Spin-1/2 fermions in quantum field theory

First, recall that 4-vectors transform under Lorentz transformations, $\Lambda^\mu{}_\nu$, as $p'^{\,\mu} = \Lambda^\mu{}_\nu p^\nu$, where $\Lambda \in$ SO(3,1) satisfies $\Lambda^\mu{}_\nu g_{\mu\rho}\Lambda^\rho{}_\lambda = g_{\nu\lambda}$.* A Lorentz transformation corresponds to a rotation by θ about an axis $\hat{\boldsymbol{n}}$ $[\vec{\boldsymbol{\theta}} \equiv \theta\hat{\boldsymbol{n}}]$ and a boost, $\vec{\boldsymbol{\zeta}} = \hat{\boldsymbol{v}}\tanh^{-1}|\vec{\boldsymbol{v}}|$, where $\vec{\boldsymbol{v}}$ is the corresponding velocity. Under the same Lorentz transformation, a generic field transforms as:

$$\Phi'(x') = M_R(\Lambda)\Phi(x)\,,$$

where $M_R \equiv \exp\left(-i\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{J}} - i\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{K}}\right)$ are $N \times N$ representation matrices of the Lorentz group. Defining $\vec{\boldsymbol{J}}_+ \equiv \frac{1}{2}(\vec{\boldsymbol{J}} + i\vec{\boldsymbol{K}})$ and $\vec{\boldsymbol{J}}_- \equiv \frac{1}{2}(\vec{\boldsymbol{J}} - i\vec{\boldsymbol{K}})$,

$$[J^i_\pm\,,\,J^j_\pm] = i\epsilon^{ijk}J^k_\pm\,, \qquad [J^i_\pm\,,\,J^j_\mp] = 0\,.$$

Thus, the representations of the Lorentz algebra are characterized by (j_1, j_2), where the j_i are half-integers. $(0,0)$ is a scalar and $(\frac{1}{2},\frac{1}{2})$ is a four-vector. Of interest to us here are the spinor representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$.

*In our conventions, $g_{\mu\nu} = \mathrm{diag}(1\,,\,-1\,,\,-1\,,\,-1)$.

$(\frac{1}{2},0)$: $\quad M=\exp\left(-\frac{i}{2}\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}-\frac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)$, but also $(M^{-1})^{\mathsf{T}}=i\sigma^2 M(i\sigma^2)^{-1}$

$(0,\frac{1}{2})$: $[M^{-1}]^\dagger=\exp\left(-\frac{i}{2}\vec{\boldsymbol{\theta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}+\frac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right)$, but also $M^*=i\sigma^2[M^{-1}]^\dagger(i\sigma^2)^{-1}$

since $(i\sigma^2)\vec{\boldsymbol{\sigma}}(i\sigma^2)^{-1}=-\vec{\boldsymbol{\sigma}}^*=-\vec{\boldsymbol{\sigma}}^{\mathsf{T}}$

Transformation laws of 2-component fields

$$\begin{aligned}
\xi'_\alpha &= M_\alpha{}^\beta\,\xi_\beta\,,\\
\xi'^{\,\alpha} &= [(M^{-1})^{\mathsf{T}}]^\alpha{}_\beta\,\xi^\beta\,,\\
\xi'^{\,\dagger\dot\alpha} &= [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}\,\xi^{\dagger\dot\beta}\,,\\
\xi'^{\,\dagger}_{\dot\alpha} &= [M^*]_{\dot\alpha}{}^{\dot\beta}\xi^\dagger_{\dot\beta}\,.
\end{aligned}$$

We use $i\sigma^2=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)=\epsilon^{\alpha\beta}=\epsilon^{\dot\alpha\dot\beta}$ and $(i\sigma^2)^{-1}=-i\sigma^2=\epsilon_{\alpha\beta}=\epsilon_{\dot\alpha\dot\beta}$ to raise and lower spinor indices: $\xi^\alpha=\epsilon^{\alpha\beta}\,\xi_\beta\,;\ \xi^\dagger_{\dot\alpha}=\epsilon_{\dot\alpha\dot\beta}\,\xi^{\dagger\,\dot\beta}$, etc. Dotted and undotted indices are related by hermitian conjugation: $\xi^\dagger_{\dot\alpha}\equiv(\xi_\alpha)^\dagger$.

Finally, we introduce the σ-matrices:

$$\sigma^{\mu}_{\alpha\dot\beta} = (I_2\,;\,\vec{\boldsymbol\sigma})\,, \qquad \overline{\sigma}^{\mu\,\dot\alpha\beta} = (I_2\,;\,-\vec{\boldsymbol\sigma})\,,$$

where I_2 is the 2×2 identity matrix. The spinor index structure derives from the relations:

$$M^\dagger\overline{\sigma}^\mu M = \Lambda^\mu{}_\nu\overline{\sigma}^\nu\,, \qquad M^{-1}\sigma^\mu(M^{-1})^\dagger = \Lambda^\mu{}_\nu\sigma^\nu\,.$$

For example, $(M^\dagger)^{\dot\alpha}{}_{\dot\beta}\overline{\sigma}^{\mu\dot\beta\gamma}M_\gamma{}^\delta = \Lambda^\mu{}_\nu\overline{\sigma}^{\nu\,\dot\alpha\delta}$. Note that the matrix M and its inverse have the same spinor index structure. Some useful identities:

$$\sigma^\mu_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\overline{\sigma}^{\mu\,\dot\beta\beta}\,, \qquad \overline{\sigma}^{\mu\,\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}\,.$$

The utility of σ^μ is that Lorentz 4-vectors can be built from spinor bilinears:

$$\begin{aligned}\chi'^{\,\alpha}(x')\sigma^\mu_{\alpha\dot\beta}\xi'^{\,\dagger\,\dot\beta}(x') &= \chi^\alpha(x)[M^{-1}\sigma^\mu(M^{-1})^\dagger]_{\alpha\dot\beta}\xi^{\dagger\,\dot\beta}(x)\\ &= \Lambda^\mu{}_\nu\,\chi(x)^\alpha\sigma^\nu_{\alpha\dot\beta}\xi^{\dagger\,\dot\beta}(x)\,.\end{aligned}$$

Spinor indices can be suppressed as long as one adopts a summation convention where we contract indices as follows:

$$ {}^{\alpha}{}_{\alpha} \qquad \text{and} \qquad {}_{\dot\alpha}{}^{\dot\alpha} \,. $$

For example,

$$ \begin{aligned} \xi\eta &\equiv \xi^\alpha \eta_\alpha , \\ \xi^\dagger \eta^\dagger &\equiv \xi^\dagger_{\dot\alpha} \eta^{\dagger\,\dot\alpha} , \\ \xi^\dagger \overline{\sigma}^\mu \eta &\equiv \xi^\dagger_{\dot\alpha} \overline{\sigma}^{\mu\dot\alpha\beta} \eta_\beta , \\ \xi\sigma^\mu \eta^\dagger &\equiv \xi^\alpha \sigma^\mu_{\alpha\dot\beta} \eta^{\dagger\,\dot\beta} . \end{aligned} $$

In particular, for anticommuting spinors,

$$ \eta\xi \equiv \eta^\alpha \xi_\alpha = -\xi_\alpha \eta^\alpha = +\xi^\alpha \eta_\alpha = \xi\eta \,. $$

Note the behavior of spinor products under hermitian conjugation:

$$ (\xi\Sigma\eta)^\dagger = \eta^\dagger \Sigma_r \xi^\dagger \,, \quad (\xi\Sigma\eta^\dagger)^\dagger = \eta\Sigma_r \xi^\dagger \,, \quad (\xi^\dagger\Sigma\eta)^\dagger = \eta^\dagger \Sigma_r \xi \,, $$

where in each case Σ stands for any sequence of alternating σ and $\overline{\sigma}$ matrices, and Σ_r is obtained from Σ by reversing the order of all of the σ and $\overline{\sigma}$ matrices.

From the sigma matrices, one can construct the antisymmetrized products:

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{i}{4}\left(\sigma^\mu_{\alpha\dot\gamma}\overline{\sigma}^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\overline{\sigma}^{\mu\dot\gamma\beta}\right) ,$$

$$(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \equiv \frac{i}{4}\left(\overline{\sigma}^{\mu\dot\alpha\gamma}\sigma^\nu_{\gamma\dot\beta} - \overline{\sigma}^{\nu\dot\alpha\gamma}\sigma^\mu_{\gamma\dot\beta}\right) .$$

We may write the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ transformation matrices, respectively, as:

$$M = \exp\left(-\tfrac{i}{2}\theta^{\mu\nu}\sigma_{\mu\nu}\right) ,$$

$$(M^{-1})^\dagger = \exp\left(-\tfrac{i}{2}\theta^{\mu\nu}\overline{\sigma}_{\mu\nu}\right) ,$$

where $\theta^{\mu\nu}$ is antisymmetric, with $\theta^{ij} = \epsilon^{ijk}\theta^k$ and $\theta^{i0} = \zeta^i$. Consider a pure boost of an on-shell spinor from its rest frame to the frame where $p^\mu = (E_{\boldsymbol{p}}\,,\,\vec{\boldsymbol{p}})$, with $E_{\boldsymbol{p}} = (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$. Setting $\theta^{ij} = 0$,

$$M = \exp\left(-\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \sqrt{\frac{p\cdot\sigma}{m}} = \frac{(E_{\boldsymbol{p}} + m)I_2 - \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}}{\sqrt{2m(E_{\boldsymbol{p}} + m)}} ,$$

$$(M^{-1})^\dagger = \exp\left(+\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}\right) = \sqrt{\frac{p\cdot\overline{\sigma}}{m}} = \frac{(E_{\boldsymbol{p}} + m)I_2 + \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}}{\sqrt{2m(E_{\boldsymbol{p}} + m)}} .$$

Useful identities and Fierz relations

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = -\delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\delta\delta_\beta^\gamma, \qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta} = -\delta_{\dot\alpha}^{\dot\gamma}\delta_{\dot\beta}^{\dot\delta} + \delta_{\dot\alpha}^{\dot\delta}\delta_{\dot\beta}^{\dot\gamma},$$

$$\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}_\mu^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta},$$

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}, \qquad \overline{\sigma}^{\mu\dot\alpha\alpha}\overline{\sigma}_\mu^{\dot\beta\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta},$$

$$[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha^\beta,$$

$$[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta_{\dot\beta}^{\dot\alpha},$$

$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho = g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa,$$

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu + g^{\nu\rho}\overline{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa,$$

where $\epsilon^{0123} = -\epsilon_{0123} = +1$ in our conventions. Computations of cross sections and decay rates often require traces of alternating products of σ and $\overline{\sigma}$ matrices:

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu] = \mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu] = 2g^{\mu\nu},$$

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}\right),$$

$$\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}\right).$$

Traces involving an odd number of σ and $\overline{\sigma}$ matrices cannot arise, since there is no way to connect the spinor indices consistently.

We shall deal with both commuting and anticommuting spinors, which we shall denote generically by z_i. Then, the following identities hold

$$\begin{aligned}
z_1 z_2 &= -(-1)^A z_2 z_1 \\
z_1^\dagger z_2^\dagger &= -(-1)^A z_2^\dagger z_1^\dagger \\
z_1 \sigma^\mu z_2^\dagger &= (-1)^A z_2^\dagger \overline{\sigma}^\mu z_1 \\
z_1 \sigma^\mu \overline{\sigma}^\nu z_2 &= -(-1)^A z_2 \sigma^\nu \overline{\sigma}^\mu z_1 \\
z_1^\dagger \overline{\sigma}^\mu \sigma^\nu z_2^\dagger &= -(-1)^A z_2^\dagger \overline{\sigma}^\nu \sigma^\mu z_1^\dagger \\
z_1^\dagger \overline{\sigma}^\mu \sigma^\rho \overline{\sigma}^\nu z_2 &= (-1)^A z_2 \sigma^\nu \overline{\sigma}^\rho \sigma^\mu z_1^\dagger \,,
\end{aligned}$$

where $(-1)^A = +1[-1]$ for commuting [anticommuting] spinors. Finally, the Fierz identities are given by:

$$\begin{aligned}
(z_1 z_2)(z_3 z_4) &= -(z_1 z_3)(z_4 z_2) - (z_1 z_4)(z_2 z_3) \,, \\
(z_1^\dagger z_2^\dagger)(z_3^\dagger z_4^\dagger) &= -(z_1^\dagger z_3^\dagger)(z_4^\dagger z_2^\dagger) - (z_1^\dagger z_4^\dagger)(z_2^\dagger z_3^\dagger) \,, \\
(z_1 \sigma^\mu z_2^\dagger)(z_3^\dagger \overline{\sigma}_\mu z_4) &= -2(z_1 z_4)(z_2^\dagger z_3^\dagger) \,, \\
(z_1^\dagger \overline{\sigma}^\mu z_2)(z_3^\dagger \overline{\sigma}_\mu z_4) &= \quad 2(z_1^\dagger z_3^\dagger)(z_4 z_2) \,, \\
(z_1 \sigma^\mu z_2^\dagger)(z_3 \sigma_\mu z_4^\dagger) &= \quad 2(z_1 z_3)(z_4^\dagger z_2^\dagger) \,.
\end{aligned}$$

Free field theories involving fermions

The $(\frac{1}{2},0)$ spinor field $\xi_\alpha(x)$ describes a neutral Majorana fermion. The free-field Lagrangian is:

$$\mathscr{L} = i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi - \tfrac{1}{2}m(\xi\xi + \xi^\dagger\xi^\dagger)\,,$$

which is hermitian up to a total divergence since we can rewrite the above Lagrangian as

$$\mathscr{L} = \tfrac{1}{2}i\xi^\dagger\overline{\sigma}^\mu\overleftrightarrow{\partial}_\mu\xi - \tfrac{1}{2}m(\xi\xi + \xi^\dagger\xi^\dagger) + \text{total divergence}\,,$$

where $\xi^\dagger\overline{\sigma}^\mu\overleftrightarrow{\partial}_\mu\xi \equiv \xi^\dagger\overline{\sigma}^\mu(\partial_\mu\xi) - (\partial_\mu\xi)^\dagger\overline{\sigma}^\mu\,\xi$.

Generalizing to a multiplet of two-component fermion fields, $\xi_{\alpha i}(x)$, labeled by flavor index i.

$$\mathscr{L} = i\hat{\xi}^{\dagger\,i}\overline{\sigma}^\mu\partial_\mu\hat{\xi}_i - \tfrac{1}{2}M^{ij}\hat{\xi}_i\hat{\xi}_j - \tfrac{1}{2}M_{ij}\hat{\xi}^{\dagger\,i}\hat{\xi}^{\dagger\,j}\,,$$

where hermiticity implies that $M_{ij} \equiv (M^{ij})^*$ is a *complex symmetric* matrix.

To identify the physical fermion fields, we express the so-called *interaction eigenstate fields*, $\hat{\xi}_{\alpha i}(x)$, in terms of *mass-eigenstate fields* $\xi(x) = \Omega^{-1}\hat{\xi}(x)$, where Ω is unitary and chosen such that

$$\Omega^{\mathsf{T}} M\,\Omega = \boldsymbol{m} = \mathrm{diag}(m_1, m_2, \ldots),$$

where the m_i are non-negative real numbers. In linear algebra, this is called the Takagi-diagonalization of a complex symmetric matrix M. To compute the values of the diagonal elements of $\boldsymbol{m}$, one may simply note that

$$\Omega^{\mathsf{T}} M M^\dagger \Omega^* = \boldsymbol{m}^2.$$

$MM^\dagger$ is hermitian, and thus it can be diagonalized by a unitary matrix. Thus, the m_i of the Takagi diagonalization are the non-negative square-roots of the eigenvalues of $MM^\dagger$. In terms of the mass eigenstate fields,

$$\mathscr{L} = i\xi^{\dagger\, i}\overline{\sigma}^\mu \partial_\mu \xi_i - \tfrac{1}{2} m_i (\xi_i \xi_i + \xi^{\dagger\, i}\xi^{\dagger\, i})\,.$$

Example: the see-saw mechanism

The see-saw Lagrangian is given by:

$$\mathcal{L} = i\left(\overline{\psi^1}\,\overline{\sigma}^\mu \partial_\mu \psi_1 + \overline{\psi^2}\,\overline{\sigma}^\mu \partial_\mu \psi_2\right) - M^{ij}\psi_i\psi_j - M_{ij}\overline{\psi^i}\,\overline{\psi^j}\,,$$

where

$$M^{ij} = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix},$$

and (without loss of generality) m_D and M are positive. The Takagi diagonalization of this matrix is $\Omega^T M \Omega = M_D$, where

$$\Omega = \begin{pmatrix} i\cos\theta & \sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}, \qquad M_D = \begin{pmatrix} m_- & 0 \\ 0 & m_+ \end{pmatrix},$$

with $m_\pm = \frac{1}{2}\left[\sqrt{M^2 + 4m_D^2} \pm M\right]$ and $\sin 2\theta = 2m_D/\sqrt{M^2 + 4m_D^2}$.

If $M \gg m_D$, then the corresponding fermion masses are $m_- \simeq m_D^2/M$ and $m_+ \simeq M$, while $\sin\theta \simeq m_D/M$. The mass eigenstates, χ_i are given by $\psi_i = U_i{}^j \chi_j$; *i.e.* to leading order in m_d/M,

$$i\chi_1 \simeq \psi_1 - \frac{m_D}{M}\psi_2 \,, \qquad\qquad \chi_2 \simeq \psi_2 + \frac{m_D}{M}\psi_1 \,.$$

Indeed, one can check that:

$$\tfrac{1}{2} m_D(\psi_1\psi_2 + \psi_2\psi_1) + \tfrac{1}{2} M\psi_2\psi_2 + \text{h.c.} \simeq \tfrac{1}{2}\left[\frac{m_D^2}{M}\chi_1\chi_1 + M\chi_2\chi_2 + \text{h.c.}\right] \,,$$

which corresponds to a theory of two Majorana fermions—one very light and one very heavy (the see-saw).

In any theory containing a multiplet of fields, one can check for the existence of global symmetries. The simplest case is a theory of two-component $(\frac{1}{2}, 0)$ fermion fields χ and η, with the free-field Lagrangian,

$$\mathscr{L} = i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi + i\eta^\dagger\overline{\sigma}^\mu\partial_\mu\eta - m(\chi\eta + \chi^\dagger\eta^\dagger)\,.$$

This Lagrangian possesses a U(1) global symmetry, $\chi \to e^{i\theta}\chi$ and $\eta \to e^{-i\theta}\eta$. That is, χ and η are oppositely charged. The corresponding mass matrix is $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$. Performing the Takagi-diagonalization yields two degenerate two-component fermions of mass m. However, the corresponding mass-eigenstates are not eigenstates of charge.

This is the analog of a free field theory of a complex scalar boson Φ with a mass term $m^2|\Phi|^2$. Writing $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$, we can write Lagrangian in terms of ϕ_1 and ϕ_2 with a diagonal mass term. But, ϕ_1 and ϕ_2 do not correspond to states of definite charge.

Together, χ and $\eta^\dagger$ constitute a single (four-component) Dirac fermion.

More generally, consider a collection charged Dirac fermions represented by pairs of two-component interaction eigenstate fields $\hat{\chi}_{\alpha i}(x)$, $\hat{\eta}^i_\alpha(x)$, with

$$\mathscr{L} = i\hat{\chi}^{\dagger i}\overline{\sigma}^\mu\partial_\mu\hat{\chi}_i + i\hat{\eta}^\dagger_i\overline{\sigma}^\mu\partial_\mu\hat{\eta}^i - M^i{}_j\hat{\chi}_i\hat{\eta}^j - M_i{}^j\hat{\chi}^{\dagger i}\hat{\eta}^\dagger_j\,,$$

where M is a complex matrix with matrix elements $M^i{}_j$, and $M_i{}^j \equiv (M^i{}_j)^*$.

Introduce the mass eigenstate fields χ_i and η^i and the unitary matrices L and R, such that $\hat{\chi}_i = L_i{}^k\chi_k$ and $\hat{\eta}^i = R^i{}_k\eta^k$ and

$$L^{\mathsf{T}}MR = \boldsymbol{m} = \mathrm{diag}(m_1, m_2, \ldots),$$

where the m_i are non-negative real numbers. This is the singular value decomposition of a complex matrix. Noting that $R^\dagger(M^\dagger M)R = \boldsymbol{m}^2$, the diagonal elements of $\boldsymbol{m}$ are the non-negative square roots of the corresponding eigenvalues of $M^\dagger M$. In terms of the mass eigenstate fields,

$$\mathscr{L} = i\chi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\chi_i + i\eta^\dagger_i\overline{\sigma}^\mu\partial_\mu\eta^i - m_i(\chi_i\eta^i + \chi^{\dagger i}\eta^\dagger_i)\,.$$

Fermion–scalar interactions

The most general set of interactions with the scalars of the theory $\hat{\phi}_I$ are then given by:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}\hat{Y}^{Ijk}\hat{\phi}_I\hat{\psi}_j\hat{\psi}_k - \tfrac{1}{2}\hat{Y}_{Ijk}\hat{\phi}^I\hat{\psi}^{\dagger\, j}\hat{\psi}^{\dagger\, k}\,,$$

where $\hat{Y}_{Ijk} = (\hat{Y}^{Ijk})^*$ and $\hat{\phi}^I = (\hat{\phi}_I)^*$. The flavor index I runs over a collection of real scalar fields $\hat{\varphi}_i$ and pairs of complex scalar fields $\hat{\Phi}_j$ and $\hat{\Phi}^j \equiv (\hat{\Phi}_j)^*$ [where a complex field and its conjugate are counted separately]. The Yukawa couplings $\hat{Y}^{Ijk}$ are symmetric under interchange of j and k.

The mass-eigenstate basis ψ is related to the interaction-eigenstate basis $\hat{\psi}$ by a unitary transformations:

$$\hat{\psi} \equiv \begin{pmatrix} \hat{\xi} \\ \hat{\chi} \\ \hat{\eta} \end{pmatrix} = U\psi \equiv \begin{pmatrix} \Omega & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & R \end{pmatrix} \begin{pmatrix} \xi \\ \chi \\ \eta \end{pmatrix}\,,$$

where Ω, L, and R are constructed as described previously. Likewise a unitary transformation yields the scalar mass-eigenstates via $\hat{\phi} = V\phi$. Thus, in terms of mass-eigenstate fields:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}Y^{Ijk}\phi_I\psi_j\psi_k - \tfrac{1}{2}Y_{Ijk}\phi^I\psi^{\dagger\, j}\psi^{\dagger\, k}\,,$$

where $Y^{Ijk} = V_J{}^I U_m{}^j U_n{}^k \hat{Y}^{Jmn}$.

Fermion–gauge boson interactions

In the gauge-interaction basis for the two-component fermions the corresponding interaction Lagrangian is given by

$$\mathscr{L}_{\rm int} = -g_a A^\mu_a \hat{\psi}^{\dagger\, i}\,\overline{\sigma}_\mu (\boldsymbol{T}^a)_i{}^j \hat{\psi}_j \,,$$

where the index a labels the (real or complex) vector bosons A^μ_a and is summed over. If the gauge symmetry is unbroken, then the index a runs over the adjoint representation of the gauge group, and the $(\boldsymbol{T}^a)_i{}^j$ are hermitian representation matrices† of the gauge group acting on the fermions. There is a separate coupling g_a for each simple group or U(1) factor of the gauge group G.

In the case of spontaneously broken gauge theories, one must diagonalize the vector boson squared mass matrix. The above form still applies where A^a_μ are gauge boson fields of definite mass, although in this case for a fixed value of a, $g_a\boldsymbol{T}^a$ is some linear combination of the original $g_a\boldsymbol{T}^a$ of the unbroken theory. Henceforth, we assume that that the A^a_μ are the gauge boson mass-eigenstate fields.

†For a $U(1)$ gauge group, the $\boldsymbol{T}^a$ are replaced by real numbers corresponding to the U(1) charges of the $(\frac{1}{2},0)$ fermions.

In terms of mass-eigenstate fermion fields,

$$\mathscr{L}_{\rm int} = -A^\mu_a \psi^{\dagger\, i}\, \overline{\sigma}_\mu (G^a)_i{}^j \psi_j \,,$$

where $G^a = g_a U^\dagger \boldsymbol{T}^a U$ (no sum over a).

The case of gauge interactions of charged Dirac fermions can be treated as follows. Consider pairs of $(\frac{1}{2},0)$ interaction-eigenstate fermions $\hat{\chi}_i$ and $\hat{\eta}^i$ that transform as conjugate representations of the gauge group (hence the difference in the flavor index heights). The Lagrangian for the gauge interactions of Dirac fermions can be written in the form:

$$\mathscr{L}_{\rm int} = -g_a A^\mu_a \hat{\chi}^{\dagger\, i}\, \overline{\sigma}_\mu (\boldsymbol{T}^a)_i{}^j \hat{\chi}_j + g_a A^\mu_a \hat{\eta}^\dagger_i\, \overline{\sigma}_\mu (\boldsymbol{T}^a)_j{}^i \hat{\eta}^j \,,$$

where the A^a_μ are gauge boson mass-eigenstate fields. Here we have used the fact that if $(\boldsymbol{T}^a)_i{}^j$ are the representation matrices for the $\hat{\chi}_i$, then the $\hat{\eta}^i$ transform in the complex conjugate representation with generator matrices $-(\boldsymbol{T}^a)^* = -(\boldsymbol{T}^a)^T$. In terms of mass-eigenstate fermion fields,

$$\mathscr{L}_{\rm int} = -A^\mu_a \left[\chi^{\dagger\, i}\, \overline{\sigma}_\mu (G^a_L)_i{}^j \chi_j - \eta^\dagger_i\, \overline{\sigma}_\mu (G^a_R)_j{}^i \eta^j\right] \,,$$

where $G^a_L = g_a L^\dagger \boldsymbol{T}^a L$ and $G^a_R = g_a R^\dagger \boldsymbol{T}^a R$ (no sum over a).

Four-component spinor notation

The correspondence between the two-component and four-component spinor language is most easily exhibited in the basis in which γ_5 is diagonal (this is called the *chiral* representation). In 2×2 blocks, the gamma matrices are given by:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot\beta} \\ \overline{\sigma}^{\mu\dot\alpha\beta} & 0 \end{pmatrix}, \quad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}.$$

The chiral projections operators are: $P_L \equiv \frac{1}{2}(1-\gamma_5)$ and $P_R \equiv \frac{1}{2}(1+\gamma_5)$.

In addition, we identify the generators of the Lorentz group in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation:[‡]

$$\tfrac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu}{}_\alpha{}^\beta & 0 \\ 0 & \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} \end{pmatrix},$$

where $\Sigma^{\mu\nu}$ satisfies the duality relation, $\gamma_5\Sigma^{\mu\nu} = \frac{1}{2}i\epsilon^{\mu\nu\rho\tau}\Sigma_{\rho\tau}$.

[‡]In most textbooks, $\Sigma^{\mu\nu}$ is called $\sigma^{\mu\nu}$. Here, we use the former symbol so that there is no confusion with the two-component definition of $\sigma^{\mu\nu}$.

A four component Dirac spinor field, $\Psi(x)$, is made up of two mass-degenerate two-component spinor fields, $\chi_\alpha(x)$ and $\eta_\alpha(x)$ as follows:

$$\Psi(x) \equiv \begin{pmatrix} \chi_\alpha(x) \\ \eta^{\dagger\,\dot{\alpha}}(x) \end{pmatrix} .$$

Note that P_L and P_R project out the upper and lower components, respectively. The Dirac conjugate field $\overline{\Psi}$ and the charge conjugate field Ψ^c are defined by

$$\overline{\Psi}(x) \equiv \Psi^\dagger A = (\eta^\alpha(x), \chi^\dagger_{\dot{\alpha}}) ,$$

$$\Psi^c(x) \equiv C\overline{\Psi}^{\mathsf{T}}(x) = \begin{pmatrix} \eta_\alpha(x) \\ \chi^{\dagger\,\dot{\alpha}}(x) \end{pmatrix} ,$$

where the Dirac conjugation matrix A and the charge conjugation matrix C satisfy

$$A\gamma^\mu A^{-1} = \gamma^{\mu\dagger} , \qquad\qquad C^{-1}\gamma^\mu C = -\gamma^{\mu\mathsf{T}} .$$

It is conventional to impose two additional conditions: (i) $\Psi = A^{-1}\overline{\Psi}^\dagger$ [which guarantees that $\overline{\Psi}\Psi$ is hermitian] and (ii) $(\Psi^c)^c = \Psi$. It follows that

$$A^\dagger = A , \qquad C^{\mathsf{T}} = -C , \qquad (AC)^{-1} = (AC)^* .$$

In the chiral representation, A and C are explicitly given by

$$A = \begin{pmatrix} 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ \delta_{\alpha}{}^{\beta} & 0 \end{pmatrix} , \qquad C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} .$$

Note the numerical equalities, $A = \gamma^0$ and $C = i\gamma^0\gamma^2$, although these identifications do not respect the structure of the undotted and dotted indices specified above.

One can relate bilinear covariants in two-component and four-component notation.

$$\overline{\Psi}_1\Psi_2 = \eta_1\xi_2 + \xi_1^\dagger\eta_2^\dagger$$

$$\overline{\Psi}_1\gamma_5\Psi_2 = -\eta_1\xi_2 + \xi_1^\dagger\eta_2^\dagger$$

$$\overline{\Psi}_1\gamma^\mu\Psi_2 = \xi_1\overline{\sigma}^\mu\xi_2 - \eta_2^\dagger\overline{\sigma}^\mu\eta_1$$

$$\overline{\Psi}_1\gamma^\mu\gamma_5\Psi_2 = -\xi_1^\dagger\overline{\sigma}^\mu\xi_2 - \eta_2^\dagger\overline{\sigma}^\mu\eta_1$$

$$\overline{\Psi}_1\Sigma^{\mu\nu}\Psi_2 = 2(\eta_1\sigma^{\mu\nu}\xi_2 + \xi_1^\dagger\overline{\sigma}^{\mu\nu}\eta_2^\dagger)$$

$$\overline{\Psi}_1\Sigma^{\mu\nu}\gamma_5\Psi_2 = -2(\eta_1\sigma^{\mu\nu}\xi_2 - \xi_1^\dagger\overline{\sigma}^{\mu\nu}\eta_2^\dagger) .$$

Relating bilinear covariants in two-component and four-component notation

$$\Psi_1(x) \equiv \begin{pmatrix} \xi_1(x) \\ \eta_1^\dagger(x) \end{pmatrix}, \qquad \Psi_2(x) \equiv \begin{pmatrix} \xi_2(x) \\ \eta_2^\dagger(x) \end{pmatrix}.$$

$\overline{\Psi}_1 P_L \Psi_2 = \eta_1 \xi_2$	$\overline{\Psi}{}^c_1 P_L \Psi^c_2 = \xi_1 \eta_2$
$\overline{\Psi}_1 P_R \Psi_2 = \xi_1^\dagger \eta_2^\dagger$	$\overline{\Psi}{}^c_1 P_R \Psi^c_2 = \eta_1^\dagger \xi_2^\dagger$
$\overline{\Psi}{}^c_1 P_L \Psi_2 = \xi_1 \xi_2$	$\overline{\Psi}_1 P_L \Psi^c_2 = \eta_1 \eta_2$
$\overline{\Psi}_1 P_R \Psi^c_2 = \xi_1^\dagger \xi_2^\dagger$	$\overline{\Psi}{}^c_1 P_R \Psi_2 = \eta_1^\dagger \eta_2^\dagger$
$\overline{\Psi}_1 \gamma^\mu P_L \Psi_2 = \xi_1^\dagger \overline{\sigma}^\mu \xi_2$	$\overline{\Psi}{}^c_1 \gamma^\mu P_L \Psi^c_2 = \eta_1^\dagger \overline{\sigma}^\mu \eta_2$
$\overline{\Psi}{}^c_1 \gamma^\mu P_R \Psi^c_2 = \xi_1 \sigma^\mu \xi_2^\dagger$	$\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = \eta_1 \sigma^\mu \eta_2^\dagger$
$\overline{\Psi}_1 \Sigma^{\mu\nu} P_L \Psi_2 = 2\, \eta_1 \sigma^{\mu\nu} \xi_2$	$\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_L \Psi^c_2 = 2\, \xi_1 \sigma^{\mu\nu} \eta_2$
$\overline{\Psi}_1 \Sigma^{\mu\nu} P_R \Psi_2 = 2\, \xi_1^\dagger \overline{\sigma}^{\mu\nu} \eta_2^\dagger$	$\overline{\Psi}{}^c_1 \Sigma^{\mu\nu} P_R \Psi^c_2 = 2\, \eta_1^\dagger \overline{\sigma}^{\mu\nu} \xi_2^\dagger$

$\Sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Note that we may also write: $\overline{\Psi}_1 \gamma^\mu P_R \Psi_2 = -\eta_2^\dagger \overline{\sigma}^\mu \eta_1$, *etc.*

For Majorana fermions defined by $\Psi_M = \Psi_M^c = C\overline{\Psi}_M^T$, the following additional conditions are satisfied:

$$\overline{\Psi}_{M1}\Psi_{M2} = \overline{\Psi}_{M2}\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\gamma_5\Psi_{M2} = \overline{\Psi}_{M2}\gamma_5\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\gamma^\mu\Psi_{M2} = -\overline{\Psi}_{M2}\gamma^\mu\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\gamma^\mu\gamma_5\Psi_{M2} = \overline{\Psi}_{M2}\gamma^\mu\gamma_5\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\Sigma^{\mu\nu}\Psi_{M2} = -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\Psi_{M1}\,,$$

$$\overline{\Psi}_{M1}\Sigma^{\mu\nu}\gamma_5\Psi_{M2} = -\overline{\Psi}_{M2}\Sigma^{\mu\nu}\gamma_5\Psi_{M1}\,.$$

In particular, if $\Psi_{M1} = \Psi_{M2} \equiv \Psi_M$, then

$$\overline{\Psi}_M\gamma^\mu\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\gamma_5\Psi_M = 0\,.$$

One additional useful result is:

$$\overline{\Psi}_{M1}\gamma^\mu P_L\Psi_{M2} = -\overline{\Psi}_{M2}\gamma^\mu P_R\Psi_{M1}\,.$$