DUE: THURSDAY, MAY 24, 2012

1. (a) Consider the Born approximation as the first term of the Born series. Show that:

(i) the Born approximation for the *forward* scattering amplitude [i.e. at $\theta = 0$] is purely real, and therefore

(ii) the Born approximation fails to satisfy the optical theorem.

Do not assume that the potential is spherically symmetric. However, you may assume that the potential is hermitian.

(b) Consider the Yukawa potential:

$$V = -\frac{ge^{-\mu r}}{r}\,.$$

In class, we computed the (first) Born approximation to the scattering amplitude. Consider now the second Born approximation; i.e., the second term in the Born series. Compute the scattering amplitude *in the forward direction*, $\theta = 0$, in the second Born approximation.[1] Check to see whether the optical theorem is now satisfied.

HINT: You will need to evaluate $\langle\vec{\boldsymbol{k}}|V(E - H_0 + i\epsilon)^{-1}V|\vec{\boldsymbol{k}}\rangle$, where $H_0 = \vec{\boldsymbol{P}}^{\,2}/(2m)$. In class, we inserted a complete set of position eigenstates in order to convert this matrix element as a multiple integral over $d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2$. However, it is easier to evaluate the matrix element by inserting a complete set of momentum eigenstates, $|\boldsymbol{k}'\rangle$. You will then only have to evaluate an integral over $d^3\boldsymbol{k}'$.

(c) Compare the magnitudes of the first and second terms of the Born series for the forward scattering amplitude. What condition do you find if you require the second term in the Born series to be smaller than the first term? Compare this condition with the one you would get for the validity of the Born approximation based on the formula derived in class.

(d) Using the first Born approximation for the scattering amplitude, compute the s and p wave phase shifts. Under what circumstances does the s-wave phase shift dominate? Is the Born approximation valid in this case?

[1] Do not attempt to compute the scattering amplitude in the second Born approximation for $\theta \neq 0$. It is extremely messy!

2. Consider the case of low-energy scattering from a spherical delta-function shell,

$$V(r) = V_0 \delta(r - a)\,,$$

where V_0 and a are constants. Calculate the scattering amplitude, $f(\theta)$, the differential cross-section and the total cross-section, under the assumption that $ka \ll 1$, so that only s-wave scattering is important.

HINT: Solve the time-independent Schrodinger equation exactly in the case of $\ell = 0$ for the radial wave function, $R(r) \equiv u(r)/r$. Consider separately the cases of $r < a$ and $r > a$. By integrating the Schrodinger equation from $r = a - \epsilon$ to $a + \epsilon$ (where $0 < \epsilon \ll 1$), show that

$$\left[\left.\frac{du}{dr}\right|_{a+\epsilon} - \left.\frac{du}{dr}\right|_{a-\epsilon} \right] = \frac{2mV_0}{\hbar^2} u(a)\,.$$

Inserting your explicit solutions for $u(r)$ for the two cases $r < a$ and $r > a$ into the equation above, you should be able to determine the s-wave phase shift. In particular, find an expression for $\tan \delta_0$ in terms of V_0 and the wave number k. Evaluate the phase shift in the limit of $ka \ll 1$ to simplify your expression and then complete the problem.

3. Low energy scattering is parameterized by two parameters: the *scattering length* a and the *effective range* r_0. In this problem, you will verify this statement.

(a) Show that in the limit of $k \to 0$, (more precisely, for $kb \ll 1$, where b is the range of the potential $V(r)$, i.e. $V(r) \approx 0$ for $r > b$):

$$k \cot \delta_0(k) = -1/a\,,$$

where a is a parameter with units of length and $\delta_0(k)$ is the s-wave phase shift. What is the cross-section in the limit of zero energy?

(b) Obtain an expression for the partial wave amplitude:

$$a_0(k) = \frac{e^{2i\delta_0} - 1}{2ik}$$

in the limit of $k \to 0$, using the results of part (a). For what values of k does $a_0(k)$ have poles? Can one associate these poles with the existence of bound states? What is the relation between the bound state energy E_b, and the scattering length? Obtain an expression for the total cross-section as a function of the energy $E = \hbar^2 k^2/(2m)$, assuming that the low energy approximation is still valid. Express your result in terms of E_b.

(c) Show that by considering the radial integral equation:

$$\text{(i)} \quad G^{\ell}_{-k}(r, r') = G^{\ell}_{k}(r, r')^*\,,$$

$$\text{(ii)} \quad A_\ell(-k, r) = (-1)^\ell A_\ell(k, r)^*\,,$$

$$\text{(iii)} \quad \exp(2i\delta_\ell(k)) = \exp(-2i\delta_\ell(-k))\,.$$

You will need (i) and (ii) to prove (iii). Using (iii), show that $\cot\delta_\ell(k)$ is an odd function of k. Assuming it has a power series about $k = 0$, show that:

$$k^{2\ell+1}\cot\delta_\ell(k) = \frac{-1}{a_\ell} + \tfrac{1}{2}r_\ell k^2 + \mathcal{O}(k^4)\,.$$

For the case of $\ell = 0$, show that a_0 and r_0 each have dimensions of length.

(d) Obtain expressions for the partial wave amplitude $a_0(k)$ and low energy cross-section in terms of the scattering length $a \equiv a_0$ and the effective range r_0 which appear in the expansion obtained in part (c).

4. In this problem, I will lead you through the steps involved in solving the scattering problem for a charged particle subject to the Coulomb potential. We shall first solve the Schrodinger equation,

$$\left(-\frac{\hbar^2\vec{\boldsymbol{\nabla}}^2}{2m} - \frac{Ze^2}{r}\right)\psi(\vec{\boldsymbol{x}}) = E\psi(\vec{\boldsymbol{x}})\,, \qquad \text{for} \quad E > 0\,. \tag{1}$$

(a) Define the dimensionless quantity,

$$\gamma \equiv -\frac{mZe^2}{\hbar^2 k}\,.$$

Let $\psi(\vec{\boldsymbol{x}}) = e^{i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{r}}}X(\vec{\boldsymbol{x}})$, with $E \equiv \hbar^2k^2/(2m)$. Inserting this result into eq. (1), derive the following differential equation for $X(\vec{\boldsymbol{x}})$,

$$\vec{\boldsymbol{\nabla}}^2 X + (2i\vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{\nabla}})X - \frac{2\gamma k}{r}X = 0\,, \tag{2}$$

where $r \equiv |\vec{\boldsymbol{x}}|$.

(b) Set up the coordinate system so that the beam is incoming along the z-direction, so that $\vec{\boldsymbol{k}} = k\hat{\boldsymbol{z}}$. Define a new variable,

$$u = kr - \vec{\boldsymbol{k}}\cdot\vec{\boldsymbol{x}} = kr(1-\cos\theta)\,. \tag{3}$$

Assume that the form of the wave function can be chosen such that $\psi(\vec{\boldsymbol{x}}) = e^{ikz}X(u)$. That is $X(\vec{\boldsymbol{x}})$ is a function of u alone.[2] In this case, show that eq. (2) now becomes

$$u\frac{d^2X}{du^2} + (1-iu)\frac{dX}{du} - \gamma X = 0\,. \tag{4}$$

<u>HINT:</u> Since $X(u)$ is an implicit function of $\vec{\boldsymbol{x}}$ [cf. eq. (3)], you can use the chain rule to express $\vec{\boldsymbol{\nabla}}X$ and $\vec{\boldsymbol{\nabla}}^2X$ in terms of dX/du and d^2X/du^2. The analysis is straightforward in Cartesian coordinates.

[2] Indeed, one can prove that this is the case for the scattering problem consisting of an incoming wave along the z direction and an outgoing spherical wave. For the present purposes, you can assume that this is true.

(c) Solve eq. (4) subject to the boundary condition that the solution for $\psi(\vec{\boldsymbol{x}})$ must be non-singular at the origin. Feel free to consult your favorite book on special functions of mathematical physics.[3] Show that the solution to eq. (4) is a confluent hypergeometric function,

$$X(u) = C\,{}_1F_1(-i\gamma, 1; iu)\,,$$

where C is a constant to be determined.

(d) To determine the constant C, consider the asymptotic behavior of $\psi(\vec{\boldsymbol{x}})$ as $r \to \infty$. Show that one can choose C such that:

$$\psi(\vec{\boldsymbol{x}}) = \psi_{inc}(\vec{\boldsymbol{x}}) + \psi_{sc}(\vec{\boldsymbol{x}})\,,$$

where the incident wave function is

$$\psi_{inc}(\vec{\boldsymbol{x}}) = \exp\left\{ikz + i\gamma \ln[k(r-z)]\right\}\left(1 + \frac{\gamma^2}{ik(r-z)}\right)\,,$$

with $z = r\cos\theta$, and the scattered wave function is

$$\psi_{sc}(\vec{\boldsymbol{x}}) = \frac{\exp\left\{ikr - i\gamma\ln[k(r-z)]\right\}}{ik(r-z)}\,\frac{\Gamma(1+i\gamma)}{\Gamma(-i\gamma)}\,.$$

<u>*HINT:*</u> You will need to find the asymptotic expansion for the confluent hypergeometric function ${}_1F_1(a, b; x)$ in an appropriate reference book (e.g., see footnote 3).

(e) Define the Coulomb scattering amplitude by:

$$\psi_{sc}(\vec{\boldsymbol{x}}) = \frac{e^{i[kr - \gamma\ln(2kr)]}}{r} f_c(\theta), \qquad \text{as } r \to \infty\,.$$

Obtain an explicit expression for $f_c(\theta)$. Express your answer in terms of the pure phase factor, $e^{2i\delta_0} \equiv \Gamma(1+i\gamma)/\Gamma(1-i\gamma)$.

(f) Compute the probability currents j_{inc} and j_{sc} and following the same procedure used in class, show that:

$$\frac{d\sigma}{d\Omega} = |f_c(\theta)|^2\,.$$

Using the expression for $f_c(\theta)$ obtained in part (d), compute the differential cross section and verify that your result coincides with the *Rutherford scattering formula.* Show that the total cross section σ diverges.

(g) Show that the poles of $f_c(\theta)$ correspond to the bound states of a hydrogenic atom with atomic number Z.

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[3] One of my favorites is N.N. Lebedev, *Special Functions and their Applications* (Dover Publications, Inc., New York, NY, 1972). The Dover books are generally not very expensive, and this book in particular is well worth the investment. Of course, you can solve eq. (4) using the standard series technique for solving differential equations, but this will require an additional investment in time.

5. Tritium (the isotope H^3), which is initially in its ground state, undergoes spontaneous beta decay, emitting an electron of maximum energy of about 17 keV. The nucleus remaining is He^3.

(a) Calculate the probability that the electron of this ion is left in a quantum state of principal quantum number $n = 2$.

(b) What is the probability that the electron of this ion is left in quantum state with $\ell \neq 0$?

In this problem, you should neglect nuclear recoil. Note the energy of the emitted electron. What is the relevant approximation? Explain.