DUE: MONDAY, MARCH 20, 2018

FINAL EXAM ALERT: The final exam will take place on Tuesday March 20, 2018 from 8–11 am in ISB 231. The exam will cover the entire course material. During the exam, you may consult Sakurai and Napolitano, your class notes (and any other handwritten notes), and any of the homework solutions, class handouts or other material that is posted on the course website.

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1. The probability density multiplied by the electron charge $-e$ can be interpreted as the current density,

$$\vec{\boldsymbol{j}}(\vec{\boldsymbol{x}}) = \frac{ie\hbar}{2\mu}\left[\psi^*(\vec{\boldsymbol{x}})\boldsymbol{\nabla}\psi(\vec{\boldsymbol{x}}) - \psi(\vec{\boldsymbol{x}})\boldsymbol{\nabla}\psi^*(\vec{\boldsymbol{x}})\right] \;.$$

of an electron of mass μ in a Coulomb potential.

(a) Evaluate the current density as a function of position for the $n = 2$, $\ell = 1$, $m = -1$ state of hydrogen. (It is particularly convenient to express the current density in spherical components.) Sketch a picture illustrating the flow of current.

(b) Calculate the current flowing in a ring of cross section dA and the magnetic moment it produces (using classical electromagnetic theory). Integrate to find the entire magnetic moment produced by the current distribution.

(c) How do your answers above change for the $n = 2$, $\ell = 1$, $m = 1$ state of hydrogen? Interpret the difference physically.

(d) Obtain the general result for the current density and the total (integrated) magnetic moment for a state of hydrogen with arbitrary n, ℓ and m. You may express the current density in terms of the corresponding hydrogen energy eigenfunction without explicitly writing out the wave function.

2. Consider the following set of expectation values for powers of the electron radius in the hydrogen atom:

$$\left\langle r^k \right\rangle \equiv \langle n\ell m|\, r^k \,|n\ell m\rangle \;.$$

(a) Derive the following recurrence relation:

$$\frac{k+1}{n^2}\left\langle r^k \right\rangle - (2k+1)a_0\left\langle r^{k-1}\right\rangle + \tfrac{1}{4}k\big[(2\ell+1)^2 - k^2\big]a_0^2\left\langle r^{k-2}\right\rangle = 0\,,$$

where a_0 is the Bohr radius. This result is valid when $k > -(2\ell+1)$.

HINT: First, show that the radial equation can be written in the following form:

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + \frac{2n}{\rho} - 1\right] u(\rho) = 0\,,$$

where ρ is a suitably rescaled dimensionless radial variable. Multiply this equation by $\rho^{k+1} du/d\rho$ and $\rho^k u$ respectively, and partially integrate the results. One can then obtain a recurrence relation for $\langle \rho^k \rangle$.

(b) Evaluate $\langle r^2 \rangle$, $\langle r \rangle$ and $\langle 1/r \rangle$.

HINT: In evaluating $\langle r^k \rangle$ for $k = -1$, use the quantum Virial Theorem, which was treated in problem 4 of Problem Set 2. For $k = 1$ and 2, use the results of part (a).

3. [EXTRA CREDIT] Consider the hydrogen atom in two dimensions. The Hamiltonian (expressed in atomic units, where $\mu = Ze^2 = 1$), is given by

$$H = \tfrac{1}{2}(P_x^2 + P_y^2) - \frac{1}{R}\,,$$

where $\vec{\boldsymbol{X}} = (X, Y)$ and $R \equiv \|\vec{\boldsymbol{X}}\| = (X^2 + Y^2)^{1/2}$. The goal of this problem is to solve for the bound state energies of H. We shall employ an algebraic technique by introducing the two dimensional Runge-Lenz vector operator, $\vec{\boldsymbol{R}} = (R_x, R_y)$, where

$$R_x = \tfrac{1}{2}(P_y L_z + L_z P_y) - \frac{X}{R}\,,$$
$$R_y = -\tfrac{1}{2}(P_x L_z + L_z P_x) - \frac{Y}{R}\,,$$

and $L_z = XP_y - YP_x$ is the two-dimensional angular momentum operator.

(a) Show that $\big[R_x, H\big] = \big[R_y, H\big] = 0$.

(b) Evaluate the commutation relations satisfied by $\{R_x, R_y, L_z\}$.

HINT: Employ the canonical commutation relations, $\big[X_i, P_j\big] = i\hbar\delta_{ij} I$, where I is the identity operator.

(c) Define $A_i \equiv (-2H)^{-1/2} R_i$, for $i = x, y$ and $A_z \equiv L_z$. Working in the subspace of the Hilbert space corresponding to the bound states, $(-H)^{-1/2}$ is a self-adjoint operator. Verify that $\{A_x, A_y, A_z\}$ satisfy the same commutation relations as the *three*-dimensional angular momentum operators.

(d) Using the well known solution of the eigenvalue problem of $\vec{\boldsymbol{L}}^2$ in three-dimensions, obtain the eigenvalues of the operator, $A_x^2 + A_y^2 + A_z^2$. Using this result, find a formula for the bound state energies of the two-dimensional hydrogen atom.

HINT: To complete the final step of part (d), prove that $A_x^2 + A_y^2 + A_z^2 = -\tfrac{1}{4}\hbar^2 I - (2H)^{-1}$, where I is the identity operator.

4. (a) Consider a system of two spin-1/2 particles (labeled 1 and 2 below). Show that

$$\mathcal{P}_1 = \tfrac{3}{4}I + \vec{\boldsymbol{S}}_{\mathbf{1}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}_{\mathbf{2}}/\hbar^2\,, \qquad \mathcal{P}_0 = \tfrac{1}{4}I - \vec{\boldsymbol{S}}_{\mathbf{1}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}_{\mathbf{2}}/\hbar^2$$

are projection operators, i.e. they obey $\mathcal{P}_i\mathcal{P}_j = \delta_{ij}\mathcal{P}_j$ (no sum over j), where I is the identity operator and the $\vec{\boldsymbol{S}}_i$ $(i = 1, 2,)$ are spin-$\frac{1}{2}$ operators for particles 1 and 2, respectively.

(b) Show that $\mathcal{P}_1$ and $\mathcal{P}_0$ project onto the spin-1 and spin-0 subspaces of the direct product space of two spin-$\frac{1}{2}$ spaces.

NOTE: This decomposition is sometimes denoted by $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$.

(c) Construct the projection operators $\mathcal{P}_\pm$ for the $j = \ell \pm \frac{1}{2}$ subspaces of the direct product space obtained by combining orbital angular momentum and spin-$\frac{1}{2}$.

HINT: You should express the $\mathcal{P}_\pm$ as linear combinations of $\vec{\boldsymbol{L}}\boldsymbol{\cdot}\vec{\boldsymbol{S}}$ and the identity operator I with coefficients that depend on ℓ.

5. Define the traceless symmetric second-rank Cartesian tensor,

$$T_{ij} = x_i x_j - \tfrac{1}{3}r^2\delta_{ij}\,,$$

where $\vec{\boldsymbol{x}} \equiv (x_1, x_2, x_3)$ and $r^2 \equiv x_1^2 + x_2^2 + x_3^2$.

(a) Write T_{12}, T_{13}, and $T_{11} - T_{22}$ as linear combinations of the components of an irreducible spherical tensor of rank 2.

(b) The expectation value,

$$Q = e\left\langle \alpha, j, m = j \;\middle|\; 3x_3^2 - r^2 \;\middle|\; \alpha, j, m = j \right\rangle\,, \tag{1}$$

is known as the quadrupole moment. In eq. (1), α denotes other unspecified quantum numbers that characterize the state. Evaluate the matrix element,

$$e\left\langle \alpha, j, m' \;\middle|\; x_1^2 - x_2^2 \;\middle|\; \alpha, j, m = j \right\rangle\,,$$

where $m' = j, j-1, j-2, \ldots$, in terms of Q and the appropriate Clebsch-Gordan coefficients.

(c) Using the Wigner-Eckart theorem, prove that a spin-$\frac{1}{2}$ particle cannot possess a static quadrupole moment.