Implications of Time-Reversal Symmetry in Quantum Mechanics

1. The time reversal operator is antiunitary

In quantum mechanics, the time reversal operator Θ acting on a state produces a state that evolves backwards in time. That is, if we consider the time evolution of a state under the assumption that the Hamiltonian is time-independent,

$$|\Psi(t)\rangle = e^{-iHt/\hbar}\,|\Psi(0)\rangle\ , \tag{1}$$

then

$$\Theta\,|\Psi(-t)\rangle = e^{-iHt/\hbar}\Theta\,|\Psi(0)\rangle\ . \tag{2}$$

Letting $t \to -t$ in eq. (1) and comparing with eq. (2) yields

$$e^{-iHt/\hbar}\Theta = \Theta e^{iHt/\hbar}\,.$$

Taking t infinitesimal yields

$$-iH\Theta = \Theta iH\,. \tag{3}$$

Suppose that Θ were a linear operator. Then, one could cancel the factors of i in eq. (2) to obtain $-H\Theta = \Theta H$. Acting on an energy eigenstate $|E_n\rangle$, we would conclude that

$$H\Theta\,|E_n\rangle = -\Theta H\,|E_n\rangle = -E_n\Theta\,|E_n\rangle\ .$$

which implies that if $|E_n\rangle$ is a state of energy E_n then $\Theta\,|E_n\rangle$ is a state of energy $-E_n$. This result would then imply the absence of a ground state, since one could generate a state of arbitrarily negative energy by choosing a state with arbitrary large positive energy.

We can avoid this dilemma by declaring Θ to be an *antilinear* operator. We first note two important properties of antilinear operators. Given an antilinear operator Θ and two states $|\Psi\rangle$ and $|\Phi\rangle$, then

$$\big(\langle\Phi|\,\Theta\big)\,|\Psi\rangle = \big[\,\langle\Phi|\,\big(\Theta\,|\Psi\rangle\big)\big]^*\,. \tag{4}$$

Second, for any complex constant c and antilinear operator Θ,

$$\Theta c\,|\Psi\rangle = c^*\Theta\,|\Psi\rangle\ . \tag{5}$$

Since Θ is a symmetry operator, we also demand that it should preserve the absolute value of any inner product. That is, Θ is an *antiunitary* operator, which in addition to satisfying eqs. (4) and (5), also satisfies,

$$\langle\Theta\Psi\,|\,\Theta\Phi\rangle = \langle\Psi\,|\,\Phi\rangle^* = \langle\Phi\,|\,\Psi\rangle\,. \tag{6}$$

Applying eq. (5) to eq. (3) then yields

$$\big[H,\Theta\big] = 0\,. \tag{7}$$

Consequently, $|E_n\rangle$ and $\Theta\,|E_n\rangle$ have the *same* energy (as expected for a symmetry operator Θ), and the existence of a ground state is preserved.

The complex conjugated one-component wave function satisfies the Schrodinger equation with $t \to -t$. It follows that

$$\Theta = UK\,, \tag{8}$$

where K is the complex conjugate operator and U is an arbitrary phase. When applied to a $(2j+1)$-component wave function that describes a particle of spin j, U is a $(2j+1)\times(2j+1)$ unitary matrix. The complex conjugate operator is antiunitary. In particular, in light of eq. (6),

$$\langle K\Psi \,|\, K\Phi\rangle = \langle\Psi\,|\,\Phi\rangle^* = \langle\Phi\,|\,\Psi\rangle\,.$$

Hence, the time-reversal operator Θ defined by eq. (8) is antiunitary as expected.[1]

2. The time reversal operator acting on states of definite angular momentum

Since the angular momentum changes sign under time reversal, the quantum mechanical angular momentum operator $\vec{\boldsymbol{J}}$ must satisfy

$$\Theta\,\vec{\boldsymbol{J}}\,\Theta^{-1} = -\vec{\boldsymbol{J}}\,. \tag{9}$$

We shall use this property to obtain an explicit form for the unitary operator U that appears in eq. (8) when acting on the $|jm\rangle$ basis,

By definition, any vector operator $\vec{\boldsymbol{V}}$ satisfies

$$U^\dagger[R(\boldsymbol{\hat{n}},\theta)]V_iU[R(\boldsymbol{\hat{n}},\theta)] = \sum_j R_{ij}V_j\,, \tag{10}$$

where $U[R(\boldsymbol{\hat{n}},\theta)] = \exp(-i\theta\boldsymbol{\hat{n}}\boldsymbol{\cdot}\vec{\boldsymbol{J}}/\hbar)$ is the unitary operator that rotates states of the Hilbert space. We choose $\vec{\boldsymbol{V}} = \vec{\boldsymbol{J}}$ and consider a rotation parameterized by $\boldsymbol{\hat{n}} = \boldsymbol{\hat{y}}$ and $\theta = \pi$. The corresponding 3×3 rotation matrix R is given by [cf. eq. (20) of the class handout entitled *Three Dimensional Rotation Matrices*]:

$$R(\boldsymbol{\hat{y}},\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{pmatrix}\,.$$

It follows from eq. (10) that

$$\exp(i\pi J_y/\hbar)J_x\exp(-i\pi J_y/\hbar) = -J_x\,, \tag{11}$$

$$\exp(i\pi J_y/\hbar)J_y\exp(-i\pi J_y/\hbar) = J_y\,, \tag{12}$$

$$\exp(i\pi J_y/\hbar)J_z\exp(-i\pi J_y/\hbar) = -J_z\,. \tag{13}$$

In class, we evaluated the matrix elements of the angular momentum operators J_i with respect to the $|j\,m\rangle$–basis. In particular, we found that the matrix elements of J_x and J_z are real and those of J_y are pure imaginary. In light of eq. (5), it follows that

$$K^{-1}J_xK = J_x\,, \qquad K^{-1}J_yK = -J_y\,, \qquad K^{-1}J_zK = J_z\,. \tag{14}$$

[1]For further details, see Section 4.4 of Sakurai and Napolitano.

Thus, if we multiply eqs. (11)–(13) on the left by $\eta^{-1}K^{-1}$ and on the right by ηK and then make use of eq. (14), it then follows that eq. (9) holds, where

$$\Theta = \eta \exp(-i\pi J_y/\hbar)K \,, \tag{15}$$

is an antiunitary operator and and η is a complex phase that can be chosen by convention to be unity.

We can confirm eq. (15) in more detail by examining the effect of operating the time reversal operator Θ on a simultaneous eigenstate of $\vec{\boldsymbol{J}}^{\,2}$ and J_z, denoted by $|j\, m\rangle$. It is convenient to rewrite eq. (9) by multiplying from the right by Θ^{-1} and the left by Θ, which yields

$$\Theta^{-1}\vec{\boldsymbol{J}}\,\Theta = -\vec{\boldsymbol{J}}\,. \tag{16}$$

It follows that

$$\Theta^{-1}\vec{\boldsymbol{J}}^{\,2}\,\Theta = \Theta^{-1}\vec{\boldsymbol{J}}\,\Theta \boldsymbol{\cdot} \Theta^{-1}\vec{\boldsymbol{J}}\,\Theta = \vec{\boldsymbol{J}}^{\,2}\,. \tag{17}$$

Using eq. (16),

$$\Theta^{-1}J_z\,\Theta\,|j\, m\rangle = -J_z\,|j\, m\rangle = -\hbar m\,|j\, m\rangle\ .$$

Multiplying both sides by Θ yields

$$J_z\big[\Theta\,|j\, m\rangle\big] = -\hbar m\big[\Theta\,|j\, m\rangle\big]\,. \tag{18}$$

Likewise, using eq. (17),

$$\Theta^{-1}\vec{\boldsymbol{J}}^{\,2}\,\Theta\,|j\, m\rangle = \vec{\boldsymbol{J}}^{\,2}\,|j\, m\rangle = \hbar j(j+1)\,|j\, m\rangle\ .$$

Multiplying both sides by Θ yields

$$\vec{\boldsymbol{J}}^{\,2}\big[\Theta\,|j\, m\rangle\big] = \hbar^2 j(j+1)\big[\Theta\,|j\, m\rangle\big]\,. \tag{19}$$

The results of eqs. (18) and (19) imply that

$$\Theta\,|j\, m\rangle = c\,|j\,,\,-m\rangle\ , \tag{20}$$

where c is some complex number. Using eq. (6) with $|\Psi\rangle = |\phi\rangle = |j\, m\rangle$, eq. (20) implies that $|c|^2 = 1$. Thus, c is a complex phase, which we can write as $c = e^{i\delta}$. In principle, δ can depend on j and m, so that

$$\Theta\,|j\, m\rangle = e^{i\delta(j,m)}\,|j\,,\,-m\rangle\ . \tag{21}$$

To determine the functional form of $\delta(j,m)$, we examine the x and y components of eq. (16). It is more convenient to write $J_\pm = J_x \pm iJ_y$ and use eq. (16) to obtain

$$\Theta^{-1}J_\pm\,\Theta = -J_\mp\,, \tag{22}$$

where the extra sign in eq. (22) arises as a result of eq. (5). In addition, recall that

$$J_\pm\,|j\, m\rangle = \hbar\big[(j\mp m)(j\pm m+1)\big]^{1/2}\,|j, m\pm 1\rangle\ . \tag{23}$$

Using eqs. (22) and (23),

$$\Theta^{-1} J_\pm \,\Theta \,|j\, m\rangle = -J_\mp \,|j\, m\rangle = -\hbar\big[(j\pm m)(j\mp m+1)\big]^{1/2}\,|j\,,\, m\mp 1\rangle\ .$$

Multiplying both sides of the above equation by Θ and using eq. (21),

$$e^{i\delta(j,m)} J_\pm \,|j\,,\,-m\rangle = -\hbar\big[(j\pm m)(j\mp m+1)\big]^{1/2} e^{i\delta(j,m\mp 1)}\,|j\,,\,-m\pm 1\rangle\ . \qquad (24)$$

We again use eq. (23) to write the left hand side of eq. (24) as

$$e^{i\delta(j,m)}\hbar\big[(j\pm m)(j\mp m+1)\big]^{1/2}\,|j\,,\,-m\pm 1\rangle\ .$$

Inserting this result into eq. (24) then yields $e^{i\delta(j,m)} = -e^{i\delta(j,m\mp 1)}$. Given the value of $\delta(j,j)$, one can obtain $\delta(j,m)$ for $m=-j\,,\,-j+1\,,\,\ldots\,,\,j-1\,,\,j$,

$$e^{i\delta(j,j-n)} = (-1)^n e^{i\delta(j,j)}\,,\qquad \text{for } n=0,1,2,\ldots,2j\,.$$

Thus, writing $n=j-m$, it follows that

$$e^{i\delta(j,m)} = (-1)^{j-m} e^{i\delta(j,j)}\,,\qquad \text{for } m=-j\,,\,-j+1\,,\,\ldots\,,\,j-1\,,\,j\,.$$

Hence, we conclude that

$$\Theta\,|j\, m\rangle = e^{i\delta(j,j)}\,(-1)^{j-m}\,|j\,,\,-m\rangle\ ,$$

where $e^{i\delta(j,j)}$ is an irrelevant j-dependent (but m-independent) phase that can be set to 1 by convention.[2] Henceforth, we shall write

$$\Theta\,|j\, m\rangle = (-1)^{j-m}\,|j\,,\,-m\rangle\ . \qquad (25)$$

Finally, we can compare the result of eq. (25) with eq. (13) of the class handout entitled, *Properties of the Wigner d-matrices*,

$$\exp(-i\pi J_y/\hbar)\,|j\, m\rangle = (-1)^{j-m}\,|j\,,\,-m\rangle\ . \qquad (26)$$

Since the complex conjugation operator K commutes[3] with $J_\pm$, J_z and $\vec{\boldsymbol{J}}^{\,2}$, it follows that $K\,|j\, m\rangle = \eta\,|j\, m\rangle$, where η is a phase that could depend on j but is independent of m. By convention, we can choose $\eta=1$, in which case $K\,|j\, m\rangle = |j\, m\rangle$. Consequently,

$$\exp(-i\pi J_y/\hbar)K\,|j\, m\rangle = (-1)^{j-m}\,|j\,,\,-m\rangle\ , \qquad (27)$$

and we can identify the antiunitary operator $\Theta=\exp(-i\pi J_y/\hbar)K$, since they have the exact same matrix elements in the $|j\, m\rangle$–basis [cf. eqs. (25) and (27)].

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[2] In some cases other choices for $e^{i\delta(j,j)}$ can be more convenient. For example, if $e^{i\delta(j,j)}=(-1)^j$ then $\Theta\,|j\, m\rangle = (-1)^{2j-m}\,|j\,,\,-m\rangle = (-1)^{2(j-m)}(-1)^m\,|j\,,\,-m\rangle = (-1)^m\,|j\,,\,m\rangle$, where we have used the fact that $j-m$ is an integer. This convention is convenient when applied to orbital angular momentum, since the spherical harmonics satisfy $Y_{\ell m}(\theta,\phi)^* = (-1)^m Y_{\ell,-m}(\theta,\phi)$, which implies that the effect of time reversal on $Y_{\ell m}(\theta,\phi)\equiv\langle\theta,\phi\mid \ell\, m\rangle$ is equivalent to complex conjugation with no additional phase factors.

[3] This result follows from eq. (14). In particular, $K^{-1}J_\pm K = J_\pm$ since $J_\pm = J_x \pm iJ_y$ are real matrices in the $|j\, m\rangle$–basis.

3. The square of the time reversal operator

Starting from eq. (25), we apply Θ twice to obtain

$$\begin{aligned}\Theta^2\,|j\,m\rangle &= \Theta\,\Theta\,|j\,m\rangle = \Theta(-1)^{j-m}\,|j\,,\,-m\rangle \\ &= (-1)^{j-m}\,\Theta\,|j\,,\,-m\rangle = (-1)^{j-m}(-1)^{j+m}\,|j\,m\rangle = (-1)^{2j}\,|j\,m\rangle\ .\end{aligned}$$

Note that the step $\Theta(-1)^{j-m} = (-1)^{j-m}\,\Theta$ is valid because $(-1)^{j-m}$ is a real number for both integral and half-integral values of j, since in either case $j-m$ is an integer.[4] Hence, we conclude that

$$\Theta^2\,|jm\rangle = (-1)^{2j}\,|jm\rangle\ . \tag{28}$$

One can also derive eq. (28) by directly employing eq. (15). First, we note that the antiunitary complex conjugation operator satisfies,

$$K = K^\dagger = K^{-1}\,,$$

which yields $K^2 = \mathbf{I}$, where $\mathbf{I}$ is the identity operator. Moreover, in light of eq. (5), one can write $Kz = z^*K$, where z is any complex number. Consequently, K commutes with $\exp(-i\pi J_y/\hbar)$ since iJ_y is a real matrix with respect to the $|j\,m\rangle$ basis. Starting with eq. (15), it then follows that

$$\begin{aligned}\Theta^2 &= \Theta\,\Theta = \eta\exp(-i\pi J_y/\hbar)K\eta\exp(-i\pi J_y/\hbar)K \\ &= \eta\eta^*\exp(-i\pi J_y/\hbar)K^2\exp(-i\pi J_y/\hbar) = \exp(-2i\pi J_y/\hbar)\,,\end{aligned}$$

after noting that $\eta\eta^* = 1$ for any complex phase. Finally,

$$\begin{aligned}\Theta^2\,|j\,m\rangle &= \exp(-2i\pi J_y/\hbar)\,|j\,m\rangle = \exp(-i\pi J_y/\hbar)\,\exp(-i\pi J_y/\hbar)\,|j\,m\rangle \\ &= (-1)^{j-m}\exp(-i\pi J_y/\hbar)\,|j\,,\,-m\rangle = (-1)^{j-m}(-1)^{j+m}\,|j\,m\rangle = (-1)^{2j}\,|j\,m\rangle\ .\end{aligned}$$

That is, we have recovered eq. (28).

Note that eq. (28) is equivalent to the relation,

$$d^{(j)}_{m'm}(2\pi) = (-1)^{2j}\,\delta_{mm'}\,, \tag{29}$$

which is derived in the class handout, *Properties of the Wigner d-matrices*. Eq. (29) implies that for bosonic systems (with integer values of j), a rotation by 2π is equivalent to the identity operator, whereas for fermionic systems (with half-odd-integer values of j), a rotation by 2π is equivalent to the *negative* of the identity operator (which implies that one must rotate by 4π to recover the initial fermionic system).

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[4] Had we adopted the phase convention of footnote 2, where $\Theta\,|j\,m\rangle = (-1)^m\,|j\,,\,-m\rangle = i^{2m}\,|j\,,\,-m\rangle$, then

$$\begin{aligned}\Theta^2\,|j\,m\rangle &= \Theta\,\Theta\,|j\,m\rangle = \Theta\,i^{2m}\,|j\,,\,-m\rangle = (-i)^{2m}\,\Theta\,|j\,,\,-m\rangle = (-i)^{2m}\,i^{-2m}\,|j\,m\rangle \\ &= (-i)^{4m}\,|j\,m\rangle = (-1)^{-2m}\,|j\,m\rangle = (-1)^{2j}(-1)^{-2(j+m)}\,|j\,m\rangle = (-1)^{2j}\,|j\,m\rangle\ ,\end{aligned}$$

after noting the relation $\Theta z = z^*\Theta$ for any complex number z, and using the fact that $j+m$ is an integer in the final step. The end result coincides with that of eq. (28), independently of the phase convention.

4. Even and odd irreducible tensor operators with respect to time reversal

An irreducible tensor operator is defined to be even or odd under time reversal if,

$$\Theta T_q^{(k)}\Theta^{-1} = \pm(-1)^q T_{-q}^{(k)}\,, \tag{30}$$

where even [odd] corresponds to the plus [minus] sign, respectively.

It is more convenient in the analysis that follows to rewrite eq. (30) as

$$T_q^{(k)} = \pm(-1)^q\,\Theta^{-1}T_{-q}^{(k)}\Theta\,. \tag{31}$$

Consider the matrix element of this operator equation with respect to the states $|\alpha\, j\, m\rangle$. Since Θ is antiunitary (so that $\Theta^{-1}=\Theta^\dagger$),

$$\langle\alpha'\, j\, m'|\Big(\Theta^{-1}T_{-q}^{(k)}\Theta\,|\alpha\, j\, m\rangle\Big) = \Big[\Big(\langle\alpha'\, j\, m'|\,\Theta^\dagger\Big)\Big(T_{-q}^{(k)}\Theta\,|\alpha\, j\, m\rangle\Big)\Big]^*\,,$$

after employing eq. (4). Hence, using eq. (31),

$$\pm(-1)^q\ \langle\alpha'\, j'\, m'|\,T_q^{(k)}\,|\alpha\, j\, m\rangle = \Big[\Big(\langle\alpha'\, j\, m'|\,\Theta^\dagger\Big)\Big(T_{-q}^{(k)}\Theta\,|\alpha\, j\, m\rangle\Big)\Big]^*\,,$$

If the theory is time-reversal invariant, then the Hamiltonian H commutes with the time reversal operator Θ. As a result, the energy eigenstate $|E_n\rangle$ and the state $\Theta\,|E_n\rangle$ have the same energy eigenvalue, as previously noted below eq. (7). Assuming that E_n is a non-degenerate state,[5] then $\Theta\,|E_n\rangle = e^{i\beta}\,|E_n\rangle$ for some complex phase $e^{i\beta}$. More generally, suppose that the maximal set of simultaneous diagonalizable operators is $\{H,\ \vec{\boldsymbol{J}}^{\,2},\ J_z\}$, and perhaps an additional set of operators A. The corresponding simultaneous eigenvalues shall be denoted collectively by $|\alpha\, j\, m\rangle$, where the α represent the energy eigenvalue and eigenvalues of the operators A if present. For a time-reversal invariant theory H is even under time reversal since $\Theta^{-1}H\Theta = H$. Likewise, we shall assume that the operators A are also even under time reversal so that $\Theta^{-1}A\Theta = A$. In this case, $\Theta\,|\alpha\, j\, m\rangle$ does not change the quantum numbers α. In particular, eq. (25) takes on the more general form,[6]

$$\Theta\,|\alpha\, j\, m\rangle = (-1)^{j-m}\,|\alpha\, j\,,\,-m\rangle\ ,$$

which implies that $\langle\alpha'\, j'\, m'|\,\Theta^\dagger = (-1)^{j-m'}\,\langle\alpha'\, j\,,\,-m'|$. It then follows that

$$\pm(-1)^q\ \langle\alpha'\, j'\, m'|\,T_q^{(k)}\,|\alpha\, j\, m\rangle = (-1)^{j-m'}(-1)^{j-m}\,\langle\alpha'\, j\,,\,-m'|\,T_{-q}^{(k)}\,|\alpha\, j\,,\,-m\rangle^*\ . \tag{32}$$

Applying the Wigner-Eckart theorem,

$$\langle\alpha'\, j\, m|\,T_q^{(k)}\,|\alpha\, j\, m\rangle = \langle jk\,;\, mq\,|\, jk\,;\, jm'\rangle\langle\alpha'\, j\|T^{(k)}\|\alpha\, j\rangle\,,$$

[5]Even in the case of degeneracies, one can form appropriate linear combinations within the degenerate subspace such that each energy eigenstate is a simultaneous eigenstate of Θ.

[6]In a theory where the time-reversal symmetry is violated, we could only conclude that

$$\Theta\,|\alpha\, j\, m\rangle = (-1)^{j-m}\sum_{\alpha'} c_{\alpha'}\,|\alpha'\, j\,,\,-m\rangle\ ,$$

for some suitable coefficients $c_{\alpha'}$.

to eq. (32), we obtain:

$$\langle jk\,;\,mq\,|\,jk\,;\,jm'\rangle\langle\alpha' j\|T^{(k)}\|\alpha\, j\rangle = \pm(-1)^{q+2j-m-m'}\langle jk\,;\,-m\,,\,-q\,|\,jk\,;\,j\,,\,-m'\rangle\langle\alpha' j\|T^{(k)}\|\alpha\, j\rangle^*\,.$$

Since the Clebsch-Gordan coefficients vanish unless $m' = q + m$, it follows that

$$(-1)^{q+2j-m-m'} = (-1)^{2(j-m)} = 1\,, \tag{33}$$

since $j - m$ is an integer. Hence,

$$\langle jk\,;\,mq\,|\,lk\,;\,jm'\rangle\langle\alpha' j\|T^{(k)}\|\alpha\, j\rangle = \pm\langle jk\,;\,-m\,,\,-q\,|\,jk\,;\,j\,,\,-m'\rangle\langle\alpha' j\|T^{(k)}\|\alpha\, j\rangle^*\,. \tag{34}$$

Finally, if we employ the following symmetry relation of the Clebsch-Gordan coefficients in eq. (34),[7]

$$\langle jk\,;\,-m\,,\,-q\,|\,jk\,;\,j\,,\,-m'\rangle = (-1)^k\langle jk\,;\,mq\,|\,jk\,;\,jm'\rangle\,,$$

then it follows that for a time-reversal invariant theory,

$$\langle\alpha' j\|T^{(k)}\|\alpha\, j\rangle = \pm(-1)^k\langle\alpha' j\|T^{(k)}\|\alpha\, j\rangle^*\,. \tag{35}$$

That is, the reduced matrix elements of a time-reversal even or odd irreducible tensor operator are either purely real or purely imaginary depending on the sign of $\pm(-1)^k$. Eq. (35) is an important result, which will be used in the next section of these notes.

5. A nonzero electric dipole moment of the neutron implies that time reversal symmetry is broken

The electric dipole operator is $\vec{\boldsymbol{\rho}} = e\vec{\boldsymbol{X}}$. In this section, we shall prove that if a neutron is observed to have a nonzero electric dipole moment, then both the parity and time reversal symmetries are separately violated.

Denote the parity operator by $\mathcal{P}$. Then,

$$\mathcal{P}^{-1}\vec{\boldsymbol{X}}\,\mathcal{P} = -\vec{\boldsymbol{X}}\,, \tag{36}$$

since $\vec{\boldsymbol{X}} \to -\vec{\boldsymbol{X}}$ under an inversion of the coordinate system. The parity operator satisfies,

$$\mathcal{P} = \mathcal{P}^{-1} = \mathcal{P}^\dagger\,,$$

so that $\mathcal{P}^2 = \mathbf{I}$. If parity is conserved, then $\mathcal{P}$ commutes with the Hamiltonian H. Hence, the neutron would be an eigenstate of parity. Since $\mathcal{P}^2 = \mathbf{I}$, the possible eigenvalues of $\mathcal{P}$ are ± 1. Denoting the neutron state by $|n\rangle$, we must have $\mathcal{P}\,|n\rangle = \pm\,|n\rangle$. (Conventionally, the positive sign is chosen, although the following argument does not depend on this choice.) It follows that

$$e\,\langle n|\,\vec{\boldsymbol{X}}\,|n\rangle = e\,\langle n|\,\mathcal{P}^{-1}\vec{\boldsymbol{X}}\mathcal{P}\,|n\rangle = -e\,\langle n|\,\vec{\boldsymbol{X}}\,|n\rangle = 0\,,$$

[7] See, e.g. eq. (15.2.11) of R. Shankar, *Principles of Quantum Mechanics*, 2nd edition (Springer Science, New York, NY, 1994). Note that we are employing the notation of Sakurai and Napolitano for the Clebsch-Gordan coefficients.

where $\mathcal{P}\,|n\rangle = \pm\,|n\rangle$ has been employed at the first step and eq. (36) was used in the second step. Hence, if parity is conserved,

$$\langle n|\,\rho\,|n\rangle = e\,\langle n|\,\vec{\boldsymbol{X}}\,|n\rangle = 0\,.$$

That is, a non-zero electric dipole moment indicates a violation of the parity symmetry of the theory.

Next, consider a time-reversal invariant theory. Note that the operator $\rho = e\vec{\boldsymbol{X}}$ is even under time-reversal since

$$\Theta\,\vec{\boldsymbol{X}}\,\Theta^{-1} = \vec{\boldsymbol{X}}\,.$$

Thus, we can apply eq. (35) with $k = 1$, which yields

$$\langle\alpha\, j\|\rho\|\alpha\, j\rangle = -\langle\alpha\, j\|\rho\|\alpha\, j\rangle^*\,. \tag{37}$$

However, ρ is an hermitian operator, which implies that its diagonal elements are real. Since the Clebsch-Gordan coefficients are real (by convention), the Wigner-Eckart theorem implies that $\langle\alpha\, j\|\rho\|\alpha\, j\rangle$ is also real. In light of eq. (37), it follows that

$$\langle\alpha\, j\|\rho\|\alpha\, j\rangle = 0\,. \tag{38}$$

Since the neutron has spin-$\frac{1}{2}$, it follows that the neutron (in its rest frame) is an eigenstate of $\vec{\boldsymbol{J}}^{\,2}$ with $j = \frac{1}{2}$. Using the Wigner-Eckart theorem, eq. (38) implies that

$$\langle n|\,\rho\,|n\rangle \propto \langle\alpha\, j\|\rho\|\alpha\, j\rangle = 0\,.$$

Hence, a non-zero static electric dipole moment indicates a violation of the time-reversal symmetry of the theory.

6. Final remarks

Both parity and time-reversal are good symmetries of quantum electrodynamics and appear to also be symmetries of the strong interactions. The weak interactions maximally violate parity symmetry, but time-reversal symmetry is only very weakly violated by the weak interactions. Indeed, the Standard Model of particle physics does predict the existence of an electric dipole moment of the neutron, but the predicted value is many orders of magnitude below the currently observed limits. Thus, if a neutron electric dipole moment were to be measured and shown to be nonzero, it would be strongly suggestive of the existence of new physics beyond the Standard Model of particle physics.

It is a curious fact that the strong interactions appear to be parity and time-reversal invariant. It turns out that there is a dimensionless parameter of the theory of of the strong interactions (called quantum chromodynamics), denoted by θ, which if non-zero would be a signal of a violation of both parity and time-reversal symmetries. The absence of an observed electric dipole moment for the neutron implies that $|\theta| < 10^{-10}$. You might think that $\theta = 0$ exactly, but since the parity and time-reversal symmetries are violated by the weak interactions, there is no justification for setting $\theta = 0$ by hand. It is currently a great mystery as to why the θ parameter of the strong interactions is so small (this is called the *strong CP problem* since in quantum field theory, a violation of time reversal symmetry implies a simultaneous violation of the charge conjugation and parity symmetries). Clever solutions to the strong CP problem lead to new physics beyond the Standard Model. One such solution requires the existence of a new extremely light spin-0 particle called the *axion*. Dedicated experiments searching for axions are currently running. So far, no evidence for the existence of axions has been found.