This is a take home exam. You may refer to the textbooks by Shankar and Baym, the class handouts and any third quantum mechanics textbook of your choosing. (If you do consult a third text, please indicate which one you used.) Any reference for integrals or other mathematical facts, and any personal handwritten notes are also OK. However, you should not collaborate with anyone else. The point value of each problem is indicated in the square brackets below; each part is worth 10 points. Completed exams should be delivered to my campus mailbox no later than 5 pm on Wednesday May 16, 2012.

There is no need to rederive results that have been previously obtained in the textbook, the class notes or the class handouts. But if you make use of any previously derived result, please cite the source of the result.

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1. [20] Consider the quantum mechanical position operator $\vec{\boldsymbol{x}}$. Denoting $r = |\vec{\boldsymbol{x}}|$, we can define the unit position operator,

$$\hat{\boldsymbol{r}} \equiv \frac{\vec{\boldsymbol{x}}}{r} \,.$$

(a) Evaluate the matrix element

$$\langle \ell' \, m' | \, \hat{\boldsymbol{r}} \, | \ell \, m \rangle \,, \tag{1}$$

as a function of ℓ, ℓ', m and m', where $|\ell \, m\rangle$ is a simultaneous eigenstate of $\vec{\boldsymbol{L}}^2$ and L_z, and $\vec{\boldsymbol{L}}$ is the orbital angular momentum operator. Find the values of ℓ, ℓ', m and m' for which $\langle \ell' \, m' | \, \hat{\boldsymbol{r}} \, | \ell \, m \rangle \neq 0$.

<u>*HINT:*</u> To evaluate eq. (1), work in the coordinate representation and write the components of $\hat{\boldsymbol{r}}$ in terms of the appropriate spherical harmonics. It is then possible to explicitly evaluate the corresponding integral and express the result as a product of two Clebsch-Gordon coefficients. Cite any formulae that you use to perform the integration.

(b) The Wigner-Eckart theorem applied to eq. (1) introduces the reduced matrix element,

$$\langle \ell' \| r^{(1)} \| \ell \rangle \,, \tag{2}$$

where $r^{(1)}_q$ ($q = -1, 0, 1$) are the elements of the vector $\hat{\boldsymbol{r}}$ in the spherical basis. Using the result of part (a), find an explicit expression for the reduced matrix element $\langle \ell' \| r^{(1)} \| \ell \rangle$ as a function of ℓ and ℓ'. In obtaining your final expression, you should evaluate any Clebsch-Gordon coefficient that appears as a function of its parameters. Explain the behavior of your expression in the case of $\ell = \ell'$.

2. [40] The potential energy of a quantum mechanical elastic ball bouncing vertically on the floor is is given by:

$$V(z) = \begin{cases} mgz\,, & \text{for } z > 0\,, \\ \infty\,, & \text{for } z = 0\,. \end{cases}$$

Treat this as a one-dimensional problem.

(a) Using the WKB approximation, determine the (unnormalized) bound state wave functions of the quantum ball.

(b) Using the WKB approximation, evaluate the bound state energy levels of the quantum ball (corresponding to the ground state and all excited states). Express each bound state energy as a numerical factor multiplied by an appropriate combination of m, g and fundamental constants.

(c) Calculate the ground state energy using the variational principle by employing the *normalized* trial wave function,

$$\Psi_0(z) = Az\,e^{-az}\,, \qquad \text{for } z \geq 0\,.$$

The constant A should be chosen such that $\Psi_0(z)$ is properly normalized. Treat a as a variational parameter. Express the ground state energy as a numerical factor multiplied by an appropriate combination of m, g and fundamental constants.

(d) In parts (b) and (c), you obtained two different estimates for the ground state energy. Which of the two is the more accurate? Explain.

3. [40] In problem 3 of problem set 3, we considered the hydrogen atom in a uniform magnetic field which points in the $\hat{\boldsymbol{z}}$-direction. The energy levels were obtained as a function of B. This is the well-known Zeeman effect. However, the terms in the Hamiltonian that are quadratic in B were neglected. We now want to see the effect of including the latter. To make the analysis simple, you may ignore the effects of electron and nuclear spin (i.e., the fine-structure and hyperfine structure can be neglected).

(a) For simplicity, we shall first consider the $n = 1$ ground state of hydrogen. Evaluate the first-order energy shift due to a uniform magnetic field $\vec{\boldsymbol{B}} = B\hat{\boldsymbol{z}}$, assuming that the term in the Hamiltonian that is quadratic in B can be neglected.

(b) Compute the *quadratic* Zeeman effect for the ground-state hydrogen atom, due to the usually neglected $e^2\vec{\boldsymbol{A}}^{\boldsymbol{2}}/(2mc^2)$ term in the Hamiltonian taken to first order in perturbation theory. Assume that the external magnetic field is uniform and points in the $\hat{\boldsymbol{z}}$-direction. Writing the energy shift as $\Delta E \equiv -\frac{1}{2}\chi\vec{\boldsymbol{B}}^{\boldsymbol{2}}$, obtain an expression for the diamagnetic susceptibility, χ.

(c) How large a magnetic field is required in order that the two contributions obtained in parts (a) and (b) are of the same order of magnitude?

<u>*NOTE:*</u> The rest-mass of the electron is $m_e c^2 \simeq 5.11 \times 10^5$ eV. Other useful numbers are:

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137}\,, \qquad \mu_B = \frac{e\hbar}{2m_e c} = 5.788 \times 10^{-9}\ \text{eV/gauss}\,.$$

(d) In the ground state of helium, the total L_z and S_z vanishes. Hence, at leading order *only* the quadratic Zeeman effect is relevant. Compute the diamagnetic susceptibility of the helium atom in its ground state, and compare with the measured value of -1.88×10^{-6} cm^3/mole.

<u>*HINT:*</u> For the ground state helium wave function, use the wave function obtained in class by the variational method [recall that the variational ground state wave function of helium is the product of two ground-state hydrogen atom wave functions, $\psi(\vec{\boldsymbol{r}}_1, \vec{\boldsymbol{r}}_2) = \psi_{100}(\vec{\boldsymbol{r}}_1)\psi_{100}(\vec{\boldsymbol{r}}_2)$, where

$$\psi_{100}(\vec{\boldsymbol{r}}) = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}\,,$$

with $Z_{\text{eff}} = 27/16$]. Thus, with hardly any additional calculation, one can write down the expression for ΔE by inspection using the results of part (b). Finally, recall that one mole consists of 6.022×10^{23} helium atoms.