Properties of the Wigner $d$-matrices

The matrix elements of Wigner’s small $d$-matrix are defined by,

$$d^{(j)}_{m'm}(\theta) \equiv \langle j m'|e^{-i\theta J_y/\hbar}|j m \rangle,$$

where $j$ is a non-negative half-integer and $m, m' = -j, -j + 1, \ldots, j - 1, j$.

**Property 1:** $d^{(j)}_{m'm}(\theta)$ are real for all values of $m, m'$ and $j$.

Recall the explicit form for the matrix elements of the angular momentum operator $J_y$ obtained in class,

$$\langle j m'|J_y|j m \rangle = \frac{-1}{2\hbar} \left[ \delta_{m',m+1}(j-m)(j+m+1)^{1/2} - \delta_{m',m-1}(j+m)(j-m+1)^{1/2} \right]. \tag{1}$$

Thus the matrix elements of $iJ_y$ are real. It then follows that the matrix elements of $e^{-i\theta J_y/\hbar}$ are also real.

**Property 2:** $d^{(j)}_{m'm}(-\theta) = d^{(j)}_{m'm}(-\theta)$ for all values of $m, m'$ and $j$.

First, note that $e^{-i\theta J_y/\hbar}$ is a unitary operator. From property 1, it follows that $e^{-i\theta J_y/\hbar}$ is real. Hence, $e^{-i\theta J_y/\hbar}$ is a real orthogonal operator, which implies that its matrix representation satisfies $d^T(\theta)d(\theta) = I$, where $I$ is the identity matrix, or equivalently, $d^T(\theta) = d^{-1}(\theta)$. Moreover, the inverse of $e^{-i\theta J_y/\hbar}$ is given by $e^{i\theta J_y/\hbar}$, which yields $d^{-1}(\theta) = d(-\theta)$. Hence, we conclude that

$$d^T(\theta) = d(-\theta).$$

The matrix elements of this equation correspond to Property 2 above.

**Property 3:** $d^{(j)}_{m'm}(\theta) = d^{(j)}_{m'-m}(\theta)$ for all values of $m, m'$ and $j$.

This property immediately follows from eq. (1), since $J_y$ is invariant under the interchange $m \rightarrow -m'$, $m' \rightarrow -m$. In particular, under this interchange,

$$\langle j, -m|J_y|j, -m' \rangle = -\frac{1}{2\hbar} \left[ \delta_{m,-m'+1}(j+m')(j-m'+1)^{1/2} - \delta_{m,-m'-1}(j-m')(j+m'+1)^{1/2} \right].$$

In the first term above, the Kronecker delta imposes $m' = m + 1$, and in the second term above, the Kronecker delta imposes $m' = m - 1$. Hence,

$$\langle j, -m|J_y|j, -m' \rangle = -\frac{1}{2\hbar} \left[ \delta_{m,-m'+1}(j+m+1)(j-m)^{1/2} - \delta_{m,-m'-1}(j-m+1)(j+m)^{1/2} \right]. \tag{2}$$

Noting that $\delta_{m',m+1} = \delta_{-m,-m'+1}$ and $\delta_{m',m-1} = \delta_{-m,-m'-1}$, it follows from eqs. (1) and (2) that,

$$\langle jm'|J_y|jm \rangle = \langle j, -m|J_y|j, -m' \rangle.$$

Upon exponentiation, Property 3 is confirmed.
Property 4: $d_{m'm'}^{(j)}(\theta) = (-1)^{m-m'}d_{mm'}^{(j)}(\theta)$ for all values of $m$, $m'$ and $j$.

From the definition of the raising and lowering operators, $J_\pm \equiv J_x \pm iJ_y$, it follows that

$$e^{-i\theta J_y/\hbar} = e^{-\theta(J_+ - J_-)/2}.$$ 

It is easy to check that the matrix representation of $J_y$ is antisymmetric. Starting from eq. (1),

\begin{align*}
\langle jm|J_y|jm'\rangle &= -\frac{1}{2}\hbar \left[ \delta_{m,m' + 1} [(j - m')(j + m' + 1)]^{1/2} - \delta_{m,m' - 1} [(j + m')(j - m' + 1)]^{1/2} \right] \\
&= -\frac{1}{2}\hbar \left[ \delta_{m',m - 1} [(j - m + 1)(j + m)]^{1/2} - \delta_{m',m + 1} [(j + m + 1)(j - m)]^{1/2} \right] \\
&= -\langle j, m'|J_y|j, m \rangle.
\end{align*}

Moreover, the matrix elements of $J_y$ are nonzero if and only if $|m - m'| = 1$. Hence, if follows that for any positive integer $k$,

1. The matrix representation of $J_y^{2k}$ is symmetric and its matrix elements vanish if $|m - m'|$ is an odd integer.

2. The matrix representation of $J_y^{2k+1}$ is antisymmetric and its matrix elements vanish if $|m - m'|$ is an even integer.

The matrix exponential, $e^{-i\theta J_y/\hbar}$, is defined via its Taylor series. In light of the two results obtained above, the validity of Property 4 follows.

Property 5: $d_{mm'}^{(j)}(2\pi) = (-1)^{2j} \delta_{mm'}$ for all values of $m$, $m'$ and $j$.

We begin by using Theorem 1, which states that if $R$ is a rotation by an angle $\theta$ about a fixed axis $\hat{n}$ and $\hat{n}' = R\hat{n}$, then

$$U[R] \vec{J} \cdot \hat{n} U^\dagger[R] = \vec{J} \cdot \hat{n}' .$$

(3)

The proof of Theorem 1 is given in Appendix A. Exponentiating eq. (3) yields

$$U[R] e^{-i\beta \vec{J} \cdot \hat{n}'/\hbar} U^\dagger[R] = e^{-i\beta \vec{J} \cdot \hat{n}/\hbar} .$$

(4)

In eq. (4), choose $\theta = 2\pi$, $\hat{n} = \hat{z}$ and $\hat{n}' = \hat{y}$. Then,

$$U[R] e^{-2\pi i J_z/\hbar} U^\dagger[R] = e^{-2\pi i J_y/\hbar} .$$

(5)

We will now evaluate eq. (5) by multiplying on the left by $\langle jm|$ and on the right by $|jm'\rangle$. We first observe that

\begin{align*}
\langle jm|e^{-2\pi i J_z/\hbar}|jm'\rangle &= e^{-2\pi im'} \delta_{mm'} = e^{-2\pi ij} e^{2\pi i(j-m')} \delta_{mm'} = e^{-2\pi ij} \delta_{mm'} = (-1)^{2j} \delta_{mm'} ,
\end{align*}
must perform a $4\pi d$ on the left hand side of eq. (5) yields, $e^{i\pi}$. That is, a $2\pi$ after noting that $j$ and $\theta$ since $H$. Hence, we have proved Property 5. In summary, we have established that

$$\langle jm|U[R]|jm'\rangle = \sum_{m_1}\sum_{m_2} \langle jm|U[R]|jm_1\rangle \langle jm_1|e^{-2\pi iJ_x/\hbar}|jm_2\rangle \langle jm_2|U^\dagger[R]|jm'\rangle$$

$$= (-1)^{2j} \sum_{m_1}\sum_{m_2} D_{mm_1}^{(j)}[R] D_{m_2m'}^{(j)^\dagger}[R] \delta_{m_1m_2}$$

$$= (-1)^{2j} (D^{(j)}[R]D^{(j)^\dagger}[R])_{mm'} = (-1)^{2j} \delta_{mm'},$$

since $D^{(j)}[R]$ is the unitary matrix representation of the operator $U[R]$, which implies that $D^{(j)}[R]D^{(j)^\dagger}[R] = I$.

Finally, multiplying on the left by $\langle jm|$ and on the right by $|jm'\rangle$ on the right hand side of eq. (5) yields,

$$\langle jm|e^{-2\pi iJ_x/\hbar}|jm'\rangle = d_{mm'}^{(j)}(2\pi).$$

Hence, we have proved Property 5. In summary, we have established that

$$d^{(j)}(2\pi) = \begin{cases} I, & \text{for integral values, } j = 0, 1, 2, \ldots \\ -I, & \text{for half-integral values, } j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \end{cases}$$

That is, a $2\pi$ rotation returns the wave function of a boson to its original value, whereas one must perform a $4\pi$ rotation to return the wave function of a fermion to its original value.

**Property 6:** $d_{mm'}^{(j)}(\pi) = (-1)^{j-m}\delta_{m,-m'}$ for all values of $m$, $m'$ and $j$.

Starting from eq. (15), we choose $\vec{V} = \vec{J}$ and consider a rotation parameterized by $\hat{n} = \hat{y}$ and $\theta = \pi$. The corresponding $3 \times 3$ rotation matrix $R$ is given by [cf. eq. (20) of the class handout entitled *Three Dimensional Rotation Matrices*]:

$$R(\hat{y}, \pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

By choosing $\hat{u}$ to be the unit vector that points in the $x$, $y$, and $z$ directions, respectively, we obtain,

$$\exp(i\pi J_y/\hbar)J_x \exp(-i\pi J_y/\hbar) = -J_x, \quad (6)$$

$$\exp(i\pi J_y/\hbar)J_y \exp(-i\pi J_y/\hbar) = J_y, \quad (7)$$

$$\exp(i\pi J_y/\hbar)J_z \exp(-i\pi J_y/\hbar) = -J_z. \quad (8)$$

If we operate on the state $|jm\rangle$ with eq. (8), we obtain

$$J_z \left[ \exp(-i\pi J_y/\hbar)|jm\rangle \right] = -\hbar m \left[ \exp(-i\pi J_y/\hbar)|jm\rangle \right].$$
Since $\vec{J}^2$ and $J_y$ commute, it follows that

$$\vec{J}^2 \left[ \exp(-i \pi J_y/\hbar)|j \, m\rangle \right] = \hbar (j + 1) \left[ \exp(-i \pi J_y/\hbar)|j \, m\rangle \right].$$

Hence, we can conclude that\(^1\)

$$\exp(-i \pi J_y/\hbar)|j \, m\rangle = e^{i \alpha(j,m)}|j \, -m\rangle,$$  \tag{9}

where $e^{i \alpha(j,m)}$ is a complex phase that can depend in principle on $j$ and $m$.

To determine $e^{i \alpha(j,m)}$, we first note that eqs. (6) and (7) can be rewritten as

$$\exp(i \pi J_y/\hbar) J_{\pm} \exp(-i \pi J_y/\hbar) = -J_{\mp},$$  \tag{10}

where $J_{\pm} \equiv J_x \pm i J_y$. Applying eq. (10) to the state $|j \, m\rangle$, and making use of eq. (9),

$$J_{\pm} \exp(-i \pi J_y/\hbar)|j \, m\rangle = -\exp(-i \pi J_y/\hbar) J_{\mp} |j \, m\rangle = -\hbar [(j \pm m)(j \mp m + 1)]^{1/2} \exp(-i \pi J_y/\hbar)|j, m \mp 1\rangle.$$

However, eq. (9) also yields

$$J_{\pm} \exp(-i \pi J_y/\hbar)|j \, m\rangle = e^{i \alpha(j,m)} J_{\pm}|j \, -m\rangle = \hbar [(j \pm m)(j \mp m + 1)]^{1/2} e^{i \alpha(j,m)}|j \, -m \pm 1\rangle.$$

Consequently, $e^{i \alpha(j,m)} = -e^{i \alpha(j,m+1)}$. Given the value of $\alpha(j,j)$, one can obtain $\alpha(j,m)$ for $m = -j, -j + 1, \ldots, j - 1, j$,

$$e^{i \alpha(j,j-n)} = (-1)^n e^{i \alpha(j,j)}, \quad \text{for } n = 0, 1, 2, \ldots, 2j.$$

Thus, writing $n = j - m$, it follows that

$$e^{i \alpha(j,m)} = (-1)^{j-m} e^{i \alpha(j,j)}, \quad \text{for } m = -j, -j + 1, \ldots, j - 1, j.$$

Hence, we conclude that

$$\exp(-i \pi J_y/\hbar)|j \, m\rangle = e^{i \alpha(j,j)} (-1)^{j-m} |j \, -m\rangle,$$  \tag{11}

and

$$d_{m'}^j(m) = \langle j \, m' | \exp(-i \pi J_y/\hbar)|j \, m\rangle = e^{i \alpha(j,j)} (-1)^{j-m} \delta_{m,-m'}.$$  \tag{12}

In Appendix B, we demonstrate that $e^{i \alpha(j,j)} = 1$ for all values of $j$. Hence,

$$d_{m'}^j(m) = (-1)^{j-m} \delta_{m,-m'},$$

which completes the proof.

One immediate consequence of $e^{i \alpha(j,j)} = 1$ is that eq. (11) is now completely determined,

$$\exp(-i \pi J_y/\hbar)|j \, m\rangle = (-1)^{j-m} |j \, -m\rangle.$$  \tag{13}

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\(^1\)Since $|j \, m\rangle$ is a normalized state and $\exp(-i \pi J_y/\hbar)$ is a unitary operator, it follows that $\exp(-i \pi J_y/\hbar)|j \, m\rangle$ is also a normalized state, in which case the constant multiplying $|j \, -m\rangle$ in eq. (9) must be a complex phase.
Eq. (13) plays an important role in the behavior of the angular momentum state $|jm\rangle$ under a time reversal transformation.\(^2\)

**An explicit form for the Wigner d-matrix**

Julian Schwinger developed the connection between the algebra of angular momentum operators and the algebra of two independent harmonic oscillators, which you explored on problems 1 and 2 of Problem Set 4. Schwinger was then able to use this formalism to derive an explicit expression for $d_{m'm}(\theta)$. The derivation is given in Section 3.9 of Sakurai and Napolitano. For completeness, we provide the final expression here.

\[
d_{m'm}(\theta) = \sum_{k=k_{\text{min}}}^{k_{\text{max}}} (-1)^{k-m+m'} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!(j-k-m')!(k-m+m')!k!} \times \left( \cos \frac{\theta}{2} \right)^{2j-2k+m-m'} \left( \sin \frac{\theta}{2} \right)^{2k-m+m'},
\]

where $k$ is an integer such that

\[ k_{\text{min}} = \min\{0, m-m'\} \geq -2j, \quad k_{\text{max}} = \max\{0, j+m, j-m'\} \leq 2j. \]

That is, in eq. (14) we sum over integer values of $k$ such that none of the arguments of the factorials in the denominator are negative.

One can now use eq. (14) to check explicitly that Properties 1–6 are satisfied.

**APPENDIX A: Proof of Theorem 1**

Consider the unitary operator, $U[R] = e^{-i\hat{n} \cdot \vec{J}/\hbar}$. For any vector operator, $\vec{V} = (V_1, V_2, V_3)$,

\[
U^\dagger[R] V_i U[R] = R_{ij} V_j,
\]

where there is an implicit sum over the repeated index $j$. In eq. (15), $R_{ij}$ are the matrix elements of the matrix $R$ corresponding to a rotation by an angle $\theta$ about an axis $\hat{n}$.\(^3\)

**Theorem 1:** Suppose that $R$ rotates a unit vector $\hat{u}$ into $\hat{u}'$; i.e., $\hat{u}' = R\hat{u}$. Then,

\[
U[R] \vec{J} \cdot \hat{u} U^\dagger[R] = \vec{J} \cdot \hat{u}'.
\]

**Proof of Theorem 1:** Since $\vec{J}$ is a vector operator, it follows from eq. (15) that

\[
U^\dagger[R] J_i U[R] = R_{ij} J_j.
\]

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\(^2\)For details, see the class handout entitled *Implications of Time-Reversal Symmetry in Quantum Mechanics.*

\(^3\)See, e.g., eq. (3.11.3) of Sakurai and Napolitano.
Multiplying eq. (17) by \( \hat{u}' \), we obtain
\[
U^\dagger[R] \mathbf{J} \cdot \hat{u}' U[R] = R_{ij} \hat{u}_i J_j .
\] (18)

But \( R \) is a real orthogonal matrix, which implies that \( R^T = R^{-1} \). Hence,
\[
R_{ij} \hat{u}_i' = R^T_{ji} \hat{u}_i = (R^{-1} \hat{u}')_j = \hat{u}_j ,
\] (19)

where we used \( \hat{u} = R^{-1} \hat{u}' \), as specified by Theorem 1. Inserting the result of eq. (19) back into eq. (18) yields,
\[
U^\dagger[R] \mathbf{J} \cdot \hat{u}' U[R] = \mathbf{J} \cdot \hat{u} .
\] (20)

Finally, we shall make use of the fact that \( U[R] \) is unitary, i.e., \( U[R] U^\dagger[R] = \mathbf{I} \), where \( \mathbf{I} \) is the identity operator. Hence, if one multiplies eq. (20) on the left by \( U[R] \) and on the right by \( U^\dagger[R] \), one obtains eq. (16).

**APPENDIX B: Proof that \( e^{i\alpha(j,j)} = 1 \)**

We shall evaluate \( e^{i\alpha(j,j)} \) by considering the Clebsch-Gordan series for the product of two \( d \)-functions [cf. eq. (14) of the class handout entitled *Clebsch-Gordan coefficients and the tensor spherical harmonics*]. Choosing the appropriate values for the relevant parameters,
\[
d^{(j_1)}_{-j_1,j_1} (\pi) d^{(j_2)}_{-j_2,j_2} (\pi) = \langle j_1 j_2 ; -j_1 , -j_2 | j_1 j_2 ; j_1 + j_2 , -(j_1 + j_2) \rangle \langle j_1 j_2 ; j_1 j_2 ; j_1 + j_2 , j_1 + j_2 \rangle 
\times d^{(j_1+j_2)}_{-j_1-j_2,j_1+j_2} (\pi) .
\] (21)

The Clebsch-Gordan factors above are trivial. In particular, the relation,
\[
\langle j_1 j_2 ; j_1 j_2 ; j_1 + j_2 , j_1 + j_2 \rangle = 1
\]
is the convention used to fix the phases of the Clebsch-Gordan coefficients. Moreover, one can show that the Clebsch-Gordan coefficients satisfy the following relation,
\[
\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m \rangle = (-1)^{j_1 + j_2 - j} \langle j_1 j_2 ; -m_1 , -m_2 | j_1 j_2 ; j , -m \rangle.
\]

It therefore follows that
\[
\langle j_1 j_2 ; j_1 j_2 ; j_1 + j_2 , j_1 + j_2 \rangle = \langle j_1 j_2 ; -j_1 , -j_2 | j_1 j_2 ; j_1 + j_2 , -(j_1 + j_2) \rangle = 1 .
\] (22)

Hence, using eq. (12), we conclude that
\[
e^{i\alpha(j,j)} e^{i\alpha(j,j)} = e^{i\alpha(j_1+j_2,j_1+j_2)} .
\] (23)

Finally, we consider the case of \( j = \frac{1}{2} \). Then,\(^4\)
\[
d^{(1/2)}_{m'm} (\pi) = \langle \frac{1}{2} m' | \exp(-i\pi J_y/\hbar) | \frac{1}{2} m \rangle = e^{-i\pi \sigma_y/2} = -i \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\] (24)

This means that \( d^{(1/2)}_{1, -\frac{1}{2}} (\pi) = -1 \). Comparing with eq. (12), it follows that \( e^{i\alpha(\frac{1}{2},\frac{1}{2})} = 1 \). In light of eq. (23), we conclude that
\[
e^{i\alpha(j,j)} = 1 , \quad \text{for all values of } j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots
\] (25)

\(^4\)In deriving eq. (24), we employed \( \exp(-i\theta \hat{\mathbf{n}} \cdot \hat{\mathbf{\sigma}}) = \mathbf{I} \cos(\frac{1}{2} \theta) - i \hat{\mathbf{n}} \cdot \hat{\mathbf{\sigma}} \sin(\frac{1}{2} \theta) \) [cf. part (b)-(iii) of problem 7 on Problem Set 1] with \( \hat{\mathbf{n}} = \hat{\mathbf{g}} \) and \( \theta = \pi \).