FINAL EXAM INSTRUCTIONS: This is an open book exam. You are permitted to consult the textbook of Sakurai and Napolitano (and any second quantum mechanics text of your choosing), your handwritten notes, and any class handouts or other material that are posted on the course website. One mathematical reference book is also permitted. No other consultations or collaborations are permitted during the exam. *In order to earn total credit for a problem solution, you must show all work involved in obtaining the solution.* However, you are *not* required to re-derive any formulae that you cite from the textbook or the class handouts.

This exam consists of ten individual parts, each of which is worth ten points. Use this information to manage your time appropriately during the exam. Please consider the hints provided for each problem—this will also help with your time management.

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1. Consider a particle of mass m subject to a one-dimensional potential of the following form,

$$V(x) = \begin{cases} \tfrac{1}{2}m\omega^2 x^2\,, & \text{for } x > 0, \\ \infty\,, & \text{for } x \leq 0. \end{cases}$$

(a) Determine the possible the bound state energy values of the particle.

(b) What is the normalized ground state wave function in the coordinate representation?

HINT: This problem can be performed with almost no work if you know the bound state energy spectrum of the one-dimensional harmonic oscillator Hamiltonian and some properties of the corresponding eigenfunctions.

2. Consider a one-dimensional problem with a Hamiltonian,

$$H = \frac{P^2}{2m} + V(X)\,,$$

where X is the position operator and P is the momentum operator. Assume that the spectrum of the Hamiltonian is discrete. Denote the normalized energy eigenstates by $|n\rangle$, and the corresponding energy eigenvalues by E_n.

(a) Show that the matrix elements of X and P, with respect to a basis consisting of orthonormal energy eigenstates, satisfy the following relation,

$$\langle k|\, P\, |n\rangle = c(k,n)\, \langle k|\, X\, |n\rangle\ ,$$

where $c(k, n)$ depends on E_k and E_n. Determine an explicit expression for $c(k, n)$.

(b) Evaluate the infinite sum,

$$S \equiv \frac{m}{\hbar^2} \sum_k (E_k - E_n) \big| \langle k | \, X \, | n \rangle \big|^2 \,,$$

where the sum is taken over the complete set of energy eigenstates, $|k\rangle$. Show that the sum is equal to a dimensionless constant and determine its value.

(c) Verify your calculation of S in part (b) in the case of the one-dimensional harmonic oscillator. That is, assuming $V(X) = \frac{1}{2} m\omega^2 X^2$, compute explicitly $\langle k | \, X \, | n \rangle$ and then use the energy eigenvalues of the one-dimensional harmonic oscillator to explicitly evaluate S.

HINTS: In part (a), evaluate the commutator, $[H, X]$ and show that it is proportional to P. Then use this result to evaluate $\langle k | \, P \, | n \rangle$. In part (b), the following trick is useful. Denote $\omega_{kn} \equiv (E_k - E_n)/\hbar$ and $X_{kn} = \langle k | \, X \, | n \rangle$. Then note that $\omega_{kn} |X_{kn}|^2 = \frac{1}{2}(\omega_{kn} - \omega_{nk}) X_{kn} X_{nk}$ (explain why this is valid). Finally, use the result of part (a). An alternative method for solving part (b) starts by considering $\langle n | \big[X, [X, H] \big] | n \rangle$ and then judiciously inserting a complete set of energy eigenstates. Feel free to use this second method if it suits you.

3. One would perhaps conclude from the lack of angular momentum in the ground state of the hydrogen atom that the electron is stationary.

(a) To show that this is not so, calculate the probability that the electron's momentum, if measured, would be found to lie in a momentum element d^3p centered at momentum $\vec{\boldsymbol{p}}$.

(b) What are the electron's mean kinetic and potential energy?

(c) Show that the results of part (b) are consistent with the quantum Virial Theorem.

HINT: See the hints for evaluating integrals that appears at the end of this exam.

4. Consider the Hamiltonian for a free particle of mass μ that is confined to a two dimensional cylindrical surface embedded in three dimensions. The cylinder is infinite in extent with a radius R centered around a symmetry axis that coincides with the z-axis. The Hamiltonian for this problem is the sum of a kinetic energy term and a rotational kinetic energy term,

$$H = \frac{P_z^2}{2\mu} + \frac{L_z^2}{2\mu R^2} \,, \tag{1}$$

where P_z and L_z are the z-components of the momentum and angular momentum operators, respectively.

(a) The Hamiltonian given by eq. (1) is invariant under which (continuous) symmetries? Identify the complete set of mutually commuting observables for this problem.

(b) Employing cylindrical coordinates (ρ, ϕ, z), where $x = \rho\cos\phi$ and $y = \rho\sin\phi$, the surface of the cylinder corresponds to $\rho = R$. Hence, the Hamiltonian of eq. (1) describes an effective two dimensional problem with coordinates z and ϕ. Write down the time-independent Schrödinger equation in the coordinate representation for this problem, and determine the allowed energy eigenvalues and the corresponding eigenfunctions of the Hamiltonian H.

Hints for evaluating integrals that appear in the solution to Problem 3

1. In evaluating three dimensional integrals, consider the possibility of using spherical coordinates.

2. If you encounter an integral of the form

$$\int_0^\infty y^p\, e^{-ay} \sin(by)\, dy\,, \tag{2}$$

first consider the integral,

$$F(a) = \int_0^\infty e^{-ay} \sin(by)\, dy\,. \tag{3}$$

This can be evaluated by writing $\sin(by) = \operatorname{Im} e^{iby}$, in which case,

$$F(a) = \operatorname{Im} \int_0^\infty e^{-(a-ib)y}\, dy\,. \tag{4}$$

The integral in eq. (4) is elementary and thus $F(a)$ easily evaluated. The integral given in eq. (2) is then related simply to the pth derivative of eq. (3) with respect to a.

3. The Gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt\,.$$

The Beta function is related to the Gamma function via

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\,.$$

A useful integral representation of the Beta function is,

$$B(p,q) = \int_0^\infty \frac{t^{p-1}\, dt}{(1+t)^{p+q}} = 2\int_0^\infty \frac{x^{2p-1}\, dt}{(1+x^2)^{p+q}}\,, \tag{5}$$

where the last integral in eq. (5) is obtained by changing variables, $t = x^2$.

4. The following integral may arise depending on your approach to part (b) of Problem 3. It is a consequence of eq. (5),

$$\int_0^\infty \frac{x^r\, dx}{(1+x^2)^s} = \frac{\Gamma\big(\tfrac{1}{2}(r+1)\big)\Gamma\big(s-\tfrac{1}{2}(r+1)\big)}{2\Gamma(s)}\,. \tag{6}$$