DUE: THURSDAY, FEBRUARY 15, 2018

MIDTERM EXAM ALERT: The midterm exam is a take-home exam that will be posted on the class website on Tuesday February 20. The completed exam must returned no later than 7 pm on Thursday February 22 to my physics department mailbox.

While working on the exam, you may refer to Sakurai and Napolitano and any second quantum mechanics textbook of your choosing. (If you do consult a second text, please indicate which one you used.) Any reference for integrals or other mathematical facts, and any personal handwritten notes are also OK. You are also free to consult any of the class handouts, including the solution sets that appear on the class website. However, you may *not* collaborate with anyone else. The exam will the first six topics of the course syllabus (and the material covered on first three problem sets of this course).

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1. Consider the one-dimensional problem of a particle moving in a delta-function potential:

$$V(x) = -A\delta(x)\,.$$

(a) Solve for the bound state energies and wave functions. Consider the cases $A > 0$ and $A < 0$ separately.

<u>*HINT*</u>: Integrate the Schrodinger equation between $-\epsilon$ and ϵ. Let $\epsilon \to 0$ and note that the derivative of the wave function is discontinuous at $x = 0$.

(b) In the case of $A > 0$, consider a scattering process where the incident wave enters from the left with $E = \hbar^2 k^2/(2m) > 0$ (where E is the energy eigenvalue of the Hamiltonian). Determine the corresponding reflection coefficient R and the transmission coefficient T as a function of k. Write the coefficients in terms of the dimensionless parameter $b \equiv E/E_g$ where $-E_g$ is the ground state energy obtained in part (a), in the case of $A > 0$. What is the behavior of $T(b)$ as $b \to -1$?

(c) In the case of $A < 0$, consider a scattering process where the incident wave enters from the left with $E = \hbar^2 k^2/(2m) > 0$. Investigate the location of any poles of the transmission amplitude in the complex k plane and the complex E plane, respectively. Explain your result in light of the fact that the repulsive delta function potential possesses no bound states.

2. A one-dimensional potential has the following form:

$$V(x) = \begin{cases} +\infty\,, & \text{for } x < 0, \\ -V_0\,, & \text{for } 0 < x < b, \\ 0\,, & \text{for } x > b, \end{cases}$$

where V_0 and b are positive constants.

(a) Find V_0 as a function of b such that there is just one bound state, of about zero binding energy, for a particle of mass M.

(b) Applying this crude model to the deuteron (a bound state of a proton and a neutron), evaluate V_0 in MeV, assuming $b = 1.3 \times 10^{-13}$ cm and $M = \frac{1}{2}m_p$ (where m_p is the proton mass).

(c) Why did I set $M = \frac{1}{2}m_p$ rather than $M = m_p$ in part (b)?

3. Consider a one-dimensional quantum mechanical problem with a time-independent Hamiltonian, H. The time evolution operator, evaluated in the coordinate basis, also known as the *propagator*, is given by,

$$G(x,t\,;\,x',0) = \langle x|\, e^{-iHt/\hbar}\, |x'\rangle = \int_{-\infty}^{\infty} dp\, \langle x\,|p\rangle\, \langle p|\, e^{-iHt/\hbar}\, |x'\rangle\ , \tag{1}$$

where we have taken the initial time to be $t_0 = 0$.

(a) The free particle Hamiltonian is given by $H = P^2/(2m)$. Evaluate the free particle propagator by explicitly performing the p-integration using eq. (1).

(b) For the one-dimensional harmonic oscillator, where $H = P^2/(2m) + \frac{1}{2}m\omega^2X^2$, can you evaluate the propagator by explicitly performing the p-integration in eq. (1)? Why is this calculation doomed to failure?

(c) Show that $G(x,t;x',0) = \langle x,t\,|\,x',0\rangle$, where the $|x,t\rangle$ are basis states in the Heisenberg representation. Deduce the following differential equation for G,

$$i\hbar\frac{\partial G}{\partial t} = \langle x,t|\, H\, |x',0\rangle\,,$$

where the boundary condition at $t = 0$ is $G(x,0;x',0) = \delta(x - x')$.

(d) Evaluate the propagator for the one-dimensional harmonic oscillator by employing the following steps. First, by using the Heisenberg equations of motion, express $P \equiv P(0)$ in terms of $X(t)$ and $X \equiv X(0)$. Then solve the differential equation obtained in part (c), subject to the boundary condition at $t = 0$.

HINT: In order to impose the boundary condition at $t = 0$, consider the limit of $G(x,t;x',0)$ as $t \to 0$ from the positive side.

(e) Check that in the limit of $\omega \to 0$, the result of part (d) reduces to the free particle propagator obtained in part (a).

4. The partition function is defined by,

$$Z(\beta) \equiv \operatorname{Tr} e^{-\beta H}\,,$$

where H is a time-independent Hamiltonian operator and β is a real positive parameter.

(a) In the case of a one-dimensional quantum mechanical problem, show that

$$Z(\beta) = \int G(x, -i\hbar\beta\,;\, x, 0)\, dx\,,$$

where the propagator is defined by,

$$G(x, t; x', 0) \equiv \langle x|\, e^{-iHt/\hbar}\, |x'\rangle\,. \tag{2}$$

(b) Show that the ground state energy E_0 is given by:

$$E_0 = \lim_{\beta\to\infty} -\frac{1}{Z}\frac{\partial Z}{\partial\beta}\,.$$

<u>*HINT*</u>: Insert a complete set of energy eigenstates into eq. (2).

(c) Using the results of part (b) of this problem and part (d) of Problem 3, compute the ground state energy of the one-dimensional harmonic oscillator.

(d) Consider the the propagator for a one-dimensional quantum system governed by a time-independent Hamiltonian with only discrete (bound state) energy levels, $\{E_n\}$, for $n = 0, 1, 2, 3, \ldots$. Using eq. (2), show that the full energy spectrum of H can be determined from[1]

$$\operatorname{Tr} e^{-iHt/\hbar} = \int_{-\infty}^{\infty} G(x, t\,;\, x, 0)\, dx = \sum_n \langle E_n|\, e^{-iHt/\hbar}\, |E_n\rangle = \sum_{n=0}^{\infty} e^{-iE_n t/\hbar}\,. \tag{3}$$

Strictly speaking, the sums on the right-hand side of eq. (3) are not convergent. However, one can give mathematical meaning to these sums by extending the time parameter to the complex plane.

In particular, let $t = -i\hbar\beta$ (where β is a positive real parameter). Using eq. (3), obtain all the energy eigenvalues of the one-dimensional harmonic oscillator.

5. Consider a particle in one dimension trapped between two impenetrable walls at $x = 0$ and $x = L$.

(a) Determine the bound state energy levels, E_n, of the particle. (Here, n labels the possible energy eigenvalues: $n = 1$ is the ground state, $n = 2$ is the first excited state, etc.).

[1] In eq. (3), the trace is expressed as a diagonal sum of matrix elements by employing two different basis choices (the coordinate basis and the energy basis, respectively). Of course, the trace is a basis-independent quantity, so one may choose any orthonormal basis to compute it.

(b) Suppose that at time $t = 0$, the state of the particle is given by the wave function

$$\psi(x, t = 0) = Ax(L - x)\,[\Theta(x) - \Theta(x - L)]\ , \tag{4}$$

where $\Theta(x)$ is the Heavyside step function and A is a normalization constant.

If an energy measurement is performed at time $t = 0$, what is the probability that the particle will be observed to be in the ground state? Find an exact expression for the probability that the particle will be observed to be in a state of energy E_n (for any positive integer n).

HINT: The following integral may be of use:

$$\int_0^\pi y^p \sin(ny)\,dy = \frac{p!}{n^{p+1}}\cos\left(\frac{p\pi}{2}\right) - \sum_{k=0}^{p} \frac{p!\,\pi^{p-k}}{(p-k)!\,n^{k+1}}(-1)^n \cos\left(\frac{k\pi}{2}\right)\,,$$

where p is a non-negative integer.

(c) Evaluate the expectation value of the Hamiltonian with respect to the wave function given in eq. (4). What is the average value of the energy at time $t = 0$?

(d) The expectation value of H can be computed by a different method than the one used in part (c). First, expand $\psi(x, 0)$ as a linear combination of energy eigenstates, and then show that the expectation value of H can be expressed as an infinite sum. Using this technique, obtain an expression for the average value of the energy at time $t = 0$, and then employ the result obtained in part (c) to determine the value of the sum,

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4}\,.$$

(e) After preparing the state given by eq. (4) at time $t = 0$, suppose that instead of performing an energy measurement at time $t = 0$, I wait a while and then make the first energy measurement at a later time $t > 0$. Do any of the results obtained in parts (b) and (c) change? Explain.

6. Consider a periodic potential in one-dimension that satisfies $V(x + \ell) = V(x)$.

(a) Show that the translation operator $T = \exp(-i\ell P/\hbar)$ commutes with the Hamiltonian:

$$H = \frac{P^2}{2m} + V(x)\,.$$

(b) We may choose the energy eigenstates to be simultaneous eigenstates of the translation operator T. Show that the general form of such eigenfunctions is:

$$\psi(x) = \exp(ipx/\hbar)u_p(x)\ .$$

where $u_p(x+\ell) = u_p(x)$. That is, the eigenfunctions are plane waves modulated by a function with the periodicity of the potential.