VOLUME TWO

CHAPTER 6

ELEMENTS AND APPLICATIONS OF THE THEORY OF ANALYTIC FUNCTIONS

INTRODUCTION

The role played by the theory of analytic functions in physics has changed considerably over the past few decades. It no longer suffices to be able to work out residue integrals; a deeper understanding of the mathematical ideas has become essential if one wants to follow current applications to physical theory. Therefore the emphasis here will be on introducing the mathematical concepts and the logical structure of the theory of analytic functions. Assuming only that the reader is familiar with the properties of complex numbers, we aim to present a self-contained account of this theory in a way that prepares one to cope with modern applications of the theory as well as those of the past.

"Imaginary" numbers were discovered in the Middle Ages in the search for a general solution of quadratic equations. It is clear from the name given them that they were regarded with suspicion. Gauss, in his doctoral thesis of 1799, gave the now familiar geometrical representation of complex numbers, and thus helped to dispel some of the mystery about them. In this century, the trend has been toward defining complex numbers as abstract symbols subject to certain formal rules of manipulation. Thus complex numbers never have taken on the "earthy" qualities of real numbers. In fact, more nearly the opposite has occurred: we have come to view real numbers abstractly as symbols obeying their own set of axioms, just like complex numbers. We now speak of *number fields*: the real field and the complex field. The axioms which define a field were stated in Chapter 3 on vector spaces.

The theory of complex numbers can be developed by viewing them as ordered pairs of real numbers, written (x, y). Let (a, b) and (c, d) be two different complex numbers, and let K be a real number. Then we define addition, multiplication of a real and a complex number, and multiplication of two complex numbers by the following rules:

1. $(a, b) + (c, d) = (a + b, c + d)$,
2. $K \cdot (a, b) = (Ka, Kb)$,
3. $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$.

From these definitions, we see that the set of all complex numbers—the complex plane—has the same mathematical structure as the set of all vectors in a plane.

This approach is followed in Landau's *Foundations of Analysis*, in which the various number systems are built up logically from Peano's five axioms; the imaginary number i is never mentioned. However, if we write the ordered pair (a, b) as $a + ib$, where $i^2 = -1$, then the above rule of complex-number multiplication is obeyed if we simply multiply out the product $(a + ib)(c + id)$ according to the usual rules of multiplication of reals. The introduction of the symbol i subsumes the ordering aspect of the ordered pair of real numbers, while extending the formal rules of arithmetic from real to complex numbers.

From the complex numbers constructed as ordered pairs of reals, where $(a, b) = a + ib$, it is possible to generalize to hypercomplex numbers of three or more components, for example $(a, b, c) = a + ib + kc$. The four-component quaternions, a type of hypercomplex number which satisfies all the rules of arithmetic except the commutative law of multiplication, are useful in dealing with rotations of a rigid body. The four 4×4 Dirac matrices, $\gamma_i (i = 1, 2, 3, 4)$, form a set of hypercomplex numbers which satisfy the anticommutative relations

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij} .$$

It can be shown that no matter how we define addition and multiplication for these hypercomplex numbers, it is impossible to retain all the usual rules of arithmetic. As Weyl points out, the complex numbers form a natural boundary for the extension of the number concept in this respect.

6.1 ANALYTIC FUNCTIONS—THE CAUCHY-RIEMANN CONDITIONS

If to each complex number z in a certain domain there corresponds another complex number w, then w is a function of the complex variable z: $w = f(z)$. If the correspondence is one to one, we can view this as a mapping from one plane (or part of it), the z-plane, to another, the w-plane. The complex functions thus defined are equivalent to ordered pairs of real functions of two variables, because w is a complex number depending on $z = x + iy$ and therefore can be written in the form

$$w(z) = u(x, y) + iv(x, y) .$$

However, this class of functions is too general for our purposes. We are interested only in functions which are differentiable with respect to the complex variable z—a restriction which is much stronger than the condition that u and v be differentiable with respect to x and y. Therefore one of our first tasks in the study of complex function theory will be to determine the necessary and sufficient conditions for a complex function to have a derivative with respect to the complex variable z. Single-valued functions of a complex variable which have derivatives throughout a region of the complex plane are called *analytic* functions. We shall restrict our attention to this special class of complex functions.

Two examples of complex functions (both written in the form $w = u + iv$) are

1. $w = z^* = x - iy$,
2. $w = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$.

Presently, we shall show that (1) is not an analytic function, but that (2) is analytic everywhere in the complex plane; i.e., its derivative exists at all points.

Before stating exactly what is meant by the derivative of a function of a complex variable, we must have a notion of *continuity* for these functions.

In the definition that follows, mention is made of the *absolute value* of a complex number, denoted by $|z|$. The reader will recall that $|z| \equiv (zz^*)^{1/2} = (x^2 + y^2)^{1/2}$. The absolute value is sometimes called the *modulus*.

Definition. A complex function $w = f(z)$ is continuous at the point z_0 if, given any $\epsilon > 0$, there exists a δ such that $|f(z) - f(z_0)| < \epsilon$, when $|z - z_0| < \delta$, or $f(z)$ is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0) .$$

This definition is formally exactly like the definition of continuity for real functions of a real variable. However, here the absolute value signs mean that whenever z lies within a *circle* of radius δ centered at z_0 in the complex z-plane, then $f(z)$ lies within a *circle* of radius ϵ centered at $f(z_0)$ in the complex w-plane. If $f(z) = u(x, y) + iv(x, y)$, then $f(z)$ is continuous at $z_0 = x_0 + iy_0$ if u and v are continuous at (x_0, y_0).

From the class of single-valued, continuous complex functions, we now want to select those that can be differentiated. Patterning the definition of a derivative after that of real analysis, we have

Definition. $f(z)$ is differentiable at the point z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

exists. We shall denote this limit, the derivative of $f(z)$ at z_0, by $f'(z_0)$.

A very important feature of the limits that occur in the definitions of continuity and the derivative is that z may approach z_0 from *any direction* on the plane. When we say the limit exists, we therefore mean that the same number must result from the limiting process regardless of how the limit is taken. This is also true in real analysis, but in that case there are only two possible directions of approach in taking the limit: from the left or the right on the real line. In real analysis, the limiting process is one-dimensional; in complex analysis, it is two-dimensional.

The equation that defines the derivative means that given any $\epsilon > 0$, there exists a δ such that

$$\left| f'(z) - \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon$$

provided $|z - z_0| < \delta$. The requirement that the ratio $[f(z) - f(z_0)]/(z - z_0)$ always tends to the same limiting value, no matter along what path z approaches z_0, is an extremely exacting condition. The theory of analytic functions contains a number of amazing theorems, and they all result from this stringent initial requirement that the functions possess "isotropic" derivatives.

A single-valued function of z is said to be *analytic* (or *regular*) at a point z_0 if it has a derivative at z_0 and at all points in some neighborhood z_0. Thus a slight distinction is drawn between differentiability and analyticity. It pays to do this, because although there exist functions which have derivatives at certain points, or even along certain curves, no interesting results can be obtained unless functions are differentiable throughout a region, i.e., unless they are analytic. Thus if we say a function is analytic on a curve, we mean that it has a derivative at all points in a two-dimensional strip containing the curve. If a function is not analytic at a point or on a curve, we say it is *singular* there.

We shall now examine the two complex functions mentioned earlier for differentiability and analyticity. We write the derivative at z_0 in the form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

by letting $z = z_0 + \Delta z$ in the original definition. For $f(z) = z^2$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0,$$

a result which is clearly independent of the path along which $\Delta z \to 0$, so $f(z) = z^2$ is differentiable and analytic everywhere. The result parallels exactly the result for the derivative of the real function $f(x) = x^2$.

On the other hand, if $f(z) = z^*$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{z_0^* + \Delta z^* - z_0^*}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z^*}{\Delta z}.$$

Now if $\Delta z \to 0$ along the real x-axis, then $\Delta z = \Delta x$ and $\Delta z^* = \Delta x^* = \Delta x$, so $f'(z_0) = +1$. However, if Δz approaches zero along the imaginary y-axis, then $\Delta z = i\Delta y$ so $\Delta z^* = -i\Delta y = -\Delta z$, so $f'(z_0) = -1$. Since at any point z_0 the limit as $z \to z_0$ depends on the direction of approach, the function is not differentiable or analytic anywhere. [As a general rule, $\Delta z^*/\Delta z = e^{-2i\theta}$, where $\theta = \tan^{-1}(\Delta y/\Delta x)$, which manifestly involves the direction of approach (θ) in taking the limit.]

Many of the theorems on differentiability in real analysis have analogs in complex analysis. For example:

1. A constant function is analytic.
2. $f(z) = z^n$ ($n = 1, 2, \cdots$) is analytic.
3. The sum, product, or quotient of two analytic functions is analytic, provided, in the case of the quotient, that the denominator does not vanish anywhere in the region under consideration.
4. An analytic function of an analytic function is analytic.

The proofs go through exactly as in the real case.

We now determine the necessary and sufficient conditions for a function $w(z) = u(x, y) + iv(x, y)$ to be differentiable at a point. First, we assume that

$w(z)$ is in fact differentiable for some $z = z_0$. Then

$$w'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right).$$

Since $w'(z_0)$ exists, it is independent of how $\Delta z \to 0$; that is, it is independent of the ratio $\Delta y / \Delta x$. If the limit is taken along the real axis, $\Delta y = 0$, and $\Delta z = \Delta x$. Then

$$w'(z_0) = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

On the other hand, if we approach the origin along the imaginary axis, $\Delta x = 0$ and $\Delta z = i \Delta y$. Now

$$w'(z_0) = \lim_{\Delta y \to 0} \left(\frac{\Delta v}{\Delta y} - i \frac{\Delta u}{\Delta y} \right) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

But by the assumption of differentiability, these two limits must be equal. Therefore, equating real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{6.1}$$

Equations (6.1) are known as the Cauchy-Riemann equations. They give a *necessary* condition for differentiability. We have determined this condition from special cases of the requirement of differentiability; therefore it is not surprising that these conditions alone are not sufficient.

The sufficient conditions for the differentiability of $w(z)$ at z_0 are, first, that the Cauchy-Riemann equations hold there, and second, that the first partial derivatives of $u(x, y)$ and $v(x, y)$ exist and be continuous at z_0.

The proof is straightforward. To begin, u is continuous at (x_0, y_0) because it is differentiable there; the partial derivatives of u are continuous by hypothesis. Under these assumptions, it follows from the calculus of functions of several variables* that

$$\begin{aligned} \Delta u &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) \\ &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \end{aligned}$$

where $\partial u / \partial x$ and $\partial u / \partial y$ are the partial derivatives evaluated at the point (x_0, y_0) and where ϵ_1 and ϵ_2 go to zero as both Δx and Δy go to zero. Using a similar formula for $v(x, y)$, we have

$$\begin{aligned} \Delta w &= w(z_0 + \Delta z) - w(z_0) = \Delta u + i \Delta v \\ &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i \left(\frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y \right). \end{aligned}$$

* See, for example, G. B. Thomas, Jr. *Calculus and Analytic Geometry*, 4th Ed., Addison-Wesley Publishing Co., 1968, Section 15-4, p. 503 Eq. 4, or W. Kaplan, *Advanced Calculus*, Addison-Wesley Publishing Co., 1953, Section 2-6, p. 84.

Now using the Cauchy-Riemann equations, which by assumption hold at the point (x_0, y_0), we have

$$\Delta w = \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + \Delta x(\epsilon_1 + i\epsilon_3) + \Delta y(\epsilon_2 + i\epsilon_4) .$$

Therefore

$$\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z} .$$

Since $|\Delta z| = [(\Delta x)^2 + (\Delta y)^2]^{1/2}$, $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$, and so $|\Delta x/\Delta z| \leq 1$ and $|\Delta y/\Delta z| \leq 1$. Since these factors are bounded, the last two terms in the above equation tend to zero with Δz because ϵ_1, ϵ_2, ϵ_3, and ϵ_4 go to zero as Δz goes to zero. Therefore at z_0

$$w'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} ; \tag{6.2}$$

the limit is independent of the path followed, so the derivative exists. Using the Cauchy-Riemann conditions, we also have

$$w'(z_0) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} . \tag{6.3}$$

Example. Consider the function z^3. We have

$$z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3) = u + iv .$$

Thus

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 6xy = -\frac{\partial u}{\partial y} .$$

Thus the Cauchy-Riemann equations hold everywhere. Since the partial derivatives are continuous, the function z^3 is, in fact, analytic everywhere. A function which is analytic in the entire complex plane is said to be an *entire* function. The derivative of z^3 may be found using Eq. (6.2) or (6.3). We obtain

$$\frac{\partial z^3}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3[(x^2 - y^2) + 2ixy] = 3z^2 ,$$

a satisfying result. As a second example, we leave it to the reader to show that the function $|z|^2 \equiv zz^*$ is differentiable only at the origin, and therefore is analytic nowhere.

One remarkable result which points to connections with physics follows immediately from the Cauchy-Riemann equations. Assuming that they hold in a region, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x} = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0 \tag{6.4}$$

if the second partial derivatives are continuous, so we can interchange the orders of differentiation in the mixed partial derivative. It follows in the same way that the function v also satisfies the two-dimensional Laplace equation. Thus both the real and imaginary parts of an analytic function with continuous second partial derivatives satisfy the two-dimensional Laplace equation. We shall later prove, using integration theory, that the second partial derivatives of an analytic function are necessarily continuous, so this qualification can be dropped. (It is interesting that these theorems about derivatives can be proved only by integration.) Any function ϕ satisfying $\nabla^2\phi = 0$ is called a *harmonic function.* If $f = u + iv$ is an analytic function, then $\nabla^2 u = \nabla^2 v = 0$, and u and v are called *conjugate* harmonic functions.

Given one of two conjugate harmonic functions, the Cauchy-Riemann equations can be used to find the other, up to a constant. For example, the function $u(x, y) = 2x - x^3 + 3xy^2$ is easily seen to be harmonic. To find its harmonic conjugate, we proceed as follows:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2 \Longrightarrow v = 2y - 3x^2y + y^3 + \phi(x) ,$$

where $\phi(x)$ is some function of x. Now, using the other Cauchy-Riemann equation, we obtain

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Longrightarrow -6xy + \phi'(x) = -6xy \Longrightarrow \phi' = 0 .$$

Thus $\phi(x)$ must be a constant, and the harmonic conjugate of u is

$$v = 2y - 3x^2y + y^3 + \text{const.}$$

Note that the function $w = u + iv = 2z - z^3 + C$ is an analytic function, as we know it must be.

Before leaving the Cauchy-Riemann conditions, let us take advantage of being physicists to present another, shorter derivation of these conditions, based on the use of infinitesimals. Let $w = u + iv$ and $w' = p + iq$. Then $\delta w = w'\delta z$, or, taking real and imaginary parts,

$$\delta u = p\delta x - q\delta y , \qquad \delta v = p\delta y + q\delta x .$$

It follows immediately that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = p , \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = q .$$

These equations are identical to the Cauchy-Riemann equations (6.1).

Continuing in this informal spirit, we may derive another closely related result which provides some insight into the meaning of analyticity. Again, let $w(z) = w(x, y) = u(x, y) + iv(x, y)$. We now show that $\partial w/\partial z^* = 0$ if and only if the Cauchy-Riemann equations hold. We shall not worry about the meaning of this derivative with respect to z^*, but just differentiate formally, treating the derivative as symbolic. Using the expressions

$$x = (z + z^*)/2 \qquad \text{and} \qquad y = (z - z^*)/2i$$

we have

$$\begin{aligned}\frac{\partial w}{\partial z^*} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z^*} \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\tfrac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right) \\ &= \tfrac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).\end{aligned}$$

If the Cauchy-Riemann equations hold, this last expression vanishes. If, on the other hand, $\partial w/\partial z^* = 0$, then both the real and imaginary parts of the last expression must vanish, so the Cauchy-Riemann equations hold.

This purely formal result, which can be made rigorous, is trying to tell us that analytic functions are independent of z^*: they are functions of z *alone*. Thus analytic functions are true functions of a *complex* variable, not just complex functions of two real variables (see, for example, Problem 1), which will in general depend on z^* as well as z according to

$$f(x, y) = f\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right).$$

6.2 SOME BASIC ANALYTIC FUNCTIONS

One of the most useful functions in the complex domain is the exponential function which we define for $z = x + iy$ by

$$e^z \equiv e^x (\cos y + i \sin y) . \tag{6.5}$$

It follows easily from this definition and our earlier work that e^z is an entire function and that

$$\frac{d}{dz} e^z = e^z .$$

The other familiar properties of exponentials, in particular, $e^{z_1+z_2} = e^{z_1} e^{z_2}$, follow readily from Eq. (6.5). We note that e^z is a periodic function of period $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z (\cos 2\pi + i \sin 2\pi) = e^z .$$

From Eq. (6.5) we see that

$$e^{iy} = \cos y + i \sin y ,$$

so it follows that

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \qquad \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

These relations suggest that for an arbitrary complex z we define

$$\cos z \equiv \frac{e^{iz} + e^{-iz}}{2}, \tag{6.6}$$

$$\sin z \equiv \frac{e^{iz} - e^{-iz}}{2i}. \tag{6.7}$$

Since

$$\frac{d}{dz} e^z = e^z ,$$

it is a simple matter to calculate the derivatives of $\cos z$ and $\sin z$. We find that

$$\frac{d}{dz} \cos z = \frac{ie^{iz} - ie^{-iz}}{2} = - \sin z ,$$

$$\frac{d}{dz} \sin z = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos z ,$$

as we might expect from experience with the real variable case. Using Eqs. (6.6) and (6.7), it is a simple matter to verify that all the familiar trigonometric identities, such as

$$\cos (z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 ,$$

continue to be valid for complex variables.

The complex functions sine and cosine may, of course, be put in the form $u(x, y) + iv(x, y)$. For example,

$$\begin{aligned} \sin z &= \frac{1}{2i} [e^{i(x+iy)} - e^{-i(x+iy)}] \\ &= \frac{1}{2i} e^{-y}(\cos x + i \sin x) - \frac{1}{2i} e^{y}(\cos x - i \sin x) \\ &= \sin x(e^y + e^{-y})/2 + i \cos x(e^y - e^{-y})/2 . \end{aligned}$$

Therefore

$$\sin z = \cosh y \sin x + i \sinh y \cos x . \tag{6.8}$$

Similarly,

$$\cos z = \cosh y \cos x - i \sinh y \sin x . \tag{6.9}$$

Setting $x = 0$, we obtain the useful relations $\sin (iy) = i \sinh y$ and $\cos (iy) = \cosh y$. We also see that the Cauchy-Riemann conditions are satisfied everywhere, as we know they must be. Other properties which follow directly from Eqs. (6.8) and (6.9) are

$$(\sin z)^* = \sin (z^*) ,$$

$$\sin (-z) = - \sin (z) ,$$

$$\sin (z + 2\pi) = \sin (z) .$$

Using the sine and cosine functions, we can define the other familiar trigonometric functions. For example,

$$\tan z = \sin z / \cos z \; ;$$

similar extensions of the real case are defined for the cotangent, secant, and cosecant. These functions differ from the sine and cosine in that they are *not* analytic everywhere. The tangent, being the ratio of two analytic functions, will be analytic everywhere *except* at points where $\cos z = 0$. Using the real and imaginary parts of the cosine, we can rewrite this condition as

$$\cosh y \cos x = 0 \,, \qquad \sinh y \sin x = 0 \,.$$

Now $\cosh y \geq 1$ for all real y, so the first equation has a solution whenever $\cos x = 0$, or $x = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \cdots$. At these points, $\sin x = \pm 1$, so the second equation requires that $\sinh y = 0$, that is, $y = 0$. Thus the tangent function is singular at the points $(2n + 1)\pi/2$, $(n = 0, \pm 1, \cdots)$ on the real axis, and only at these points. Therefore $\tan z$ becomes infinite at precisely those points where $\tan x$ (real x) becomes infinite and *only* at those points.

On the basis of the above discussion, one might be tempted to think that the complex trigonometric functions are "just the same thing" as their real counterparts. However, the reader can easily show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y \,,$$

and this expression increases *without limit* as y tends to infinity. This is in marked contrast with the real case, where $|\sin x| \leq 1$ for all real x.

The functions which we have discussed thus far all have the property that if we pick any point z_0 in the complex plane and follow any path from z_0 through the plane back to z_0, then the value of the function changes continuously along the path, returning to its original value at z_0. For example, suppose that we consider the function $w(z) = e^z$ and start at the point $z_0 = 1$, encircling the origin in the z-plane counterclockwise along the unit circle. Figure 6.1(a) shows the circular path in the z-plane, and Fig. 6.1(b) shows the corresponding path in the w-plane. [The use of two complex planes to "graph" the function $w(z)$ is often employed in complex variable theory.] We note that both paths are closed, which is just the geometrical statement of the fact that if we start at a point z_0, where the function has the value $w(z_0)$, then when we move along a closed curve back to z_0, the functional values also follow a smooth path back to $w(z_0)$.

Now for e^z this result is hardly surprising since we have *defined* e^z in such a way as to ensure this behavior, letting ourselves be guided by the properties of the real exponential function. Now if we look at another simple function, namely, the square root, we see that things do not always go so smoothly. Let us write formally

$$w(z) \equiv \sqrt{z} \equiv \sqrt{x + iy} \,.$$

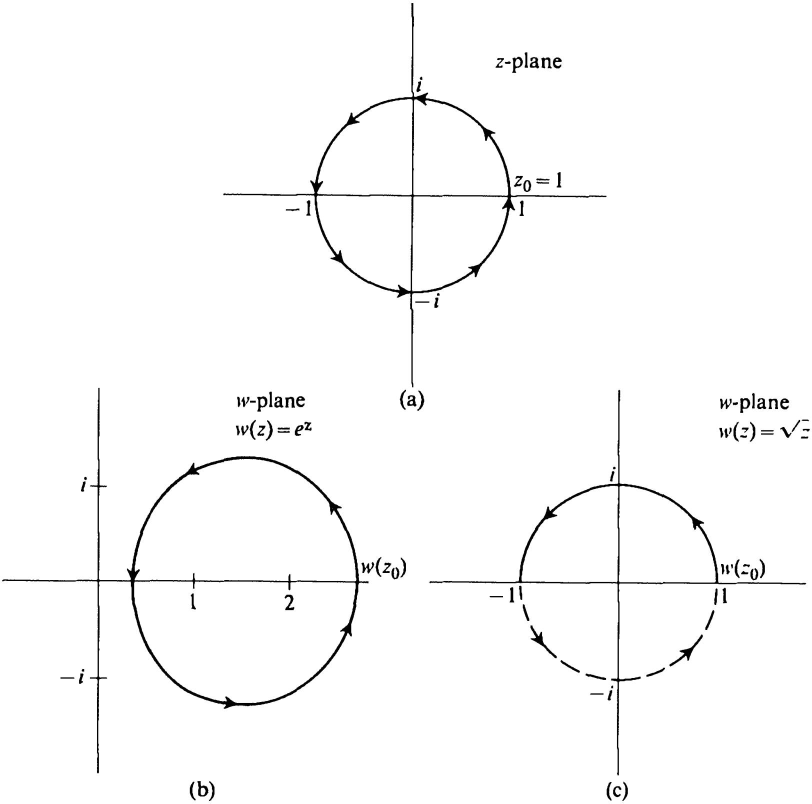

Fig. 6.1(a) A circular contour in the z-plane about the origin. **6.1(b)** The mapping of the contour of Figure 6.1(a) by the function e^z. **Fig. 1(c)** The mapping of the contour of Figure 6.1(a) by the function $\sqrt{z}$.

We observe that this definition is empty, since there is no set of operations presently at our disposal which will enable us to find $w(z)$ for some given x and y (unless $y = 0$). This is in contrast with the situation in Eq. (6.5) where all problems of evaluation can be handled by familiar real-variable operations.

Fortunately, in the case of the square root there is another possibility, namely, we can write z in polar form† as $z = re^{i\theta}$. In this form, a logical extension of the square root to the complex domain is contained in the definition

$$w(z) \equiv \sqrt{z} \equiv \sqrt{r}\, e^{i\theta/2} = \sqrt{r}\,[\cos(\theta/2) + i\sin(\theta/2)]\,.$$

† If we write $z = x + iy$ and make the familiar change to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) we obtain $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$.

Clearly, this function satisfies $w^2 = z$, which is certainly a minimum requirement for any sensible square root. Using this definition, let us vary z along the same path chosen in Fig. 6.1(a), starting at $r = 1, \theta = 0$. Figure 6.1(c) shows the corresponding path in the w-plane. Note that it is *not* a closed path; after making a complete circle around the origin in the z-plane, we arrive at the point $w = -1$ in the w-plane, not at $w = +1$. In order to get back to $w = +1$, we must let θ go from 2π to 4π; that is, make the circular trip in the z-plane one more time [see the dotted curve in Fig. 6.1(c)]. Actually this is not quite the best way to describe the situation; we do not want to think of tracing the circular path in the original z-plane a second time, but rather of tracing an identical circular path in a *different* z-plane. This corresponds to the fact that in the first circuit, θ went from 0 to 2π, whereas in the second circuit, it went from 2π to 4π.

This is not so different from the case of functions like e^z as one might imagine at first glance. We can write in polar variables

$$e^z = e^{r\cos\theta}[\cos(r\sin\theta) + i\sin(r\sin\theta)],$$

and then trace out a circular path as many times as we please ($\theta = 0 \to \theta = 2\pi$, $\theta = 2\pi \to \theta = 4\pi$, etc.). In this case we get the same values of e^z for each circuit. Therefore no information about e^z is lost if we identify the z-planes corresponding to $\theta = 0 \to \theta = 2\pi$, $\theta = 2\pi \to \theta = 4\pi$, etc., with each other. However, in the case of $w(z) = \sqrt{z}$ we need two planes, usually referred to as *Riemann sheets*, to characterize the values of $w(z)$ in a single-valued manner. Two planes are clearly sufficient: when we let θ range from 4π to 6π we obtain the same values as we did when we let θ range from 0 to 2π.

It is important to remember that the path of Fig. 6.1(a) encloses the origin. If we choose a closed path which neither encloses the origin nor intersects the positive real axis, then we also obtain a closed path in the w-plane. Fig. 6.2(a) and (b) illustrates the situation for $w(z) = \sqrt{z}$, starting from the $z_0 = \sqrt{5}\,e^{i\phi_0}$, where $\phi_0 = \tan^{-1} 2$ (we adopt the usual trigonometric convention that $\tan^{-1} x$ takes on values between 0 and $\pi/2$). In Fig. 6.2(a) we may say that we start at $z_0 = 1 + 2i$ on the *first Riemann sheet* and return to that point without encircling the origin. If we do the same thing for $z_0 = 1 + 2i$ on the *second Riemann sheet* (that is, $\phi_0 = \tan^{-1} 2 + 2\pi$), we obtain the corresponding closed curve traced out in the w-plane (Fig. 6.2(c)).

It is readily seen that the difficulties described above for $w(z) = \sqrt{z}$ will persist for any path beginning on the positive real axis and returning to the original point along a path enclosing the origin. Thus if we wish to consider $w(z) = \sqrt{z}$ in the simple fashion that we used for e^z, then we conclude that $\sqrt{z}$ is not continuous along the positive real axis and is not analytic there. However, to avoid this dilemma, we can say that when we come back to the real axis after a circuit of 2π radians, we transfer continuously onto the *second* Riemann sheet. If we go around $z = 0$ once more on the second sheet, when we return toward the positive real axis, we transfer continuously back to the *first* Riemann sheet. Thus the two sheets can be imagined to be *cut* along the positive real axis and

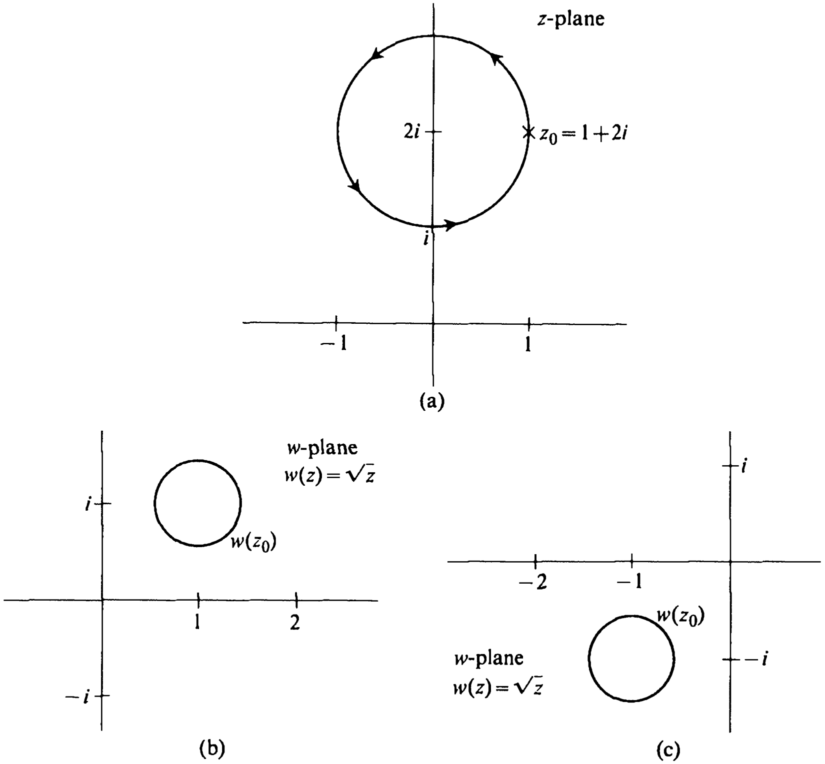

Fig. 6.2(a) A closed contour in the z-plane which does not enclose the origin. **6.2(b)** The mapping of the contour of Figure 6.2(a) by the function $\sqrt{z}$, if the contour of Figure 6.2(a) is imagined to lie on the *first* Riemann sheet. **6.2(c)** The mapping of the contour of Figure 6.2(a) by the function $\sqrt{z}$, if the contour of Figure 6.2(a) is imagined to lie on the *second* Riemann sheet.

joined in the manner illustrated in Fig. 6.3. With this construction, the function $w(z) = \sqrt{z}$ is seen to be single valued everywhere [on both sheets we set $w(0) = 0$] and analytic everywhere except at the origin, where $\sqrt{z}$ suffers from the same difficulty as does $\sqrt{x}$ in the real variable case. Thus the origin is a singular point for $w(z) = \sqrt{z}$.

In general, suppose that we have a singular point, z_0, of some function $w(z)$, and a path starting at z_1, which encircles z_0. If we must sweep through an angle

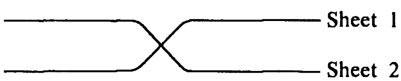

Fig. 6.3 A side view of the two Riemann sheets, looking down the real axis towards the origin, for the function $w(z) = \sqrt{z}$.

greater than 2π in order to return to the original value at z_1, then z_0 is called a *branch point* of $w(z)$. The cut which emanates from this point is called a *branch cut*. In the simple case discussed above $[w(z) = \sqrt{z}]$, the value of $w(z_1)$ with z_1 on the *second* sheet is just the negative of $w(z_1)$ with z_1 on the *first* sheet. Note that it is possible for a point z_1 on the *first* sheet to be a point at which some function $w(z)$ is analytic, whereas the point z_1 on the *second* sheet is a singular point. The function

$$w(z) = \frac{1}{i + \sqrt{z}}$$

is an example of such a function; $w(z)$ is a single-valued function over a two-sheeted Riemann surface cut along the positive real axis and joined as in Fig. 6.3. It is analytic at $z = -1\ (\phi = \pi)$ on the first Riemann sheet and is singular at $z = -1\ (\phi = 3\pi)$ on the second Riemann sheet.

In the above discussions we could, of course, have insisted that θ, the argument of z, range only through 2π radians. Then we could say that

$$w_1(z) = \sqrt{r}\, e^{i\theta/2}, \qquad 0 \leq \theta < 2\pi,$$

defines a single-valued function, analytic everywhere in the complex z-plane *except along the positive real axis* (including $z = 0$), and that

$$w_2(z) = \sqrt{r}\, e^{i(\theta+2\pi)/2} = -\sqrt{r}\, e^{i\theta/2}, \qquad 0 \leq \theta < 2\pi,$$

also defines such a function. Both $w_1(z)$ and $w_2(z)$ satisfy $w_1^2 = w_2^2 = z$ and are referred to as *single-valued branches* of $\sqrt{z}$. Clearly, if we defined

$$w_3(z) = \sqrt{r}\, e^{i(\theta+4\pi)/2}, \qquad 0 \leq \theta < 2\pi,$$

we would find $w_3(z) = w_1(z)$, so we do not obtain a new branch in this manner. We leave it as an exercise to show that one can define *three* single-valued branches of $w(z) = \sqrt[3]{z}$ and that, in this case, if one wants to define $w(z)$ as a single-valued function, analytic everywhere except at $z = 0$, a three-sheeted Riemann surface is necessary.

It should be noted that the choice of the real axis as the branch cut for $w(z) = \sqrt{z}$ was entirely arbitrary. Any other ray, say $\theta = \theta_0$, will serve equally well. The only thing which is *not* arbitrary is the choice of $z = 0$ as a branch point; $z = 0$ is a bona fide singular point for $w(z) = \sqrt{z}$, and this cannot be changed. However, in the functions $w_1(z)$ and $w_2(z)$ defined above, the singular line $\theta = 0$ is, apart from $z = 0$, a line of "man-made" singularities; we could equally well choose the line $\theta = \theta_0$ to be the singular line. For example, we could define

$$w_1(z) = \sqrt{r}\, e^{i\theta/2}, \qquad \theta_0 \leq \theta < \theta_0 + 2\pi,$$

and, similarly,

$$w_2(z) = \sqrt{r}\, e^{i(\theta+2\pi)/2} = -\sqrt{r}\, e^{i\theta/2}, \qquad \theta_0 \leq \theta < \theta_0 + 2\pi.$$

If we want a single-valued function which is analytic everywhere except at $z = 0$, then we can construct a two-sheeted Riemann surface, cut and joined along the line $\theta = \theta_0$. On the counterclockwise edge of the cut, $\sqrt{z} = \sqrt{r}\, e^{i\theta_0/2}$. After going through a circuit of 2π radians, we do not return to this value, but to $-\sqrt{r}\, e^{i\theta_0/2}$, and we pass onto the second sheet. Note that, in this case, some of the values of $w(z) = \sqrt{z}$ which were on the *first* Riemann sheet when the cut was made along $\theta = 0$ now find themselves on the *second* Riemann sheet and vice versa. This brings home the fact that the Riemann construction is merely a way to write a collection of values of a function in a single-valued manner. Distinctions between the first sheet and the second sheet are purely matters of convention. In fact, it should not be difficult for the reader to imagine that any reasonable curve from the origin to infinity could serve as an acceptable cut along which the two Riemann sheets of $w(z) = \sqrt{z}$ can be joined.

Before leaving this example, let us first propose an argument which might appear at first sight to contradict what we have been saying. Consider any point $x_0 \neq 0$ on the positive real axis. We should imagine that there exists a neighborhood of x_0 in which we can write $z = x_0 + \rho$, where ρ is some complex number, and then define $\sqrt{z}$ by the power series

$$\sqrt{z} \equiv \sqrt{x_0}\left(1 + \frac{1}{2}\frac{\rho}{x_0} - \frac{1}{8}\frac{\rho^2}{x_0^2} + \cdots\right),$$

whenever the series converges (we will see later that the series converges whenever $|\rho| < x_0$). This is a single-valued function which defines $\sqrt{z}$ continuously across a part of the positive real axis. However, this definition does not apply to the whole complex plane since the series does not converge everywhere. It turns out that it is impossible to extend (or "continue") this function to all points of the z-plane in such a way that $\sqrt{z}$ is single-valued and analytic. We will return to this point when we discuss the principle of analytic continuation later in this chapter. The continuity of the series definition across the positive real axis does not contradict our original positioning of a cut along the positive real axis, because, as we have seen above, the cut could be positioned anywhere so long as it begins at $z = 0$. In particular, the cut can be chosen so that it lies completely outside the domain of convergence of the series used to define $\sqrt{z}$ (for example, the cut could be chosen to lie along the negative real axis). We may remark in passing that we *could* successfully define e^z in this manner. The function e^x possesses an everywhere convergent power series for real x, and it is not hard to believe that a complex power series with the same coefficients $(1/n!)$ will converge *everywhere* in the complex plane.

As another example of a multivalued function, we consider the logarithm. Again using $z = re^{i\theta}$, we define

$$\log z \equiv \ln r + i\theta\,,$$

where ln denotes the usual natural logarithm of a positive real number. Note that

$$e^{\log z} = e^{\ln r + i\theta} = e^{\ln r}e^{i\theta} = re^{i\theta} = z$$

and also that

$$\log(z_1z_2) = \ln(r_1r_2) + i(\theta_1 + \theta_2) = \ln r_1 + i\theta_1 + \ln r_2 + i\theta_2 = \log z_1 + \log z_2\,,$$

so the logarithm has the main properties that one would expect by analogy with the real variable case. With the logarithm, the multivaluedness difficulties described above are more striking, since no matter how many times one encircles the origin, starting, say, at some point on the positive real axis, one *never* returns to the original value of the logarithm. The logarithm increases by $2\pi i$ on each circuit (or decreases by $2\pi i$ if one moves in the direction of decreasing θ). Thus an infinite number of Riemann sheets, each one joined to the one below it via a cut along the positive real axis, is necessary to turn $\log z$ into a single-valued function. When this is done, $\log z$ is analytic everywhere except at $z = 0$, where we assign the value $\log(z = 0) = -\infty$ on all sheets. We can also form an infinite number of single-valued branches of the logarithm:

$$w_n(z) = \ln r + i\theta + 2\pi ni\,, \qquad 0 \le \theta < 2\pi\,,$$

where $n = 0, \pm 1, \pm 2, \cdots$, and $w_n(z)$ is a single-valued function, analytic everywhere except at $z = 0$ and along the positive real axis. Just as before,

$$e^{w_n(z)} = z\,,$$

but

$$w_n(z_1 z_2) = w_n(z_1) + w_n(z_2) - 2\pi ni\,.$$

The branch

$$w_0(z) = \ln r + i\theta\,, \qquad 0 \le \theta < 2\pi\,,$$

is called the principal value or principal branch of the logarithm and is usually denoted by $\operatorname{Log} z$. We have

$$e^{\operatorname{Log} z} = z\,,$$

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2\,.$$

As before, the choice of the ray $\theta = 0$ as the line of singularities is entirely arbitrary.

From the preceding examples, it is a simple matter to build up to more complicated cases. For example, the function

$$w(z) = \sqrt{z - a}$$

is a single-valued function on the two-sheeted Riemann surface cut from a to infinity. The point $z = a$ is a branch point singularity; if we choose the branch cut to lie parallel to the real axis, we obtain the picture of Fig. 6.4. We thus may define, using the notation of Fig. 6.4,

$$w(z) \equiv |z - a|^{1/2} e^{i\theta/2}\,.$$

A more challenging problem is provided by the function

$$w(z) = \sqrt{(z - a)(z - b)}\,.$$

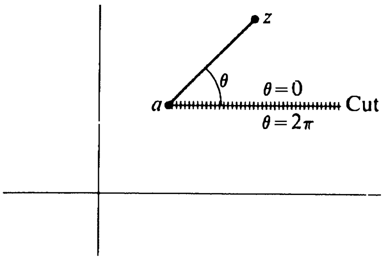

Fig. 6.4 The appropriate cut for the function $w(z) = \sqrt{z - a}$.

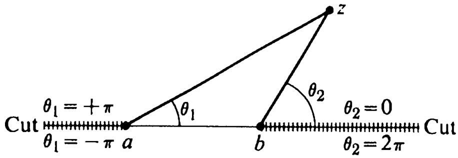

Fig. 6.5 The appropriate cuts for the function $w(z) = \sqrt{(z - a)(z - b)}$.

To simplify the geometry, we will consider the special case when a and b are real. The most obvious way to proceed is simply to put in two cuts from the branch points $z = a$ and $z = b$ as shown in Fig. 6.5. We have chosen these two simple directions for the cuts because of pictorial convenience. The left-hand cut corresponds to what we would have for the function $\sqrt{z - a}$, the right-hand cut to what we would have for $\sqrt{z - b}$. Here we take θ_1 to begin on the "bottom" of the left-hand cut at $-\pi$ and go to $+\pi$ on the "top" of the cut. At π we transfer to the second sheet and θ_1 continues to 3π, where we return to the first sheet. As we look down the negative real axis toward the origin, the two sheets are joined as shown in Fig. 6.3. Similarly, for the right-hand cut, we start at the "top" of the cut at $\theta_2 = 0$ and move counterclockwise, passing to the second sheet at $\theta_2 = 2\pi$, and finally return to the first sheet when $\theta_2 = 4\pi$. Along the positive real axis, the two sheets are joined as shown in Fig. 6.3. Using the above conventions, we define

$$w(z) \equiv |z - a|^{1/2}|z - b|^{1/2}e^{i\theta_1/2}e^{i\theta_2/2} .$$

As shown in Fig. 6.6, any point z_0 on the second sheet can be reached from the point z_0 on the first sheet either by going via the left-hand cut or the right-hand cut. These two options differ only in the sense that in the first case θ_1 increases by 2π, whereas in the second case θ_2 increases by 2π. In both cases, the value

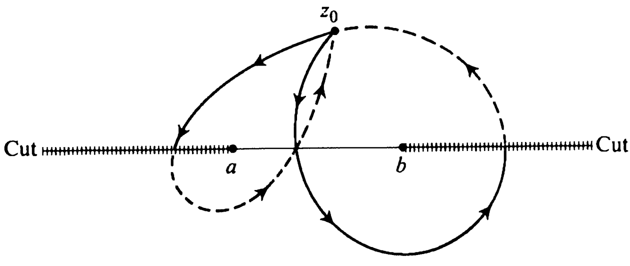

Fig. 6.6 Two different paths by which one can go from $z = z_0$ on the first sheet to $z = z_0$ on the second sheet. The function in question is $w(z) = \sqrt{(z - a)(z - b)}$.

of $w(z)$ is the same (the negative of what it would be for the corresponding point on the first sheet), as it must be if we are to have a single-valued function. With this construction, $w(z) = \sqrt{(z-a)(z-b)}$ becomes a single-valued function, analytic everywhere except at $z = a$ and $z = b$.

Note that in the above example, if we go through 2π radians in both θ_1 and θ_2, starting for example on the "top" of the right-hand cut, then we go down to the second sheet via the left-hand cut and return to our starting point on the first sheet via the right-hand cut. This suggests that it should equally well be possible to cut and join the two sheets along the real axis from a to b. The reader may find it interesting to show that this is indeed the case.

Using the above ideas, we can also obtain sensible expressions for the inverse trigonometric functions. For example, consider

$$w = \tan^{-1} z .$$

Writing this as

$$z = \tan w = \frac{1}{i}\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}},$$

we obtain readily

$$(1 - iz)e^{iw} = (1 + iz)e^{-iw},$$

and hence

$$e^{2iw} = \frac{1 + iz}{1 - iz}.$$

Taking the logarithms of both sides, we find that

$$w = \frac{1}{2i}\log\left(\frac{1 + iz}{1 - iz}\right) = \frac{1}{2i}\log\left(\frac{i - z}{i + z}\right).$$

Thus

$$\tan^{-1} z = \frac{i}{2}\log\left(\frac{i + z}{i - z}\right).$$

Just as we did in discussing the logarithm, we can speak of single-valued branches of $\tan^{-1} z$ corresponding to single-valued branches of $\log z$. The principal branch of the inverse tangent is defined in the obvious manner:

$$\mathrm{Tan}^{-1} z = \frac{i}{2}[\mathrm{Log}\,(i + z) - \mathrm{Log}\,(i - z)] .$$

Similar considerations apply to the functions $\sin^{-1} z$ and $\cos^{-1} z$.

6.3 COMPLEX INTEGRATION—THE CAUCHY-GOURSAT THEOREM

We now come to integration of complex functions, the part of the theory that makes the subject really interesting to both mathematicians and physicists. Because of the correspondence between complex numbers and two-dimensional vectors, we might expect to be able to define the line integral of a complex function along a curve in the z-plane.

Let t be a real parameter ranging from t_A to t_B, and let $z = z(t)$ be a curve, or *contour* C, in the complex plane, with endpoints $A = z(t_A)$ and $B = z(t_B)$. (See Fig. 6.7.) Now we mark off a number of points t_i between t_A and t_B, and approximate the curve by a series of straight lines drawn from each $z(t_i)$ to $z(t_{i+1})$.

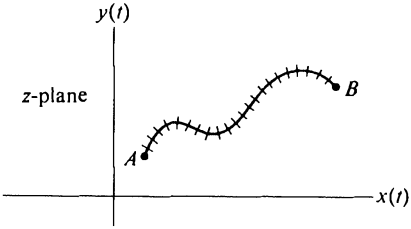

Fig. 6.7

To define the integral of a function w of a complex variable, we form the quantity

$$\lim_{|\Delta z_i|\to 0} \sum_{i=0}^{n} w(z_i)\Delta z_i \equiv \int_C w(z)\,dz\,,$$

where $\Delta z_i = z(t_{i+1}) - z(t_i)$, and $w(z_i)$ is the function evaluated at a point z_i on C between $z(t_{i+1})$ and $z(t_i)$. The sum is evaluated in the limit of an arbitrarily fine partition of the range through which the real parameter t moves as it generates the contour from A to B; that is, as $n \to \infty$, or, what is the same thing, in the limit of arbitrarily small $|\Delta z_i|$ for all i.

Writing $w(z) = u(x, y) + iv(x, y)$ and $dz = dx + i\,dy$, we have

$$\int_C w(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)\,. \tag{6.10}$$

We can also write this in parametric form. Then

$$dx = \frac{dx}{dt}\,dt\,, \qquad dy = \frac{dy}{dt}\,dt$$

and so

$$\int_C w(z)\,dz = \int_{t_A}^{t_B}\left(u\frac{dx}{dt} - v\frac{dy}{dt}\right)dt + i\int_{t_A}^{t_B}\left(u\frac{dy}{dt} + v\frac{dx}{dt}\right)dt\,.$$

For a given contour C running from A to B we define the opposite contour, written as $-C$, to be the same curve but traversed from B to A. The integral of $w(z)$ along $-C$ is clearly given by the above equation but with t_A and t_B interchanged. Thus,

$$\int_C = -\int_{-C}\,. \tag{6.11}$$

Also it follows that

$$\int_{C_1} + \int_{C_2} = \int_{C_1+C_2} . \tag{6.12}$$

If C is a closed curve that does not intersect itself, we shall always interpret $\oint_C$ to mean the integral taken counterclockwise along the closed contour C.

Another property of the integral that we shall need very often is

$$\left| \int_C f(z)\, dz \right| \leq \int_C |f(z)|\, |dz| \leq ML , \tag{6.13}$$

where M is the maximum value of $|f(z)|$ on C, and the length of C is L. The first inequality in Eq. (6.13) is a generalization of $|z_1 + z_2| \leq |z_1| + |z_2|$, the triangle inequality; both inequalities are derived exactly as in the real case.

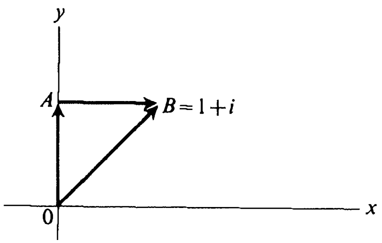

Fig. 6.8

Before presenting the general theorems on the integration of functions of a complex variable, we work out two examples of such integrals using only the results obtained so far. First, we evaluate the integral

$$I = \int_C \sin z\, dz$$

over the two paths shown in Fig. 6.8: (1) $C_1 = OB$, (2) $C_2 = OA + AB$. Since

$$\sin z = \cosh y \sin x + i \sinh y \cos x ,$$

we have, using Eq. (6.10),

$$I \equiv \int_C \sin z\, dz = \int_C [\cosh y \sin x\, dx - \sinh y \cos x\, dy] + i \int_C [\cosh y \sin x\, dy + \sinh y \cos x\, dx] .$$

Along the curve C_1, $x = y$. Therefore

$$I_1 = (1 + i) \int_0^1 \cosh x \sin x\, dx - (1 - i) \int_0^1 \sinh x \cos x\, dx = (1 - \cosh 1 \cos 1) + i(\sinh 1 \sin 1) .$$

Now we compute I along C_2. Along the path from O to A, $x = 0$ and $dx = 0$, and along the path from A to B, $y = 1$ and $dy = 0$. Therefore

$$\begin{aligned} I_2 = \int_{C_2} \sin z\, dz &= -\int_0^1 \sinh y\, dy + \int_0^1 \cosh 1 \sin x\, dx + i\int_0^1 \sinh 1 \cos x\, dx \\ &= 1 - \cosh 1 \cos 1 + i \sinh 1 \sin 1 = I_1 . \end{aligned}$$

The integral from O to B is the same for both paths. In fact, we shall prove later that it is the same for any path whatsoever—it depends only on the two endpoints. Also, the definite integral around the closed contour consisting of C_1 and $-C_2$ (that is, C_2 traveled backward) is zero. We shall show that this result holds for any function which is analytic on and inside the closed contour.

Note that if we evaluate formally, according to the rule of real calculus,

$$\begin{aligned} I_1 = I_2 = \int_0^{1+i} \sin z\, dz &= -\cos z\Big]_0^{1+i} \\ &= 1 - \cosh 1 \cos 1 + i \sinh 1 \sin 1 , \end{aligned}$$

where we have used Eq. (6.9). We shall also prove this "fundamental theorem of complex calculus," which holds in any region in which the integrand is analytic.

As a second example, let us integrate the function $f(z) = z^*$ counterclockwise around the unit circle centered at the origin. The values of z on this curve are given by $z = e^{i\theta}$, $\theta = 0$ to 2π. Therefore

$$I = \oint_C z^*\, dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta}\, d\theta = 2\pi i .$$

Since $zz^* = 1$ on C, we also obtain the result

$$\oint_C \frac{1}{z}\, dz = 2\pi i .$$

Neither integral around the closed contour is zero. The reason, as we shall see, is that z^* is not analytic anywhere, and therefore not within and on C, and z^{-1} is not analytic at $z = 0$, which is within C.

Both these examples are explained by

Cauchy's Theorem. If a function $f(z)$ is analytic within and on a closed contour C, *and $f'(z)$ is continuous throughout this region*, then

$$\oint_C f(z)\, dz = 0 .$$

We shall give two proofs of this theorem.

Proof 1.

$$\oint_C f(z)\, dz = \oint_C (u\, dx - v\, dy) + i\oint_C (u\, dy + v\, dx) .$$

To evaluate the two line integrals on the right, we use Green's theorem for line integrals. It states that if the derivatives of P and Q are continuous functions within and on a closed contour C, then

$$\oint_C (P\,dx + Q\,dy) = \int_R\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

where R is the surface bounded by C. By hypothesis, $f'(z)$ is continuous, so the first partial derivatives of u and v are also continuous; then Green's theorem yields

$$\oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx)$$
$$= \int_R\int \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx\,dy + i\int_R\int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy\,.$$

But since the Cauchy-Riemann equations hold, the integrands above all vanish. Therefore

$$\oint_C f(z)\,dz = 0\,. \qquad \text{QED}$$

Proof 2. Cauchy's theorem may also be proved if we use Stokes's theorem, which is closely related to Green's theorem and is perhaps more familiar. We write

$$\oint_C f(z)\,dz = \oint_C \mathbf{F}\cdot d\mathbf{l} + i\oint_C \mathbf{G}\cdot d\mathbf{l}\,,$$

where

$$\mathbf{F} = u\mathbf{i} - v\mathbf{j}\,, \qquad \mathbf{G} = v\mathbf{i} + u\mathbf{j}\,, \qquad \text{and} \qquad d\mathbf{l} = dx\mathbf{i} + dy\mathbf{j}\,.$$

Let S be the region interior to and including C. Since the Cauchy-Riemann conditions hold throughout S, it follows that $(\nabla \times \mathbf{F})_z = 0$ and $(\nabla \times \mathbf{G})_z = 0$ throughout S, where the subscript z identifies the $\mathbf{k}$ component of the curl; for example,

$$(\nabla \times \mathbf{F})_z = \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} = 0\,,$$

by virtue of the Cauchy-Riemann equations. Now, using Stokes's theorem, the validity of which depends on the continuity of the four first partials of u and v,

$$\oint_C f(z)\,dz = \oint_C \mathbf{F}\cdot d\mathbf{l} + i\oint_C \mathbf{G}\cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F})\cdot d\mathbf{S} + i\int_S (\nabla \times \mathbf{G})\cdot d\mathbf{S}$$
$$= \int_S (\nabla \times \mathbf{F})_z\,dS + i\int_S (\nabla \times \mathbf{G})_z\,dS = 0\,,$$

where $d\mathbf{S} = \mathbf{k}\,dS$. QED

It is possible to prove Cauchy's theorem without assuming the continuity of $f'(z)$. This is because any function which is analytic in a region *necessarily* has a continuous derivative. In fact, we shall prove that an analytic function has derivatives of all orders, and therefore all its derivatives are continuous, the continuity of the nth derivative being a consequence of the existence of the derivative of order $n + 1$. But we shall only be able to establish this result on higher derivatives *after* we have shown that the continuity of $f'(z)$ is not needed in the proof of Cauchy's theorem.

This relaxation or weakening of the hypotheses under which

$$\oint_C f(z)\,dz = 0$$

is therefore of the utmost importance—it is, in fact, the centerpiece of the theory of analytic functions. Some authors (never mathematicians) define an analytic function as a differentiable function *with a continuous derivative*. Then the central result of the theory follows trivially, as we have seen in the previous theorem. But this is a mathematical fraud of cosmic proportions. It was Goursat who first proved that the condition that $f'(z)$ be continuous is superfluous. It is Goursat's result that really distinguishes the theory of integration of a function of a complex variable from the theory of line integrals in the real plane. Although the theorem is often simply called Cauchy's theorem, it is the "-Goursat" half that gives it real mathematical power. In our proof we follow the presentations of Franklin and of Knopp.

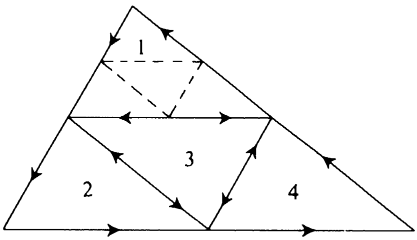

Fig. 6.9

Cauchy-Goursat Theorem. If a function $f(z)$ is analytic within and on a closed contour C, then $\oint_C f(z)\,dz = 0$.

Proof. We shall first prove the theorem for a triangular region; then it is very easily extended to an arbitrary region. Let R denote the closed region consisting of the points interior to and on the triangle bounded by the closed contour T of total length L. Since $f(z)$ is analytic in R, $f'(z)$ exists throughout R, and therefore, $f(z)$ is continuous in R. We now begin subdividing R into smaller triangles as shown in Fig. 6.9. Each subtriangle is similar to the original triangle but its sides (and perimeter) are only one-half as long. The boundaries of the subtriangles are denoted by T_i $(i = 1, 2, 3, 4)$. Clearly

$$\oint_T f(z)\,dz = \sum_{i=1}^{4} \oint_{T_i} f(z)\,dz\,,$$

where all contours are traversed in a counterclockwise direction. All three of the boundaries of triangle 3 in Fig. 6.9 are traversed in both directions and therefore cancel out. Applying the triangle inequality to this equation, we obtain

$$\left|\oint_T f(z)\,dz \leq \sum_{i=1}^{4} \left|\oint_{T_i} f(z)\,dz\right| .$$

The object of the proof is to show that the quantity on the left is arbitrarily small.

Now let C_1 denote the triangle which contributes the largest term to the above sum. Then we have

$$\left|\oint_T f(z)\,dz\right| \leq 4 \left|\oint_{C_1} f(z)\,dz\right| ,$$

where the length of $C_1 \equiv L_1 = L/2$. We now repeat this process on the subtriangle bounded by C_1. That is, we find a contour C_2, bounding a "sub-subtriangle," such that

$$\left|\oint_{C_1} f(z)\,dz\right| \leq 4 \left|\oint_{C_2} f(z)\,dz\right| ,$$

where the length of $C_2 \equiv L_2 = L_1/2 = L/2^2$. If the subdivision is repeated n times, we obtain a nested sequence of triangular contours C_n, such that

$$\left|\oint_T f(z)\,dz\right| \leq 4^n \left|\oint_{C_n} f(z)\,dz\right| , \tag{6.14}$$

where the length of $C_n \equiv L_n = L/2^n$. In order to finish the proof, we have to show that

$$\left|\oint_{C_n} f(z)\,dz\right|$$

is decreasing with n more rapidly than 4^n is increasing.

We let R_n denote the closed region consisting of C_n and the interior points of the (sub)$^{n\text{th}}$-triangle bounded by C_n. Clearly, each point of the region R_{n+1} is a point of R_n, and as n goes to infinity this nested sequence of closed sets closes down on a single point z_0 which is in each R_n, and R itself. (If R is the continent of Africa and we select R_n as that subregion which contains the biggest lion in Africa, we have an algorithm for capturing a big lion—we simply build a cage about the point z_0.)

Since $f'(z)$ exists, it follows by definition that for any $\epsilon > 0$, there exists a δ, such that when $0 < |z - z_0| < \delta$,

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \leq \epsilon .$$

Now consider the function $g(z)$ defined by

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & \text{for } z \neq z_0 , \\ 0 & \text{for } z = z_0 . \end{cases}$$

Note that $|g(z)| \leq \epsilon$ if $|z - z_0| < \delta$; $g(z)$ is therefore continuous at $z = z_0$.

Now $f(z)$ is given for all z in R by

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)g(z);$$

we use this relation to evaluate $\oint_{C_n} f(z)\,dz$. The first two terms are entire functions of z (z_0 is constant) with *derivatives* that are *continuous* everywhere. Therefore we may apply the earlier version of Cauchy's theorem to deduce that

$$\oint_{C_i} [f(z_0) + (z - z_0)f'(z_0)]\,dz = 0\,, \qquad \text{for all } i\,.$$

Therefore

$$\oint_{C_n} f(z)\,dz = \oint_{C_n} (z - z_0)g(z)\,dz\,.$$

We now subdivide enough times (i.e., we choose n large enough) so that $2^n > L/\delta$. Then $L_n = L/2^n < \delta$. Furthermore, for any point z on C_n, $|z - z_0| < L_n < \delta$, since z_0 is inside R_n, and the distance from any interior point to any point on the boundary of a triangle is clearly less than the perimeter of the triangle. Therefore, since $|z - z_0| < \delta$, $|g(z)| < \epsilon$, and $|(z - z_0)g(z)| \le L_n\epsilon$. Consequently,

$$\left|\oint_{C_n} f(z)\,dz\right| = \left|\oint_{C_n} (z - z_0)g(z)\,dz\right| \le L_n^2\epsilon = \epsilon(L/2^n)^2 = \epsilon L^2/4^n\,,$$

where we have used Eq. (6.13). It now follows from Eq. (6.14) that

$$\left|\oint_T f(z)\,dz\right| \le \epsilon L^2\,.$$

Since L is the fixed finite perimeter of the triangular region R, and ϵ is arbitrary, we can make the quantity ϵL^2 smaller than any preassigned number ϵ'. Thus

$$\left|\oint_T f(z)\,dz\right| = 0\,, \qquad \text{and hence} \qquad \oint_T f(z)\,dz = 0\,.$$

This proves the Cauchy-Goursat theorem for triangular contours.

We shall not give a formal proof of the extension of this result to arbitrary regions, because the method and result are simple and clear. Given an arbitrary C, we inscribe a polygon in C. Any polygon may be decomposed into a sum of triangles, so we know the theorem holds for polygons of any number of sides. It is clear that the difference

$$\left|\oint_C f(z)\,dz - \oint_P f(z)\,dz\right|\,,$$

where P is the perimeter of the polygon inscribed in C, can be made arbitrarily small by simply choosing a polygon with a sufficiently large number of sides. This establishes the Cauchy-Goursat theorem for a region of arbitrary shape.

Throughout the proof we have tacitly assumed that the region R is a *simply-connected* region. This means that any closed contour in R encloses only points belonging to R. Suppose, however, that R were a region with one or more subregions "punched out." Then it would be possible to construct curves around these holes in such a way that the curves would lie entirely in R, but enclose points *not* belonging to R. Such regions are called *multiply-connected*. Cauchy's theorem does not hold for arbitrary contours in multiply-connected regions.

6.4 CONSEQUENCES OF CAUCHY'S THEOREM

The hardest work is behind us; we turn now to an examination of some of the main consequences of Cauchy's theorem.

Path Independence

We first prove that if $f(z)$ is analytic in the region R and C_1 and C_2 lie in R and have the same endpoints, then

$$\int_{C_1} f\,dz = \int_{C_2} f\,dz\,.$$

The proof follows immediately by applying Cauchy's theorem to the closed contour consisting of C_2 and $-C_1$ as shown in Fig. 6.10;

$$\int_{C_2} + \int_{-C_1} = 0 \Longrightarrow \int_{C_2} = -\int_{-C_1} = \int_{C_1}$$

by Eq. (6.11).

Fundamental Theorem of Calculus

From our discussion of path independence it follows that the equation

$$F(z) \equiv \int_{z_0}^{z} f(z')\,dz'$$

defines a unique function of z if $f(z')$ is analytic throughout the region containing the path between z_0 and z.

Theorem. $F(z)$ is analytic and $F'(z) = f(z)$.

Proof.

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(z')\,dz'\,,$$

where the path from z to $(z + \Delta z)$ may be taken to be a straight line. We can write

$$f(z) = \frac{f(z)}{\Delta z}\int_{z}^{z+\Delta z} dz' = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,dz'\,,$$

and it follows that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(z') - f(z)]\,dz'\,.$$

Now $f(z)$ is continuous because it is analytic; therefore, for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|z' - z| < \delta$, then $|f(z') - f(z)| < \epsilon$. Now take $0 < |\Delta z| < \delta$. Then

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \epsilon\frac{1}{|\Delta z|}\int_{z}^{z+\Delta z} |dz'| = \epsilon\,.$$

That is,

$$F'(z) \equiv \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) ,$$

so $F(z)$ is analytic and its derivative is $f(z)$.

Thus the integral $F(z)$ of an analytic function $f(z)$ is an analytic function of its upper limit, provided the path of integration is confined to a region R within which the integrand is analytic. The fundamental theorem of calculus follows immediately from this result.

$$\int_a^b f(z)\, dz = \int_{z_0}^b f(z)\, dz - \int_{z_0}^a f(z)\, dz = F(b) - F(a) ,$$

where a and b are points in R, and $F'(z) = f(z)$, that is, $F(z)$ is an antiderivative of $f(z)$. We have already noticed that this method of evaluating integrals worked in a special case: the integral of $\sin z$ from $a = 0$ to $b = 1 + i$ (Section 6.3).

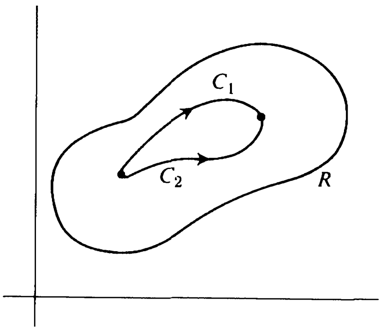

Fig. 6.10

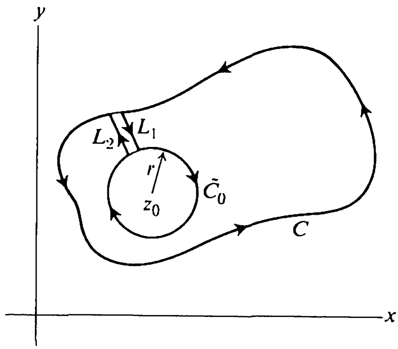

Fig. 6.11

Cauchy's Integral Formula

We now prove one of the most useful results in all mathematical physics.

Theorem. If $f(z)$ is analytic within and on a closed contour C, then for *any* point z_0, interior to C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz . \tag{6.15}$$

Proof. Inside the contour C, draw a circle $\tilde{C}_0$ of radius r about z_0, and consider the contour shown in Fig. 6.11. It consists of the circle $\tilde{C}_0$ and the contour C joined by two straight line segments, L_1 and L_2, which lie arbitrarily close to each other. Let us call this entire contour C'. Now consider

$$\oint_{C'} \frac{f(z)}{z - z_0}\, dz = \oint_C \frac{f(z)}{z - z_0}\, dz + \int_{L_1} \frac{f(z)}{z - z_0}\, dz + \oint_{\tilde{C}_0} \frac{f(z)}{z - z_0}\, dz + \int_{L_2} \frac{f(z)}{z - z_0}\, dz .$$

Inside C', $f(z)/(z-z_0)$ is analytic, so by the Cauchy-Goursat theorem,

$$\oint_{C'} \frac{f(z)}{z-z_0}\,dz = 0 .$$

Now, as we bring the line segments L_1 and L_2 arbitrarily close together,

$$\int_{L_1} \frac{f(z)}{z-z_0}\,dz \to -\int_{L_2} \frac{f(z)}{z-z_0}\,dz ,$$

since the lines are traversed in opposite directions. Thus in this limit we have

$$\oint_{C'} \frac{f(z)}{z-z_0}\,dz = 0 = \oint_{C} \frac{f(z)}{z-z_0}\,dz + \oint_{\tilde{C}_0} \frac{f(z)}{z-z_0}\,dz ,$$

so that

$$\oint_{C} \frac{f(z)}{z-z_0}\,dz = -\oint_{\tilde{C}_0} \frac{f(z)}{z-z_0}\,dz .$$

At this point we note that considered as a contour in its own right, i.e., not just as a part of C', $\tilde{C}_0$ is traversed in a clockwise direction. Let us therefore define $C_0 = -\tilde{C}_0$ so that C_0 is a counterclockwise contour (as is C). Then we may write

$$\oint_{C} \frac{f(z)}{z-z_0}\,dz = \oint_{C_0} \frac{f(z)}{z-z_0}\,dz .$$

(Note that for the purposes of what we have just done, C_0 need not be a circle; it could equally well be *any* closed contour lying completely inside C and oriented in the same sense as C.) We may rewrite the last equation as

$$\oint_{C} \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_{C_0} \frac{dz}{z-z_0} + \oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz .$$

We now use the fact that C_0 is a circle to write $z - z_0 = re^{i\theta}$ on C_0. Thus the first integral on the right becomes

$$\oint_{C_0} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = 2\pi i , \qquad \text{for all } r > 0 \text{ within } C .$$

Cauchy's formula will therefore be established if we can show that the second integral vanishes for some choice of the contour C_0. The continuity of $f(z)$ at z_0 tells us that for all $\epsilon > 0$ there exists a δ such that if $|z - z_0| \leq \delta$, then $|f(z) - f(z_0)| < \epsilon$. So by taking $r = \delta$, we satisfy the equation $|z - z_0| = \delta$, which in turn implies that

$$\left|\oint_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \leq \oint_{C_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz| < \frac{\epsilon}{\delta}(2\pi\delta) = 2\pi\epsilon .$$

Thus by taking r small enough, but still greater than zero, the absolute value of the integral can be made smaller than any preassigned number. Thus

$$\oint_{C} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) .$$

This result gives us another hint of the amazingly strong inner structure of analytic functions. It means that if a function is analytic within and on a contour C, its value at every point inside C is determined by its values on the bounding curve C. There is a familiar equivalent result from electrostatics: If a real-valued function $u(x, y)$ is fixed on some boundary, and if $\nabla^2 u = 0$, then u is determined everywhere inside the boundary. An analytic function is built out of a pair of such harmonic functions. We could, if we liked, study the "theory of harmonic functions" instead of "analytic function theory".

Derivatives of Analytic Functions

Using Cauchy's integral formula, we can prove that all the derivatives of an analytic function are analytic. The corresponding result for real variables fails: a function which is once differentiable in some region is not necessarily infinitely differentiable in that region. The function $f(x) = x|x|$, for example, has as its derivative $f'(x) = 2|x|$, which is continuous everywhere; $f'(x)$ is not differentiable at the origin, however.

If we differentiate both sides of Cauchy's integral formula, interchanging the orders of integration and differentiation, we get

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,dz\,. \tag{6.16}$$

Since z_0 is any point inside C, we may take it as a variable. To establish this formula in a rigorous manner, note that by using Cauchy's integral formula, we may write

$$\begin{aligned} f'(z_0) = \lim_{z_1\to z_0}\frac{f(z_1) - f(z_0)}{z_1 - z_0} &= \frac{1}{2\pi i}\lim_{z_1\to z_0}\oint_C\left[\frac{f(z)}{z - z_1} - \frac{f(z)}{z - z_0}\right]\frac{dz}{z_1 - z_0} \\ &= \frac{1}{2\pi i}\lim_{z_1\to z_0}\oint_C \frac{f(z)}{(z - z_1)(z - z_0)}\,dz\,. \end{aligned}$$

Hence

$$\begin{aligned} f'(z_0) - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,dz &= \frac{1}{2\pi i}\lim_{z_1\to z_0}\oint_C f(z)\left[\frac{1}{(z - z_1)(z - z_0)} - \frac{1}{(z - z_0)^2}\right]dz \\ &= \frac{1}{2\pi i}\lim_{z_1\to z_0}(z_1 - z_0)\oint_C \frac{f(z)}{(z - z_1)(z - z_0)^2}\,dz\,. \end{aligned}$$

Calling $z_1 - z_0 = \epsilon e^{i\theta}$, we have

$$\left|f'(z_0) - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,dz\right| \le \frac{1}{2\pi}\lim_{\epsilon\to 0}\epsilon\oint_C \frac{|f(z)||dz|}{|(z - z_0) - \epsilon e^{i\theta}||z - z_0|^2}\,.$$

Replacing $|z - z_0|$ by its minimum value, say μ, and $|f(z)|$ by its maximum value M, we obtain

$$\left|f'(z_0) - \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\,dz\right| \le \frac{1}{2\pi}\frac{ML}{\mu^2}\lim_{\epsilon\to 0}\frac{\epsilon}{\mu - \epsilon} = 0\,,$$

where L is the length of the contour. Thus we have proved Eq. (6.16). Repeating the process, we obtain

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3}\, dz\,,$$

and, in general, for the nth derivative,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz\,. \tag{6.17}$$

This result is readily established by induction in the same manner we used to prove Eq. (6.16). Thus $f(z)$ has derivatives of all orders within C. The kth derivative of $f(z)$ is continuous within C because the $(k + 1)$ derivative exists. Thus if we write $f(z) = u(x, y) + iv(x, y)$, the partial derivatives of u and v of all orders are continuous whenever f is analytic. We can therefore drop in our derivation of Eq. (6.4) the *requirement* that the second partial derivatives be continuous: they are guaranteed to be continuous because f is analytic.

Liouville's Theorem

Theorem. If $f(z)$ is entire and $|f(z)|$ is bounded for all values of z, then $f(z)$ is a constant.

Proof. From Cauchy's integral formula, we have found that

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz\,.$$

If we take C to be the circle $|z - z_0| = r_0$, then

$$|f'(z_0)| \le \left|\frac{1}{2\pi i}\right| \oint_{C_0} \frac{|f(z)|}{|(z - z_0)^2|}\, |dz|$$
$$< \frac{1}{2\pi r_0^2} M 2\pi r_0 = \frac{M}{r_0}\,,$$

where $|f(z)| < M$ within and on C_0. Therefore $|f'(z_0)| < M/r_0$, and we may take r_0 as large as we like because $f(z)$ is entire. So taking r_0 large enough, we can make $|f'(z_0)| < \epsilon$, for any preassigned ϵ. That is, $|f'(z_0)| = 0$, which implies that $f'(z_0) = 0$ for all z_0, so $f(z_0) =$ constant. QED

Example. The entire functions sin z and cos z must not be bounded. It is clear from Eqs. (6.8) and (6.9) that they are not.

Fundamental Theorem of Algebra

This theorem, which is difficult to prove algebraically, follows easily from Liouville's theorem, and provides a remarkable tie-up between analysis and algebra. We include the proof because of its great simplicity and beauty.

Theorem. If $P(z) = a_0 + a_1 z + \cdots + a_m z^m$ is a polynomial in z of degree one or greater, then the equation $P(z) = 0$ has at least one root.

Proof. Assume the contrary, namely that $P(z) \neq 0$ for any z. Then the function $1/P(z)$ is entire. Furthermore $|1/P(z)| \to 0$ as $|z| \to \infty$ so $|1/P(z)|$ is bounded for all z. Therefore, by Liouville's theorem, $1/P(z) =$ const, a contradiction, since $P(z)$ is of degree one or greater. Hence $P(z) = 0$, for at least one value of z. QED

6.5 HILBERT TRANSFORMS AND THE CAUCHY PRINCIPAL VALUE

It is often the case that in the study of some physical system one has to deal with complex-valued functions—indices of refraction, susceptibilities, scattering amplitudes, impedances, etc.—which have a physical meaning only when the argument of the function (which might, for example, be a frequency or an energy) takes on *real* values. In many cases it is possible to obtain, from the laws governing the system, information about the general properties of such functions when the argument is *complex*; for example, it may be that the function is analytic in some region of the complex plane. Since experimental data can only be obtained for real values of the argument, it is of interest to see whether we can use general properties such as analyticity to deduce relations between real quantities of direct physical significance. The key to such a program can be found in the study of Hilbert transform pairs, which we shall investigate in this section.

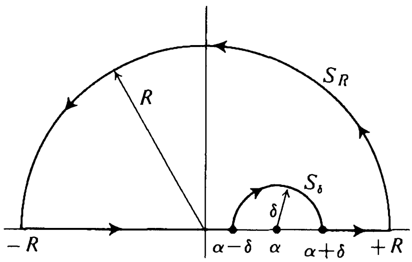

Fig. 6.12 The contour, C, used to obtain Eq. (6.18). The radius, R, of the semi-circle, S_r, may be made as large as necessary, and the radius, S, of the semi-circle, S_δ, may be made as small as we please.

Let us begin by considering a function $f(z)$, which is analytic in the upper half of the complex plane, and which is such that $|f(z)| \to 0$ as $|z| \to \infty$ in the upper half-plane. (Note that the only function which can satisfy these conditions in the *entire* plane is $f \equiv 0$). Now consider the contour integral

$$\oint_C \frac{f(z)}{z-\alpha}\,dz\,,$$

where C is the contour shown in Fig. 6.12 and α is real. By assumption, $f(z)$ is analytic within and on C; so is $1/(z-\alpha)$. Thus

$$\oint_C \frac{f(z)}{z-\alpha}\,dz = 0\,.$$

Let us break this up as follows:

$$\oint_C \frac{f(z)}{z-\alpha}\,dz \equiv \int_{-R}^{\alpha-\delta} \frac{f(x)}{x-\alpha}\,dx + \int_{S_\delta} \frac{f(z)}{z-\alpha}\,dz + \int_{\alpha+\delta}^{R} \frac{f(x)}{x-\alpha}\,dx + \int_{S_R} \frac{f(z)}{z-\alpha}\,dz = 0\,.$$

Here δ is the radius of the small semicircle S_δ, centered at $x = \alpha$, and R is the radius of the large semicircle S_R, centered at the origin, as shown in Fig. 6.12. The radius δ can be chosen as small as we please, and R can be chosen as large as we please. In the limit of arbitrarily small δ, the quantity

$$\int_{-R}^{\alpha-\delta} \frac{f(x)}{x-\alpha}\,dx + \int_{\alpha+\delta}^{R} \frac{f(x)}{x-\alpha}\,dx$$

is called the *principal-value integral* of $f(x)/(x-\alpha)$ and is denoted by

$$P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx\,.$$

We will say more about this integral below. Now, along the large semicircle S_R, we set $z = Re^{i\theta}$, so that

$$\int_{S_R} \frac{f(z)}{z-\alpha}\,dz = i\int_0^{\pi} \frac{f(Re^{i\theta})}{Re^{i\theta}-\alpha}\,Re^{i\theta}\,d\theta\,,$$

and hence

$$\left|\int_{S_R} \frac{f(z)}{z-\alpha}\,dz\right| \le \frac{R}{|R-\alpha|}\int_0^{\pi} |f(Re^{i\theta})|\,d\theta\,,$$

since $|Re^{i\theta}-\alpha| = [R^2+\alpha^2-2R\alpha\cos\theta]^{1/2} \ge [R^2+\alpha^2-2R\alpha]^{1/2} = |R-\alpha|$. But as $R\to\infty$, $|f(z)|\to 0$ and $R/|R-\alpha|\to 1$. Therefore the integral over the semicircle of radius R can be made arbitrarily small by choosing R sufficiently large. Thus we may write

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx = -\int_{S_\delta} \frac{f(z)}{z-\alpha}\,dz = -f(\alpha)\int_{S_\delta} \frac{dz}{z-\alpha} - \int_{S_\delta} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz\,,$$

where we have added and subtracted the term $\int_{S_\delta}[f(\alpha)/(z-\alpha)]\,dz$. Setting $z-\alpha = \delta e^{i\theta}$ in the first integral on the right-hand side of this equation, we find that

$$-f(\alpha)\int_{S_\delta} \frac{dz}{z-\alpha} = -if(\alpha)\int_{\pi}^{0} d\theta = i\pi f(\alpha)\,.$$

Thus

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx = i\pi f(\alpha) - \int_{S_\delta} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz\,.$$

Since $f(z)$ is continuous at $z = \alpha$, the argument used in deriving Cauchy's integral formula tells us that this last integral over S_δ vanishes. Hence

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx = i\pi f(\alpha)\;.$$

For the sake of brevity, we write this simply as

$$P\int_{-\infty}^{\infty} \frac{f(x)}{x-\alpha}\,dx = i\pi f(\alpha)\;, \tag{6.18}$$

where $f(x)$ is a complex-valued function of a real variable. We may write it as

$$f(x) \equiv f_R(x) + if_I(x)\;.$$

Equating real and imaginary parts in Eq. (6.18), we get

$$f_R(\alpha) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f_I(x)}{x-\alpha}\,dx\,, \tag{6.19a}$$

$$f_I(\alpha) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f_R(x)}{x-\alpha}\,dx\;. \tag{6.19b}$$

Any pair of functions which satisfy Eqs. (6.19a) and (6.19b) is called a *Hilbert transform pair*. Note that these equations tell us that if $f_I(x) \equiv 0$, then $f_R(x) \equiv 0$.

The principal-value integral is seen to be a way of avoiding singularities on a path of integration: one integrates to within δ of the singularity in question, skips over the singularity, and begins integrating again a distance δ beyond the singularity. This prescription enables one to make sense out of integrals like

$$\int_{-R}^{R} \frac{dx}{x}\,.$$

One would like this integral to be zero, since we are integrating an odd function over a symmetric domain. However, unless we insert a P in front of this integral, the singularity at the origin makes the integral meaningless. Following the prescription for principal-value integrals, we can easily evaluate the above integral. We have

$$P\int_{-R}^{R} \frac{dx}{x} = \lim_{\delta\to 0}\left[\int_{-R}^{-\delta} \frac{dx}{x} + \int_{\delta}^{R} \frac{dx}{x}\right].$$

In the first integral on the right-hand side, set $x = -y$. Then

$$P\int_{-R}^{R} \frac{dx}{x} = \lim_{\delta\to 0}\left[\int_{R}^{\delta} \frac{dy}{y} + \int_{\delta}^{R} \frac{dx}{x}\right].$$

The sum of the two integrals inside the brackets is zero, since

$$\int_{R}^{\delta} = -\int_{\delta}^{R}\,.$$

Thus

$$P\int_{-R}^{R} \frac{dx}{x} = 0\,.$$

In a similar manner, we may evaluate

$$P\int_{-R}^{R}\frac{dx}{x-a}$$

when $-R < a < R$. As above, we write

$$P\int_{-R}^{R}\frac{dx}{x-a} = \lim_{\delta\to 0}\left[\int_{-R}^{a-\delta}\frac{dx}{x-a} + \int_{a+\delta}^{R}\frac{dx}{x-a}\right].$$

Again setting $x = -y$ in the first integral on the right-hand side, we find that

$$\begin{aligned} P\int_{-R}^{R}\frac{dx}{x-a} &= \lim_{\delta\to 0}\left[\int_{R}^{\delta-a}\frac{dy}{y+a} + \ln(R-a) - \ln\delta\right] \\ &= \lim_{\delta\to 0}\left[\ln\delta - \ln(R+a) + \ln(R-a) - \ln\delta\right]. \end{aligned}$$

Thus

$$P\int_{-R}^{R}\frac{dx}{x-a} = \ln\left(\frac{R-a}{R+a}\right), \qquad -R < a < R. \tag{6.20}$$

For the case

$$P\int_{-R}^{R}\frac{f(x)}{x-a},$$

the result of Eq. (6.20) leads to

$$P\int_{-R}^{R}\frac{f(x)}{x-a}\,dx = P\int_{-R}^{R}\frac{f(a)}{x-a}\,dx + P\int_{-R}^{R}\frac{f(x)-f(a)}{x-a}\,dx,$$

or

$$P\int_{-R}^{R}\frac{f(x)}{x-a}\,dx = f(a)\ln\left(\frac{R-a}{R+a}\right) + P\int_{-R}^{R}\frac{f(x)-f(a)}{x-a}\,dx. \tag{6.21}$$

It will often happen that the second integral on the right-hand side of Eq. (6.21) will not be singular at $x = a$ [for example, this will be the case if $f(x)$ is differentiable at $x = a$] so the P symbol there can be dropped. We leave it to the reader to obtain the closely related result:

$$P\int_{0}^{R}\frac{f(x)}{x^2-a^2}\,dx = f(a)\frac{1}{2a}\ln\left(\frac{R-a}{R+a}\right) + P\int_{0}^{R}\frac{f(x)-f(a)}{x^2-a^2}\,dx. \tag{6.22}$$

To illustrate the use of the principal-value method and also the use of Eqs. (6.19a) and (6.19b), we consider the function $f(z) = 1/(z+i)$. This function satisfies all the hypotheses made in deriving Eqs. (6.19a) and (6.19b). We see that

$$f_R(x) = \frac{x}{x^2+1}, \qquad f_I(x) = -\frac{1}{x^2+1}.$$

Let us examine Eq. (6.19b). We write it as

$$\lim_{R\to\infty} P\int_{-R}^{R}\frac{f_R(x)}{x-\alpha}\,dx = -\pi f_I(\alpha).$$

We want to see if $f(z) = 1/(z + i)$ satisfies this relation. Using Eq. (6.21), we have

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{f_R(x)}{x - \alpha}\, dx = \lim_{R\to\infty}\left[f_R(\alpha) \ln\left(\frac{R - \alpha}{R + \alpha}\right) + P\int_{-R}^{R} \frac{f_R(x) - f_R(\alpha)}{x - \alpha}\, dx \right]. \tag{6.23}$$

Now, since $f_R(x) = x/(x^2 + 1)$, we find that

$$\frac{f_R(x) - f_R(\alpha)}{x - \alpha} = \frac{1 - \alpha x}{(\alpha^2 + 1)(x^2 + 1)},$$

so we may drop the principal-value sign on the right-hand side of Eq. (6.23). Also,

$$\lim_{R\to\infty} \ln\left(\frac{R - \alpha}{R + \alpha}\right) = 0,$$

so Eq. (6.23) becomes

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{f_R(x)}{x - \alpha}\, dx = \frac{1}{\alpha^2 + 1} \lim_{R\to\infty} \int_{-R}^{R} \frac{1 - \alpha x}{x^2 + 1}\, dx = \frac{1}{\alpha^2 + 1} \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2 + 1},$$

since $x/(x^2 + 1)$ is an odd function. Thus

$$\lim_{R\to\infty} P\int_{-R}^{R} \frac{f_R(x)}{x - \alpha}\, dx = \frac{2}{\alpha^2 + 1} \lim_{R\to\infty} \tan^{-1} R = \frac{\pi}{\alpha^2 + 1} = -\pi f_I(\alpha),$$

so Eq. (6.19b) is indeed satisfied. We leave it to the reader to show that Eq. (6.19a) is also satisfied. Clearly Eq. (6.21) is very useful in conjunction with Eqs. (6.19a) and (6.19b) because under the assumptions made in deriving these equations, $f(z)$ is differentiable at $z = \alpha$. Thus

$$f_R(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f_I(x) - f_I(\alpha)}{x - \alpha}\, dx, \tag{6.24a}$$

$$f_I(\alpha) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f_R(x) - f_R(\alpha)}{x - \alpha}\, dx, \tag{6.24b}$$

where, in case of any ambiguity,

$$\int_{-\infty}^{\infty} \equiv \lim_{R\to\infty} \int_{-R}^{R}.$$

These methods can sometimes be used to evaluate real definite integrals. For example, consider the function $f(z) = e^{iz}$. This function is analytic everywhere, and if we write $z = Re^{i\theta}$, then $|f(z)| \to 0$ as $R \to \infty$ for all θ such that $0 < \theta < \pi$. This is not quite what we used above to show that the integral around a large semicircle of $f(z)/(z - \alpha)$ vanishes (although it would be the same if we had $0 \le \theta \le \pi$). However, the reader can show for $f(z) = e^{iz}$ that the contribution from the large semicircle vanishes and Eqs. (6.19a) and (6.19b) are satisfied. In this case, $f_R(x) = \cos x$ and $f_I(x) = \sin x$, so using Eq.

(6.24a), we obtain

$$\cos\alpha = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin x - \sin\alpha}{x-\alpha}\,dx\,.$$

Since $\sin x - \sin\alpha = 2\sin\frac{1}{2}(x-\alpha)\cos\frac{1}{2}(x+\alpha)$, we see that indeed there is no singularity of the integrand at $x = \alpha$. For the special case $\alpha = 0$, we find that

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx\,,$$

that is,

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx = \pi\,.$$

From this result, we also obtain by symmetry

$$\int_{0}^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}\,.$$

Here we see that the oscillations of $\sin x$, when x is large, make the integral converge, even though

$$\int_{1}^{\infty}\frac{1}{x}\,dx$$

diverges. This is analogous to the fact that the *alternating* series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\;(=\ln 2)$$

converges, whereas $\sum_{n=1}^{\infty}(1/n)$ diverges.

6.6 AN INTRODUCTION TO DISPERSION RELATIONS

As mathematical results, the equations derived in the previous section are interesting in their own right. However, a scientist naturally wants to know if there are any physical systems to which these results can be applied. What we shall now show is that under fairly broad physically motivated assumptions, one can find physical quantities which possess the analytic properties necessary for them to satisfy a Hilbert transform relation. In our detailed applications, we will focus our attention on electromagnetic theory, but many of our results will be more general than this.

We begin by considering a physical system for which an input, $I(t)$, is related to a response, $R(t)$, in the following *linear* manner:

$$R(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(t-t')I(t')\,dt'\,. \tag{6.25}$$

For example, $I(t')$ might be the electric field at a time t', and $R(t)$ might be the resulting polarization field at time t. We have assumed that G depends only on

$t - t'$ because we want the system to respond to a sharp input at t_0, $I(t') = I_0\delta(t' - t_0)$, in the same way it would respond to a sharp input at $t_0 + \tau$, that is, at a time τ later. For the first case, we have

$$R_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t - t')I_0\delta(t' - t_0)\, dt' = \frac{1}{\sqrt{2\pi}} I_0 G(t - t_0) .$$

For the second case, we have

$$R_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t - t')I_0\delta(t' - t_0 - \tau)\, dt' = \frac{1}{\sqrt{2\pi}} I_0 G(t - t_0 - \tau) ,$$

or, in other words,

$$R_2(t + \tau) = \frac{1}{\sqrt{2\pi}} I_0 G(t - t_0) = R_1(t) .$$

Thus if we shift the input by τ, then we shift the response by τ.

Now, what can we say about $G(\tau)$ on general physical grounds? First we see that an input at t should not give rise to a response at times prior to t, that is, $G(\tau) = 0$ for $\tau < 0$, so

$$R(t) = \int_{-\infty}^{t} G(t - t')I(t')\, dt' ,$$

which shows that the response at t is the weighted linear superposition of all inputs *prior* to t. This is the *causality requirement*. The possibility that $G(\tau)$ is singular for any finite τ is excluded since the response from a sharp input, $I(t') = I_0\delta(t' - t_0)$, is

$$R(t) = \frac{1}{\sqrt{2\pi}} I_0 G(t - t_0) , \qquad t > t_0 ,$$

and since on physical grounds we require that this response always be finite, $G(\tau)$ is finite for all τ. Furthermore, we make the physically reasonable assumption that the effect of an input in the *remote* past does not appreciably influence the present. This may be stated as the requirement that $G(\tau) \to 0$ as $\tau \to \infty$ since, from the previous equation, it amounts to the assumption that the response to any impulse dies down after a sufficiently long time (i.e., any system has some dissipative mechanism).

Now consider the Fourier transform of Eq. (6.25). Using the convolution theorem (see Section 5.7), we find that

$$r(\omega) = g(\omega)i(\omega) ,$$

where

$$r(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} R(t)e^{i\omega t}\, dt , \qquad g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t)e^{i\omega t}\, dt ,$$

$$i(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t)e^{i\omega t}\, dt .$$

In electromagnetic theory, where I is the applied electric field $\mathbf{E}$, and R is the polarization field $\mathbf{P}$, it is customary to denote $g(\omega)$ by $\chi(\omega)$, which is referred to as the electric susceptibility. Thus

$$P(\omega) = \chi(\omega)E(\omega) .$$

By assuming that $G(\tau)$ satisfies

$$\int_0^\infty |G(\tau)|\, d\tau < \infty ,$$

we can guarantee the existence of a bounded $g(\omega)$ for all ω. We may now summarize our physically motivated assumptions on $G(\tau)$:

a) $G(\tau)$ is bounded for all τ;
b) $|G(\tau)|$ is integrable, so $G(\tau) \to 0$ faster than $1/\tau$ as $\tau \to \infty$;
c) $G(\tau) = 0$ for $\tau < 0$.

We may remark that (a) and (b) taken together imply that $G(\tau)$ is square integrable and hence (see Section 9.6) $g(\omega)$ is square integrable.

We now want to show that we can extend $g(\omega)$ into the complex z-plane in such a way that $g(z)$ satisfies the conditions under which we derived the Hilbert transform pair [Eqs. (6.19a) and (6.19b)] of the previous section. First, since $G(\tau) = 0$ for $\tau < 0$, we write

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t)e^{i\omega t}\, dt .$$

We extend this relation into the complex plane by using the definition

$$g(z) \equiv \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t)e^{izt}\, dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t)e^{i\omega t}e^{-\omega' t}\, dt ,$$

where we have written $z = \omega + i\omega'$. We now restrict our attention to the upper half-plane ($\omega' > 0$) where, because of the causality requirement given in assumption (c) above, ($t > 0$ in the above integral), the term $e^{-\omega' t}$ is a *decaying* exponential. For $0 < \theta < \pi$, we have

$$|g(z)| \leq \frac{1}{\sqrt{2\pi}} M_G \int_0^\infty e^{-[|z| \sin\theta]t}\, dt ,$$

where we have replaced $G(t)$ by its maximum value, M_G [assumption (a) above]. Thus

$$|g(z)| \leq \frac{M_G}{\sqrt{2\pi}\, |z| \sin\theta} ,$$

which tends to zero when $|z| \to \infty$. For $\theta = 0$ or π, we have

$$g(\omega, \omega' = 0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t)e^{i\omega t}\, dt .$$

Since $G(t)$ is square integrable, so is $g(\omega, \omega' = 0)$ as a function of ω (see Section 9.6), and hence $|g(\omega, \omega' = 0)|$ tends to zero as $\omega \to \infty$. Thus in any direction in the upper half-plane, $|g(z)| \to 0$ as $|z| \to \infty$.

Now we want to show that $g(z)$ is analytic in the upper half-plane. Using

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t) e^{izt}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t) e^{i\omega t} e^{-\omega' t}\, dt\,, \tag{6.26}$$

we see that for $\omega' > 0$,

$$\frac{d^n g}{dz^n} = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t) \frac{d^n}{dz^n} e^{izt}\, dt = \frac{i^n}{\sqrt{2\pi}} \int_0^\infty t^n G(t) e^{i\omega t} e^{-\omega' t}\, dt\,, \tag{6.27}$$

since in this case the integrals in both (6.26) and (6.27) are uniformly convergent because of the term $e^{-\omega' t}(\omega' > 0, t > 0)$. Thus $g(z)$ is analytic in the upper half-plane $(\omega' > 0)$. However, it is clear that our assumptions on $G(t)$ do not enable us to extend the domain of analyticity to $\omega' \geq 0$. Nevertheless, we can say that $g(z)$ is bounded on the real axis, so the only singularities when $\omega' = 0$ will be of the branch point variety, and even then such branch singularities as $1/\sqrt{z}$ or $\log z$ are excluded by the boundedness requirement. The reader can see by looking back at the derivation of the original pair of Hilbert transform equations [Eqs. (6.19a) and (6.19b)] that it can be modified to include bounded branch point singularities on the real axis by taking a small semicircular detour around any such point. Thus Eqs. (6.19a) and (6.19b) remain unaltered by the presence of such singularities (the branches can always be chosen to avoid the upper half-plane). To eliminate the possibility of these branch singularities, one would have to assume an exponential type falloff of $G(\tau)$ as $\tau \to \infty$. Hence for any $g(z)$ arising from a $G(t)$ which satisfies assumptions (a), (b), and (c), we may write

$$g_R(\omega) = \frac{1}{\pi} P \int_{-\infty}^\infty \frac{g_I(\bar\omega)}{\bar\omega - \omega}\, d\bar\omega\,, \tag{6.28a}$$

$$g_I(\omega) = -\frac{1}{\pi} P \int_{-\infty}^\infty \frac{g_R(\bar\omega)}{\bar\omega - \omega}\, d\bar\omega\,. \tag{6.28b}$$

Thus by making a few very reasonable assumptions about the system in question, we can show that the real and imaginary parts of the physical quantity $g(\omega)$ are intimately related to each other for *real* values of the argument by what is essentially a dispersion relation. The key assumption is the *causality requirement*; we may say that causality implies the existence of dispersion relations in the case we have considered. In actual practice, one often restricts the term "dispersion relation" to mean an integral relation between two observable quantities which involves only an integration over values of the argument which are physically meaningful. Thus in Eqs. (6.28a) and (6.28b) only positive frequencies are accessible to experiment, so they are not directly useful as they stand. However, $G(t)$ is real, so we may proceed as follows:

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t) e^{izt}\, dt\,,$$

$$g^*(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty G^*(t) e^{-iz^*t}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^\infty G(t) e^{-iz^*t}\, dt = g(-z^*)\,.$$

Thus we have $g^*(z) = g(-z^*)$, which is often referred to as the *reality condition*. If z is real $(z = \omega)$, we find that

$$g_R(\omega) - ig_I(\omega) = g_R(-\omega) + ig_I(-\omega) ,$$

or

$$g_R(\omega) = g_R(-\omega) , \tag{6.29a}$$

$$g_I(\omega) = -g_I(-\omega) , \tag{6.29b}$$

that is, g_R is an *even* function of ω and g_I is an *odd* function of ω. Note that if Eqs. (6.29a) and (6.29b) are satisfied, then the function

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} \, d\omega$$

is a real function.

Now in Eq. (6.28a), let us write

$$g_R(\omega) = \frac{1}{\pi} P \int_{-\infty}^{0} \frac{g_I(\bar{\omega})}{\bar{\omega} - \omega} \, d\bar{\omega} + \frac{1}{\pi} P \int_{0}^{\infty} \frac{g_I(\bar{\omega})}{\bar{\omega} - \omega} \, d\bar{\omega} .$$

In the first integral, we let $\bar{\omega} \to -\bar{\omega}$. Thus

$$g_R(\omega) = \frac{1}{\pi} P \int_{\infty}^{0} \frac{g_I(-\bar{\omega})}{\bar{\omega} + \omega} \, d\bar{\omega} + \frac{1}{\pi} P \int_{0}^{\infty} \frac{g_I(\bar{\omega})}{\bar{\omega} - \omega} \, d\bar{\omega} .$$

Using Eq. (6.29b), we finally obtain

$$g_R(\omega) = \frac{2}{\pi} P \int_{0}^{\infty} \frac{\bar{\omega} g_I(\bar{\omega})}{\bar{\omega}^2 - \omega^2} \, d\bar{\omega} , \tag{6.30a}$$

and in an identical manner,

$$g_I(\omega) = -\frac{2\omega}{\pi} P \int_{0}^{\infty} \frac{g_R(\bar{\omega})}{\bar{\omega}^2 - \omega^2} \, d\bar{\omega} . \tag{6.30b}$$

These expressions involve only positive, experimentally accessible frequencies. For the electric susceptibility, for example, we have

$$\chi_R(\omega) = \frac{2}{\pi} P \int_{0}^{\infty} \frac{\bar{\omega} \chi_I(\bar{\omega})}{\bar{\omega}^2 - \omega^2} \, d\bar{\omega} , \tag{6.31a}$$

$$\chi_I(\omega) = -\frac{2\omega}{\pi} P \int_{0}^{\infty} \frac{\chi_R(\bar{\omega})}{\bar{\omega}^2 - \omega^2} \, d\bar{\omega} . \tag{6.31b}$$

Equations (6.31a) and (6.31b) were first derived by H. A. Kramers and R. de L. Kronig and are referred to as the Kramers-Kronig dispersion relations.

Now, according to electromagnetic theory, we may write

$$n^2(\omega) = 1 + 4\pi\chi(\omega) ,$$

where $n(\omega)$ is the (complex) index of refraction. Since $\chi(z)$ is analytic in the upper half-z-plane, so is $n^2(z)$, and the function

$$n(z) = \sqrt{1 + 4\pi\chi(z)} \tag{6.32}$$

is also analytic in this region *if* $1 + 4\pi\chi(z)$ has no zeroes in the upper half-plane. If $1 + 4\pi\chi(z)$ vanishes for some $z = z_0$ in the upper half-plane, then according to the reality condition

$$\chi^*(z_0) = \chi(-z_0^*) ,$$

we find that $1 + 4\pi\chi(z)$ also vanishes at $z = -z_0^*$. Since $-z_0^*$ lies in the upper half-plane with the imaginary part equal to Im z_0, there will be a cut in the upper half-plane running from $-z_0^*$ to z_0. In this case, the dispersion relations of Eqs. (6.31a) and (6.31b) would have to be modified. We will assume that $n(z)$ has no zeroes in the upper half-plane and is therefore analytic in this region. However, because of Eq. (6.32), $|n(z)|$ does not tend to zero as $|z| \to \infty$. In fact,

$$n(z) \to 1 \qquad \text{as} \qquad |z| \to \infty$$

in the upper half-plane. This necessitates a modification in the treatment of the term coming from the large semicircle of Fig. 6.12 in obtaining Eq. (6.18). We have, in the case of $n(z)$,

$$\int_{S_R} \frac{n(z)}{z - \alpha}\, dz \xrightarrow{|z|\to\infty} \int_0^\pi \frac{dz}{z} = i\pi$$

in the notation of the previous section (see Fig. 6.12). Thus, for the case of $n(z)$, Eq. (6.18) must be replaced by

$$i\pi n(\omega) = i\pi + P\int_{-\infty}^{\infty} \frac{n(\bar\omega)}{\bar\omega - \omega}\, d\bar\omega .$$

Separating real and imaginary parts, we obtain

$$n_R(\omega) = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{n_I(\bar\omega)}{\bar\omega - \omega}\, d\bar\omega , \tag{6.33a}$$

$$n_I(\omega) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{n_R(\bar\omega)}{\bar\omega - \omega}\, d\bar\omega . \tag{6.33b}$$

Making use of the reality condition, we can write these equations as

$$n_R(\omega) = 1 + \frac{2}{\pi} P\int_0^{\infty} \frac{\bar\omega n_I(\bar\omega)}{\bar\omega^2 - \omega^2}\, d\bar\omega , \tag{6.34a}$$

$$n_I(\omega) = -\frac{2\omega}{\pi} P\int_0^{\infty} \frac{n_R(\bar\omega)}{\bar\omega^2 - \omega^2}\, d\bar\omega . \tag{6.34b}$$

The quantity $\mu(\omega) \equiv (2\omega/c)n_I(\omega)$, when c is the speed of light in vacuum, is called the *absorption coefficient* and is the inverse of the distance a wave

$$\psi(x, t) = A \exp\left\{i\omega\left[\frac{n(\omega)}{c}x - t\right]\right\}$$

travels before its intensity drops to $1/e$ of its value at $x = 0$. In terms of μ, we have

$$n_R(\omega) = 1 + \frac{c}{\pi} P\int_0^{\infty} \frac{\mu(\bar\omega)}{\bar\omega^2 - \omega^2}\, d\bar\omega . \tag{6.35}$$

Thus the real part of the index of refraction is completely specified by knowing the absorption coefficient at all frequencies! It is from Eq. (6.35) that dispersion relations derive their name. Equation (6.35) relates a substance's absorption to its dispersive effects, i.e., to the way the real index of refraction varies with frequency. It is this variation with frequency which produces the well-known separation (dispersion) of different wavelengths of light by a prism.

In more recent times, the term dispersion relation has continued to be used to denote any relationship between real and imaginary parts of a physical quantity (a scattering amplitude in quantum mechanics, for example) which has the general appearance of a Hilbert transform. Note that since $\mu(\omega)$ must be positive for all frequencies on physical grounds (i.e., we do not expect to find waves which grow in time as they pass through a substance), Eq. (6.35) specifies that

$$n_R(0) = 1 + \frac{c}{\pi} P \int_0^\infty \frac{\mu(\bar{\omega})}{\bar{\omega}^2}\, d\bar{\omega}\,, \tag{6.36}$$

so we see that $n_R(0) > 1$. Thus the familiar static dielectric constant ϵ, given by $\epsilon = n^2(0)$ is always greater than unity. Since $\chi_I(0) = 0$, we see that the integral in Eq. (6.36) converges without our using the principal-value technique. Thus using $n(0) = \sqrt{\epsilon}$, we may write Eq. (6.36) as

$$\sqrt{\epsilon} - 1 = \frac{c}{\pi} \int_0^\infty \frac{\mu(\bar{\omega})}{\bar{\omega}^2}\, d\bar{\omega}\,.$$

This relation is known as a "sum rule" for the absorption coefficient; it relates the weighted integral over all values of the absorption coefficient to a simple, experimentally accessible constant.

The derivation of the dispersion relation for the index of refraction exhibits certain features which are often encountered in deriving dispersion relations. Namely, it often happens that the quantity in question does not tend toward zero as $|z|$ tends toward infinity, and, furthermore, one is not usually fortunate enough to know the precise behavior of the quantity as $|z|$ tends to infinity, except, for example, to say that it is *bounded* for large values of $|z|$. In this case, we can proceed as follows. Let α_0 be some point on the real axis at which $f(z)$ is analytic. Then the function

$$\frac{f(z) - f(\alpha_0)}{z - \alpha_0} \equiv \phi(z)$$

is not singular at $z = \alpha_0$, and $|\phi(z)| \to 0$ as $|z| \to \infty$. Also, if $f(z)$ is analytic in the upper half-plane, so is $\phi(z)$, and we can write a dispersion relation for $\phi(z)$:

$$i\pi\left[\frac{f(\alpha) - f(\alpha_0)}{\alpha - \alpha_0}\right] = P \int_{-\infty}^\infty \frac{f(x) - f(\alpha_0)}{(x - \alpha)(x - \alpha_0)}\, dx\,.$$

But

$$\frac{1}{(x - \alpha)(x - \alpha_0)} = \frac{1}{\alpha - \alpha_0}\left[\frac{1}{x - \alpha} - \frac{1}{x - \alpha_0}\right].$$

Therefore, our dispersion relation takes the form

$$i\pi f(\alpha) = i\pi f(\alpha_0) + (\alpha - \alpha_0) P \int_{-\infty}^{\infty} \frac{f(x)\, dx}{(x - \alpha)(x - \alpha_0)} - f(\alpha_0) P \int_{-\infty}^{\infty} \frac{dx}{x - \alpha} + f(\alpha_0) P \int_{-\infty}^{\infty} \frac{dx}{x - \alpha_0} .$$

According to the work of the previous section, these last two principal-value integrals vanish, so we have just

$$i\pi f(\alpha) = i\pi f(\alpha_0) + (\alpha - \alpha_0) P \int_{-\infty}^{\infty} \frac{f(x)\, dx}{(x - \alpha)(x - \alpha_0)} .$$

Separating the real and imaginary parts, we finally obtain

$$f_R(\alpha) = f_R(\alpha_0) + \frac{1}{\pi} (\alpha - \alpha_0) P \int_{-\infty}^{\infty} \frac{f_I(x)}{(x - \alpha)(x - \alpha_0)} dx , \qquad (6.37a)$$

$$f_I(\alpha) = f_I(\alpha_0) - \frac{1}{\pi} (\alpha - \alpha_0) P \int_{-\infty}^{\infty} \frac{f_R(x)}{(x - \alpha)(x - \alpha_0)} dx . \qquad (6.37b)$$

Relations of the type of Eqs. (6.37a) and (6.37b) are referred to as *once-subtracted dispersion relations*. For them to be of use in a particular physical problem, one must have a means of determining, say, $f_R(\alpha_0)$ for *some* α_0 in addition to possessing the usual information required by an ordinary dispersion relation of the type of Eq. (6.19a). If the properties of $f(z)$ for large $|z|$ are even "worse" than assumed above [for example, suppose that $|f(z)/z|$ tended toward a nonzero constant as $|z|$ tended toward infinity], then one could introduce more subtraction points, α_1, α_2, etc, in a similar manner.

We have already seen that causality implies certain analyticity properties; we conclude this section by showing that the converse is also true, namely, analyticity implies causality! We will do this by using the analytic properties of $n(z)$ to show that electromagnetic signals will not propagate in any medium faster than the speed of light. Consider a wave front traveling in the x-direction in a dielectric medium with a complex index of refraction, $n(z)$. Assume that at $x = 0$ there is no disturbance before $t = 0$, that is, $\phi(0, t) = 0$ for $t < 0$. We can write a general wave, as $\phi(x, t)$, a superposition of plane waves of all frequencies:

$$\phi(x, t) = \int_{-\infty}^{\infty} \phi(\omega) \exp\left\{i\omega\left[\frac{n(\omega)}{c} x - t\right]\right\} d\omega .$$

Note that

$$\phi(0, t) = \int_{-\infty}^{\infty} \phi(\omega) e^{-i\omega t}\, d\omega ,$$

so

$$\phi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(0, t) e^{i\omega t}\, dt . \qquad (6.38)$$

We may now use Eq. (6.38) to define $\phi(z)$ for z complex:

$$\phi(z) \equiv \frac{1}{2\pi} \int_0^{\infty} \phi(0, t) e^{i\omega t} e^{-\omega' t}\, dt ,$$

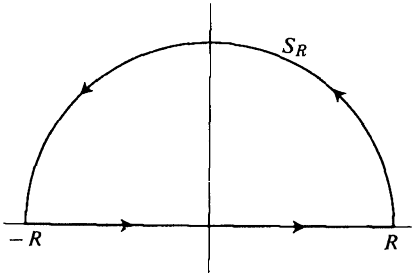

Fig. 6.13 The contour, C, used in evaluating Eq. (6.39). R may become as large as we please.

where we have made use of the fact that $\psi(0, t) = 0$ for $t < 0$ and have written, as usual, $z = \omega + i\omega'$. By our previous arguments, $\psi(z)$ is analytic in the upper half-plane and tends in norm toward zero as $|z|$ tends toward infinity in this region. Now let us consider

$$\oint_C \psi(z) \exp\left[iz\left(\frac{n(z)}{c} x - t\right)\right] dz$$

around the contour shown in Fig. 6.13. Since $\psi(z)$ and $n(z)$ are analytic in the upper half-plane,

$$\oint_C \psi(z) \exp\left[iz\left(\frac{n(z)}{c} x - t\right)\right] dz = 0 .$$

Thus

$$\int_{-R}^{R} \psi(\omega) \exp\left[i\omega\left(\frac{n(\omega)}{c} x - t\right)\right] d\omega + \int_{S_R} \psi(z) \exp\left[iz\left(\frac{n(z)}{c} x - t\right)\right] dz = 0 . \tag{6.39}$$

Calling the second integral in Eq. (6.39) I_R, we have

$$|I_R| \leq \int_0^\pi |\psi(z)| \exp\left[-R \sin\theta \left(\frac{n_R}{c} x - t\right) - R\cos\theta \frac{n_I}{c} x\right] R\, d\theta ,$$

where R is some arbitrarily large number. Now as $R \to \infty$, $n_I \to 0$ and $n_R \to 1$, as we have seen above. Thus, under these circumstances,

$$|I_R| \leq 2R \int_0^{\pi/2} |\psi(z)| \exp\left[-R\left(\frac{x}{c} - t\right) \sin\theta\right] d\theta .$$

But $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$. (To see this, note that $g(\theta) = \sin\theta - 2\theta/\pi$ vanishes at $\theta = 0$ and $\theta = \pi/2$. It is positive at $\theta = \pi/4$, so if it is to become zero or negative inside $[0, \pi/2]$, it must take on a minimum in this region. However, $g''(\theta)$ is always *negative* in $[0, \pi/2]$ so the function cannot possibly take on a minimum value. This can also be seen by drawing a graph of $\sin\theta$ and $2\theta/\pi$.) Therefore, if $(x/c - t)$ is positive,

$$|I_R| \leq 2R \int_0^{\pi/2} |\psi(z)| \exp\left[-\frac{2R}{\pi}\left(\frac{x}{c} - t\right) \theta\right] d\theta .$$

Now $|\psi(z)| \to 0$ as $R \to \infty$, so let us write for large R, $|\psi(z)| \leq aR^{-\lambda}(\lambda > 0)$. Then

$$|I_R| \leq 2aR^{1-\lambda} \int_0^{\pi/2} \exp\left[-\frac{2}{\pi} R \left(\frac{x}{c} - t\right)\theta\right] d\theta ,$$

or

$$|I_R| \leq \frac{\pi a}{R^\lambda[(x/c) - t]}\left[1 - \exp\left\{-R\left(\frac{x}{c} - t\right)\right\}\right].$$

If we assume that $x/c > t$, then as $R \to \infty$, we see that $|I_R| \to 0$. Note that if $x/c < t$, we cannot draw this conclusion because of the *growing* exponential in the above inequality. Thus as $R \to \infty$, Eq. (6.39) becomes

$$\psi(x, t) = \int_{-\infty}^{\infty} \psi(\omega) \exp\left\{i\omega\left[\frac{n(\omega)}{c} x - t\right]\right\} d\omega = 0$$

for $x/c > t$. Thus we reach the satisfying conclusion that if no signal is present at $x = 0$ when $t = 0$, then there will be no signal at $x = x_0 > 0$ before $t = x_0/c$, that is, a signal can propagate with at most the speed of light, c, even though $c/n(\omega)$ may be greater than c, since $n(\omega)$ is known experimentally to become less than 1 at high frequencies [it clearly *cannot* do so at very low frequencies, since we have already shown that $n(0) \geq 1$].

6.7 THE EXPANSION OF AN ANALYTIC FUNCTION IN A POWER SERIES

We now come to one of the most important applications of the Cauchy-Goursat theorem, namely, the possibility of expanding an analytic function in a power series. The main result may be stated as follows:

Laurent's theorem. Let $f(z)$ be analytic throughout the closed annular region between the two circles C_1 and C_2 with common center z_0. Then at each point in this annulus

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n , \tag{6.40}$$

with the series converging uniformly in any closed region, R, lying wholly within the annulus. Here

$$A_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' , \tag{6.41}$$

for $n = 0, \pm 1, \pm 2, \cdots$, and C is any closed contour in the annulus which encircles z_0.

Proof. Consider the contour K enclosing the region R as shown in Fig. (6.14); it may be written symbolically as $K = C_1 + L_1 - C_2 + L_2$. Here we adhere to the convention that simple circular contours are always traversed in a counterclockwise direction. Since the inner circle is traversed in a clockwise direction

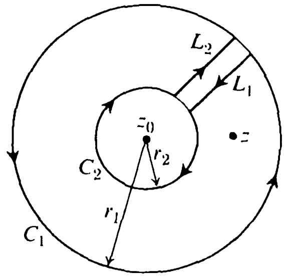

Fig. 6.14

when considered as a part of K, we write it is $-C_2$, where C_2 is a conventionally oriented curve. Since $f(z)$ is analytic within and on K,

$$2\pi i f(z) = \oint_K \frac{f(z')}{z' - z} dz' = \int_{C_1} \frac{f(z')}{z' - z} dz' + \int_{L_1} \frac{f(z')}{z' - z} dz + \int_{-C_2} \frac{f(z')}{z' - z} dz' + \int_{L_2} \frac{f(z')}{z' - z} dz' ,$$

where $z \in R$. By the same argument used in deriving Cauchy's integral formula we see that if L_1 and L_2 are taken to be arbitrarily close together the integrals along L_1 and L_2 cancel, and

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{z' - z} dz' , \tag{6.42}$$

where we have used Eq. (6.11). Equation (6.42) is the starting point for the proof of Laurent's theorem. To proceed further we make use of the identity (for $\alpha \neq 0$)

$$\frac{1}{\alpha - \beta} = \frac{1}{\alpha} + \frac{\beta}{\alpha(\alpha - \beta)} .$$

The second term is just $(\beta/\alpha)[1/(\alpha - \beta)]$, that is, β/α times the originally expanded quantity, so upon iterating N times we get

$$\frac{1}{\alpha - \beta} = \sum_{n=0}^{N} \frac{\beta^n}{\alpha^{n+1}} + \left(\frac{\beta}{\alpha}\right)^{N+1} \frac{1}{\alpha - \beta} . \tag{6.43}$$

Keeping this last equation in mind, let us write Eq. (6.42) in the form

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')}{[(z' - z_0) - (z - z_0)]} dz' + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')}{[(z - z_0) - (z' - z_0)]} dz' .$$

Using Eq. (6.43) we can rewrite this as

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{N}\oint_{C_1}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}f(z')\,dz' + \frac{1}{2\pi i}\oint_{C_1}\left[\frac{z-z_0}{z'-z_0}\right]^{N+1}\frac{f(z')}{z'-z}dz'$$
$$+ \frac{1}{2\pi i}\sum_{m=0}^{N}\oint_{C_2}\frac{(z'-z_0)^m}{(z-z_0)^{m+1}}f(z')\,dz' + \frac{1}{2\pi i}\oint_{C_2}\left[\frac{z'-z_0}{z-z_0}\right]^{N+1}\frac{f(z')}{z-z'}dz'. \tag{6.44}$$

According to by now familiar arguments the contours C_1 and C_2 in the first and third terms of the above equation may be replaced by any contour C about z_0 which is contained in the annulus bounded by C_1 and C_2. Making the change of index $m = n + 1$ in the third term, we obtain

$$f(z) = \sum_{n=-N-1}^{N} A_n(z-z_0)^n + R_N(z),$$

where A_n is given by Eq. (6.41) and

$$R_N(z) = \frac{1}{2\pi i}\oint_{C_1}\left[\frac{z-z_0}{z'-z_0}\right]^{N+1}\frac{f(z')}{z'-z}dz' + \frac{1}{2\pi i}\oint_{C_2}\left[\frac{z'-z_0}{z-z_0}\right]^{N+1}\frac{f(z')}{z-z'}dz'.$$

If we can show that the magnitude of $R_N(z)$ can be made less than any pre-as signed ϵ for N sufficiently large, where N is independent of z, then the proof will be complete. Letting r_1 be the radius of C_1 and r_2 be the radius of C_2, we have

$$|R_N(z)| \le \frac{1}{2\pi}\left[\int_0^{2\pi}\frac{|z-z_0|^{N+1}}{r_1^{N+1}}\frac{|f(z')|}{|z'-z|}r_1\,d\theta + \int_0^{2\pi}\frac{r_2^{N+1}}{|z-z_0|^{N+1}}\frac{|f(z')|}{|z-z'|}r_2\,d\theta\right].$$

Now we define

$$M_1 \equiv \operatorname*{Max}_{z\in C_1}|f(z)|, \qquad M_2 \equiv \operatorname*{Max}_{z\in C_2}|f(z)|,$$
$$l_1 \equiv \operatorname*{Max}_{z\in R}|z-z_0|, \qquad l_2 \equiv \operatorname*{Min}_{z\in R}|z-z_0|,$$
$$d_1 \equiv \operatorname*{Min}_{\substack{z\in R\\ z'\in C_1}}|z'-z|, \qquad d_2 \equiv \operatorname*{Min}_{\substack{z\in R\\ z'\in C_2}}|z'-z|.$$

Since R is a domain within the annulus bounded by C_1 and C_2,

$$l_1 < r_1, \qquad l_2 > r_2, \tag{6.45}$$

Thus,

$$|R_N(z)| \le \frac{M_1 r_1}{d_1}\left(\frac{l_1}{r_1}\right)^{N+1} + \frac{M_2 r_2}{d_2}\left(\frac{r_2}{l_2}\right)^{N+1}. \tag{6.46}$$

But according to Eq. (6.45), $(l_1/r_1) < 1$ and $(r_2/l_2) < 1$, independent of z, so the magnitude of $R_N(z)$ can be made as small as we please for N sufficiently large. Since the bound of Eq. (6.46) is independent of z we have completed the proof of the uniform convergence of the series in Eq. (6.40). This series is known as the *Laurent expansion* of $f(z)$ about the point z_0.

Example 1. Consider the function

$$f(z) = \frac{z^3 + 2z^2 + 4}{(z-1)^3} .$$

We shall obtain the Laurent expansion of $f(z)$ about the singular point $z = 1$. Writing $z \equiv (z-1) + 1$, we have

$$\begin{aligned} f(z) &= \frac{[(z-1)+1]^3 + 2[(z-1)+1]^2 + 4}{(z-1)^3} \\ &= 1 + \frac{5}{z-1} + \frac{7}{(z-1)^2} + \frac{7}{(z-1)^3} . \end{aligned}$$

Clearly this is the Laurent expansion of $f(z)$ about $z = 1$, and the reader is urged to verify this by evaluating the coefficients A_n of Eq. (6.41) which are non-vanishing for this particular $f(z)$.

Example 2. We now consider the Laurent series for a less trivial case, namely the function $\cosh(z + 1/z)$. The hyperbolic cosine is an entire function. Its argument, $z + 1/z$, is analytic everywhere except at the origin, and therefore $\cosh(z + 1/z)$ is analytic everywhere except at the origin. Thus we can pick C_2 to be an arbitrarily small circle about the origin and C_1 to be an arbitrarily large circle about the origin. Then

$$\cosh\left(z + \frac{1}{z}\right) = \sum_{-\infty}^{\infty} A_n z^n , \quad \text{where} \quad A_n = \frac{1}{2\pi i}\oint_C \frac{\cosh(z' + 1/z')}{(z')^{n+1}} dz' ,$$

and C is any closed contour about the origin. Let C be the unit circle. Then $z' = e^{i\theta}$ on C, so the integral becomes

$$\begin{aligned} A_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(2\cos\theta) e^{-in\theta}\, d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta . \end{aligned}$$

This integral may be evaluated by using the integral representation of the Bessel function, which will be discussed in Section 6.9. We will obtain A_n by still another method later in this section but will give the result here for the sake of completeness:

$$\begin{aligned} A_{2n} &= \sum_{m=0}^{\infty} \frac{1}{m!(m + 2|n|)!} , \\ A_{2n+1} &= 0 \quad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots \end{aligned} \tag{6.47}$$

Example 3. To illustrate another kind of problem involving Laurent expansions, consider the function $f(z) = (z^2 - 1)^{-1/2}$. According to the discussion in

Section 6.2, this function has branch points at $z = -1$ and $z = +1$. We can choose the cut to run between these two points along the real axis, defining a single-valued branch by

$$f(z) = (\rho_1\rho_2)^{-1/2} e^{-i\phi_1/2} e^{-i\phi_2/2} ,$$

where $0 \leq \phi_1 < 2\pi$, $0 \leq \phi_2 < 2\pi$, $\rho_1 = |z - 1|$, $\rho_2 = |z + 1|$ and $(\rho_1 \rho_2)^{-1/2}$ denotes the positive square root of $1/\rho_1\rho_2$ (see Fig. (6.15)). With these definitions $f(z)$ is analytic everywhere except on the real axis between $z = -1$ and $z = +1$. Thus, if we take C_2 to be a circle centered at $z = 0$ with radius infinitesimally larger than 1 and C_1 to be a circle of arbitrarily large radius, we can obtain a Laurent expansion of $f(z)$ in the annulus defined by C_1 and C_2. As usual

$$(z^2 - 1)^{-1/2} = \sum_{n=-\infty}^{\infty} A_n z^n ,$$

where

$$A_n = \frac{1}{2\pi i} \oint_C \frac{(z'^2 - 1)^{-1/2}}{(z')^{n+1}} dz' .$$

For $n \geq 0$ we choose C to be a circle of arbitrarily large radius; it is clear that for this contour the relevant integral vanishes, so $A_n = 0$ for $n \geq 0$. To deal with the case of negative n, we choose as contour any circle with radius greater than 1 (see Fig. 6.15). According to our results on path independence, this contour can be deformed into a "dogbone" contour as shown in Fig. 6.15. The

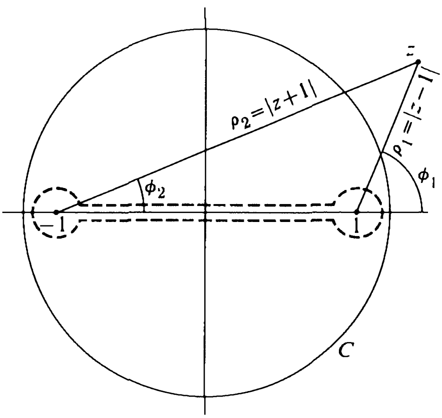

Fig. 6.15 The circular contour C is the starting point for the evaluation of the Laurent series coefficients, A_n of $(z^2 - 1)^{-1/2}$ when n is negative. ρ_1, ρ_2, ϕ_1, and ϕ_2 define the single-valued branch of this function, and the dashed "dogbone" contour is the deformation of C which enables us to evaluate the coefficients.

contribution to A_n from the infinitesimally small circles at the ends of the bone is vanishingly small, so we have simply

$$A_n = \frac{1}{2\pi i}\left[\int_1^{-1} \frac{1}{\sqrt{1-x^2}} e^{-i\pi/2} x^m \, dx + \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} e^{-i(\pi+2\pi)/2} x^m \, dx\right]$$
$$= \frac{1}{\pi}\int_{-1}^{1} \frac{x^m}{\sqrt{1-x^2}} \, dx \,,$$

where $m = -n - 1$ (note that for the range of n in question m is never negative). Clearly, for m odd (n even) A_n vanishes. For m even (n odd) the integral is elementary; we have

$$A_n = \frac{2}{\pi}\int_0^1 \frac{x^m}{\sqrt{1-x^2}} \, dx = \frac{2}{\pi}\int_0^{\pi/2} \sin^m \theta \, d\theta = \frac{m!}{2^m[(m/2)!]^2} \,.$$

Thus,

$$(z^2 - 1)^{-1/2} = \sum_{m=0}^{\infty}{}' \frac{m!}{2^m[(m/2)!]^2} \frac{1}{z^{m+1}} \,,$$

where the prime indicates that only even values of m are included in the summation. Setting $m = 2\nu$, we have finally

$$(z^2 - 1)^{-1/2} = \sum_{\nu=0}^{\infty} \frac{(2\nu)!}{4^\nu(\nu!)^2} \frac{1}{z^{2\nu+1}} = \frac{1}{z} + \frac{1}{2}\frac{1}{z^3} + \frac{3}{8}\frac{1}{z^5} + \frac{5}{16}\frac{1}{z^7} + \cdots \,.$$

This result is not surprising: it is just what we would have obtained by expanding $(z^2 - 1)^{-1/2}$ using real variable techniques. Note that in keeping with our definition of the single-valued branch of $(z^2 - 1)^{-1/2}$ the Laurent series gives a value of the square root which is positive on the positive real axis and negative on the negative real axis.

Until now we have spoken of singularities as points where the function in question is not analytic, but have made no attempt to classify singular points. There are differences, however, as we might expect if we look at the singular point $z = 0$ of the three functions $1/z$, $1/z^2$, and $\cosh(z + 1/z)$. We feel that the singularity of $1/z^2$ is a little worse than that of $1/z$ and that the singularity of $\cosh(z + 1/z)$ is terrible. A way of classifying the singularities is provided by the Laurent series of each function.

Suppose that $f(z)$ is analytic in a domain R except at $z = z_0$. Expand $f(z)$ in a Laurent series about z_0 inside R:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n} \,.$$

If $b_n = 0$ for $n = N + 1, N + 2, \cdots, \infty$, that is, if the series of negative powers of $(z - z_0)$ terminates with the Nth power, then $f(z)$ is said to have a *pole of order* N at z_0. The functions $1/z$ and $1/z^2$ have poles of order 1 and 2 respectively. But the Laurent expansion of $\cosh(z + 1/z)$ has an *infinite* number of negative powers of z; so does the Laurent expansion of $e^{1/z}$ as we shall soon see. The singularities of these functions at the origin are called *essential* singularities. They cannot be removed by multiplying the function by some finite power of z, as is the case with the poles of $1/z$ and $1/z^2$. Notice, however, that the presence of an infinite number of negative powers of z in a Laurent series does not guarantee the presence of an essential singularity. Example 3 above shows that a branch cut singularity can also give rise to an infinite number of negative powers of z in a Laurent series. Only if we know that the singularity in question is confined within an arbitrarily small region can we conclude that an infinite number of negative powers of z implies the existence of an essential singularity. A function with no essential singularities in a region—although it may have poles there—is said to be a *meromorphic* function.

A particularily important consequence of Laurent's theorem arises when there are *no singularities* contained within the inner circle C_2, that is, if $f(z)$ is analytic everywhere inside C_1. Using Laurent's theorem, we can write

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n ,$$

where

$$A_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' . \tag{6.48}$$

But if $f(z')$ is analytic inside C_1, then for $n = -1, -2, \cdots$, the integrand in Eq. (6.48) is analytic within and on C (since C lies within C_1). Therefore, according to the Cauchy-Goursat theorem, $A_n = 0$ for $n \leqslant -1$. Furthermore, for $n \geqslant 0$, we have, using Eq. (6.17),

$$\frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz' = \frac{1}{n!} f^{(n)}(z_0) ,$$

where $f^{(n)}(z_0)$ denotes the nth derivative of $f(z)$ at $z = z_0$ and $f^{(0)}(z) \equiv f(z_0)$. Thus,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z - z_0)^n . \tag{6.49}$$

This result is known as Taylor's theorem, and the series in Eq. (6.49) is called Taylor's series for $f(z)$ about z_0. This very important result may be stated formally as follows:

Taylor's Theorem. If $f(z)$ is analytic at all points interior to a circle C centered about z_0 then in any closed region contained wholly inside C

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)\,(z - z_0)^n , \tag{6.50}$$

and the series converges uniformly.

We can now express the analytic function $f(z)$ as a uniformly convergent series of analytic functions (it is a simple matter to show that *any* uniformly convergent series of analytic functions is analytic). Because of the uniformity of the convergence, it can readily be shown that the integral of $f(z)$ along any path in the region of convergence of the power series in Eq. (6.49) can be evaluated by integrating the series term-by-term (see Problem 6.28). Using Eq. (6.17), we see that this also means that the derivatives of $f(z)$ can be evaluated similarly by term-by-term differentiation. Thus Eq. (6.49) provides $f(z)$ and all its derivatives throughout the region of convergence.

Examples

1. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \text{for} \quad |z| < \infty .$

2. $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad \text{for} \quad |z| < \infty .$

3. $(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n, \qquad \text{for} \quad |z| < 1 .$

These series all reduce to familiar results for real values of z.

It is often possible to obtain the *Laurent* expansion of some function by using a related Taylor series. Thus, Example 1, above, gives us

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} \qquad (|z| > 0) .$$

This show clearly that $e^{1/z}$ has an essential singularity at $z = 0$. A slightly more substantial example is provided by the Taylor expansion of cosh z. It is a simple matter to show that

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

For $|z| < \infty$. Thus for $|z| > 0$ we have

$$\cosh (z + 1/z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (z + 1/z)^{2n} .$$

Using the binomial theorem, we find

$$\cosh (z + 1/z) = \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \frac{z^{2(n-m)}}{(2n-m)!\, m!} = \sum_{n=0}^{\infty} \sum_{\mu=-n}^{n} \frac{z^{2\mu}}{(n+\mu)!\,(n-\mu)!} .$$

If in Eq. (6.50) we interchange the order of summation and then make the change of variable $\nu = n - \mu$, we obtain

$$\cosh (z + 1/z) = \sum_{\mu=-\infty}^{\infty} z^{2\mu} \left[\sum_{\nu=0}^{\infty} \frac{1}{(\nu + 2|\mu|)!\, \nu!} \right],$$

in agreement with the result of Eqs. (6.47).

The region in which a Taylor series expansion of a function about a point z_0 is valid is limited by the presence of a singularity of $f(z)$. We have seen that if this singularity is at the point α_0, the Taylor's expansion holds within a circular region of radius $|\alpha_0 - z_0|$ about z_0. Now the Taylor expansion gives the values of the analytic function and all its derivatives at every point in the region of analyticity. In particular, these quantities are known at a point z_1 (see Fig. 6.16) near the region's border. We may now use the point z_1 as a point about which to expand the function in another Taylor series. We can do this because $f^{(n)}(z_1)$ is known for all n from the first Taylor expansion about z_0. The radius of convergence of this second series expansion about z_1 is determined by the distance from z_1 to the nearest singularity. Continuing in this way (Fig. 6.16), we can determine the function throughout the entire plane except the points at which it is singular. To get started, we need only know the values of the analytic function in some *region*, however small. This process is called *analytic continuation*. It is as though a paleontologist could reconstruct a whole dinosaur from the fossil remains of a single toenail.

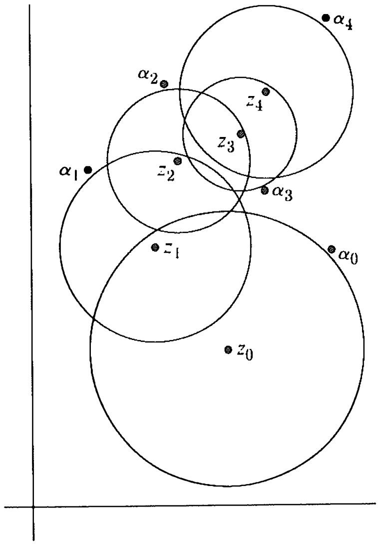

Fig. 6.16 A sequence of Taylor expansions which analytically continue a function originally known in some region around z_0. The first expansion about z_0 is limited in its radius of expansion by the singularity at α_0. The next Taylor expansion about z_1 (inside the first expansion's radius of convergence) is limited by a second singularity at α_1, and so on.

The process of analytic continuation is the best demonstration of the rigid inner structure of *analytic* functions. Again we see how interdependent are the values of an analytic function: its values in *any* region on the plane determine its values *everywhere* that it is analytic. This blueprint for the construction of all the values of an analytic function also demonstrates the necessity of the property that all derivatives of analytic functions are analytic. For if this were not the case, we could not continue a function analytically by a chain of Taylor expansions. To do this requires that we be able to approximate the function arbitrarily well, and this in turn means that all its derivatives must exist so the Taylor series can be extended to achieve arbitrary accuracy.

An immediate consequence of the fact that an analytic function may be continued is that a function which is zero along any curve in a region R, throughout which it is analytic, must be zero everywhere in R. This means that if two functions $f(z)$ and $g(z)$ are equal on any curve inside a simply-connected region R in which both are single-valued and analytic, then they are equal everywhere in R, since $f(z) - g(z)$ is zero on the curve and therefore throughout R. It will be recalled that the analytic functions e^z, $\sin z$, $\sinh z$, etc., were defined for complex arguments in a way that reduced to the usual definitions on the real axis. The principle discussed above shows that these functions could not have been defined otherwise away from the real axis and still be analytic. Furthermore, this explains why all the familiar identities satisfied by the functions for real values continue to hold throughout the complex plane.

It should also be noted that the process of analytic continuation is closely related to the problem of multivalued functions and Riemann surfaces discussed in Section 6.2. Suppose that we are given an analytic power series for a function in some region, for example,

$$w(z) = \sqrt{z} = \sqrt{1 - (1 - z)} = 1 - \tfrac{1}{2}(1 - z) - \tfrac{1}{8}(1 - z)^2 + \cdots$$

in the region consisting of the interior of a circle of unit radius about the point $z = 1$. If we try to continue such a function along certain paths in the z-plane (in the case above, any path enclosing the origin), it may happen that upon returning to the original region of definition, we do *not* return to the original values of the function. This leads in a natural way to the construction of Riemann surfaces of the type discussed in Section 6.2.

6.8 RESIDUE THEORY—EVALUATION OF REAL DEFINITE INTEGRALS AND SUMMATION OF SERIES

There is really nothing fundamentally new in this section. All the theorems have been proved; here we just apply them in certain ways to determine the values of some real definite integrals.

The Residue Theorem. The integral of $f(z)$ around a closed contour C containing a finite number n of singular points of $f(z)$ equals the sum of n integrals of $f(z)$ about n circles, each enclosing one (and only one) of the n singular points.

Proof. If we apply Cauchy's theorem to the region shown in Fig. 6.17, we obtain

$$\int_C f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz = 0.$$

Note that the contours C_j are traversed clockwise in Fig. (6.17). It follows that

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} R_j\,,$$

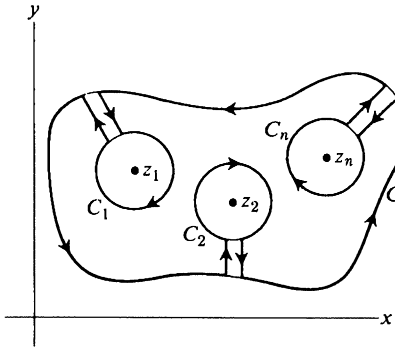

Fig. 6.17

where $R_j = (1/2\pi i) \oint_{C_j} f(z)\, dz$ is called the *residue* at the point z_j. In this equation for R_j the integral sign has its conventional meaning, that is, the contours C_j are traversed counterclockwise.

There is nothing new in this theorem except the name "residue." To compute the residues we shall often use Cauchy's integral formula or the formula for the derivative of an analytic function derived from Cauchy's formula by differentiation.

There is another way to compute residues, however, which is sometimes useful. We expand $f(z)$ in a Laurent series about the singular point z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n} .$$

Now $b_1 = (1/2\pi i) \oint_{C_0} f(z)\, dz = R_0$. It is instructive to check this by integrating both sides of the Laurent series expansion about a circle which includes z_0 but no other singularities. Interchanging summation and integration, which is permissible because these series converge uniformly, we get

$$\oint_{C_0} f(z)\, dz = \sum_{n=0}^{\infty} a_n \oint_{C_0} (z - z_0)^n\, dz + \sum_{n=1}^{\infty} b_n \oint_{C_0} \frac{dz}{(z - z_0)^n} .$$

The first integral vanishes for all n because $(z - z_0)^n$ is analytic. For $n = 1$, the second integral gives $2\pi i b_1$ by Cauchy's integral formula. For $n > 1$, we write $b_n \equiv g(z)$; then by Eq. (6.17)

$$\oint_{C_0} \frac{b_n\, dz}{(z - z_0)^n} = \oint_{C_0} \frac{g(z)\, dz}{(z - z_0)^n} = \frac{2\pi i}{(n - 1)!} g^{(n-1)}(z_0) = 0 ,$$

since $g(z) = b_n = \text{const}$. Thus $(1/2\pi i) \oint_{C_0} f(z)\, dz = b_1 =$ the residue at z_0, so the residue at a point can be found by expanding the function about the point in a Laurent series and picking out the coefficient of the term in $(z - z_0)^{-1}$. There is nothing mysterious about this; it is a simple consequence of Cauchy's integral formula.

Examples.

1. $I = \oint_C e^{1/z}\, dz$, C = unit circle about the origin.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}, \qquad \text{so } b_1 = 1 .$$

Therefore $I = 2\pi i$.

2. $I = \oint_C \frac{3z^2 + 2}{z(z+1)}\, dz$,

where C is the circle: $|z| = 3$. The quickest way to do this integral is to write it as

$$\oint_C = \oint_{C_0} + \oint_{C_1} = \oint_{C_0} \frac{(3z^2+2)/(z+1)}{z}\, dz + \oint_{C_1} \frac{(3z^2+2)/z}{(z+1)}\, dz ,$$

where C_0 is a little circle inside C enclosing $z = 0$, but not $z = -1$, and C_1 is a little circle inside C enclosing $z = -1$, but not $z = 0$. Then using Cauchy's integral formula to evaluate both integrals, we obtain $I = 2\pi i(2 - 5) = -6\pi i$.

3. $I = \oint_C \frac{3z + 2}{z(z+1)^3}\, dz$,

where C is the circle $|z| = 3$. We break up the integral as before:

$$I = \oint_{C_0} \frac{(3z+2)/(z+1)^3}{z}\, dz + \oint_{C_1} \frac{(3z+2)/z}{(z+1)^3}\, dz$$

$$= 2\pi i[2] + 2\pi i \left[\frac{f''(z)}{2!}\right]_{z=-1}$$

where $f(z) = (3z + 2)/z$. Therefore $f''(-1) = -4$, so the residue at $z = -1$ is -2, thus $I = 2\pi i(2 - 2) = 0$.

In this last example we evaluated the residue of a function of the form $f(z)/(z - z_0)^n$ at the singular point z_0 by using Cauchy's formula for the derivative of an analytic function. In general, if $f(z)$ is analytic within and on some contour C surrounding z_0, then the residue of the function $f(z)/(z - z_0)^n$ at z_0 is

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^n}\, dz = \frac{f^{n-1}(z_0)}{(n-1)!} .$$

An instructive way to view this formula is to expand $f(z)$ in a Taylor series about z_0. This is possible because $f(z)$ is analytic in some region about z_0, since the singularities are isolated. Then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots + \frac{f^{n-1}(z_0)(z - z_0)^{n-1}}{(n-1)!} + \cdots .$$

If we now form the quantity $f(z)/(z - z_0)^n$ by dividing both sides by $(z - z_0)^n$, the coefficient of the term in $1/(z - z_0)$, which is just the residue of $f(z)/(z - z_0)^n$, is $f^{(n-1)}(z_0)/(n - 1)!$, in agreement with the above.

We now use these techniques to evaluate various definite integrals.

a) $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$, where R is a rational function, that is,

$$R = \frac{a_1 \cos\theta + a_2 \sin\theta + a_3 \cos^2\theta + \cdots}{b_1 \cos\theta + b_2 \sin\theta + b_3 \cos^2\theta + b_4 \sin^2\theta + \cdots}.$$

To evaluate this integral, let $z = e^{i\theta}$, so that

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

and $d0 = -i(dz/z)$. The integral becomes

$$-i \oint_C R\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\frac{dz}{z},$$

where C is the unit circle.

Example.

$$I = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta}, \qquad a > 1.$$

$$I = -i \oint_{|z|=1} \frac{1}{\left(a + \frac{z}{2} + \frac{1}{2z}\right)} \frac{dz}{z} = -2i \oint_{|z|=1} \frac{dz}{z^2 + 2az + 1}.$$

The denominator can be factored into $(z - \alpha)\ (z - \beta)$, where

$$\alpha = -a + (a^2 - 1)^{1/2}, \qquad \beta = -a - (a^2 - 1)^{1/2}.$$

Since $a > 1$, it follows readily that $|\alpha| < 1$ and $|\beta| > 1$. Thus the integrand has one singularity, at $z = \alpha$, within the unit circle, and

$$I = -2i(2\pi i)\frac{1}{\alpha - \beta} = \frac{2\pi}{(a^2 - 1)^{1/2}}.$$

Next we consider integrals of the form:

b) $\int_{-\infty}^{\infty} R(x)\, dx$, where $R(x)$ is a rational function (i.e., a ratio of two polynomials), without poles on the real axis.

If there are no poles on the real axis, then this integral exists if the degree of the denominator of $R(x)$ is at least two units higher than the degree of the numerator. This means that $|R(z)| \to 1/|z^2|$ as $|z| \to \infty$. Now

$$\oint_C R(z)\, dz = \int_{-\rho}^{\rho} R(x)\, dx + \int_{\text{semicircle}} R(z)\, dz$$

(see Fig. 6.18). As $\rho \to \infty$, the closed contour C encloses all the singularities of $R(z)$ in the upper half-plane, so

$$\oint_C R(z)\, dz = 2\pi i \sum_{y>0} \operatorname{Res} R(z),$$

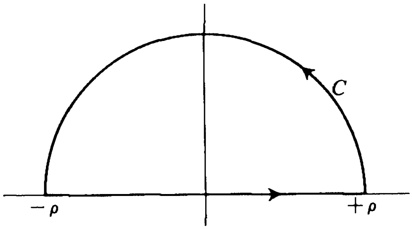

Fig. 6.18

where this notation means to form the sum of all the residues of $R(z)$ in the upper half-plane. Now in the limit of large $|z|$,

$$\left|\int_{\text{semicircle}} R(z)\, dz\right| \leq \int \frac{|\text{const}|\, \rho\, d\theta}{\rho^2} = \frac{|\text{const}|}{\rho},$$

which goes to zero as $\rho \to \infty$. Thus

$$\oint_C R(z)\, dz = \int_{-\infty}^{\infty} R(x)\, dx = 2\pi i \sum_{y>0} \text{Res}\, R(z)\,.$$

When we say that an integral over a "contour at infinity" vanishes, we really mean, of course, that given any $\epsilon > 0$ there exists a *finite* contour C such that $\left|\oint_C f(z)\, dz\right| < \epsilon$. If this is so, we say that the integral over the contour at ∞ is zero; all contours are really finite.

Example.

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}\,.$$

To find the zeros of the denominator in the upper half-plane, we must solve $z^2 = -1$. We get $z = \pm i$. Only the root $+i$ is in the upper half-plane. Therefore

$$I = \oint \frac{dz}{(z+i)^2(z-i)^2} = \frac{2\pi i}{1!} \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)\Bigg]_{z=i}$$

$$= (2\pi i)\cdot(-2)\cdot\frac{1}{(z+i)^3}\Bigg]_{z=i} = \frac{\pi}{2}\,.$$

c) Another very important class of integrals is $\int_{-\infty}^{\infty} R(x) e^{ix}\, dx$. This is the Fourier integral of the rational function $R(x)$. Its real and imaginary parts determine the integrals:

$$\int_{-\infty}^{\infty} R(x) \cos x\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} R(x) \sin x\, dx\,.$$

We retain the assumption that there are no poles on the real axis. Here, too, we consider the integral $\oint_C R(z) e^{iz}\, dz$ over a semicircle. Since $|e^{iz}| = e^{-y} \leq 1$

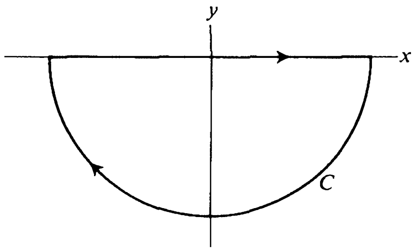

Fig. 6.19

for any point in the upper half-plane, this integral will exist if the rational function $R(z)$ has a zero of at least order two at infinity. Then, just as before, the integral over the infinite semicircle vanishes, leaving

$$\oint_C R(z)e^{iz}\,dz = \int_{-\infty}^{\infty} R(x)e^{ix}\,dx = 2\pi i \sum_{y>0} \operatorname{Res}\,[R(z)e^{iz}]\,.$$

Example.

$$I = \int_{-\infty}^{\infty} \frac{e^{ikr}}{k^2 + \mu^2}\,dk\,.$$

If $r > 0$, then letting $x = rk$, we find that the integral becomes

$$I = r\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + (\mu r)^2}\,dx = r\oint \frac{e^{iz}}{z^2 + (\mu r)^2}\,dz = 2\pi i r \sum_{y>0} \operatorname{Res}\left(\frac{e^{iz}}{z^2 + (\mu r)^2}\right)$$

$$= 2\pi i r \left.\frac{e^{iz}}{z + i\mu r}\right]_{z=i\mu r} = \pi\,\frac{e^{-\mu r}}{\mu}\,.$$

Note that

$$I = \int_{-\infty}^{\infty} \frac{\cos kr}{k^2 + \mu^2}\,dk$$

and hence the result above is actually independent of the sign of r. Therefore

$$\int_{-\infty}^{\infty} \frac{e^{ikr}}{k^2 + \mu^2}\,dk = \pi\,\frac{e^{-\mu|r|}}{\mu}\,.$$

Another way to see this is to compute the integral assuming $r < 0$. Then we must take the contour in the *lower* half-plane, since $|e^{ikr}| = e^{-(\operatorname{Im} k)r}$, which is bounded only for $\operatorname{Im}(k) < 0$, if $r < 0$. As before, for a large enough semicircle (Fig. 6.19),

$$\oint_C = \int_{-\infty}^{\infty} = -2\pi i \sum_{y<0} \operatorname{Res}\,.$$

The minus sign arises because the contour C is traversed clockwise. Thus

$$I = -2\pi i r \left.\frac{e^{iz}}{z - i\mu r}\right]_{z=-i\mu r} = \frac{\pi e^{\mu r}}{\mu}\,,$$

which for $r < 0 = \pi e^{-\mu|r|}/\mu$. Here the residue was computed at the only pole in the lower half-plane, $z = -i\mu r$.

d) There arise integrals of the form $\int_{-\infty}^{\infty} R(x)e^{i\alpha x}\,dx$, where the rational function $R(x)$ has a zero of order one at infinity. The integrals

$$\int_{-\infty}^{\infty} R(x)\cos\alpha x\,dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} R(x)\sin\alpha x\,dx$$

are the real and imaginary parts, respectively, of this integral.

From the preceding discussion, it is not clear that these integrals exist. Jordan's lemma, which we now prove, tells us that they do. Again we assume that there are no poles on the real axis.

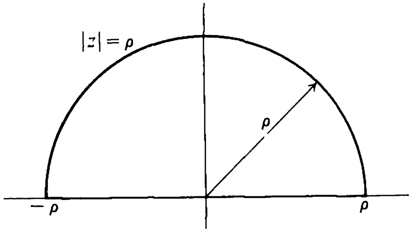

Fig. 6.20

Jordan's Lemma. If as $\rho \to \infty$ (in Fig. 6.20), $|R(z)| \to 0$ uniformly in θ for $0 < \theta < \pi$, then $\lim_{\rho\to\infty} \int_{|z|=\rho} R(z)e^{i\alpha z}\,dz = 0$, for $\alpha > 0$.

Proof. We have almost proved this result in the section on dispersion relations. There, however, the analyticity of $R(z)$ in the upper half-plane was used. Here we use the hypothesis on the uniformity of the approach of $|R(z)|$ to zero. Let $M(\rho)$ be the maximum of $|R(z)|$ on the semicircle $|z| = \rho$. Then the statement that $|R(z)| \to 0$ uniformly means that $|R(z)| \leq M(\rho)$, where $\lim_{\rho\to\infty} M(\rho) = 0$, *independent of* θ.

Let

$$I = \int_{|z|=\rho} R(z)e^{i\alpha z}\,dz\ .$$

Then

$$|I| \leq M(\rho)\int_0^{\pi} |e^{i\alpha(\rho\cos\theta + i\rho\sin\theta)}|\,|\rho i e^{i\theta}|\,d\theta$$

$$= \rho M(\rho)\int_0^{\pi} e^{-\alpha\rho\sin\theta}\,d\theta\ .$$

Treating this integral exactly as in Section 6.6, we obtain

$$|I| < \pi M(\rho)\,\frac{(1 - e^{-\alpha\rho})}{\alpha} \to 0 \qquad \text{as } \rho \to \infty\ ,$$

since $\alpha > 0$, and $M(\rho) \to 0$ as $\rho \to \infty$. Thus we arrive at the formula

$$\int_{-\infty}^{\infty} R(x)e^{i\alpha x}\,dx = 2\pi i \sum_{y>0} \text{Res}\,[R(z)e^{i\alpha z}]$$

for $\alpha > 0$. For $\alpha < 0$, the derivation works only in the lower half-plane $(y < 0)$, and changes in the above formula must be made accordingly.

Example.

$$\begin{aligned} I &= \int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+\mu^2}\,d\mathbf{k} \\ &= \int_0^{2\pi} k\,d\phi \int_0^{\pi} k\sin\theta\,d\theta \int_0^{\infty} dk\,\frac{e^{ik|r|\cos\theta}}{k^2+\mu^2} \\ &= 2\pi\int_{-1}^{1} dt \int_0^{\infty} dk\,\frac{k^2 e^{ik|r|t}}{k^2+\mu^2} \\ &= \frac{2\pi}{i|r|}\int_0^{\infty} dk\,\frac{k(e^{ik|r|}-e^{-ik|r|})}{k^2+\mu^2} \\ &= \frac{2\pi}{i|r|}\int_{-\infty}^{\infty} dk\,\frac{ke^{ik|r|}}{k^2+\mu^2}. \end{aligned}$$

Jordan's lemma tells us that this integral for complex k vanishes over the infinite semicircle. Therefore

$$I = \frac{2\pi}{i|r|}\,2\pi i \sum_{y>0} \operatorname{Res}\frac{ke^{ik|r|}}{k^2+\mu^2} = \frac{4\pi^2}{|r|}\,\frac{i\mu e^{i\cdot i\mu|r|}}{2i\mu} = \frac{2\pi^2}{|r|}\,e^{-\mu|r|}, \qquad (6.51)$$

since there is only one pole in the upper half-plane $(k = +i\mu)$.

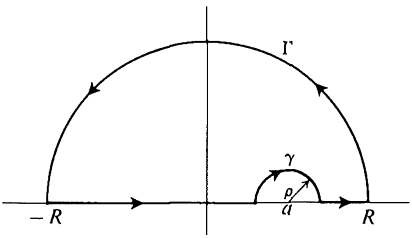

Fig. 6.21

e) We have not as yet allowed the integrand to have poles on the real axis. If $Q(z)$ is a meromorphic function in the upper half-plane, if it only has poles of order one (i.e., simple poles) on the real axis, and if $Q(z)$ behaves at infinity like any of the integrands of class b, c, or d, then the techniques of these cases can be extended to evaluate integrals of the form

$$P\int_{-\infty}^{\infty} Q(x)\,dx,$$

where the P stands for Cauchy's principal value, which arises because of the presence of the poles on the real axis.

Let $Q(z)$ have a single simple pole on the real axis at $z = a$. Consider the indented contour C (Fig. 6.21) consisting of the small semicircle γ about point a, the large semicircle Γ about the origin, and the two straight line segments on the real axis from $-R$ to $a - \rho$ and from $a + \rho$ to R. We take γ small enough so it encloses only the pole at a; Γ is taken large enough to enclose all the poles in the upper half-plane and large enough so the integral over Γ approaches 0 as $R \to \infty$. We have

$$\left(\int_\Gamma + \int_{-R}^{a-\rho} + \int_\gamma + \int_{a+\rho}^{R}\right) Q(z)\,dz = 2\pi i \sum_{y>0} \operatorname{Res} Q(z)\,.$$

Taking the limit as $R \to \infty$, we get

$$\left[\lim_{R\to\infty}\left(\int_{-R}^{a-\rho} + \int_{a+\rho}^{R}\right) + \int_\gamma\right] Q(z)\,dz = P\int_{-\infty}^{\infty} Q(x)\,dx + \int_\gamma Q(z)\,dz$$
$$= 2\pi i \sum_{y>0} \operatorname{Res} Q(z)\,.$$

We now consider $\int_\gamma Q(z)\,dz$. On γ, $z = a + \rho e^{i\theta}$, so that

$$\int_\gamma Q(z)\,dz = \int_\pi^0 Q(a + \rho e^{i\theta})\rho e^{i\theta} i\,d\theta\,.$$

Since $Q(z)$ has a simple pole at $z = a$, it contains the factor $(z-a)^{-1}$. We may therefore write $Q(z) = \phi(z)/(z-a) + \psi(z)$, where $\phi(z)$ and $\psi(z)$ are analytic at and near $z = a$. Clearly $\psi(z)$ does not contribute to the integral over γ as $\rho \to 0$, so since $z - a = \rho e^{i\theta}$, we have

$$\int_\gamma Q(z)\,dz = \int_\pi^0 \phi(a + \rho e^{i\theta}) i\,d\theta\,.$$

We now expand ϕ in a Taylor series about a (it is analytic there):

$$\phi(a + \rho e^{i\theta}) = \phi(a) + \text{terms in } \rho\,.$$

Therefore, letting $\rho \to 0$, we get

$$\int_\gamma Q(z)\,dz = \int_\pi^0 \phi(a) i\,d\theta = -i\pi\phi(a)\,.$$

Now $\phi(a)$ is the residue of $Q(z) = \phi(z)/(z-a)$ at $z = a$, so the final answer may be written as

$$P\int_{-\infty}^{\infty} Q(x)\,dx = 2\pi i \sum_{y>0} \operatorname{Res} Q(z) + \pi i \sum_{y=0} \operatorname{Res} Q(z)\,,$$

where $\sum_{y=0} \operatorname{Res} Q(z)$ denotes the sum of the residues of $Q(z)$ at each of its simple poles on the real axis (generalizing from one to several simple poles).

Example.

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx\,.$$

The value of this integral may be determined from the following simple result:

$$P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i \sum_{y=0} \operatorname{Res}\left(\frac{e^{iz}}{z}\right) = \pi i\,.$$

Here the only pole is on the real axis at $z = x = 0$.

Equating real and imaginary parts, we have

$$P\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx = 0\,,$$

which is trivial because the integrand is an odd function, and

$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi\,,$$

a result derived previously from Hilbert transform theory. In this formula the P is superfluous because the integrand has no pole. Many integrals involving $\sin x$ or $\cos x$ may be evaluated, as we have done here, by replacing the trigonometric function with e^{iz}, and later taking real and imaginary parts.

f) Next we consider integrals of the form $\int_0^\infty x^{\lambda-1} R(x)\,dx$, where $R(z)$ is rational, analytic at $z = 0$, and has no poles on the positive real axis, and where $|z^\lambda R(z)| \to 0$ uniformly as $|z| \to 0$ and as $|z| \to \infty$. Since the case of integral λ can be handled by the methods described earlier, we will assume that λ is not equal to an integer.

This problem involves branch points and branch cuts because $z^{\lambda-1}$ is not in general, a single-valued function. The power function $z^{\lambda-1}$ has a branch point at the origin. Let $z^{\lambda-1}$ denote the following branch of the power function:

$$z^{\lambda-1} = \exp[(\lambda-1)\log z] = \exp[(\lambda-1)(\log r + i\theta)]\,,$$

where $0 < \theta < 2\pi$, $r > 0$. Then

$$z^{\lambda-1} = r^{\lambda-1}e^{i(\lambda-1)\theta}\,, \qquad 0 < \theta < 2\pi, r > 0\,.$$

The branch cut has been chosen to be the positive real axis. For $\theta = 0$, the power function has the value

$$z^{\lambda-1} = r^{\lambda-1} = x^{\lambda-1}\,.$$

Now consider the contour integral $\oint_{C'} z^{\lambda-1}R(z)\,dz$, where the closed contour C' is shown in Fig. 6.22. Here C' consists of a small circle about $z = 0$, whose radius will later be shrunk to zero, a large circle whose radius will later be expanded to ∞, and two integrals along the positive real axis in opposite directions and on opposite sides of the branch cut. Since the integrand is discontinuous along the branch cut, these two integrals will not cancel. To do the integral, the phase of $z^{\lambda-1}$ must be prescribed everywhere. We have done this above by defining it to be 0 on the positive real axis so that $z^{\lambda-1} = x^{\lambda-1}$ there, which is the usual convention. Therefore, on the line just below the real axis, $z^{\lambda-1} = x^{\lambda-1}e^{i2\pi(\lambda-1)}$. The integrals over the little and big circles vanish, because $|z^\lambda R(z)| \to 0$ as $|z| \to 0$, and as $|z| \to \infty$. Therefore

$$\oint_{C'} z^{\lambda-1}R(z)\,dz = -\int_0^\infty e^{2\pi i(\lambda-1)}x^{\lambda-1}R(x)\,dx + \int_0^\infty x^{\lambda-1}R(x)\,dx\,,$$

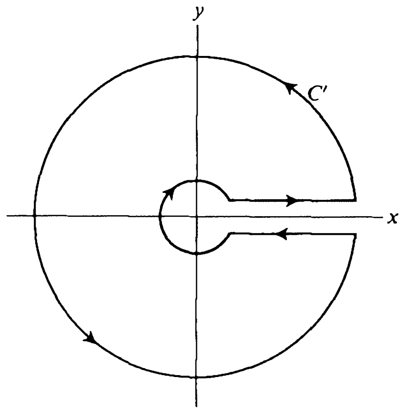

Fig. 6.22

where the first integral on the right is the contribution from the line below the positive x-axis and the second is the contribution from the line above the x-axis. Together these integrals give

$$[1 - e^{2\pi i(\lambda-1)}]\int_0^\infty x^{\lambda-1}R(x)\,dx = \left(\frac{-2i\sin\pi\lambda}{e^{-\pi i\lambda}}\right)\int_0^\infty x^{\lambda-1}R(x)\,dx\,.$$

Since

$$\oint_{C'} z^{\lambda-1}R(z)\,dz = 2\pi i \sum_{\text{inside } C'} \text{Res}\,[z^{\lambda-1}R(z)\,]\,,$$

we obtain

$$\int_0^\infty x^{\lambda-1}R(x)\,dx = \frac{-\pi e^{-\pi i\lambda}}{\sin\pi\lambda}\sum_{\text{inside } C'} \text{Res}\,[z^{\lambda-1}R(z)\,] \qquad (6.52)$$

$$= \frac{\pi(-1)^{\lambda-1}}{\sin\pi\lambda}\sum_{\text{inside } C'} \text{Res}\,[z^{\lambda-1}R(z)\,]\,,$$

since $e^{-i\pi\lambda} = (-1)^\lambda$. This is as far as we can go without choosing a specific function $R(z)$. A very simple example of this type is the following integral:

$$I = \int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx\,, \qquad 0 < \lambda < 1\,.$$

After evaluating it, we shall use it to prove some results involving the beta and gamma functions.

First since $0 < \lambda < 1$, $|z^\lambda/(1+z)| \to 0$ as $|z| \to \infty$ and as $z \to 0$. Therefore

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx = \frac{\pi(-1)^{\lambda-1}}{\sin\pi\lambda}\,\text{Res}\left(\frac{z^{\lambda-1}}{1+z}\right)_{z=-1} = \frac{\pi}{\sin\pi\lambda}\,.$$

The beta function $\beta(x, y)$ is defined as

$$\beta(x, y) \equiv \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,, \qquad x, y > 0\,, \qquad (6.53)$$

where $\Gamma(x)$ is the gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt\,. \tag{6.54}$$

The proof of the identity in Eq. (6.53) may be found in many books on analysis. We note that

$$\beta(x, 1-x) = \Gamma(x)\Gamma(1-x) = \int_0^1 t^{x-1}(1-t)^{-x}\,dt\,, \qquad \text{if } 0 < x < 1.$$

If in the last integral we make the change of variable

$$t = \frac{u}{1+u} = 1 - \frac{1}{1+u}\,,$$

it becomes

$$\int_0^\infty \frac{u^{x-1}}{1+u}\,du\,,$$

which is precisely the integral we have just evaluated. Thus

$$\beta(x, 1-x) = \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}\,, \qquad \text{for } 0 < x < 1\,. \tag{6.55}$$

This formula may be extended by analytic continuation to all z in the complex plane. In particular, for $x = \frac{1}{2}$, we have $[\Gamma(\frac{1}{2})]^2 = \pi$. Therefore

$$\pi^{1/2} = \Gamma(\tfrac{1}{2}) \equiv \int_0^\infty t^{-1/2}e^{-t}\,dt = 2\int_0^\infty e^{-u^2}\,du = \int_{-\infty}^\infty e^{-u^2}\,du\,, \tag{6.56}$$

where we have made the substitution $t = u^2$.

We close this section on residue theory with an illustration of how series can be summed by contour integration. The result we shall obtain depends on the fact that $\pi \cot \pi z$ has poles of order one at the zeros of $\sin \pi z$, (that is, at $z = n, n = 0, \pm 1, \pm 2, \cdots$), and the fact that the residue at each of these poles is 1.

Theorem. Let $f(z)$ be a meromorphic function and let C be a contour which encloses the zeros of $\sin \pi z$, located at $z = \rho, \rho + 1, \cdots, n$. If we assume that the poles of $f(z)$ and $\sin \pi z$ are distinct, then

$$\sum_{m=\rho}^{n} f(m) = \frac{1}{2\pi i}\oint_C \pi \cot \pi z\, f(z)\,dz - \sum_{\substack{\text{poles of } f(z)\\ \text{inside } C}} \operatorname{Res}\,[\pi \cot(\pi z)\, f(z)\,]\,. \tag{6.57}$$

Proof.

$$\oint_C \pi \cot \pi z f(z)\,dz = 2\pi i \sum (\text{Residues at all the poles of the integrand})$$

$$= 2\pi i\left[\sum_{m=\rho}^{n} f(m) + \sum_{\substack{\text{poles of } f(z)\\ \text{inside } C}} \operatorname{Res}\,[\pi \cot \pi z\, f(z)\,]\right]. \quad \text{QED}$$

Example. We shall use this theorem to establish the equivalence of Langevin's function $(\coth x - 1/x)$ and the sum

$$\sum_{n=1}^{\infty} \frac{2x}{x^2 + n^2\pi^2} \,.$$

We shall then use this result to derive an infinite *product* expression for $\sin\theta$, which is of considerable interest in its own right.

Letting $f(z) = 2x/(x^2 + z^2\pi^2)$, and using the formula above, we obtain

$$\sum_{m=-N}^{N} \frac{2x}{x^2 + m^2\pi^2} = \frac{1}{2\pi i}\oint_C \pi \cot \pi z\, f(z)\, dz - \sum_{\substack{\text{poles of } f(z)\\ \text{inside } C}} \text{Res}\,[\pi \cot \pi z\, f(z)] \,,$$

where C is a closed contour, say, a rectangle, enclosing the points $z = -N, -N+1, \cdots, 0, 1, \cdots, N-1, N$. Now let the length and width of the rectangle C approach ∞. As this happens,

$$\left|\frac{1}{2\pi i}\oint_C \pi \cot \pi z\, f(z)\, dz\right| \leq \tfrac{1}{2}\oint_C |\cot \pi z| \left|\frac{2x}{x^2 + z^2\pi^2}\right| |dz| \to 0 \,.$$

To see that this integral vanishes as $z \to \infty$, we observe that

$$|\cot \pi z| = \frac{|\cos \pi z|}{|\sin \pi z|} = \left(\frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y}\right)^{1/2} \,.$$

Now, arbitrarily high accuracy can be achieved in summing the series by choosing the rectangle so that its vertical sides cross the x-axis at a large enough half-integer, for example $(10^{97} + \frac{1}{2})$, where $\cos \pi(10^{97} + \frac{1}{2}) = 0$ and $\sin \pi(10^{97} + \frac{1}{2}) = 1$. Then, over these sides of the rectangle

$$|\cot \pi z| = \left|\left(\frac{\sinh^2 \pi y}{1 + \sinh^2 \pi y}\right)^{1/2}\right| = |\tanh \pi y| \leq 1 \,.$$

Over the horizontal sides of the rectangle, $\lim_{z\to\infty} |\cot \pi z| = 1$. Thus the integrand goes as $|1/z^2|$ as $|z| \to \infty$, and the integral vanishes.

If we take an infinite rectangular contour, then

$$\sum_{m=-\infty}^{\infty} \frac{2x}{x^2 + m^2\pi^2} = -\,\text{Res}\left[\frac{(\pi \cot \pi z)\, 2x}{x^2 + z^2\pi^2}\right]_{z=\pm ix/\pi}$$

$$= -\frac{2x}{\pi}\left[\frac{\cot \pi(ix/\pi)}{2ix/\pi} + \frac{\cot \pi(-ix/\pi)}{-2ix/\pi}\right] = 2i \cot ix = 2 \coth x \,.$$

Therefore

$$2\sum_{m=1}^{\infty} \frac{2x}{x^2 + m^2\pi^2} + \frac{2}{x} = 2 \coth x$$

or

$$\coth x - \frac{1}{x} = \sum_{m=1}^{\infty} \frac{2x}{x^2 + m^2\pi^2} \,, \tag{6.58}$$

which establishes the result stated at the outset.

Aside from illustrating this summation technique, this particular result is not of much interest. However, if we integrate both sides from 0 to x, we get

$$\sum_{m=1}^{\infty} \ln\left(1 + \frac{x^2}{m^2\pi^2}\right) = \ln\left(\frac{\sinh x}{x}\right).$$

Hence

$$\ln\left(\frac{\sinh x}{x}\right) = \sum_{m=1}^{\infty} \ln\left(1 + \frac{x^2}{m^2\pi^2}\right) = \ln \prod_{m=1}^{\infty}\left(1 + \frac{x^2}{m^2\pi^2}\right),$$

so

$$\frac{\sinh x}{x} = \prod_{m=1}^{\infty}\left(1 + \frac{x^2}{m^2\pi^2}\right). \tag{6.59}$$

We may extend this result to all z in the complex plane by analytic continuation. Then setting $x = i\theta$ (θ real), we obtain

$$\sin\theta = \theta \prod_{n=1}^{\infty}\left(1 - \frac{\theta^2}{n^2\pi^2}\right). \tag{6.60}$$

This infinite product formula displays *explicitly* all the zeros of $\sin\theta$. It represents the complete factorization of the Taylor series. It can, in fact, be taken as the definition of the sine function.

By equating coefficients of the θ^3 term of both sides of the above equation, we obtain a useful sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{6.61}$$

This is a special value of the Riemann zeta function,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \tag{6.62}$$

There are many other special tricks for evaluating integrals and summing series. We have surveyed only the principal techniques, but the reader will have no trouble understanding any particular evaluation procedure he may encounter if he understands this section. Problem 31 provides applications of all the basic techniques of residue calculus developed above.

6.9 APPLICATIONS TO SPECIAL FUNCTIONS AND INTEGRAL REPRESENTATIONS

In this section we use the formula for the derivative of an analytic function to find generating functions for certain special functions from their Rodrigues formulas. Also, we shall derive integral representations for Bessel functions and Legendre polynomials.

Bessel Functions

The function $e^{(1/2)z(w-1/w)}$ is analytic everywhere in the w-plane except at $w = 0$, so it can be expanded in a Laurent series in any annulus $R_2 < |w| < R_1$, no matter how small $R_2(>0)$ or how large R_1. Denoting the expansion coefficients by $J_n(z)$, we have

$$e^{(1/2)z(w-1/w)} = \sum_{n=-\infty}^{\infty} J_n(z)w^n . \tag{6.63}$$

The expansion coefficients are functions of the complex variable z. The notation chosen for them anticipates the fact that they will turn out to be Bessel functions of integral order. In other words, $e^{(1/2)z(w-1/w)}$ is a generating function for Bessel functions.

We now prove that the functions $J_n(z)$ do satisfy Bessel's differential equation:

$$z^2 \frac{d^2J_n}{dz^2} + z\frac{dJ_n}{dz} + (z^2 - n^2)J_n = 0 . \tag{6.64}$$

Along the way to this result we shall find an integral representation for Bessel functions and also an explicit formula for them.

By Laurent's theorem,

$$J_n(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-w_0)^{n+1}}\,dw , \tag{6.65}$$

where C is any closed contour in the annulus. We take C to be the unit circle: $w = e^{i\theta}$. Then in Eq. (6.65), $w_0 = 0$ and $f(w) = e^{(1/2)z(w-1/w)}$. On the unit circle $w - 1/w = 2i\sin\theta$. Therefore

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin\theta}e^{-in\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - z\sin\theta)}\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(n\theta - z\sin\theta)\,d\theta - \frac{i}{2\pi}\int_{-\pi}^{\pi}\sin(n\theta - z\sin\theta)\,d\theta . \end{aligned}$$

But the second integral is zero because the integrand is odd in θ. Since the first integrand is even in θ,

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)\,d\theta . \tag{6.66}$$

This is an integral representation for the function $J_n(z)$, which we now prove is a solution of Bessel's equation by substituting it into Eq. (6.64). The verification is trickier than one might expect. We therefore resist the temptation to "leave it to the reader." He is, of course, welcome to have a try at it before reading further. We have

$$J_n'(z) = \frac{1}{\pi}\int_0^{\pi}\sin\theta\sin(n\theta - z\sin\theta)\,d\theta \tag{6.67a}$$

$$\begin{aligned} &= -\frac{1}{\pi}\cos\theta\sin(n\theta - z\sin\theta)\Big]_0^{\pi} \\ &\quad + \frac{1}{\pi}\int_0^{\pi}\cos\theta\cos(n\theta - z\sin\theta)(n - z\cos\theta)\,d\theta , \end{aligned} \tag{6.67b}$$

where we have integrated by parts. The first term is zero. Differentiating Eq. (6.67a) again, we have

$$J_n''(z) = -\frac{1}{\pi}\int_0^\pi \sin^2\theta \cos(n\theta - z\sin\theta)d\theta .$$

We now form the quantity $z^2(J_n'' + J_n) + zJ_n' - n^2J_n$, using Eq. (6.67b) for $J_n'(x)$. It must be identically zero if the $J_n(z)$ are Bessel functions. We get

$$\frac{1}{\pi}\int_0^\pi \cos(n\theta - z\sin\theta)[z^2 - z^2\sin^2\theta + z\cos\theta(n - z\cos\theta) - n^2]\,d\theta$$

$$= \frac{-n}{\pi}\int_0^\pi [\cos(n\theta - z\sin\theta)(n - z\cos\theta)\,]\,d\theta .$$

The function $\sin(n\theta - z\sin\theta)$ is an antiderivative of the integrand. It vanishes at both 0 and π, so the proof is complete: the $J_n(z)$ are solutions of Bessel's equation. It is easy to show that

$$\begin{aligned} J_0(0) &= 1 , & J_n(0) &= 0 \quad \text{for } n \neq 0 , \\ J_1'(0) &= \tfrac{1}{2} , & J_n'(0) &= 0 \quad \text{for } n \neq 1 , \end{aligned} \tag{6.68}$$

so these functions are indeed that solution of Bessel's equation, known as Bessel functions of nth (integral) order of the *first* kind (which are analytic at the origin).

Usually, the solution of Bessel's equation is expressed as an infinite series, not as an integral. We now derive the infinite series solution from the generating function:

$$e^{(1/2)z(w-1/w)} = e^{(1/2)zw}e^{-z/2w}$$

$$= \sum_{r=0}^\infty \frac{(z/2)^r w^r}{r!} \sum_{m=0}^\infty \frac{(-z/2)^m w^{-m}}{m!} = \sum_{n=-\infty}^\infty w^n J_n(z) .$$

To obtain $J_n(z)$, the coefficient of the term in $w^n (n \geq 0)$, we multiply each term in w^{-m} in the second series by the term w^{n+m} in the first series, and then sum over all m:

$$J_n(z)w^n = \sum_{m=0}^\infty \frac{(z/2)^{n+m}w^{n+m}(-z/2)^m w^{-m}}{(n+m)!\,m!}$$

$$= \sum_{m=0}^\infty \left[\frac{(-1)^m (z/2)^{2m+n}}{(n+m)!\,m!}\right] w^n .$$

Thus

$$J_n(z) = \sum_{m=0}^\infty \frac{(-1)^m (z/2)^{2m+n}}{(n+m)!\,m!} . \tag{6.69}$$

This is the infinite series solution for integral $n \geq 0$. The reader may show that, for integral n,

$$J_{-n}(z) = (-1)^n J_n(z) . \tag{6.70}$$

Thus these solutions are linearly dependent, and there must exist another set of linearly independent solutions. They are called the Neumann functions, and are denoted by $N_n(z)$. A whole family of equations and solution functions may be constructed from Bessel's equations and Bessel functions by permitting the index n to be nonintegral and half-integral. Also, various linear combinations of Bessel and Neumann functions of real and imaginary argument give rise to the modified Bessel functions and the Hankel functions. The spherical Bessel functions (which we discuss in Chapter 7) are still another set of functions that are defined in terms of the Bessel functions. Each of these sets of functions is a Sturm-Liouville system. It is not our purpose to provide an exhaustive (and exhausting) account of them, but merely to derive and discuss the fundamental results so the reader can dig the particular facts he needs for his work out of a treatise on the subject.

Legendre polynomials

Previously we have shown that the Legendre polynomials, defined by the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\,,$$

are an orthogonal set on the interval $[-1, 1]$ and satisfy Legendre's equation:

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0\,.$$

It was mentioned in Section 1.8, but not proved, (except for P_0, P_1, and P_2), that a generating function for the Legendre polynomials is

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)t^n\,, \qquad 0 < t < 1\,, \qquad |x| \le 1\,. \tag{6.71}$$

In the comparison of this equation with Eq. (1.101), the reader will note that we have set $t \equiv r'/r$ and $x \equiv \cos\theta$. We now prove this relation for all n. We begin by expressing the Rodrigues formula as a contour integral by making use of the formula for the nth derivative of an analytic function:

$$P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x^2)^n = \frac{(-1)^n}{2^n}\frac{1}{2\pi i}\oint_C \frac{(1-z^2)^n}{(z-x)^{n+1}}\,dz\,. \tag{6.72}$$

Here we have used the fact that the function $(1-z^2)^n$ is entire (C is a closed contour that encloses the point x). This integral representation of the Legendre polynomials is known as Schläfli's integral formula. If we now form the series $S = \sum_{n=0}^{\infty} P_n(x)t^n$, using this integral representation for $P_n(x)$, and interchange the order of summation and integration, we get

$$S = \frac{1}{2\pi i}\oint_C \frac{dz}{z-x}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}t^n\left(\frac{1-z^2}{z-x}\right)^n.$$

This is a simple geometric series, which converges and is easily summed if

$$\left|\frac{t}{2}\frac{1-z^2}{z-x}\right| < 1\,.$$

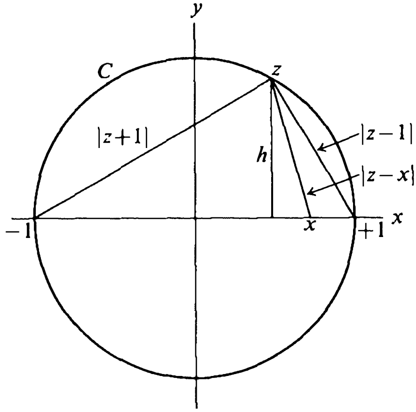

Fig. 6.23

We must show, however, that a contour C, enclosing the point x, can always be chosen so that this is true for all x and t in their prescribed ranges. Let C be the unit circle; z is a point anywhere on C, and x is on the real axis inside or on C as shown in Fig. 6.23. The area A of the right triangle with vertices at -1, z, and $+1$ is given by

$$A = \tfrac{1}{2}\,|z+1|\,|z-1| = \tfrac{1}{2}\,|1-z^2|\,.$$

It is also given by $\frac{1}{2}(2h) = h$. But $|z-x| \geq h$ for all z and x, so

$$|z-x| \geq h = A = \tfrac{1}{2}\,|1-z^2| \Longrightarrow \left|\tfrac{1}{2}\cdot\frac{1-z^2}{z-x}\right| \leq 1\,.$$

Consequently,

$$\left|\frac{t}{2}\,\frac{1-z^2}{z-x}\right| < 1$$

since $t < 1$. A purely algebraic proof is also possible, but this geometric one is simpler. We now sum the geometric series to obtain

$$S = \frac{1}{2\pi i}\oint_C \frac{dz}{z-x}\,\frac{1}{1-\left[\dfrac{-t}{2}\,\dfrac{(1-z^2)}{(z-x)}\right]} = -\frac{1}{2\pi i}\oint_C \frac{2/t\,dz}{z^2-\dfrac{2}{t}z-\left(1-\dfrac{2}{t}x\right)}\,.$$

The denominator has two roots;

$$z_\pm = \frac{1}{t} \pm \left(\frac{1}{t^2}-\frac{2x}{t}+1\right)^{1/2} = \frac{1}{t}\,[1 \pm (1-2xt+t^2)^{1/2}]\,.$$

Remembering that $0 < t < 1$ and $-1 \leq x \leq 1$, it is easy for us to show that $z_+ > 1$ and $-1 < z_- < 1$, so z_- is the root enclosed by the contour C. Evaluating the residue at z_-, we have

$$S = \sum_{n=0}^{\infty} P_n(x)t^n = \frac{-1}{2\pi i}\,\frac{2}{t}\,2\pi i\,\frac{1}{z_- - z_+} = \frac{1}{(1-2xt+t^2)^{1/2}}\,. \qquad \text{QED}$$

Table 6.1

PROPERTIES OF ORTHOGONAL FUNCTIONS ARISING FROM STURM-LIOUVILLE SYSTEMS

Name and physical application	Rodrigues formula	Generating function
Legendre polynomials: 1) Multipole expansion 2) ∇^2 in spherical coordinates	$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$	$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}$ $(0 < t < 1)$
Hermite polynomials: Quantum oscillator	$H_n(x) = (-1)^n e^{x^2}\frac{d^n e^{-x^2}}{dx^n}$	$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n = e^{-t^2+2tx}$ $(t > 0)$
Laguerre polynomials: Hydrogen atom (radial equation)	$L_n(x) = e^x\frac{d^n}{dx^n}x^n e^{-x}$	$\sum_{n=0}^{\infty}\frac{L_n(x)}{n!}t^n = \frac{e^{-xt/(1-t)}}{1-t}$ $(0 < t < 1)$
	Series representation	
Bessel's function (of integral order) ∇^2 in cylindrical coordinates	$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m(x/2)^{2m+n}}{(n+m)!\,m!}$	$\sum_{n=-\infty}^{\infty} J_n(x)t^n = e^{(1/2)x(t-1/t)}$ $(t > 0)$
Trigonometric functions: Classical oscillator	$f_n = \sin nx = \sum_{m=0}^{\infty}(-1)^m\frac{(nx)^{2m+1}}{(2m+1)!}$ $g_n = \cos nx = \sum_{m=0}^{\infty}(-1^m)\frac{(nx)^{2m}}{(2m)!}$	

Differential equation	Differential equation in Sturm-Liouville form	Orthonormality
$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$	$\frac{d}{dx}((1 - x^2)P_n') + n(n+1)P_n = 0$	$\int_{-1}^{1} P_n P_m \, dx = \delta_{nm} \frac{2}{2n+1}$
$H_n'' - 2xH' + 2nH_n = 0$	$\frac{d}{dx}(e^{-x^2}H_n') + 2ne^{-x^2}H_n = 0$	$\int_{-\infty}^{\infty} H_n H_m e^{-x^2} \, dx = \delta_{nm}\sqrt{\pi}2^n n!$
$xL_n'' + (1 - x)L_n' + nL_n = 0$	$\frac{d}{dx}(xe^{-x}L_n') + ne^{-x}L_n = 0$	$\int_{0}^{\infty} L_n L_m e^{-x} \, dx = \delta_{nm}(n!)^2$
$x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n = 0$	$\frac{d}{dx}(xJ_n') + \left(x - \frac{n^2}{x}\right)J_n = 0$	See Problem 5.23
$f_n'' + n^2 f_n = 0$ $g_n'' + n^2 g_n = 0$	$\frac{d}{dx}(f_n') + n^2 f_n = 0$ $\frac{d}{dx}(g_n') + n^2 g_n = 0$	$\int_{-\pi}^{\pi} f_n f_m \, dx = \delta_{nm}\pi$ $\int_{-\pi}^{\pi} g_n g_m \, dx = \delta_{nm}\pi$ $\int_{-\pi}^{\pi} f_n g_m \, dx = 0$

In exactly the same way, the generating functions for the Hermite polynomials and the Laguerre polynomials can be derived from their Rodrigues formulas (see Problems 36 and 37).

Let us return to Schläfli's integral representation of the Legendre polynomials:

$$P_n(x) = \frac{1}{2\pi i}\oint_C \frac{(z^2-1)^n}{2^n(z-x)^{n+1}}\,dz\,. \tag{6.73}$$

It can be proved directly from this integral representation that the $P_n(x)$ satisfy Legendre's differential equation. But we have already proved that the $P_n(x)$ as given by Rodrigues' formula satisfy Legendre's equation, so we omit this proof. Here we derive another integral representation: a real integral.

For the contour C, we take a circle about x of radius $|(x^2-1)^{1/2}|$. Then for any point z on C, we may put $z = x + (x^2-1)^{1/2}e^{i\theta}$, where θ increases from $-\pi$ to π. Making this substitution in Schläfli's formula, we obtain

$$P_n(x) = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\left[\frac{(x-1+(x^2-1)^{1/2}e^{i\theta})(x+1+(x^2-1)^{1/2}e^{i\theta})}{2(x^2-1)^{1/2}e^{i\theta}}\right]^n i\,d\theta\,,$$

where we have written $(z^2-1)^n$ in the form $[(z-1)(z+1)]^n$ before substituting $z = x + (x^2-1)^{1/2}e^{i\theta}$. Now everything in square brackets above simplifies after a little algebra to $x + (x^2-1)^{1/2}\cos\theta$, an even function of θ. Thus we have a real integral representation of the Legendre polynomials due to Laplace:

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}[x + (x^2-1)^{1/2}\cos\theta]^n\,d\theta\,, \qquad \text{for } |x| \le 1\,. \tag{6.74}$$

Note how obvious it is from this representation that $P_n(1) = 1$, and $P_n(-1) = (-1)^n$. It is instructive to compute P_0, P_1, and P_2 from this formula. Although $(x^2-1)^{1/2}$ is pure imaginary for $|x| < 1$, all the terms in $(x^2-1)^{1/2}$ vanish when the integral is performed because $\int_0^\pi \cos^m\theta\,d\theta = 0$ for odd m.

We conclude this section with Table 6.1. It summarizes some of the more useful information concerning the special functions of mathematical physics.

PROBLEMS

1. Let

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^2} \qquad \text{for } z \ne 0\,, \quad f(0) = 0\,.$$

Determine where, if anywhere, this function is a) differentiable, b) analytic.

2. Show that the complex numbers z_1, z_2, z_3 lie on a straight line if and only if $(z_1-z_3)/(z_2-z_3)$ is a real number.

[*Hint*: This problem can be done very simply if one thinks of the complex numbers geometrically (as vectors). It can be done algebraically also, but the geometric proof is certainly easier.]

3. Prove that $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is a harmonic function and find a conjugate harmonic function v. Find a complex function of the complex variable z such that $f(z) = u + iv$.

4. a) Determine all the values of the constants a, b, c, d for which the polynomial $u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic, i.e., satisfies the two-dimensional Laplace equation in the entire plane.
   b) Find a harmonic conjugate, $v(x, y)$, of $u(x, y)$.
   c) Find an analytic function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.

5. A complex function $E^*(z)$ is defined by $E^*(z) = E_x(x, y) - iE_y(x, y)$, where E_x and E_y are the components of the electric field in two dimensions. Show that the static Maxwell equations in free space imply that $E^*(z)$ is an analytic function of z.

6. Show that if we write $z = re^{i\theta}$, the Cauchy-Riemann equations become, in terms of r and θ,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

7. Analyze the function $z^{1/3}$ in terms of the Riemann sheet concepts discussed in Section 6.2.

8. Using the results of Problem 6.6, show that $\log z$, $z^{1/2}$, and $z^{1/3}$ are analytic everywhere on their appropriate Riemann sheets except at $z = 0$.

9. Show that the two-sheeted Riemann surface for $w(z) = \sqrt{(z - a)(z - b)}$ can be cut and joined along the line segment $[a, b]$ of the real axis (assume for convenience that a and b are real). Make an appropriate definition of $w(z)$ in terms of the polar representation of complex numbers, including a definition of the range of the phases.

10. Find the Riemann surface on which $\sqrt{z - a} + \sqrt{z - b}$ (a, b real and positive) is a single-valued function, analytic except at $z = a$ and $z = b$.

11. Find the Riemann surface on which $\sqrt{(z - 1)(z - 2)(z - 3)}$ is a single-valued function, analytic except at $z = 1, 2, 3$.

12. Show that $w(z) = z^\alpha$ $(0 < \alpha < 1)$, where α is irrational, can be made single-valued only on an infinite-sheeted Riemann surface. Discuss how the sheets should be connected.

13. Show that, according to the definitions of Section 6.2,

$$\tan(\tan^{-1} z) = z, \qquad \log(e^z) = z + 2\pi ni.$$

14. *The mean-value theorem.* Prove that for charge-free two-dimensional space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* circle centered on that point. Do this by considering the electrostatic potential as the real part of an analytic function.

15. Find all the singularities of $\tanh z$.

16. Explain why one of the following definite integrals is meaningful and the other meaningless. Evaluate the one that makes sense, writing the answer in the form $a + bi$.

$$\text{a)} \int_{-1}^{1} z^*\,dz, \qquad \text{b)} \int_0^i \sin 2z\,dz.$$

17. Compute, using Cauchy's integral formula,

a) $\oint_C \frac{e^z\,dz}{z - \pi i/2}$, C = boundary of a square with sides $x = \pm 2$, $y = \pm 2$.

b) $\oint_C \frac{dz}{z^2 + 2}$, C = circle of radius 1, center i.

(Both contours are oriented counterclockwise.)

18. Show that

a) $\oint_C \frac{z^4 + 2z + 1}{(z - z_0)^4}\,dz = 8\pi i z_0$, where C is any closed contour containing z_0.

b) $\oint_C \frac{\cosh z}{z^{n+1}}\,dz = \frac{2\pi i}{n!}\frac{[1 + (-1)^n]}{2}$, where C is any closed contour containing the origin.

19. *Cauchy's integral formula for a polynomial.* Let $P(z)$ be a polynomial. Using Cauchy's Theorem, prove that

$$\frac{1}{2\pi i}\oint_C \frac{P(z)}{z - a}\,dz = P(a)\ ,$$

where C encloses the point a. [*Hint*: Consider

$$Q(z) = \frac{P(z) - P(a)}{z - a},$$

which is also a polynomial (why?), and evaluate the integral you want from this expression.]

20. Prove that if $f(z)$ is analytic within and on a closed contour C, and α and β are two distinct points within C, then

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - \alpha)(z - \beta)}\,dz = \frac{f(\alpha)}{\alpha - \beta} + \frac{f(\beta)}{\beta - \alpha}.$$

From this result, deduce Liouville's theorem, which states that a function must be a constant. [*Hint*: If for arbitrary α and β, $f(\alpha) = f(\beta)$, then $f(z) = \text{const.}$]

21. *Morera's theorem.* Morera's theorem is a kind of converse to the Cauchy-Goursat theorem. It states that if $f(z)$ is continuous in a region R and if $\oint f(z)\,dz = 0$ for any C inside R, then $f(z)$ is analytic inside R. Prove this theorem. Morera's theorem gives us another way of establishing the analyticity of a function. We can use it instead of checking the Cauchy-Riemann conditions *and* the sometimes troublesome additional requirement for analyticity, the continuity of the derivative of $f(z) = u + iv$, that is, continuity of the four first partial derivatives of u and v.

22. *Conformal mappings.* Consider the analytic function $w = f(z)$ as transforming the complex z-plane into a complex w-plane. Suppose that two curves in the z-plane, F_z^1 and F_z^2, meet at an angle α at the point z_0 in the z-plane. Prove that the transformed curves in the w-plane, F_w^1 and F_w^2, meet at the same angle in the w-plane if $f'(z_0) \neq 0$. The requirement that $f'(z_0) \neq 0$ is essential for conformality at z_0. Consider the analytic function $w = z^2$, for which $w'(0) = 0$. The coordinate axes themselves, which pass through the origin and are separated by an angle of 90°, are mapped into lines that are separated by 180°. The angle between any two straight lines through the origin in the z-plane will be doubled in the w-plane. Thus the mapping is not conformal at the origin. However, it is conformal everywhere else.

23. *Poisson's formula*

a) Let $f(z)$ be an analytic function within and on the circle C of radius a. Prove that

$$f(z') = \frac{1}{2\pi i}\oint_C \left[\frac{f(z)}{z - z'} - \frac{f(z)}{z - (a^2/z'^*)}\right] dz\,,$$

if $z' = re^{i\phi}$, $r < a$, in polar coordinates located at the center of C.

b) From the above formula, derive Poisson's formula:

$$f(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - \phi)} f(ae^{i\theta})\, d\theta\,.$$

c) From Poisson's formula, prove that if a function is analytic throughout and on a circle, then the value of the function at the center of the circle, $f(0)$, is the average of its boundary values on the circle (mean value theorem).

d) Now, starting with the equation

$$f(z') = \frac{1}{2\pi i}\oint_C \left[\frac{f(z)}{z - z'} + \frac{f(z)}{z - (a^2/z'^*)}\right] dz\,,$$

[which differs from the equation of part (a) by the + sign, but holds for the same reason], derive [using part (c) eventually]

$$f(z') = f(0) - \frac{iar}{\pi}\int_0^{2\pi} \frac{\sin(\theta - \phi) f(ae^{i\theta})}{a^2 + r^2 - 2ar\cos(\theta - \phi)}\, d\theta\,.$$

e) Letting $f(z') = f(re^{i\phi}) = u(r, \phi) + iv(r, \phi)$, etc., deduce from (d) the formulas

$$u(a, \phi) = u(0) + \frac{1}{2\pi} P\int_0^{2\pi} v(a, \theta) \cot\left(\frac{\theta - \phi}{2}\right) d\theta\,,$$

$$v(a, \phi) = v(0) - \frac{1}{2\pi} P\int_0^{2\pi} u(a, \theta) \cot\left(\frac{\theta - \phi}{2}\right) d\theta\,.$$

These formulas express the real part of an analytic function on a circle in terms of its imaginary part on the circle, and vice versa. Many further applications of these formulas are possible.

24. *An alternative approach to Hilbert transforms*

a) Show that if $f(z)$ is analytic in the upper half z-plane, then if $a = \alpha + i\beta$ $(\beta > 0)$,

$$\frac{1}{2\pi i}\oint_C \left[\frac{f(z)}{z - a} + \frac{f(z)}{z - a^*}\right] dz = f(a)\,,$$

where C is a semicircle of arbitrarily large radius with the real axis for its base.

b) If $|f(z)| \to 0$ as $|z| \to \infty$ in the upper half-plane, then show that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \left[\frac{f(x)}{x - a} + \frac{f(x)}{x - a^*}\right] dx = f(a)\,.$$

Combine fractions to obtain

$$\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{(x - \alpha) f(x)}{(x - \alpha)^2 + \beta^2}\, dx = f(a)\,.$$

c) If $f(z) = u(x, y) + iv(x, y)$, show that (b) reduces to

$$u(\alpha, \beta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x - \alpha) v(x, 0)}{(x - \alpha)^2 + \beta^2}\, dx\,, \quad v(\alpha, \beta) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x - \alpha) u(x, 0)}{(x - \alpha)^2 + \beta^2}\, dx\,.$$

d) Hence show that if we let $\beta = 0$,

$$u(\alpha, 0) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x, 0)}{x - \alpha}\, dx\,, \qquad v(\alpha, 0) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{u(x, 0)}{x - \alpha}\, dx\,,$$

which is, apart from notation, the Hilbert transform pair derived in Section 6.5 [Eqs. (6.19a) and (6.19b)].

25. Consider the integral

$$I_\pm = \lim_{\epsilon\to 0} \int_{-A}^{+B} \frac{f(x)}{x \pm i\epsilon}\, dx\,,$$

where A and B are positive real numbers and $\epsilon > 0$.

a) Show that $I_\pm$ can be written as

$$I_\pm = \lim_{\epsilon\to 0} \int_{-A}^{-\delta} \frac{xf(x)}{x^2 + \epsilon^2}\, dx + \lim_{\epsilon\to 0} \int_{+\delta}^{+B} \frac{xf(x)}{x^2 + \epsilon^2}\, dx + \lim_{\epsilon\to 0} \int_{-\delta}^{+\delta} \frac{xf(x)}{x^2 + \epsilon^2}\, dx \mp i \lim_{\epsilon\to 0} \epsilon \int_{-A}^{B} \frac{f(x)}{x^2 + \epsilon^2}\, dx\,,$$

where δ is a small positive number.

b) Use plausible arguments (of the type found, for example, in Section 5.3) to show that $I_\pm$ can be written as

$$I_\pm = \int_{-A}^{-\delta} \frac{f(x)}{x}\, dx + \int_{\delta}^{+B} \frac{f(x)}{x}\, dx + f(0) \lim_{\epsilon\to 0} \int_{-\delta}^{\delta} \frac{x\, dx}{x^2 + \epsilon^2} \mp i f(0) \lim_{\epsilon\to 0} \epsilon \int_{-A}^{B} \frac{dx}{x^2 + \epsilon^2}\,.$$

c) Show that the third term of the equation in (b) vanishes by symmetry. Doing the fourth integral explicitly, show that

$$I_\pm = P\int_{-A}^{B} \frac{f(x)}{x}\, dx \mp i f(0) \lim_{\epsilon\to 0} \left[\tan^{-1}\left(\frac{B}{\epsilon}\right) + \tan^{-1}\left(\frac{A}{\epsilon}\right)\right].$$

d) Let $\epsilon \to 0$ and use the definition of the δ-function to write, finally,

$$\lim_{\epsilon\to 0} \int_{-A}^{+B} \frac{f(x)}{x \pm i\epsilon}\, dx = P\int_{-A}^{+B} \frac{f(x)}{x}\, dx \mp i\pi f(0) = P\int_{-A}^{B} \frac{f(x)}{x}\, dx \mp i\pi \int_{-A}^{B} \delta(x) f(x)\, dx\,.$$

This is often written, rather cryptically, as the symbolic formula

$$\lim_{\epsilon\to 0} \frac{1}{x \pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x)\,.$$

26. Solve the integral equation

$$\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)}{x - t}\, dx = \frac{1}{1 + t^2} \equiv -v(t)$$

by finding the Hilbert transform of $v(t)$. Find a complex function $f(z) = u(x, y) + iv(x, y)$ such that $u(x, 0) \equiv u(x) = Re(f(z))$ and $v(x, 0) \equiv v(x) = Im(f(z))$ and verify that this complex function has the two properties (what are they?) that ensure that $u(x)$ and $v(x)$ are a Hilbert transform pair.

27. Derive a dispersion relation for $f(z)$ when $|f(z)/z|$ tends to a constant as $|z| \to \infty$.

28. Prove that a uniformly convergent series of analytic functions can be integrated term by term. Hence show that a uniformly convergent series of analytic functions is an analytic function.
[*Hint*: Use Morera's theorem of Problem 21.]

29. Find the order of the pole at the origin, the residue there, and the integral around a (small) path C enclosing the origin, but no other singularities, for each of the following functions:

a) $\cot z$, b) $\csc^2 z \log(1 - z)$, c) $\dfrac{z}{\sin z - \tan z}$.

30. What are the positions and natures of the singularities and the residues at these singularities of the following functions in the z-plane, excluding the point at infinity?

a) $f_1(z) = \dfrac{\cot \pi z}{(z-1)^2}$, b) $f_2(z) = \dfrac{1}{z(e^z - 1)}$.

[*Note*: Both these functions have nonisolated and hence essential singularities at infinity because the origin for $f_1(1/z)$ and $f_2(1/z)$ is a limit point of poles.]

31. Problems on the evaluation of real definite integrals by contour integration:

a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{(a^2 - b^2)^{1/2}}, \quad a > b > 0.$

b) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}[a - (a^2 - b^2)^{1/2}], \quad a > b > 0.$

c) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}, \quad a > b > 0.$

d) $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$ e) $\displaystyle\int_0^{\infty} \frac{x^2\, dx}{(a^2 + x^2)^3} = \frac{\pi}{16a^3}.$

f) $\displaystyle\int_0^{\infty} \frac{\sin x\, dx}{x(a^2 + x^2)} = \frac{\pi}{2a^2}(1 - e^{-a}).$ g) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi.$

h) $\displaystyle\int_0^{\infty} \frac{x^{2a-1}}{b^2 + x^2}\, dx = \frac{\pi b^{2(a-1)}}{2} \csc \pi a, \quad 0 < a < 1.$

i) $\displaystyle\int_0^{\infty} \frac{\log x}{b^2 + x^2}\, dx = \frac{\pi \log b}{2b}.$

j) $\displaystyle P\int_{-\infty}^{\infty} \frac{1}{(\omega' - \omega_0)^2 + a^2}\,\frac{1}{\omega' - \omega}\, d\omega' = \frac{\pi}{a}\,\frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + a^2}.$

k) $\displaystyle\int_0^{\infty} \frac{dx}{x^3 + a^3} = \frac{2\pi}{3a^2\sqrt{3}}.$

l) Consider the real integral

$$G(x, x', \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}e^{-k^2\tau}\, dk,$$

where τ is real. In this case it is easy to show that

$$G(x, x', \tau) = \frac{1}{(4\pi\tau)^{1/2}}\, e^{-(x-x')^2/4\tau}.$$

Prove that this result still holds when τ is pure imaginary. [*Hint*: The value of the integral

$$\int_0^{\infty} e^{-iu^2}\, du$$

may be found by considering the integral of e^{-z^2} around the boundary of the circular sector $0 \le \theta \le \pi/4$, $0 \le r \le R$. In the limit $R \to \infty$, *prove* that the contribution to the integral over the circular arc goes to zero. The contributions over the remaining straight segments of the contour provide the value of the integral needed in the problem.]

Note: This contour integral provides the values of the two real definite integrals:

$$\int_{-\infty}^{\infty} \sin(x^2)\,dx = \int_{-\infty}^{\infty} \cos(x^2)\,dx = (\pi/2)^{1/2},$$

by taking real and imaginary parts. By making the change of variable $x^2 = t$, this result can be transformed into

$$\int_0^{\infty} \frac{\sin t}{\sqrt{t}}\,dt = \int_0^{\infty} \frac{\cos t}{\sqrt{t}}\,dt = (\pi/2)^{1/2}.$$

32. We have seen how the function $\pi \cot \pi z$ may be used to sum certain series. But it is useless if the series is an alternating one. Show how alternating series may be summed by using the function $\pi \csc(\pi z)$ in place of $\pi \cot(\pi z)$. Use the general result to prove that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \pi^2/12.$$

33. Consider the problem of evaluating the integral of $z^{1/2}$ from A to D along the circular path C, shown in Fig. 6.24. We take the cut to lie along the ray $\theta = \alpha$ and define $z^{1/2} \equiv r^{1/2}e^{i\theta/2}$ for $\alpha \leq \theta < \alpha + 2\pi$. Compare this with the value obtained by integrating $z^{1/2}$ from A to D along the "keyhole" path $ABCD$. Do the results agree? Should they agree? Why?

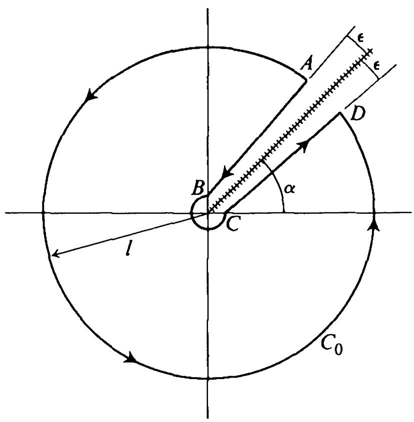

Fig. 6.24 The contours for Problem 6-34. The angles, ε, for the "keyhole" path are arbitrarily small. The radius of the circle, C_0, is l, and the cut is along the ray $\theta = \alpha$.

34. Consider the following response function, $G(t)$, which vanishes for negative values of the argument and which, for positive values, is given by

$$G(t) = G_0 \frac{\sin^2 \mu t}{t^{3/2}}.$$

Note that this function satisfies conditions (a), (b), (c) at the beginning of Section

6.6. Show that the Fourier transform, $g(\omega)$, is given by

$$g(\omega) = \frac{i-1}{2} G_0[\sqrt{\omega} - \tfrac{1}{2}\sqrt{\omega + 2\mu} - \tfrac{1}{2}\sqrt{\omega - 2\mu}]\,, \qquad \omega \geq 2\mu\,;$$
$$= \frac{i-1}{2} G_0[\sqrt{\omega} - \tfrac{1}{2}\sqrt{\omega + 2\mu}] + \frac{i+1}{2} G_0[\tfrac{1}{2}\sqrt{2\mu - \omega}]\,, \qquad 0 \leq \omega \leq 2\mu\,,$$
$$g(-\omega) = g^*(\omega)\,.$$

Show that the complex-valued function of z which reduces to the above result on the real axis is

$$g(z) = \frac{i-1}{2} G_0[\sqrt{z} - \tfrac{1}{2}\sqrt{z + 2\mu} - \tfrac{1}{2}\sqrt{z - 2\mu}]\,.$$

Thus we see that $g(z)$ has branch points on the real axis,a possibility which is *not* excluded by conditions (a), (b), and (c). Note that we can choose the cuts to lie in the lower half-plane, so $g(z)$ is analytic in the upper half-plane.
[*Hint*: the integrals mentioned in the note following Problem 6.31 will be of use.]

35. The θ-function (or step function) is defined as

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0\,, \\ \tfrac{1}{2} & \text{for } x = 0\,, \\ 1 & \text{for } x > 0\,. \end{cases}$$

Prove

a) $\theta(x) = \dfrac{1}{2} + \dfrac{1}{2\pi i} P\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\omega}\, d\omega\,,$ b) $\dfrac{d\theta(x)}{dx} = \delta(x)\,.$

[*Hint*: See Eq. (5.63)]

36. The Laguerre polynomials are generated by the generating function

$$\psi(x, t) = \frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n\,, \qquad 0 < t < 1\,.$$

They are given directly by the formula

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x})\,, \qquad 0 \leqq x \leqq \infty\,,$$

analogous to Rodrigues' formula for Legendre polynomials.

a) By differentiating the generating function with respect to x [getting $(1 - t)\psi' = - t\psi$], derive the recursion relation

$$L_n' - nL_{n-1}' = -nL_{n-1}\,.$$

By differentiating $\psi(x, t)$ with respect to t, another recursion relation can be derived. Show that

$$L_{n+1} - (2n + 1 - x)L_n + n^2 L_{n-1} = 0\,.$$

b) From these two recursion relations, derive Laguerre's differential equation

$$xL_n'' + (1 - x)L_n' + nL_n = 0\,.$$

c) Show that $L_n(0) = n!$

d) Derive the generating function from Rodrigues' formula by contour integration.

37. In Section 5.10, we defined the Hermite polynomials in terms of the generating function

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n , \qquad t > 0 ,$$

and deduced from this the Rodrigues formula:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} .$$

Now prove the converse: that Rodrigues' formula implies the generating function. Enroute, establish the integral representation

$$H_n(x) = \frac{(-1)^n n!}{2\pi i} e^{x^2} \oint_C \frac{e^{-z^2}}{(z-x)^{n+1}} dz ,$$

where C encloses the point x.

38. Prove the integral representation for the Hermite polynomials:

$$H_n(x) = \frac{2^n}{\pi^{1/2}} \int_{-\infty}^{\infty} (x + it)^n e^{-t^2} dt .$$

Use this result to derive an explicit series for the Hermite polynomials:

$$H_n(x) = \sum_{\substack{r=0 \\ (2r \leq n)}}^{n} \frac{(-1)^r n!}{r!(n-2r)!} (2x)^{n-2r} .$$

39. Show that
 a) $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$
 b) $\Gamma(n+1) = n!$, where n is a positive integer.

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