \sqrt{2}\,J_+A^{(1)}_{-1}) + J_zA^{(0)}_0|\alpha\, j\, m\rangle \\ &= -\tfrac{1}{2}\sqrt{2}\,\hbar\,[(j-m)(j+m+1)]^{1/2}\,\langle\alpha'\, j\, m+1|A^{(1)}_+|\alpha\, j\, m\rangle \\ &\quad +\tfrac{1}{2}\sqrt{2}\,\hbar\,[(j+m)(j-m+1)]^{1/2}\,\langle\alpha'\, j\, m-1|A^{(1)}_-|\alpha\, j\, m\rangle \\ &\quad +\hbar m\langle\alpha'\, j\, m|A^{(1)}_0|\alpha\, j\, m\rangle\,. \end{aligned} \tag{10}$$

Using the Wigner-Eckart theorem,

$$\langle\alpha'\, j\, m'|A^{(1)}_q|\alpha\, j\, m\rangle = \langle j\, 1\,;\, m\, q\,|\, j\, 1\,;\, j\, m'\rangle\frac{\langle\alpha'\, j\|A^{(1)}\|\alpha\, j\rangle}{\sqrt{2j+1}}\,, \qquad \text{for } q = \pm 1\,,\, 0\,,$$

in eq. (10), it follows that

$$\langle\alpha'\, j\, m|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle = c\,\frac{\langle\alpha'\, j\|A^{(1)}\|\alpha, j\rangle}{\sqrt{2j+1}}\,, \tag{11}$$

where the constant c is explicitly given by:

$$\begin{aligned} c = &-\tfrac{1}{2}\sqrt{2}\,\hbar\,[(j-m)(j+m+1)]^{1/2}\,\langle j\, 1\,;\, m\, 1\,|\, j\, 1\,;\, j\,,\, m+1\rangle \\ &+\tfrac{1}{2}\sqrt{2}\,\hbar\,[(j+m)(j-m+1)]^{1/2}\,\langle j\, 1\,;\, m\,,\, -1\,|\, j\, 1\,;\, j\,,\, m-1\rangle + \hbar m\langle j\, 1\,;\, m\, 0\,|\, j\, 1\,;\, j\, m\rangle\,. \end{aligned}$$

Hence, the constant c is independent of α, α' and the choice of vector operator $\vec{\boldsymbol{A}}$. One can therefore evaluate c by plugging in $\vec{\boldsymbol{A}} = \vec{\boldsymbol{J}}$ in eq. (11). Since,

$$\langle\alpha'\, j\, m|\vec{\boldsymbol{J}}^{\,2}|\alpha\, j\, m\rangle = \hbar^2 j(j+1)\delta_{\alpha\alpha'}\,,$$

it follows from eqs. (9) and (11) that

$$c = \sqrt{2j+1}\,\frac{\langle\alpha\, j\, m|\vec{\boldsymbol{J}}^{\,2}|\alpha\, j\, m\rangle}{\langle\alpha\, j\|J^{(1)}\|\alpha\, j\rangle} = \hbar\sqrt{j(j+1)}\,. \tag{12}$$

Step 3: Finally, in light of the results of the first two steps, we shall derive the Projection Theorem given by eq. (4).

Applying the Wigner-Eckart theorem to the vector operators $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{J}}$, respectively,

$$\langle\alpha'\, j\, m'|A_q^{(1)}|\alpha\, j\, m\rangle = \langle j\, 1\, ;\, m\, q\,|\, j\, 1\, ;\, j\, m'\rangle \frac{\langle\alpha'\, j\|A^{(1)}\|\alpha\, j\rangle}{\sqrt{2j+1}}\,, \tag{13}$$

$$\langle\alpha\, j\, m'|J_q^{(1)}|\alpha\, j\, m\rangle = \langle j\, 1\, ;\, m\, q\,|\, j\, 1\, ;\, j\, m'\rangle \frac{\langle\alpha\, j\|J^{(1)}\|\alpha\, j\rangle}{\sqrt{2j+1}}\,, \tag{14}$$

for $q = \pm 1\,,\,0$. Note that we have set $\alpha' = \alpha$ in eq. (14), since the corresponding matrix elements would otherwise vanish. Assuming that $j \neq 0$, then dividing eq. (13) by eq. (14) yields

$$\frac{\langle\alpha'\, j\, m'|A_q^{(1)}|\alpha\, j\, m\rangle}{\langle\alpha\, j\, m'|J_q^{(1)}|\alpha\, j\, m\rangle} = \frac{\langle\alpha'\, j\|A^{(1)}\|\alpha\, j\rangle}{\langle\alpha\, j\|J^{(1)}\|\alpha\, j\rangle}\,. \tag{15}$$

Using eqs. (9), (11) and (12),

$$\langle\alpha\, j\|J^{(1)}\|\alpha\, j\rangle = \hbar\sqrt{j(j+1)(2j+1)}\,, \tag{16}$$

$$\langle\alpha'\, j\|A^{(1)}\|\alpha\, j\rangle = \frac{1}{\hbar}\sqrt{\frac{2j+1}{j(j+1)}}\,\langle\alpha'\, j\, m|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle\,. \tag{17}$$

which confirms eq. (5). Hence, eq. (15) can be written as

$$\frac{\langle\alpha'\, j\, m'|A_q^{(1)}|\alpha\, j\, m\rangle}{\langle\alpha\, j\, m'|J_q^{(1)}|\alpha\, j\, m\rangle} = \frac{\langle\alpha'\, j\, m'|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle}{\hbar^2 j(j+1)}\,.$$

This result yields the Projection Theorem,[2]

$$\langle\alpha'\, j\, m'|A_q^{(1)}|\alpha\, j\, m\rangle = \frac{\langle\alpha'\, j\, m|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle}{\hbar^2 j(j+1)}\,\langle j\, m'|J_q^{(1)}|j\, m\rangle\,, \tag{18}$$

under the assumption that $j \neq 0$. In contrast, if $j = 0$, then $\langle\alpha'\, 0\, 0|A_q^{(1)}|\alpha\, 0\, 0\rangle = 0$, since the vector operator acting on $|\alpha\, 0\, 0\rangle$ imparts one unit of angular momentum, thereby producing a state that is orthogonal to $|\alpha'\, 0\, 0\rangle$.

One can rewrite eq. (18) in a more suggestive form. First we note that $\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}$ is a scalar operator and hence it commutes with $\vec{\boldsymbol{J}}$. Then, we can write,

$$\begin{aligned}\langle\alpha'\, j\, m'|(\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}})\vec{\boldsymbol{J}}|\alpha\, j\, m\rangle = \langle\alpha'\, j\, m'|\vec{\boldsymbol{J}}(\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}})|\alpha\, j\, m\rangle &= \sum_{j'',m''}\langle j\, m'|\vec{\boldsymbol{J}}|j''\, m''\rangle\langle\alpha'\, j''\, m''|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle \\ &= \langle j\, m'|\vec{\boldsymbol{J}}|j\, m\rangle\langle\alpha'\, j\, m|\vec{\boldsymbol{J}}\cdot\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle\,, \end{aligned} \tag{19}$$

[2]Note that $\langle\alpha\, j\, m'|J_q^{(1)}|\alpha\, j\, m\rangle$ is independent of the quantum numbers α, so we can omit an explicit reference to α in this matrix element.

after using eq. (2) to conclude that $\langle\alpha'\, j''\, m''|\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}|\alpha\, j\, m\rangle$ is proportional to $\delta_{jj''}\delta_{mm''}$. Comparing eqs. (18) and (19), it follows that the Projection Theorem can be rewritten in the following form,

$$\langle\alpha'\, j\, m'|A_q^{(1)}|\alpha\, j\, m\rangle = \frac{\langle\alpha'\, j\, m|(\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{A}})J_q^{(1)}|\alpha\, j\, m\rangle}{\hbar^2 j(j+1)} \tag{20}$$

An equivalent way to write eq. (20) is,

$$\langle\alpha'\, j\, m'|\vec{\boldsymbol{A}}\,|\alpha\, j\, m\rangle = \left\langle\alpha'\, j\, m\left|\left(\frac{\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}}{\vec{\boldsymbol{J}}^{\,2}}\right)\vec{\boldsymbol{J}}\right|\alpha\, j\, m\right\rangle, \tag{21}$$

where the factor of $\vec{\boldsymbol{J}}^{\,2}$ in the denominator of eq. (21) when acting on the state $|j\, m\rangle$ yields the eigenvalue $\hbar^2 j(j+1)$. Note that $(\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}/\vec{\boldsymbol{J}}^{\,2})\vec{\boldsymbol{J}}$ is the projection of the vector $\vec{\boldsymbol{A}}$ along the direction of $\vec{\boldsymbol{J}}$. This is the origin of the name of the Projection Theorem. We again remind the reader that the Projection Theorem is valid for $j \neq 0$. In the case of $j=0$, $\langle\alpha'\, 0\, 0|\vec{\boldsymbol{A}}\,|\alpha\, 0\, 0\rangle = 0$, as previously noted.

The interpretation of eq. (21) is as follows. The expectation value of the vector operator $\vec{\boldsymbol{A}}$ with respect to the $|j, m\rangle$ basis corresponds to the classical precession of the vector $\vec{\boldsymbol{A}}$ about the angular momentum vector $\vec{\boldsymbol{J}}$. With this picture in mind, it is clear that the average value of the components of $\vec{\boldsymbol{A}}$ perpendicular to $\vec{\boldsymbol{J}}$ vanish. The components of $\vec{\boldsymbol{A}}$ parallel to $\vec{\boldsymbol{J}}$ are given by the projection of $\vec{\boldsymbol{A}}$ in the direction of $\vec{\boldsymbol{J}}$. Thus, the expectation value of the vector operator $\vec{\boldsymbol{A}}$ classically precessing about $\vec{\boldsymbol{J}}$ is given by

$$\langle\vec{\boldsymbol{A}}\rangle = \left\langle\left(\frac{\vec{\boldsymbol{J}}\boldsymbol{\cdot}\vec{\boldsymbol{A}}}{\vec{\boldsymbol{J}}^{\,2}}\right)\vec{\boldsymbol{J}}\right\rangle, \tag{22}$$

corresponding to the projection of $\vec{\boldsymbol{A}}$ along the direction of $\vec{\boldsymbol{J}}$. This is precisely the form of the quantum mechanical Projection Theorem given in eq. (21).

Finally, let us return to eq. (3) and note that if we apply this equation to $\vec{\boldsymbol{J}}$ then,

$$\langle\alpha'\, j'\, m'|J_q^{(1)}|\alpha\, j\, m\rangle = \langle j\, 1\,;\, m\, q\,|\, j\, 1\,;\, j'\, m'\rangle\,\frac{\langle\alpha'\, j'\|J^{(1)}\|\alpha\, j\rangle}{\sqrt{2j+1}}, \tag{23}$$

Then, dividing eqs. (3) and (23) [assuming that $\langle\alpha'\, j'\, m'|J_q^{(1)}|\alpha\, j\, m\rangle \neq 0$], it follows that

$$\langle\alpha'\, j'\, m'|A_q^{(1)}|\alpha\, j\, m\rangle = \frac{\langle\alpha'\, j'|\|A^{(1)}\|\alpha\, j\rangle}{\langle\alpha'\, j'|\|J^{(1)}\|\alpha\, j\rangle}\,\langle\alpha'\, j'\, m'|J_q^{(1)}|\alpha\, j\, m\rangle\,. \tag{24}$$

As noted below eq. (3), $\langle\alpha'\, j'\, m'|A_q^{(1)}|\alpha\, j\, m\rangle$ does not vanish in general as long as the corresponding Clebsch-Gordan coefficient, $\langle j\, 1\,;\, m\, q\,|\, j\, 1\,;\, j'\, m'\rangle \neq 0$. This latter requirement implies that $|j=j'|=0$ or 1, excluding the case of $j=j'=0$.

Thus, there are two generic cases of interest. In the case of $j=j'\neq 0$, the reduced matrix elements in eq. (24) can be evaluated explicitly [cf. eqs. (16) and (17)], which yields the Projection Theorem [eq. (18)]. In the case of $|j-j'|=1$, $\langle\alpha'\, j'\, m'|A_q^{(1)}|\alpha\, j\, m\rangle$ is given by eq. (24), but this result one cannot be developed any further. Nevertheless, eq. (24) demonstrates that the matrix elements of any vector operator are proportional to the corresponding matrix elements of the angular momentum operator $\vec{\boldsymbol{J}}$.