APPENDIX A

STATIONARY PHASE AND SADDLE POINT METHODS

A.1 INTRODUCTION

The intent of this appendix is to provide a simple approximate solution for the integral

$$\int_{A}^{B} g(x)\, e^{ikf(x)}\, dx. \tag{A.1}$$

A.2 THE METHOD OF STATIONARY PHASE

In one dimension, the solution can be found by reducing the Eq. (A.1) to the Fresnel Integral

$$F = \int_{-\infty}^{\infty} e^{iax^2} dx = \sqrt{\frac{\pi}{2a}}(1+i). \tag{A.2}$$

To understand the solution to come, let us look at the real and imaginary parts of the integrand of

$$F = \sqrt{\frac{\pi}{2a}}(1+i).$$

The main contribution to the real part of F

$$F = \int_{-\infty}^{\infty} \cos(ax^2) dx = \sqrt{\frac{\pi}{2a}} \tag{A.3}$$

comes from the interval, $-\sqrt{\frac{\pi}{2a}} < x < \sqrt{\frac{\pi}{2a}}$, and the rest cancels out because of the oscillations of the cosine function (see Figure A.1a).

Free Space Optical Systems Engineering: Design and Analysis, First Edition. Larry B. Stotts.

Companion website: www.wiley.com\go\stotts\free_space_optical_systems_engineering

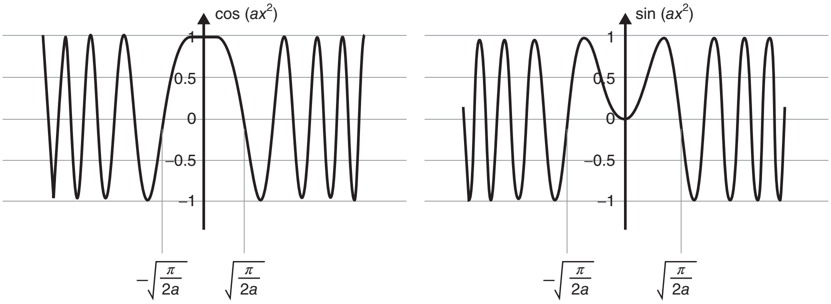

FIGURE A.1 Oscillatory nature of the (a) Cosine function and (b) Sine function.

Referring to Figure A.1b, it is plausible that the imaginary part of F is given by

$$F = \int_{-\infty}^{\infty} \sin(ax^2)dx \approx \sqrt{\frac{\pi}{2a}} \tag{A.4}$$

over the same interval $-\sqrt{\frac{\pi}{2a}} < x < \sqrt{\frac{\pi}{2a}}$, with the rest canceling out because of the oscillations of the sine function.

Looking again at Eq. (A.1), let us see how it will assume the form of a Fresnel Integral. The exponent $kf(x)$ might vary rapidly over most of the x-regime, but let us assume that it is "stationary" around $x = x_0$. This means that the first derivate $\frac{\partial f}{\partial x}$ equals 0 at $x = x_0$ and that we can approximate function $f(x)$ by the equation

$$f(x) \approx f(x_0) + \frac{1}{2!}\frac{\partial^2 f(x_0)}{\partial x^2}. \tag{A.5}$$

Given the above, the result of the integration will depend on

(a) $g(x)$
(b) $\cos[kf(x)]$
(c) the width of the unusually wide maximum of $\cos[kf(x)]$ at $x = x_0$.

Item (c) in the above list will be narrow if the bend of $f(x)$ at $x = x_0$ is sharp, which depends on $\frac{\partial^2 f(x_0)}{\partial x^2}$. The function $g(x)$ should change only a little during one oscillation, which is achieved by a large k (see Figure A.2). If $f(x)$ is stationary only once within an interval $A < x < B$, then we will have a contribution only from there. If so, it does not make a difference to the answer if we extend the integration limits from (A, B) to $(-\infty, \infty)$ as long as $kf(x)$ does not have another stationary point outside the interval (A, B) .

Given the above is true, we have

$$\int_A^B g(x)\, e^{ikf(x)}dx \approx \int_{-\infty}^{\infty} g(x)\, e^{ikf(x)}dx \approx \int_{-\infty}^{\infty} g(x)e^{ik\left[f(x_0)+\frac{1}{2!}\frac{\partial^2 f(x_0)}{\partial x^2}\right]}dx. \tag{A.6}$$

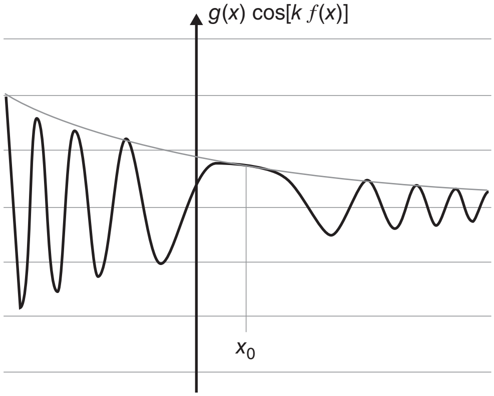

FIGURE A.2 Notional plot of $g(x)$ cos$[kf(x)]$.

Expanding $g(x)$ into a Taylor Series and integrating, we have

$$\int_A^B g(x)e^{ikf(x)}dx \approx \sqrt{\frac{2\pi}{\left(\frac{\partial^2 f(x_0)}{\partial x^2}\right)}}g(x_0)e^{ik\mathrm{f}(x_0)+\frac{\pi}{4}}. \tag{A.7}$$

If $\frac{\partial f}{\partial x}$ has more than one zero, then

$$\int_A^B g(x)e^{ikf(x)}dx \approx \sum_n \sqrt{\frac{2\pi}{\left(\frac{\partial^2 f(x_n)}{\partial x^2}\right)}}g(x_n)e^{ikf(x_n)+\frac{\pi}{4}}. \tag{A.8}$$

A.3 SADDLE POINT METHOD

This method is essentially the same as the method of stationary phase, except it applies to the two-dimensional version of the previous integral. That is, we are interested in the approximate solution to the integral

$$\int_{A_x}^{B_x}\int_{A_y}^{B_y} g(x,y)\,e^{ikf(x,y)}dx\,dy. \tag{A.9}$$

Following in essence the same development as above, we can show that

$$\int_{A_x}^{B_x}\int_{A_y}^{B_y} g(x,y)\,e^{ikf(x,y)}\,dx\,dy \approx \frac{2\pi g(x_0,y_0)\,e^{ikf(x_0,y_0)+\frac{i\pi}{2}}}{k\sqrt{\left(\frac{\partial^2 f(x_0,y_0)}{\partial x^2}\right)\left(\frac{\partial^2 f(x_0,y_0)}{\partial y^2}\right)-\left(\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}\right)}}, \tag{A.10}$$

where x_0 and y_0 are solutions to the equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$, respectively.