CHAPTER **ONE**

QUANTUM FIELD THEORY IN THE HEISENBERG PICTURE

Quantum field theory is the fundamental theory of particle physics. In this chapter, we summarize its general features as the preliminaries for the succeeding chapters, though it is supposed that the readers are familiar with quantum field theory.

1.1 GENERAL REMARKS ON QUANTUM FIELD THEORY

In this section, we make some general remarks on quantum field theory.

1.1.1 Basic concepts

A **quantum field**, or simply a **field**, is a fundamental object in quantum field theory. Mathematically, a quantum field is a finite set of operator-valued generalized functions (distributions or hyperfunctions) of spacetime coordinates x^μ. When we discuss a field generically, it is denoted by $\varphi(x)$ (or $\varphi_A(x)$ if we discuss many fields). For particular fields, we use various symbols such as $\phi(x)$, $\psi(x)$, etc.

Quantum field theory is formulated so as to meet the requirement of special relativity (except for quantum gravity). That is, the theory is invariant under translations and Lorentz transformations. The Poincaré invariance of the theory is discussed in some detail in the next subsection.

Quantum fields satisfy certain partial differential equations, which are called **field equations**. The field equation should be *local*, that is, every field involved in it depends only on *one* spacetime point. If a field by itself satisfies linear differential equations of the same number as that of its components, it is called **free**; otherwise it is **interacting**. That is, an interacting field satisfies nonlinear differential equations, which necessarily involve some products of fields at the *same* spacetime point. Since, however, a field is, apart from its operator nature, not an ordinary function but a *generalized function*, which is not defined point by point, a product of fields at the same spacetime point is *not well-defined* in general. This causes a very difficult problem, called **ultraviolet divergence**. There are several attitudes for coping with this trouble.

1. One gives up field equations, and discusses only the general framework of quantum field theory. This standpoint is called **axiomatic field theory**. This approach, however, makes it impossible to formulate the theory concretely.
2. One first replaces the singular product of fields by a well-defined one by means of point-splitting or some other regularization method. After applying a certain device of removing divergent pieces (called renormalization), one takes a limit to reproduce the original singular product formally. But this approach is successful only in some simple lower-dimensional models.
3. In order to deal with realistic theories concretely, therefore, one must appeal to some approximation method. In perturbation theory, one can obtain an explicit solution in the form of a (non-convergent) series expansion. After making renormalization, one can define the singular product of fields *in each order* of the perturbation series.

We do not wish to discuss this very difficult problem here. Since our primary interest is to develop concrete formalisms of realistic theories without approximation, we simply *postulate* that *the product of fields at the same spacetime point exists uniquely* without specifying how to define it explicitly. Here the uniqueness means that *the ordering of field operators*[1] *in the product is totally irrelevant* apart from an overall statistical signature factor. We emphasize that this statement does not mean to neglect the (anti)commutators of fields at the same spacetime point, but claims that the operator ordering at the same spacetime point is *meaningless*. Only under this understanding, we can reasonably develop the Lagrangian canonical formalism, which is presented in Sec.1.2.

Canonical quantization is carried out by using commutation relations or anticommutation relations according as φ_A is **bosonic** (i.e., obeying Bose statistics)

[1] The word "field operator" is used for emphasizing the operator nature of a field.

or **fermionic** (i.e., obeying Fermi one). Since any two fields at two different point (anti)commute at the equal time, Lorentz invariance implies that

$$[\varphi_A(x), \varphi_B(y)]_\mp = 0 \quad \text{for} \quad (x-y)^2 < 0, \tag{1.1.1 - 1}$$

We call Eq. (1) **local (anti)commutativity** or **Einstein causality**, because it means that no action can propagate faster than light.

The operand of field operators is called a **state vector**, or simply a **state**. It is denoted by $|\cdot\rangle$. For two states $|f\rangle$ and $|g\rangle$, their **inner product** is denoted by $\langle g|f\rangle$ and has the property

$$\langle g|f\rangle = \langle f|g\rangle^*. \tag{1.1.1 - 2}$$

We sometimes write $\langle f|$ without considering the inner product. We call $\langle f|$ a **bra-vector** and correspondingly $|f\rangle$ a **ket-vector**.

The totality of states is called a **state-vector space**, and it is usually denoted by $\mathcal{V}$. It is a complex linear space equiped with the inner product. If

$$\langle f|f\rangle > 0 \quad \text{for any} \quad |f\rangle \neq 0, \tag{1.1.1 - 3}$$

then $\mathcal{V}$ is a **Hilbert space** (completion should be understood). In this case, probabilistic interpretation is obvious. But one should note that probabilistic interpretability does *not* necessarily require the validity of Eq. (3) in the whole $\mathcal{V}$. If there are some states in $\mathcal{V}$ which do not satisfy Eq. (3), $\mathcal{V}$ is called an **indefinite-metric Hilbert space**. Its properties are described in detail in the Appendix. In the indefinite-metric Hilbert space, $\langle f|f\rangle$ is called **norm** in abuse of language (**positive norm** if $\langle f|f\rangle > 0$, **zero norm** if $\langle f|f\rangle = 0$, and **negative norm** if $\langle f|f\rangle < 0$). We always assume that $\mathcal{V}$ is **non-degenerate**, that is, if $|f\rangle$ is zero norm then there exists $|g\rangle$ in $\mathcal{V}$ such that $\langle g|f\rangle \neq 0$.

Field operators are represented in $\mathcal{V}$. In general, there are infinitely many inequivalent representations. It is postulated that there exists a unique distinguished state, called the **vacuum**, which should be Poincaré invariant. It is denoted by $|0\rangle$. Any state in $\mathcal{V}$ is essentially constructible by applying field operators on $|0\rangle$ (postulate of **cyclicity**).

Given an operator A, $\langle g|A|f\rangle$ is called **matrix element** of A, though it does not necessarily obey the matrix multiplication law if $\mathcal{V}$ is not a Hilbert space. If

$$\langle g|A^\dagger|f\rangle = \langle f|A|g\rangle^*, \tag{1.1.1 - 4}$$

then $A^\dagger$ is called **hermitian conjugate** of A. If $A^\dagger = A$, A is **hermitian**, and if $A^\dagger = A^{-1}$, A is **unitary**. We do not strictly distinguish hermitian and self-adjoint because we hardly pay attention to the domain in which A is defined.

1.1.2 Poincaré invariance

Quantum field theory is invariant under the **Poincaré group**, which is the totality of translations and Lorentz transformations. Under a Lorentz transformation plus a translation

$$x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu + a^\mu \quad (\text{or } x' = Lx + a) \tag{1.1.2 – 1}$$

preserving the Minkowski metric ($\eta_{\mu\nu} L^\mu{}_\sigma L^\nu{}_\tau = \eta_{\sigma\tau}$), a field $\varphi(x)$ transforms as

$$\varphi(x) \to \varphi'(x') = s(L)\varphi(x), \tag{1.1.2 – 2}$$

where $s(L)$ stands for a finite-dimensional matrix representation[2] of the Lorentz transformation $L \in SO(3,1)$. The above transformation is induced by a unitary operator $U(a, L)$ in the following way:

$$U^{-1}(a, L)\varphi(x)U(a, L) = \varphi'(x) = s(L)\varphi(L^{-1}(x - a)) \tag{1.1.2 – 3}$$

with the composition rule

$$U(a_1, L_1)U(a_2, L_2) = U(L_1 a_2 + a_1, L_1 L_2). \tag{1.1.2 – 4}$$

It is convenient to consider the infinitesimal transformation, for which one can neglect higher order terms. The infinitesimal versions of Eqs. (1) and (2) are

$$x'^\mu = x^\mu + \varepsilon^\mu{}_\nu x^\nu + \varepsilon^\mu \tag{1.1.2 – 5}$$

with $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}$ and

$$\varphi'_\alpha(x') = [\delta_\alpha^{\ \beta} - \frac{i}{2}\varepsilon^{\mu\nu}(s_{\mu\nu})_\alpha^{\ \beta}]\varphi_\beta(x) \tag{1.1.2 – 6}$$

[2] For the fields having a half-odd spin, $s(L)$ should be understood as the representation of the universal covering group, $SL(2, \mathbf{C})$, of $SO(3,1)$.

with $s_{\mu\nu} = -s_{\nu\mu}$, respectively. Here, owing to the composition rule $s(L_1)s(L_2) = s(L_1L_2)$, the matrices $s_{\mu\nu}$ satisfy

$$[s_{\mu\nu}, s_{\lambda\rho}] = i(\eta_{\nu\lambda}s_{\mu\rho} - \eta_{\mu\lambda}s_{\nu\rho} + \eta_{\mu\rho}s_{\nu\lambda} - \eta_{\nu\rho}s_{\mu\lambda}). \qquad (1.1.2-7)$$

Likewise, the infinitesimal version of $U(a, L)$ is

$$1 + i\epsilon^{\mu}P_{\mu} - \frac{i}{2}\epsilon^{\mu\nu}M_{\mu\nu}; \qquad (1.1.2-8)$$

P_μ and $M_{\mu\nu}(= -M_{\nu\mu})$ are called **translation generators** and **Lorentz generators**, respectively, and they are altogether called **Poincaré generators**. Then the infinitesimal form of Eq.(3) becomes

$$[iP_{\mu}, \varphi_{\alpha}(x)] = \partial_{\mu}\varphi_{\alpha}(x), \qquad (1.1.2-9)$$

$$[iM_{\mu\nu}, \varphi_{\alpha}(x)] = [(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\delta_{\alpha}^{\ \beta} - i(s_{\mu\nu})_{\alpha}^{\ \beta}]\varphi_{\beta}(x). \qquad (1.1.2-10)$$

From the composition rule, Eq.(4), we see that the Poincaré generators satisfy the following commutation relations:

$$[P_{\mu}, P_{\nu}] = 0, \qquad (1.1.2-11)$$

$$[M_{\mu\nu}, P_{\lambda}] = i(\eta_{\nu\lambda}P_{\mu} - \eta_{\mu\lambda}P_{\nu}), \qquad (1.1.2-12)$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(\eta_{\nu\lambda}M_{\mu\rho} - \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\rho}M_{\mu\lambda}). \qquad (1.1.2-13)$$

This algebra is called **Poincaré algebra**. One can confirm the consistency between Eqs. (9),(10) and Eqs. (11)-(13) in the sense of the **Jacobi identity**

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \equiv 0. \qquad (1.1.2-14)$$

The operators which commute with all generators are called **Casimir operators**. The Casimir operators of the Poincaré algebra are P^2 and $W \equiv -w^2$, where we set

$$w^{\rho} \equiv \frac{1}{2}\epsilon^{\rho\lambda\mu\nu}P_{\lambda}M_{\mu\nu}, \qquad (1.1.2-15)$$

which commutes with P_σ.

The representations of the Poincaré group are constructed in the following way. Since the translation group is abelian, its irreducible representations are one-dimensional. The representation is specified by the eigenvalue p_μ of P_μ. Hence the Lorentz transformations which change p_μ should be excluded. The totality of Lorentz transformations which leave p_μ invariant is called a **little group** on p_μ. Taking advantage of the technique of induced representations, we have only to construct the irreducible representation of the little group on p_μ. According as p_μ is timelike, lightlike, or spacelike, the corresponding little group is isomorphic to the three-dimensional orthogonal group $SO(3)$, the two-dimensional Euclidean group $E(2)$, or the three-dimensional Lorentz group $SO(2,1)$.

The algebra corresponding to a little group can be found in the following way. Since P_μ takes the value p_μ, we substitute $P_\mu = p_\mu$ into Eq. (12). Then, since the commutator vanishes, the rhs. also must vanish. Thus the generators of the little group are the linear combinations $c^{\mu\nu}M_{\mu\nu}$ which are consistent with

$$c^{\mu\nu}(\eta_{\nu\lambda}p_\mu - \eta_{\mu\lambda}p_\nu) = 0. \qquad (1.1.2-16)$$

The commutators satisfied by them are calculated from Eq. (13). For example, if $p_\mu = (p_0 \neq 0, 0, 0, 0)$, the generators of the little group are M_{kl}, and they satisfy the algebra of $SO(3)$; its representation defines spin. The case in which p_μ is lightlike is discussed in Sec.2.2.2.

From the physical reason, it is postulated that $p^2 \geq 0$ and $p_0 \geq 0$. This requirement is called **spectrum condition**. Accordingly, the little group on a spacelike p_μ is unphysical.

1.2 LAGRANGIAN CANONICAL FORMALISM

In this section, we review the general theory of Lagrangian canonical formalism, that is, starting with the Lagrangian density we define canonical variables, their canonical conjugates, and the Hamiltonian, and then set up the canonical commutation relations. We also discuss the symmetry properties of the theory.

1.2.1 Lagrangian density and field equations

We consider a set of fields $\{\varphi_A(x)\}$; they are called **primary fields**. The Lagrangian canonical formalism is based on an **action**

$$I \equiv \int d^4x \mathcal{L}(x), \tag{1.2.1 - 1}$$

where I is bosonic, hermitian, and Poincaré invariant. The **Lagrangian density** $\mathcal{L}(x)$ is a local function of primary fields $\varphi_A(x)$, that is , it is constructed from $\varphi_A(x)$ and their derivatives *at the same spacetime point.*[1]

In order for the canonical formalism to be applicable, $\mathcal{L}(x)$ is assumed to contain no second or higher derivatives of $\varphi_A(x)$ and to be at most quadratic with respect to first derivatives of $\varphi_A(x)$. Furthermore, when $\mathcal{L}(x)$ is a polynomial in $\varphi_A(x)$, one always drops a constant term and eliminates linear terms by redefining fields. Then the quadratic part of $\mathcal{L}(x)$ is called **free Lagrangian density**, and the remainder is called **interaction Lagrangian density**. Parameters appearing in the latter are called **coupling constants**.

In Eq.(1), the integration volume is the whole four-dimensional spacetime because any other spacetime region is non-invariant under translations. Hence in considering a variation of I, we need not take account of that of the integration volume. Then the **variational principle** $\delta I = 0$ yields

$$\begin{aligned} 0 = \delta I &= \int d^4x \delta\mathcal{L}(x) \\ &= \int d^4x \left[\delta\varphi_A \frac{\partial \mathcal{L}}{\partial \varphi_A} + \delta\partial_\mu\varphi_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_A)}\right], \end{aligned} \tag{1.2.1 - 2}$$

[1] If $\mathcal{L}(x)$ is not local, then we have a non-local theory, which is known to have fundamental difficulties.

where summation over A should be understood. Using $\delta\partial_\mu\varphi_A = \partial_\mu\delta\varphi_A$ and integrating by parts, we obtain **Euler-Lagrange equations**,

$$\frac{\partial\mathcal{L}}{\partial\varphi_A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)} = 0, \qquad (1.2.1-3)$$

which are called **field equations**.

In the above, an important assumption is that $\delta\varphi_A(x)$ is local and vanishes sufficiently rapidly at infinity in any direction. Hence any term in $\mathcal{L}(x)$ expressible as $\partial_\mu(\,\cdot\,)^\mu$, which is called **total divergence**, can *always* be discarded.[2] It is therefore unreasonable to consider a surface term,[3] which could arise from total divergence owing to the Gauss theorem, in the action I.

1.2.2 Canonical quantization in the ordinary case

In this subsection, we consider canonical quantization in the ordinary case. A more general case is discussed in the next subsection. The most general case is dealt with in Addendum 1.A.

We take φ_A as **canonical variables** and define their **canonical conjugates** by

$$\pi^A \equiv (\partial/\partial\dot{\varphi}_A)\mathcal{L}(\varphi, \partial\varphi, \dot{\varphi}). \qquad (1.2.2-1)$$

Since $\mathcal{L}$ is at most quadratic in $\dot{\varphi}_A \equiv \partial_0\varphi_A$, π^A is linear in $\dot{\varphi}_B$. We now assume that the simultaneous "linear" equations, Eq.(1), can be solved with respect to $\dot{\varphi}_B$. That is, we assume that it is possible to write

$$\dot{\varphi}_A = \Phi_A(\varphi, \partial\varphi, \pi). \qquad (1.2.2-2)$$

Then, since π^A's are independent, we can set up the **canonical (anti)commutation relations**

$$[\varphi_A(x), \varphi_B(y)]_{\mp,0} = 0, \qquad (1.2.2-3)$$

$$[\pi^A(x), \varphi_B(y)]_{\mp,0} = -i\delta^A{}_B\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad (1.2.2-4)$$

$$[\pi^A(x), \pi^B(y)]_{\mp,0} = 0. \qquad (1.2.2-5)$$

2 A total-divergence term may be meaningful in the "effective" Lagrangian density, which is a non-local function of primary fields. Any quantity F is total divergence in the sense $F = \partial_\mu(\partial^\mu\Box^{-1}F)$ if non-local quantities are admitted.

3 It is true that it is often necessary to introduce surface terms, but one should note that such circumstances can arise only after the representation of field operators is considered.

Here we use the following notation:

$$[\varphi_A(x), \varphi_B(y)]_{\mp,0} \equiv \varphi_A(x)\varphi_B(y) - \epsilon_{(AB)}\varphi_B(y)\varphi_A(x) \qquad (1.2.2-6)$$

with $\epsilon_{(AB)} = -1$ if both φ_A and φ_B are fermionic and $\epsilon_{(AB)} = +1$ otherwise (indices in parentheses are irrelevant to the summation convention); the subscript 0 of a bracket means to set $x^0 = y^0$ and $\delta(\boldsymbol{x} - \boldsymbol{y})$ is the spatial three-dimensional delta function.

The **Hamiltonian** H is defined by

$$H \equiv \int d\boldsymbol{x}\, \mathcal{H}(x) \qquad (1.2.2-7)$$

with

$$\mathcal{H}(x) \equiv \dot{\varphi}_B(x)\pi^B(x) - \mathcal{L}(x). \qquad (1.2.2-8)$$

Then the following **Heisenberg equations** hold:

$$[iH, \varphi_A(x)] = \dot{\varphi}_A(x), \qquad (1.2.2-9)$$

$$[iH, \pi^A(x)] = \dot{\pi}^A(x). \qquad (1.2.2-10)$$

Their validity is shown in the following way.

From Eqs.(2)-(5), we have

$$[\dot{\varphi}_A(x), \varphi_B(y)]_{\mp,0} = -i\frac{\partial\Phi_A}{\partial\pi^B}\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad (1.2.2-11)$$

$$[\pi^A(x), \dot{\varphi}_B(y)]_{\mp,0} = -i\left(\frac{\partial\Phi_B}{\partial\varphi_A} + \frac{\partial\Phi_B}{\partial(\partial_k\varphi_A)}\partial_k\right)^y \delta(\boldsymbol{x} - \boldsymbol{y}), \qquad (1.2.2-12)$$

where the superscript y indicates that $\boldsymbol{y}$ should be used inside the parentheses. Noting Eq.(1), we obtain

$$[\mathcal{L}(x), \varphi_A(y)]_0 = -i\epsilon_{(AB)}\frac{\partial\Phi_B}{\partial\pi^A}\pi^B\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad (1.2.2-13)$$

$$\begin{aligned}[\mathcal{L}(x), \pi^A(y)]_0 =& i\left(\frac{\partial\mathcal{L}}{\partial\varphi_A} + \frac{\partial\mathcal{L}}{\partial(\partial_k\varphi_A)}\partial_k\right)^x \delta(\boldsymbol{x} - \boldsymbol{y}) \\ &+ i\left(\frac{\partial\Phi_B}{\partial\varphi_A} + \frac{\partial\Phi_B}{\partial(\partial_k\varphi_A)}\partial_k\right)^x \delta(\boldsymbol{x} - \boldsymbol{y})\cdot\pi^B, \qquad (1.2.2-14)\end{aligned}$$

and then

$$[\mathcal{H}(x), \varphi_A(y)]_0 = -i\dot{\varphi}_A\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad (1.2.2-15)$$

$$[\mathcal{H}(x), \pi^A(y)]_0 = -i\left(\frac{\partial\mathcal{L}}{\partial\varphi_A} + \frac{\partial\mathcal{L}}{\partial(\partial_k\varphi_A)}\partial_k\right)^x \delta(\boldsymbol{x} - \boldsymbol{y}). \qquad (1.2.2-16)$$

Since the field equation, Eq.(1.2.1–3), is rewritten as

$$\frac{\partial \mathcal{L}}{\partial \varphi_A} - \partial_k \frac{\partial \mathcal{L}}{\partial(\partial_k \varphi_A)} = \dot{\pi}^A, \tag{1.2.2 – 17}$$

Eqs.(15) and (16) yield Eqs.(9) and (10), respectively.

Finally, we demonstrate that the canonical (anti)commutation relations, Eqs.(3)–(5), are invariant under field redefinition. Let $\{\varphi_A\}$ and $\{\varphi'_A\}$ are two systems of canonical variables, each of which is expressible in terms of the other through a set of local relations involving no derivatives:

$$\varphi_A = f_A(\varphi'), \quad \varphi'_A = g_A(\varphi); \tag{1.2.2 – 18}$$

then the canonical quantizations induced by both systems are mutually equivalent, as shown below.

The canonical conjugates are, of course, given by

$$\pi^A \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_A}, \quad \pi'^A \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}'_A}. \tag{1.2.2 – 19}$$

Since Eq.(18) implies

$$\dot{\varphi}_A = \dot{\varphi}'_B \frac{\partial f_A}{\partial \varphi'_B}, \quad \dot{\varphi}'_A = \dot{\varphi}_B \frac{\partial g_A}{\partial \varphi_B}, \tag{1.2.2 – 20}$$

we have

$$\frac{\partial f_C}{\partial \varphi'_A}\frac{\partial g_B}{\partial \varphi_C} = \frac{\partial g_C}{\partial \varphi_A}\frac{\partial f_B}{\partial \varphi'_C} = \delta^A{}_B, \tag{1.2.2 – 21}$$

and therefore

$$\pi^A = \frac{\partial g_B}{\partial \varphi_A}\pi'^B, \quad \pi'^A = \frac{\partial f_B}{\partial \varphi'_A}\pi^B. \tag{1.2.2 – 22}$$

Setting up the canonical (anti)commutation relations for $\{\varphi_A\}$, we calculate (anti) commutators with respect to $\{\varphi'_A\}$. We then have

$$[\varphi'_A(x), \varphi'_B(y)]_{\mp,0} = [g_A(\varphi(x)), g_B(\varphi(y))]_{\mp,0} = 0, \tag{1.2.2 – 23}$$

$$\begin{aligned}[\pi'^A(x), \varphi'_B(y)]_{\mp,0} &= \frac{\partial f_C}{\partial \varphi'_A}(x) \cdot [\pi^C(x), g_B(\varphi(y))]_{\mp,0} \\ &= -i\frac{\partial f_C}{\partial \varphi'_A}\frac{\partial g_B}{\partial \varphi_C}\delta(\mathbf{x}-\mathbf{y}) \\ &= -i\delta^A{}_B\delta(\mathbf{x}-\mathbf{y}),\end{aligned} \tag{1.2.2 – 24}$$

$$\begin{aligned}[\pi'^A(x), \pi'^B(y)]_{\mp,0} &= \frac{\partial f_C}{\partial \varphi'_A}(x)\left[\pi^C(x), \frac{\partial f_D}{\partial \varphi'_B}(y)\right]_{\mp,0}\pi^D(y) - \epsilon_{(AB)}(A \leftrightarrow B, x \leftrightarrow y) \\ &= -i\left[\frac{\partial f_C}{\partial \varphi'_A}\frac{\partial^2 f_D}{\partial \varphi_C \partial \varphi'_B} - \epsilon_{(AB)}(A \leftrightarrow B)\right]\pi^D\delta(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{1.2.2 – 25}$$

Since

$$\frac{\partial^2 f_D}{\partial\varphi_C \partial\varphi'_B} = \frac{\partial g_E}{\partial\varphi_C} \frac{\partial^2 f_D}{\partial\varphi'_E \partial\varphi'_B}, \qquad (1.2.2-26)$$

Eq.(21) implies

$$\frac{\partial f_C}{\partial\varphi'_A} \frac{\partial^2 f_D}{\partial\varphi_C \partial\varphi'_B} = \frac{\partial^2 f_D}{\partial\varphi'_A \partial\varphi'_B}. \qquad (1.2.2-27)$$

We thus obtain

$$[\pi'^A(x), \pi'^B(y)]_{\mp,0} = 0. \qquad (1.2.2-28)$$

1.2.3 Canonical quantization with multiplier fields

It is rather stringent to assume that Eq.(1.2.2–1) is solvable with respect to $\dot\varphi_A$; we often encounter the situation that the rhs. of Eq.(1.2.2–1) does not involve some of $\dot\varphi_A$'s, which we denote by $\dot\varphi_\alpha$. Here α may run over only some of , but not all of, components of a field. In such a case, the trouble is resolved by considering a new Lagrangian density having the following form:

$$\mathcal{L} = \mathcal{L}_0(\varphi, \partial\varphi, \dot\varphi) + F_\alpha(\varphi, \partial\varphi, \dot\varphi) b^\alpha + G(b), \qquad (1.2.3-1)$$

where $\mathcal{L}_0$ corresponds to the previous Lagrangian density, F_α is linear in $\dot\varphi$ and $\det(\partial/\partial\dot\varphi_\beta)F_\alpha \neq 0$. Since the new field b^α plays the role of a Lagrange multiplier, it is sometimes called a **multiplier field**. We encounter such Lagrangian densities as Eq.(1) in gauge theories and in quantum gravity.

The field equations are

$$\frac{\partial\mathcal{L}}{\partial\varphi_A} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_A)} = 0, \qquad (1.2.3-2)$$

$$F_\alpha + \frac{\partial G}{\partial b^\alpha} = 0. \qquad (1.2.3-3)$$

Canonical variables are φ_A but *not* b^α. The canonical conjugates are

$$\pi^A \equiv \frac{\partial\mathcal{L}}{\partial\dot\varphi_A} = \frac{\partial\mathcal{L}_0}{\partial\dot\varphi_A} + \frac{\partial F_\alpha}{\partial\dot\varphi_A} b^\alpha. \qquad (1.2.3-4)$$

By assumption, the matrix $((\partial/\partial\dot\varphi_\beta)F_\alpha)$ is invertible. It is therefore possible to solve Eq.(4) with respect to $\dot\varphi_A\,(A \neq \alpha)$ and b^α. Note that $\dot\varphi_A\,(A = \alpha)$ is expressible in terms of other variables through Eq.(3). We set up the canonical (anti)commutation relations, Eqs.(1.2.2–3)–(1.2.2–5). In the following, we check the consistency with the Heisenberg equations, Eqs.(1.2.2–9) and (1.2.2–10).

Since from Eq.(4) b^α is expressible as a function of φ, $\partial\varphi$, and π, we can *eliminate* all b^α from $\mathcal{L}$ and the field equations. Then we can write Eq.(1.2.2–2) for *all* A's, and hence proceed in the same way as before. That is, we obtain Eqs.(1.2.2–15) and (1.2.2–16), and therefore the Heisenberg equations. The only difference from the ordinary case is that the Heisenberg equations, Eq.(1.2.2–9), hold not as trivial identities $\dot{\varphi}_A = \dot{\varphi}_A$, but those corresponding to $A = \alpha$ are equivalent to field equations, Eq.(3).

1.2.4 Noether theorem

As discussed in Sec.1.1.2, the theory is invariant under the Poincaré group. In general, the theory may have other invariance properties. If I is invariant under certain transformations of primary fields, the theory is said to have a **symmetry** corresponding to those transformations. There are discrete symmetries and continuous ones, but we are interested only in the latter.

A continuous symmetry can be discussed by considering the infinitesimal transformation, which is characterized by an infinitesimal parameter ε. We set

$$\delta^\varepsilon \varphi_A(x) \equiv \varphi'_A(x') - \varphi_A(x), \tag{1.2.4 – 1}$$

$$\delta_*^\varepsilon \varphi_A(x) \equiv \varphi'_A(x) - \varphi_A(x), \tag{1.2.4 – 2}$$

whence

$$\delta^\varepsilon \varphi_A = \delta_*^\varepsilon \varphi_A + \delta^\varepsilon x^\mu \cdot \partial_\mu \varphi_A, \tag{1.2.4 – 3}$$

where $x'^\mu \equiv x^\mu + \delta^\varepsilon x^\mu$. The symmetry considered is called **spacetime symmetry** if $\delta^\varepsilon x^\mu \neq 0$ and **internal symmetry** if vanishing; $\delta^\varepsilon - \delta_*^\varepsilon$ is called the **orbital part** of the symmetry. Poincaré invariance is, of course, a spacetime symmetry.

Although the invariance of I does not necessarily imply

$$\mathcal{L}'(x') = \mathcal{L}(x), \tag{1.2.4 – 4}$$

we here restrict our consideration only to this case.[4] Correspondingly, the Jacobian $\det \partial_\mu x'^\nu$ is unity, whence $\partial_\mu \delta^\varepsilon x^\mu = 0$. We rewrite Eq.(4) as

$$\delta^\varepsilon \mathcal{L} = \delta_*^\varepsilon \mathcal{L} + \delta^\varepsilon x^\mu \cdot \partial_\mu \mathcal{L} = 0 \tag{1.2.4 – 5}$$

[4] General case is discussed in Sec.5.2.4.

with

$$\delta_*^\epsilon \mathcal{L} = \delta_*^\epsilon \varphi_A \cdot \frac{\partial \mathcal{L}}{\partial \varphi_A} + \delta_*^\epsilon \partial_\mu \varphi_A \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_A)}. \tag{1.2.4-6}$$

Here, from the definition, Eq.(2), ∂_μ commutes with δ_*^ϵ. It is important to note that δ_*^ϵ is a particular case of a variation.[5] Hence, the form invariance of I automatically implies the form invariance of the field equations.

We now define the **Noether current** by

$$\epsilon J^\mu \equiv \delta_*^\epsilon \varphi_A \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_A)} + \delta^\epsilon x^\mu \cdot \mathcal{L}. \tag{1.2.4-7}$$

It satisfies the conservation law

$$\partial_\mu J^\mu = 0. \tag{1.2.4-8}$$

Indeed, with the help of Eqs.(1.2.3–2),(5), and (6), we have

$$\begin{aligned} \epsilon \partial_\mu J^\mu &= \delta_*^\epsilon \varphi_A \cdot \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_A)} + \partial_\mu \delta_*^\epsilon \varphi_A \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_A)} + \delta^\epsilon x^\mu \cdot \partial_\mu \mathcal{L} \\ &= \delta_*^\epsilon \varphi_A \cdot \frac{\partial \mathcal{L}}{\partial \varphi_A} + \delta_*^\epsilon \partial_\mu \varphi_A \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_A)} - \delta_*^\epsilon \mathcal{L} \\ &= 0. \end{aligned} \tag{1.2.4-9}$$

The existence of the conserved current J^μ is known as the **Noether theorem**.

From Eq.(8), we obtain

$$\partial_0 \int d\mathbf{x}\, J^0 = - \int d\mathbf{x}\, \partial_k J^k = 0, \tag{1.2.4-10}$$

that is, the **charge**

$$Q \equiv \int d\mathbf{x}\, J^0(x) \tag{1.2.4-11}$$

is time-independent. Here, however, we have assumed that J^k vanishes at spatial infinity and that the integral in Eq.(11) is convergent.

If Eq.(11) is not convergent, Q is ill-defined as charge operator, that is, we cannot consider eigenstates of Q nor expectation values of Q. But we can still meaningfully consider

[5] We forbid a *non-local* transformation of φ_A, which causes a serious operator-ordering problem in the transformed Lagrangian density.

$$[iQ, F(y)]_{\mp} \equiv i \int d\boldsymbol{x} [J^0(x), F(y)]_{\mp} \qquad (1.2.4-12)$$

for any *local* quantity $F(y)$, because $[J^0(x), F(y)]_{\mp}$ vanishes identically if $|\boldsymbol{x}|$ is sufficiently large.

We now calculate $[iQ, \varphi_A(y)]_{\mp}$ by means of the canonical (anti)commutation relations. Since Eq.(7) for $\mu = 0$ is

$$\varepsilon J^0 = \delta_{\bullet}^{\epsilon}\varphi_B \cdot \pi^B + \delta^{\epsilon} x^0 \cdot \mathcal{L}, \qquad (1.2.4-13)$$

we have

$$\begin{aligned}[i\varepsilon J^0(x), \varphi_A(y)]_{\mp 0} =&\delta_{\bullet}^{\epsilon}\varphi_A \cdot \delta(\boldsymbol{x}-\boldsymbol{y}) + i\epsilon_{(AB)}[\delta_{\bullet}^{\epsilon}\varphi_B(x), \varphi_A(y)]_{\mp 0}\pi^B(x) \\ &+ \delta^{\epsilon}x^0 \cdot \epsilon_{(AB)}\frac{\partial \Phi_B}{\partial \pi^A}\pi^B \delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (1.2.4-14)\end{aligned}$$

with the aid of Eq.(1.2.2–13). Since $\delta^{\epsilon}\varphi_B$ can be written in terms of primary fields without using time derivatives, it (anti)commutes with $\varphi_A(y)$ at the equal time. Accordingly, we have

$$\begin{aligned}[\delta_{\bullet}^{\epsilon}\varphi_B(x), \varphi_A(y)]_{\mp 0} &= -\delta^{\epsilon}x^{\mu} \cdot [\partial_{\mu}\varphi_B(x), \varphi_A(y)]_{\mp 0} \\ &= i\delta^{\epsilon}x^0 \cdot \frac{\partial \Phi_B}{\partial \pi^A}\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (1.2.4-15)\end{aligned}$$

owing to Eqs.(3) and (1.2.2–11). Thus the last two terms of Eq.(14) cancel out. We therefore obtain

$$\varepsilon[iQ, \varphi_A(y)]_{\mp} = \delta_{\bullet}^{\epsilon}\varphi_A(y), \qquad (1.2.4-16)$$

that is, Q is the **symmetry generator.**

1.2.5 Poincaré generators

Since the time coordinate plays a special role in the canonical quantization, one may feel uneasy about the manifest covariance of the theory. But Poincaré invariance is manifestly guaranteed in the canonical formalism as seen below.

For translations, since

$$\delta_{\bullet}^{\epsilon}\varphi_A = -\varepsilon^{\mu}\partial_{\mu}\varphi_A, \quad \delta^{\epsilon}x^{\nu} = \varepsilon^{\mu}\delta^{\nu}{}_{\mu}, \qquad (1.2.5-1)$$

the Noether theorem implies that the **canonical energy-momentum** (or **stress**) **tensor,**

$$\mathcal{T}^{\nu}{}_{\mu} \equiv \partial_\mu \varphi_A \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi_A)} - \delta^{\nu}{}_{\mu}\mathcal{L}, \qquad (1.2.5-2)$$

is conserved:

$$\partial_\nu \mathcal{T}^{\nu}{}_{\mu} = 0. \qquad (1.2.5-3)$$

Hence the translation generators

$$P_\mu \equiv \int d\boldsymbol{x}\, \mathcal{T}^{0}{}_{\mu}(x) \qquad (1.2.5-4)$$

satisfy

$$[iP_\mu, \varphi_A(x)] = \partial_\mu \varphi_A(x). \qquad (1.2.5-5)$$

We expect that P_μ's are well-defined. Then they are energy-momentum operators. Especially, P_0 is nothing but the Hamiltonian H.

For Lorentz transformations, since

$$\delta_*^{\epsilon} \varphi_A = -(1/2)\varepsilon^{\mu\nu}[(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta_A^{\ B} - i(s_{\mu\nu})_A^{\ B}]\varphi_B, \qquad (1.2.5-6)$$

$$\delta^{\epsilon} x^{\lambda} = (1/2)\varepsilon^{\mu\nu}(x_\mu \delta^{\lambda}_{\ \nu} - x_\nu \delta^{\lambda}_{\ \mu}), \qquad (1.2.5-7)$$

the Noether theorem implies that the **angular-momentum tensor,**

$$\begin{aligned}\mathcal{M}^{\lambda}{}_{\mu\nu} \equiv& [(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta_A^{\ B} - i(s_{\mu\nu})_A^{\ B}]\varphi_B \cdot \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \varphi_A)} - (x_\nu \delta^{\lambda}_{\ \nu} - x_\nu \delta^{\lambda}_{\ \mu})\mathcal{L} \\ =& x_\mu \mathcal{T}^{\lambda}{}_{\nu} - x_\nu \mathcal{T}^{\lambda}{}_{\mu} - i(s_{\mu\nu})_A^{\ B}\varphi_B \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \varphi_A)} \\ =& -\mathcal{M}^{\lambda}{}_{\nu\mu},\end{aligned} \qquad (1.2.5-8)$$

is conserved:

$$\partial_\lambda \mathcal{M}^{\lambda}{}_{\mu\nu} = 0. \qquad (1.2.5-9)$$

Hence the Lorentz generators

$$M_{\mu\nu} \equiv \int d\boldsymbol{x}\, \mathcal{M}^{0}{}_{\mu\nu}(x) = -M_{\nu\mu} \qquad (1.2.5-10)$$

satisfy

$$[iM_{\mu\nu}, \varphi_A(x)] = [(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta_A^{\ B} - i(s_{\mu\nu})_A^{\ B}]\varphi_A(x). \qquad (1.2.5-11)$$

Thus Eqs.(1.1.2–9) and (1.1.2–10) have been reproduced in the framework of the Lagrangian canonical formalism.

Since $\mathcal{T}_{\mu\nu}$ is not necessarily symmetric, it is convenient to symmetrize it in the following way. First, we set

$$\mathcal{S}^{\lambda}{}_{\mu\nu} \equiv -i(s_{\mu\nu})_A{}^B \varphi_B \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \varphi_A)} = -\mathcal{S}^{\lambda}{}_{\nu\mu}, \qquad (1.2.5-12)$$

and seek for the quantity $\mathcal{F}_{\lambda\mu\nu}$ satisfying

$$\mathcal{F}_{\lambda\mu\nu} = -\mathcal{F}_{\mu\lambda\nu}, \qquad (1.2.5-13)$$

$$\mathcal{F}_{\lambda\mu\nu} - \mathcal{F}_{\lambda\nu\mu} = \mathcal{S}_{\lambda\mu\nu}. \qquad (1.2.5-14)$$

The solution is

$$\mathcal{F}_{\lambda\mu\nu} \equiv (1/2)(\mathcal{S}_{\lambda\mu\nu} - \mathcal{S}_{\mu\lambda\nu} + \mathcal{S}_{\nu\mu\lambda}). \qquad (1.2.5-15)$$

Then, since Eqs.(9) and (8) imply

$$\begin{aligned} 0 =& \partial^\lambda \mathcal{M}_{\lambda\mu\nu} = \delta^\lambda{}_\mu \mathcal{T}_{\lambda\nu} - \delta^\lambda{}_\nu \mathcal{T}_{\lambda\mu} + \partial^\lambda \mathcal{S}_{\lambda\mu\nu} \\ =& \mathcal{T}_{\mu\nu} + \partial^\lambda \mathcal{F}_{\lambda\mu\nu} - (\mathcal{T}_{\nu\mu} + \partial^\lambda \mathcal{F}_{\lambda\nu\mu}), \end{aligned} \qquad (1.2.5-16)$$

we see

$$\Theta_{\mu\nu} = \Theta_{\nu\mu} \qquad (1.2.5-17)$$

if we set

$$\Theta_{\mu\nu} \equiv \mathcal{T}_{\mu\nu} + \partial^\lambda \mathcal{F}_{\lambda\mu\nu}. \qquad (1.2.5-18)$$

Furthermore, Eqs.(3) and (13) imply

$$\partial^\mu \Theta_{\mu\nu} = 0. \qquad (1.2.5-19)$$

We call $\Theta_{\mu\nu}$ the **symmetric energy-momentum tensor**. By using $\Theta_{\mu\nu}$, we can rewrite Eqs.(4) and (10) as

$$P_\mu = \int d\boldsymbol{x}\, \Theta_{0\mu}(x), \qquad (1.2.5-20)$$

$$M_{\mu\nu} = \int d\boldsymbol{x}\, [x_\mu \Theta_{0\nu}(x) - x_\nu \Theta_{0\mu}(x)], \qquad (1.2.5-21)$$

respectively.

We now proceed to considering the Poincaré algebra. Since Eq.(5) implies

$$[iP_\mu, F(x)] = \partial_\mu F(x) \qquad (1.2.5-22)$$

for any local operator $F(x)$, we have

$$[P_\mu, P_\nu] = -i\int d\boldsymbol{x}\,\partial_\mu \Theta_{0\nu} = 0, \qquad (1.2.5-23)$$

where use has been made of Eq.(19) for $\mu = 0$. Likewise, we have

$$[M_{\mu\nu}, P_\lambda] = i\int d\boldsymbol{x}(x_\mu \partial_\lambda \Theta_{0\nu} - x_\nu \partial_\lambda \Theta_{0\mu}). \qquad (1.2.5-24)$$

For $\lambda = k$, therefore, partial integrations lead us to

$$[M_{\mu\nu}, P_k] = i(\eta_{\nu k} P_\mu - \eta_{\mu k} P_\nu). \qquad (1.2.5-25)$$

For $\lambda = 0$, owing to Eq.(19) we have

$$\begin{aligned}[M_{\mu\nu}, P_0] =& i\int d\boldsymbol{x}(-x_\mu \partial^k \Theta_{k\nu} + x_\nu \partial^k \Theta_{k\mu}) \\ =& i\int d\boldsymbol{x}(\Theta_{\mu\nu} - \eta_{\mu 0}\Theta_{0\nu} - \Theta_{\nu\mu} + \eta_{\nu 0}\Theta_{0\mu}) \\ =& i(\eta_{\nu 0} P_\mu - \eta_{\mu 0} P_\nu). \end{aligned} \qquad (1.2.5-26)$$

Thus Eqs.(1.1.2–11) and (1.1.2–12) have been reproduced in the framework of the Lagrangian canonical formalism.

Unfortunately, however, a similar method does not work for establishing Eq.(1.1.2–13). But, except for $[M_{0k}, M_{0l}]$, we can reproduce it by direct calculation. Since

$$\mathcal{M}^0{}_{kl} = [(x_k\partial_l - x_l\partial_k)\delta_B^{\ C} - i(s_{kl})_B^{\ C}]\varphi_C \cdot \pi^B, \qquad (1.2.5-27)$$

the canonical commutation relations yield

$$[\mathcal{M}^0{}_{kl}(x), \pi^A(y)] = i[(x_k\partial_l^{\,x} - x_l\partial_k^{\,x})\delta_B^{\ A} - i(s_{kl})_B^{\ A}]\delta(\boldsymbol{x}-\boldsymbol{y})\cdot\pi^B(x), \quad (1.2.5-28)$$

whence

$$[iM_{kl}, \pi^A] = [(x_k\partial_l - x_l\partial_k)\delta_B^{\ A} + i(s_{kl})_B^{\ A}]\pi^B. \qquad (1.2.5-29)$$

Because $\mathcal{L}$ is Lorentz scalar, we should have

$$[iM_{\mu\nu}, \mathcal{L}] = (x_\mu \partial_\nu - x_\nu \partial_\mu)\mathcal{L}. \qquad (1.2.5-30)$$

Since

$$\mathcal{M}^0{}_{\lambda\rho} = [(x_\lambda \partial_\rho - x_\rho \partial_\lambda)\delta_A^{\ B} - i(s_{\rho\lambda})_A^{\ B}]\varphi_B \cdot \pi^A - (x_\lambda \delta_\rho^{\ 0} - x_\rho \delta_\lambda^{\ 0})\mathcal{L}, \quad (1.2.5-31)$$

we can calculate $[M_{kl}, M_{\lambda\rho}]$ by using Eqs.(11), (29), (30). In this way, after integrating by parts, we obtain

$$\begin{aligned} i[M_{kl}, M_{\lambda\rho}] = \int d\boldsymbol{x} \Big\{ & [(x_\lambda \partial_\rho - x_\rho \partial_\lambda)(x_k \partial_l - x_l \partial_k) \\ & - (x_k \partial_l - x_l \partial_k)(x_\lambda \partial_\rho - x_\rho \partial_\lambda)]\varphi_A \cdot \pi^A \\ & + \left[(s_{kl})_A^{\ B}(s_{\lambda\rho})_B^{\ C} - (s_{\lambda\rho})_A^{\ B}(s_{kl})_B^{\ C}\right] \varphi_C \pi^A \\ & + \left[x_k(\eta_{l\lambda}\delta_\rho^{\ 0} - \eta_{l\rho}\delta_\lambda^{\ 0}) - x_l(\eta_{k\lambda}\delta_\rho^{\ 0} - \eta_{k\rho}\delta_\lambda^{\ 0})\right] \mathcal{L} \Big\}. \end{aligned} \quad (1.2.5-32)$$

Then, with the aid of Eq.(1.1.2–7), it is straightforward to show that

$$i[M_{kl}, M_{\lambda\rho}] = -\eta_{l\lambda} M_{k\rho} + \eta_{k\lambda} M_{l\rho} - \eta_{k\rho} M_{l\lambda} + \eta_{l\rho} M_{k\lambda}. \qquad (1.2.5-33)$$

1.3 FREE FIELD THEORIES

In this section, after summarizing basic properties of the invariant singular functions, we describe some free field theories as the simplest examples of quantum field theory. Since they are elementary, we here present main results only.

1.3.1 Invariant singular functions

A free field, $\varphi(x)$, having a mass $m(\geq 0)$ usually satisfies the **Klein-Gordon equation**

$$(\Box + m^2)\varphi(x) = 0. \qquad (1.3.1-1)$$

There are two independent c-number Lorentz-invariant solutions to the Klein-Gordon equation:

$$\Delta(x; m^2) \equiv \frac{-i}{(2\pi)^3} \int d^4p\, \epsilon(p_0)\delta(p^2 - m^2)e^{-ipx}, \qquad (1.3.1-2)$$

$$\Delta^{(1)}(x; m^2) \equiv \frac{1}{(2\pi)^3} \int d^4p\, \delta(p^2 - m^2)e^{-ipx}. \qquad (1.3.1-3)$$

They are called **invariant delta functions**. They are not ordinary functions but generalized functions, whence they are also called **singular functions**. Both are real; $\Delta(x; m^2)$ is odd while $\Delta^{(1)}(x; m^2)$ is even under the sign change of x^μ. The first one, $\Delta(x; m^2)$, is the commutator function appearing in four-dimensional commutation relations, and correspondingly it vanishes in the spacelike region $x^2 < 0$. Furthermore, it is the solution to the following (singular) Cauchy problem:

$$(\Box + m^2)\Delta(x; m^2) = 0, \qquad (1.3.1-4)$$

$$\Delta(x; m^2)|_0 = 0, \qquad (1.3.1-5)$$

$$\partial_0\Delta(x; m^2)|_0 = -\delta(\mathbf{x}), \qquad (1.3.1-6)$$

where the symbol $|_0$ means to set $x^0 = 0$. On the other hand, $\Delta^{(1)}(x; m^2)$ has no such properties. The explicit expressions[1] for the invariant delta functions are as follows:

$$\Delta(x;m^2) = -\frac{1}{2\pi}\epsilon(x^0)\left[\delta(x^2) - \frac{mJ_1(m\sqrt{x^2})}{2\sqrt{x^2}}\theta(x^2)\right], \quad (1.3.1-7)$$

$$\Delta^{(1)}(x;m^2) = \frac{m}{4\pi\sqrt{x^2}}N_1(m\sqrt{x^2})\theta(x^2) + \frac{m}{2\pi^2\sqrt{-x^2}}K_1(m\sqrt{-x^2})\theta(-x^2), \quad (1.3.1-8)$$

where J_n and N_n are Bessel functions and K_n is a modified Bessel function. Near the lightcone $x^2 = 0$, they behave like

$$\Delta(x;m^2) = -(2\pi)^{-1}\epsilon(x^0)\left\{\delta(x^2) + m^2\left[-\frac{1}{4} + O(x^2)\right]\theta(x^2)\right\}, \quad (1.3.1-9)$$

$$\Delta^{(1)}(x^2;m^2) = -\frac{1}{2\pi^2}\left\{\mathrm{P}\frac{1}{x^2} - \frac{m^2}{4}\left[\log m^2|x^2| + O(1)\right]\right\}, \quad (1.3.1-10)$$

where P stands for the Cauchy principal value.

The positive/negative-frequency delta functions $\Delta^{(+)}(x;m^2)$ and $\Delta^{(-)}(x;m^2)$ are defined by

$$\Delta^{(\pm)}(x;m^2) = \mp(2\pi)^{-3}\int d^4p\,\theta(\pm p_0)\delta(p^2 - m^2)e^{-ipx}. \quad (1.3.1-11)$$

They are related to $\Delta(x;m^2)$ and $\Delta^{(1)}(x;m^2)$ through

$$i\Delta(x;m^2) = \Delta^{(+)}(x;m^2) + \Delta^{(-)}(x;m^2), \quad (1.3.1-12)$$

$$\Delta^{(1)}(x;m^2) = \Delta^{(+)}(x;m^2) - \Delta^{(-)}(x;m^2), \quad (1.3.1-13)$$

and they are mutually related through

$$\Delta^{(+)}(-x;m^2) = -\Delta^{(-)}(x;m^2), \quad (1.3.1-14)$$

$$[\Delta^{(+)}(x;m^2)]^* = -\Delta^{(-)}(x;m^2). \quad (1.3.1-15)$$

[1] The expressions in the complex D-dimensional spacetime are as follows:

$$\Delta_D(x;m^2) = -\frac{\epsilon(x^0)m^{(D/2)-1}}{2^{D/2}\pi^{(D/2)-1}}(\sqrt{x^2})^{1-(D/2)}J_{1-(D/2)}(m\sqrt{x^2})\theta(x^2),$$

$$\Delta_D^{(1)}(x;m^2) = \frac{m^{(D/2)-1}}{2^{D/2}\pi^{(D/2)-1}}[-(\sqrt{x^2})^{1-(D/2)}N_{1-(D/2)}(m\sqrt{x^2})\theta(x^2) + \frac{2}{\pi}(\sqrt{-x^2})^{1-(D/2)}K_{1-(D/2)}(m\sqrt{-x^2})\theta(-x^2)].$$

Lorentz-invariant solution to the equation

$$(\Box + m^2)G(x) = \delta^4(x) \qquad (1.3.1-16)$$

are called the **Green's functions** (of the Klein-Gordon operator). The following five Green's functions are of interest:

$$\overline{\Delta}(x;m^2) \equiv -(1/2)\epsilon(x^0)\Delta(x;m^2), \qquad (1.3.1-17)$$

$$\Delta_R(x;m^2) \equiv -\theta(x^0)\Delta(x;m^2), \qquad (1.3.1-18)$$

$$\Delta_A(x;m^2) \equiv \theta(-x^0)\Delta(x;m^2), \qquad (1.3.1-19)$$

$$i\Delta_F(x;m^2) \equiv \overline{\Delta}(x;m^2) + (i/2)\Delta^{(1)}(x;m^2), \qquad (1.3.1-20)$$

$$-i\Delta_{\overline{F}}(x;m^2) \equiv \overline{\Delta}(x;m^2) - (i/2)\Delta^{(1)}(x;m^2). \qquad (1.3.1-21)$$

We call $\Delta_R(x;m^2)$ the **retarded Green's function**, $\Delta_A(x;m^2)$ **advanced Green's function**, and $\Delta_F(x;m^2)$ the **causal Green's function** or **Feynman propagator**. The causal nature of $\Delta_F(x;m^2)$ becomes manifest by writing

$$\Delta_F(x;m^2) = \theta(x^0)\Delta^{(+)}(x;m^2) - \theta(-x^0)\Delta^{(-)}(x;m^2). \qquad (1.3.1-22)$$

The momentum representation of some of Green's functions are as follows:

$$\Delta_R(x;m^2) = -(2\pi)^{-4}\int d^4p\,\frac{e^{-ipx}}{p^2 - m^2 + i0p_0}, \qquad (1.3.1-23)$$

$$\Delta_A(x;m^2) = -(2\pi)^{-4}\int d^4p\,\frac{e^{-ipx}}{p^2 - m^2 - i0p_0}, \qquad (1.3.1-24)$$

$$\Delta_F(x;m^2) = i(2\pi)^{-4}\int d^4p\,\frac{e^{-ipx}}{p^2 - m^2 + i0}. \qquad (1.3.1-25)$$

In the massless ($m = 0$) case, the invariant delta functions are particularly called **invariant D functions**, and expressed by using D in place of Δ. From Eqs.(9) and (10), we have

$$D(x) = -(2\pi)^{-1}\epsilon(x^0)\delta(x^2), \qquad (1.3.1-26)$$

$$D^{(1)}(x) = -\frac{1}{2\pi^2}\mathrm{P}\frac{1}{x^2}. \qquad (1.3.1-27)$$

In gauge theories, $D(x)$ is very important; it is called the **Pauli-Jordan D function**. From Eqs.(26) and (27), it is straightforward to see

$$D^{(\pm)}(x) = \mp\frac{1}{4\pi^2}\cdot\frac{1}{x^2 \mp i0x^0}. \qquad (1.3.1-28)$$

As for the Green's functions, we have

$$D_R(x) = (2\pi)^{-1}\theta(x^0)\delta(x^2), \tag{1.3.1 – 29}$$

$$D_A(x) = (2\pi)^{-1}\theta(-x^0)\delta(x^2), \tag{1.3.1 – 30}$$

$$D_F(x) = -\frac{1}{4\pi^2}\cdot\frac{1}{x^2 - i0}. \tag{1.3.1 – 31}$$

1.3.2 Free real scalar field

We consider a free **real** (or **neutral**) **scalar field** $\phi(x)$ having a mass $m(\geq 0)$. Its Lagrangian density is given by

$$\mathcal{L}_S \equiv \frac{1}{2}(\partial^\mu\phi\cdot\partial_\mu\phi - m^2\phi^2). \tag{1.3.2 – 1}$$

Its field equation is, of course, the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0. \tag{1.3.2 – 2}$$

The canonical conjugate of ϕ is $\pi = \dot{\phi}$, and canonical quantization is carried out. Then it is easy to calculate the four-dimensional commutator

$$[\phi(x), \phi(y)] = i\Delta(x - y; m^2). \tag{1.3.2 – 3}$$

The canonical energy-momentum tensor is

$$\mathcal{T}_{\mu\nu} \equiv \partial_\mu\phi\cdot\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}_S, \tag{1.3.2 – 4}$$

which coincides with the symmetric energy-momentum tensor because ϕ is spinless. Poincaré generators are constructed in a straightforward way. In particular, the Hamiltonian is

$$H \equiv \frac{1}{2}\int d\boldsymbol{x}\,[\pi^2 + (\boldsymbol{\partial}\varphi)^2 + m^2\phi^2]. \tag{1.3.2 – 5}$$

Since Eq.(2) is linear, it is solved by means of Fourier transform. That is, we have the momentum-space representation

$$\phi(x) = (2\pi)^{-3/2}\int d\boldsymbol{p}(2\hat{p}_0)^{-1/2}\left[a(\hat{p})e^{-i\hat{p}x} + a^\dagger(\hat{p})e^{i\hat{p}x}\right], \tag{1.3.2 – 6}$$

where $\hat{p}_0 \equiv \sqrt{\boldsymbol{p}^2 + m^2}$ and $\hat{p}_\mu = (\hat{p}_0, p_k)$, so that $\hat{p}^2 = m^2$. The **annihilation operator** $a(\hat{p})$ and the **creation operation** $a^\dagger(\hat{p})$ satisfy the commutation relations

$$[a(\hat{p}), a^\dagger(\hat{q})] = \delta(\boldsymbol{p} - \boldsymbol{q}), \qquad (1.3.2-7)$$

$$[a(\hat{p}), a(\hat{q})] = [a^\dagger(\hat{p}), a^\dagger(\hat{q})] = 0. \qquad (1.3.2-8)$$

The **vacuum** $|0\rangle$ is defined by

$$a(\hat{p})|0\rangle = 0. \qquad (1.3.2-9)$$

The Hamiltonian is rewritten as

$$H = \frac{1}{2}\int d\boldsymbol{p}\,\hat{p}_0\left[a(\hat{p})a^\dagger(\hat{p}) + a^\dagger(\hat{p})a(\hat{p})\right]. \qquad (1.3.2-10)$$

Unfortunately, $H|0\rangle$ is divergent. Hence we subtract $\langle 0|H|0\rangle$ from H; then the redefined Hamiltonian is

$$H = \int d\boldsymbol{p}\,\hat{p}_0 a^\dagger(\hat{p})a(\hat{p}), \qquad (1.3.2-11)$$

which satisfies

$$H|0\rangle = 0, \qquad (1.3.2-12)$$

and all other eigenvalues of H are positive definite.

In a free field theory, one can count the absolute number of particles. The **number operator** N is defined by

$$N \equiv \int d\boldsymbol{p}\, a^\dagger(\hat{p})a(\hat{p}), \qquad (1.3.2-13)$$

whose eigenvalues are positive semi-definite. The space spanned by all eigenstates of N is (after completion) called the **Fock space**. Fock representation is characterized by the existence of a well-defined (absolute) number operator.

1.3.3 Free complex scalar field

We consider a free **complex** (or **charged**) **scalar field** $\phi(x)$ having a mass m. Its Lagrangian density is given by

$$\mathcal{L}_{\mathrm{CS}} = \partial^\mu\phi^\dagger\cdot\partial_\mu\phi - m^2\phi^\dagger\phi. \qquad (1.3.3-1)$$

If we set $\sqrt{2}\phi = \phi_1 + i\phi_2$, this theory reduces to the theory of two free real scalar fields ϕ_1 and ϕ_2, but it is more convenient not to do so. We regard ϕ and $\phi^\dagger$ as *independent* canonical variables; we therefore have $\pi = \dot{\phi}^\dagger$ and $\pi^\dagger = \dot{\phi}$.

The field equation is, of course,

$$(\Box + m^2)\phi = 0. \tag{1.3.3 – 2}$$

The four-dimensional commutation relations are

$$[\phi(x), \phi^\dagger(y)] = i\Delta(x - y; m^2), \tag{1.3.3 – 3}$$

$$[\phi(x), \phi(y)] = 0. \tag{1.3.3 – 4}$$

The canonical energy-momentum tensor is

$$\mathcal{T}_{\mu\nu} = \partial_\mu\phi \cdot \partial_\nu\phi^\dagger + \partial_\mu\phi^\dagger \cdot \partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}_{\mathrm{CS}}. \tag{1.3.3 – 5}$$

This theory has an internal symmetry; $\mathcal{L}_{\mathrm{CS}}$ is invariant under the **phase transformation**

$$\phi \rightarrow \phi' = e^{-i\theta}\phi, \qquad \phi^\dagger \rightarrow \phi'^\dagger = e^{i\theta}\phi^\dagger, \tag{1.3.3 – 6}$$

where θ is a real parameter. The corresponding Noether current is

$$J^\mu \equiv i(\phi^\dagger\partial^\mu\phi - \phi\partial^\mu\phi^\dagger). \tag{1.3.3 – 7}$$

Hence the generator of the phase transformation is

$$Q \equiv i\int d\boldsymbol{x}(\phi^\dagger\partial_0\phi - \phi\partial_0\phi^\dagger); \tag{1.3.3 – 8}$$

indeed

$$[iQ, \phi(x)] = -i\phi(x), \tag{1.3.3 – 9}$$

$$[iQ, \phi^\dagger(x)] = i\phi^\dagger(x). \tag{1.3.3 – 10}$$

1.3.4 Free Dirac field

The field of spin 1/2 is called a **Dirac field** or a **spinor field**. Usually, it is a complex field[2] and consists of four components.[3] It obeys Fermi statistics. The free Dirac field $\psi(x)$ satisfies the **Dirac equation**

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \qquad (1.3.4-1)$$

where the **gamma matrices** γ^μ are 4×4 matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad (1.3.4-2)$$

so that ψ satisfies the Klein-Gordon equation. We employ the convention $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{k\dagger} = -\gamma^k$, whence $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$. We set

$$\gamma_5 \equiv i(4!)^{-1}\epsilon_{\mu\nu\sigma\tau}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau = i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad (1.3.4-3)$$

whence $\{\gamma_5, \gamma^\mu\} = 0$ and ${\gamma_5}^\dagger = \gamma_5$, and

$$\sigma^{\mu\nu} \equiv (1/4)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \qquad (1.3.4-4)$$

The sixteen matrices

$$\{1,\ \gamma^\mu,\ 2i\sigma^{\mu\nu},\ \gamma^\mu\gamma_5,\ -i\gamma_5\} \qquad (1.3.4-5)$$

are linearly independent. Hence the representation of Eq.(2) by 4×4 matrices is irreducible.[4]

The Dirac equation is derived from the Lagrangian density

$$\mathcal{L}_\mathrm{D} \equiv \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi, \qquad (1.3.4-6)$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$. Although $\mathcal{L}_\mathrm{D}$ is not hermitian, we can hermitize it by integrating one half of the first term by parts.[5] The canonical variables are ψ_α ($\alpha = 1, 2, 3, 4$)

[2] If it is a real field, it is called a **Majorana field**.

[3] If massless, it can consist of only two components.

[4] In D dimensions, the irreducible representation of Eq.(2) is given by $2^{[D/2]} \times 2^{[D/2]}$ matrices, where $[k]$ denotes the largest integer not larger than k. If D is odd, there is no γ_5. For D even, $\gamma_5 = \prod_{\mu=0}^{D-1}\gamma^\mu$ if $D/2$ is odd and $\gamma_5 = i\prod_{\mu=0}^{D-1}\gamma^\mu$ otherwise.

[5] In the hermitized case, canonical quantization should be carried out by means of the Dirac method (see Addendum 1.A).

only, and their canonical conjugates are $\pi_\alpha = -\psi_\alpha^\dagger$ (the minus sign is due to the fermionic nature of ψ). The four-dimensional anticommutation relations are

$$\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = iS_{\alpha\beta}(x-y;m), \qquad (1.3.4-7)$$

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)\} = 0, \qquad (1.3.4-8)$$

where

$$S(z;m) \equiv (i\gamma^\mu \partial_\mu + m)\Delta(z;m^2). \qquad (1.3.4-9)$$

The infinitesimal Lorentz transformation matrix $s_{\mu\nu}$ of the Dirac theory is given by $i\sigma_{\mu\nu}$. Since $\mathcal{L}_{\mathrm{D}}$ vanishes when the Dirac equation is used, the Poincaré generators simply become

$$P_\mu = i\int d\boldsymbol{x}\,\psi^\dagger \partial_\mu \psi, \qquad (1.3.4-10)$$

$$M_{\mu\nu} = i\int d\boldsymbol{x}\,\psi^\dagger (x_\mu\partial_\nu - x_\nu\partial_\mu + \sigma_{\mu\nu})\psi. \qquad (1.3.4-11)$$

The Dirac theory is invariant under the phase transformation

$$\psi \;\to\; \psi' = e^{-i\theta}\psi, \qquad \psi^\dagger \;\to\; \psi'^\dagger = \psi^\dagger e^{i\theta}, \qquad (1.3.4-12)$$

and if $m = 0$ under the **chiral transformation**

$$\psi \;\to\; \psi' = \exp(-i\theta\gamma_5)\psi, \qquad \psi^\dagger \;\to\; \psi'^\dagger = \psi^\dagger \exp(i\theta\gamma_5). \qquad (1.3.4-13)$$

The corresponding Noether currents are

$$J^\mu = \bar{\psi}\gamma^\mu\psi, \qquad (1.3.4-14)$$

$$J_5{}^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi, \qquad (1.3.4-15)$$

respectively.

1.4 GENERAL CONSIDERATION IN THE HEISENBERG PICTURE

In Sec.1.2, we have discussed the general framework of the operator formalism of quantum field theory. In this section, we introduce state vectors to represent field operators. Of course, it is natural that the notion of states should have no explicit dependence on spacetime coordinates, as long as the theory contains no external force. Nevertheless, it is often made to describe the theory by transferring the time dependence of field operators to state vectors totally (**Schrödinger picture**) or partially (**interaction picture**, etc.). The original description is, correspondingly, called **Heisenberg picture** in order to discriminate it from other pictures. One should not forget, however, that only the Heisenberg picture is fundamental and that any other picture is derived from it. In this sense, the construction of the theory in the Heisenberg picture is the most important from the theoretical point of view.

1.4.1 Two-point functions

As mentioned in Sec.1.1.1, we postulate the unique existence of a distinguished state, called **vacuum**, which is denoted by $|0\rangle$. It is a state belonging to discrete spectrum, normalized to unity, i.e.,

$$\langle 0|0\rangle = 1, \tag{1.4.1 - 1}$$

and is Poincaré-invariant, i.e.,

$$P_\mu|0\rangle = 0, \qquad M_{\mu\nu}|0\rangle = 0. \tag{1.4.1 - 2}$$

We admit the case in which Eqs.(1) and (2) do not uniquely characterize $|0\rangle$. Its precise characterization is given later.

As shown in Sec.1.3, the four-dimensional (anti)commutator of field operators can be explicitly calculated in free field theories. It is no longer possible to do so if the interaction Lagrangian density is present. It is, however, possible to investigate the structure of the **vacuum expectation value**, $\langle 0| \cdots |0\rangle$, of the four-dimensional (anti)commutator by means of the general postulates of quantum field theory.

For simplicity of description, we consider a (non-free) real scalar field $\phi(x)$. We first investigate

$$W(x,y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle, \tag{1.4.1-3}$$

which is called the **two-point function**. From Eq.(1.1.2-3) we can write

$$\phi(x) = U(x,1)\phi(0)U^{-1}(x,1). \tag{1.4.1-4}$$

Furthermore, since P_μ's are mutually commuting, Eq.(1.1.2-8) leads us to

$$U(x,1) = \exp ix^\mu P_\mu \tag{1.4.1-5}$$

for any finite value of x^μ. Substituting Eqs.(4) and (5) into Eq.(3) and using Eq.(2), we obtain

$$W(x,y) = \langle 0|\phi(0)e^{-i(x-y)P}\phi(0)|0\rangle. \tag{1.4.1-6}$$

If the state-vector space $\mathcal{V}$ is a Hilbert space, the eigenstates of P_μ form a complete set, that is, we can formally write[1]

$$P_\mu|n\rangle = p_\mu{}^{(n)}|n\rangle, \tag{1.4.1-7}$$

$$\sum_n |n\rangle\langle n| = 1, \tag{1.4.1-8}$$

where $\sum_n$ includes integration over $p_\mu{}^{(n)}$. Inserting Eq.(8) into Eq.(6), we have

$$W(x,y) = \sum_n \exp\left[-i(x-y)p^{(n)}\right] |\langle n|\phi(0)|0\rangle|^2 . \tag{1.4.1-9}$$

Because of the spectrum condition $(p^{(n)})^2 \geq 0$ with $p_0{}^{(n)} \geq 0$, we may write Eq.(9) as

$$W(x,y) = \int d^4p\, e^{-i(x-y)p}(2\pi)^{-3}\theta(p_0)\rho(p^2), \tag{1.4.1-10}$$

where

$$\rho(p^2) = \int_0^\infty ds\, \delta(s-p^2)\rho(s) \geq 0. \tag{1.4.1-11}$$

Using Eq.(1.3.1-11), we obtain

$$W(x,y) = \int_0^\infty ds\, \Delta^{(+)}(x-y;s)\rho(s). \tag{1.4.1-12}$$

[1] Strictly speaking, we should consider wave-packet states rather than non-normalizable eigenstates of P_μ.

This formula is called the **spectral representation** of $W(x,y)$, and $\rho(s)$ is called **a spectral function**.

From Eq.(12), we immediately obtain

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = i\int_0^\infty ds\,\Delta(x-y;s)\rho(s). \qquad (1.4.1-13)$$

The **time-ordered product** is defined by

$$\mathrm{T}\phi(x)\phi(y) \equiv \theta(x^0-y^0)\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x), \qquad (1.4.1-14)$$

for which

$$\langle 0|\mathrm{T}\phi(x)\phi(y)|0\rangle = \int_0^\infty ds\,\Delta_F(x-y;s)\rho(s). \qquad (1.4.1-15)$$

We differentiate Eq.(13) by x^0 and set $x^0 = y^0$. Then, owing to the canonical commutation relation[2] and Eq.(1.3.1-6), we have

$$-i\delta(\boldsymbol{x}-\boldsymbol{y}) = -i\int_0^\infty ds\,\delta(\boldsymbol{x}-\boldsymbol{y})\rho(s), \qquad (1.4.1-16)$$

namely,

$$\int_0^\infty ds\,\rho(s) = 1. \qquad (1.4.1-17)$$

If particle contents of the theory are known, the spectrum condition can be made more precise. If there are only one kind of scalar particles having a physical mass m_r, which is generally different from m, then we have $p^2 = m_r{}^2$ for one-particle states and $p^2 \geq (2m_r)^2$ for many-particle states. In this case, we can write

$$\rho(s) = Z\delta(s-m_r{}^2) + \sigma(s)\theta(s-4m_r{}^2) \qquad (1.4.1-18)$$

with $Z \geq 0$ and $\sigma(s) \geq 0$. Substituting Eq.(18) into Eq.(17), we find

$$1 - Z = \int_{4m_r{}^2}^\infty ds\,\sigma(s), \qquad (1.4.1-19)$$

whence

$$0 \leq Z \leq 1. \qquad (1.4.1-20)$$

[2] We assume that the interaction Lagrangian density does not contain $\dot{\phi}$.

If $Z = 1$, then $\sigma(s) \equiv 0$, that is, we have

$$\langle 0|\phi(x)\phi(y)|0\rangle = \Delta^{(+)}(x-y; m_r^{\,2}), \qquad (1.4.1-21)$$

whence

$$\langle 0|(\Box + m_r^{\,2})\phi(x)\cdot(\Box + m_r^{\,2})\phi(y)|0\rangle = 0. \qquad (1.4.1-22)$$

The metric positivity, Eq.(1.1.1-3), therefore, implies

$$(\Box + m_r^{\,2})\phi(x)|0\rangle = 0. \qquad (1.4.1-23)$$

Then the separating property of vacuum (see Corollary A.2-6 in Appendix A.2) implies

$$(\Box + m_r^{\,2})\phi(x) = 0. \qquad (1.4.1-24)$$

Thus $\phi(x)$ is a free field, that is, as long as $\phi(x)$ is non-free, we have $Z < 1$.

Since the coefficient of the discrete spectrum is $Z < 1$, it is convenient to *re*-normalize it to unity. We therefore consider

$$\phi^{(r)}(x) \equiv Z^{-1/2}\phi(x), \qquad (1.4.1-25)$$

and call it a **renormalized field**. Correspondingly, Z is called a (wave-function) **renormalization constant**.

In the above discussion, we have assumed that the canonical commutation relations are consistent with the general principles of the theory including the metric positivity of $\mathcal{V}$. But such an optimistic standpoint may cause troubles. Consider a current

$$j^\mu \equiv \bar{\psi}\gamma^\mu\psi. \qquad (1.4.1-26)$$

If we naively apply the canonical anticommutation relations, we find

$$[j^0(x), j^k(y)]_0 = 0. \qquad (1.4.1-27)$$

On the other hand, the spectral representation consistent with $\partial_\mu j^\mu = 0$ is

$$\langle 0|[j^\mu(x), j^\nu(y)]|0\rangle = i\int_0^\infty ds\, \pi(s)(-s\eta^{\mu\nu} - \partial^\mu\partial^\nu)\Delta(x-y; s), \qquad (1.4.1-28)$$

where the metric positivity implies

$$\pi(s) \geq 0 \quad (\pi(s) \not\equiv 0). \qquad (1.4.1-29)$$

Hence we have

$$\begin{aligned}\langle 0|[j^0(x), j^k(y)]_0|0\rangle &= i\int_0^\infty ds\,\pi(s)\partial^k\delta(\mathbf{x}-\mathbf{y}) \\ &\neq 0 \end{aligned} \qquad (1.4.1-30)$$

in contradiction with Eq.(27)[Got 55, Sch 59].

The widely accepted resolution of the difficulty is to distrust the canonical result, Eq.(27). Indeed, if one calculates $[j^0, j^k]_0$ by point-splitting, one finds that it does not vanish at the coinciding limit. Such a non-canonical term as Eq.(30) is generally called a **Schwinger term** [Sch 59]. However, the Schwinger term is a very pathological concept; it violates the Jacobi identity [Joh 66] and its expression is generally dependent on the method of calculation. Furthermore, no physical result follows from the Schwinger term. In Sec.5.8.3, a possible resolution which respects the canonical result is proposed: The appearance of the Schwinger term is due to the negligence of gravity. Only in the two-dimensional spacetime, the Schwinger term is significant [cf. Eq.(2.5.3-15)] because it is finite and because there is no gravity.

1.4.2 Asymptotic fields

In particle-physics experiments, what are really observed are not fields but particles. In the scattering experiments, two particles collide with each other and after complicated interactions some particles are observed. Both before and after the collision, particles are so distantly located from each other that they can be regarded as free.[3] That is, in the remote past and in the remote future compared with the interaction time, we encounter only free particles. It is thus quite important to describe free particle nature in the asymptotic regions $x^0 \to \pm\infty$.

As a simple example, we consider a scalar field $\phi^{(r)} = Z^{-1/2}\phi$ satisfying a (renormalized) field equation

$$(\Box + m_r{}^2)\phi^{(r)}(x) = J(x), \qquad (1.4.2-1)$$

where we assume that $J(x)$ has *no discrete spectrum* on mass shell. We can integrate Eq.(1) as

$$\phi^{(r)}(x) = \phi^{\rm in}(x) + \int d^4y\,\Delta_R(x-y; m_r{}^2)J(y), \qquad (1.4.2-2)$$

3 If there are massless particles, special care is required.

$$\phi^{(r)}(x) = \phi^{\text{out}}(x) + \int d^4y\, \Delta_A(x-y; m_r^2) J(y), \qquad (1.4.2-3)$$

where

$$(\Box + m_r^2)\phi^{\text{as}}(x) = 0 \qquad (1.4.2-4)$$

with $\phi^{\text{as}} = \phi^{\text{in}}$ or ϕ^{out}. The integral equations, Eqs.(2) and (3), are called **Yang-Feldman equations** [Yan 50].

From the definitions, Eqs.(1.3.1-18) and (1.3.1-19), of Δ_R and Δ_A, we can infer that

$$\phi^{(r)}(x) \to \phi^{\text{in}}(x) \quad \text{as} \quad x^0 \to -\infty, \qquad (1.4.2-5)$$

$$\phi^{(r)}(x) \to \phi^{\text{out}}(x) \quad \text{as} \quad x^0 \to +\infty. \qquad (1.4.2-6)$$

In this sense, $\phi^{\text{in}}(x)$ and $\phi^{\text{out}}(x)$ are called an **in-field** and an **out-field**, respectively. They are altogether called **asymptotic fields** and denoted by $\phi^{\text{as}}(x)$.

From Eq.(1.3.1-18) we have

$$\partial_0\Delta_R(x-y; m_r^2) = -\theta(x^0-y^0)\partial_0\Delta(x-y; m_r^2) - \delta(x^0-y^0)\Delta(x-y; m_r^2), \qquad (1.4.2-7)$$

but the last term vanishes identically. Hence we can infer

$$\partial_0\phi^{(r)}(x) \to \partial_0\phi^{\text{in}}(x) \quad \text{as} \quad x^0 \to -\infty. \qquad (1.4.2-8)$$

It should be noted, however, that the same thing is no longer true for $\partial_0{}^2\phi^{(r)}$ because we then encounter a non-vanishing term

$$-\delta(x^0-y^0)\partial_0\Delta(x-y; m_r^2) = \delta^4(x-y). \qquad (1.4.2-9)$$

The equal-time commutators concerning ϕ^{in} and $\partial_0\phi^{\text{in}}$ are the same as those concerning $\phi^{(r)}$ and $\partial_0\phi^{(r)}$ because the second term of Eq.(2) essentially vanishes as $x^0 \to -\infty$. Combining this result with Eq.(4), we obtain

$$[\phi^{\text{in}}(x), \phi^{\text{in}}(y)] = i\Delta(x-y; m_r^2). \qquad (1.4.2-10)$$

Likewise for ϕ^{out}. Thus ϕ^{as} is a free field. But we must remember that ϕ^{as} is a *non-local* field in the Heisenberg picture.

Although the above reasoning clarifies what the asymptotic field is, the mathematical meaning of Eqs.(5) and (6) is not clear. The asymptotic equality between

$\phi^{(r)}(x)$ and $\phi^{as}(x)$ can be defined in the following sense [Leh 55]: For any two states $|f\rangle$ and $|g\rangle \in \mathcal{V}$,

$$\langle f|\phi^{(r)}(x) - \phi^{as}(x)|g\rangle \ \to\ 0 \quad \text{as} \quad x^0 \ \to\ \mp\infty. \qquad (1.4.2-11)$$

Such convergence as above is called **weak convergence** in mathematics, in order to discriminate it from the norm convergence in the mathematical sense (**strong convergence**). It should be noted that weak convergence is meaningful even if $\mathcal{V}$ is an indefinite-metric Hilbert space. Weak convergence is really "weak" because the weak limit of a product of two operators is *not*, in general, equal to the product of the weak limits of those operators. Indeed, a product of primary fields may have an asymptotic field different from any primary-field's asymptotic fields. In general, if the two-point function $\langle 0|\Phi_1(x)\Phi_2(y)|0\rangle$ for *local* operators $\Phi_1(x)$ and $\Phi_2(y)$ has a discrete spectrum, then there exist asymptotic fields ${\Phi_1}^{as}$ and ${\Phi_2}^{as}$ such that

$$\langle 0|{\Phi_1}^{as}(x){\Phi_2}^{as}(y)|0\rangle = \ \text{discrete spectrum of}\ \ \langle 0|\Phi_1(x)\Phi_2(y)|0\rangle. \qquad (1.4.2-12)$$

Since $[{\Phi_1}^{as}(x), {\Phi_2}^{as}(y)]_{\mp}$ is a c-number [Gre 62, Rob 62], it is equal to the discrete spectrum of $\langle 0|\,[\Phi_1(x), \Phi_2(y)]_{\mp}\,|0\rangle$.

The (anti)commutator between a symmetry generator Q and a local operator $\Phi(x)$ is not spoiled by taking weak limit, provided that the symmetry is not broken. That is, if

$$[iQ, \Phi(x)]_{\mp} = \Psi(x), \qquad (1.4.2-13)$$

then we have

$$[iQ, \Phi^{as}(x)]_{\mp} = \Psi^{as}(x). \qquad (1.4.2-14)$$

In particular, from the commutators with the Poincaré generators, we obtain the covariance of asymptotic fields under the Poincaré algebra. From Eq.(14) we see that Q is expressible as an integral over a quadratic function of asymptotic fields.

1.4.3 Asymptotic states and asymptotic completeness

Since asymptotic fields are free fields, we can construct their Fock representation on the basis of the vacuum $|0\rangle$. Since the vacuum of the Fock space is unique, this fact uniquely characterizes $|0\rangle$ even when its Poincaré invariance cannot do so.

The state-vectors of the above Fock space are called **asymptotic states**. We call an in-field asymptotic state an **in-state** and an out-field one an **out-state**. The Fock space spanned by all in-states is denoted by $\mathcal{V}^{\text{in}}$; $\mathcal{V}^{\text{out}}$ is similarly defined. In the axiomatic field theory [Haa 58, Haa 59, Rue 62], it is rigorously proved that the (wave-packet) asymptotic states exist as strong limits of some states in $\mathcal{V}$, provided that $\mathcal{V}$ is a Hilbert space and that there exists a finite gap between the discrete spectrum and the continuous one. Therefore, both $\mathcal{V}^{\text{in}}$ and $\mathcal{V}^{\text{out}}$ are subspaces of $\mathcal{V}$.

We now introduce a very crucial postulate, called **asymptotic completeness**: We postulate that $\mathcal{V}^{\text{in}} = \mathcal{V}$. Then the PCT theorem (see Appendix A.2) implies $\mathcal{V}^{\text{out}} = \mathcal{V}$. We thus have

$$\mathcal{V}^{\text{in}} = \mathcal{V} = \mathcal{V}^{\text{out}}. \tag{1.4.3 - 1}$$

The asymptotic completeness is important because *it is the only known general principle which uniquely determines the representation space of field operators.* We may say that the asymptotic completeness is a certain minimality requirement because any other representation space must be *larger*, that is, it contains non-asymptotic extra states in addition to all asymptotic states. The choice of such extra states is not only necessarily model-dependent but also often *"author-dependent"* (see Sec.2.5). This fact may lead authors to futile controversy. Furthermore, extra states are usually devoid of particle contents, whence it is quite questionable in what way they are observed, unless one can invent a mechanism by which they become unobservable.

Under the asymptotic completeness, any operator is (at least formally) expandable into a series of normal products of asymptotic fields, where **normal product** is a product of free fields such that any annihilation operator lies on the right of any creation operator. This is because we can always choose expansion coefficients in such a way that all matrix elements of that operator in terms of asymptotic states coincide with those of that series. In this sense, under the asymptotic completeness, *any operator is expressible in terms of asymptotic fields.*

Since $\mathcal{V}^{\text{in}} = \mathcal{V}^{\text{out}}$, the transformation between the in-states $|\text{in}\rangle$ and the out-states $|\text{out}\rangle$ is a unitary transformation S, called **S-matrix**, because it is symbolically written as

$$S = \{\langle \text{out}|\text{in}\rangle\}. \tag{1.4.3 - 2}$$

More precisely, S should be called **S-operator**; it is defined by

$$\varphi^{out}(x) = S^{-1}\varphi^{in}(x)S, \tag{1.4.3 – 3}$$

with $S^{-1} = S^{\dagger}$ and

$$S|0\rangle = |0\rangle. \tag{1.4.3 – 4}$$

The S-operator S commutes with any symmetry generator Q because of the time independence of Q. Especially, S commutes with the Poincaré generators. Since one-particle states are uniquely specified by the eigenvalue of P_μ and spin component, the one-particle restriction of S is a unit operator. The expression for Q in terms of φ^{out} has the same form as that in term of φ^{in}, because if $Q = F(\varphi^{\text{in}})$ then

$$\begin{aligned} F(\varphi^{\text{out}}) &= F(S^{-1}\varphi^{\text{in}}S) = S^{-1}F(\varphi^{\text{in}})S \\ &= S^{-1}QS = Q. \end{aligned} \tag{1.4.3 – 5}$$

Hence we may write $Q = F(\varphi^{\text{as}})$.

1.4.4 Reduction formula

The S-operator introduced in Sec.1.4.3 is the concept of practical importance because the transition probabilities of particle reactions are essentially given by the absolute squares of its matrix elements. It is therefore very important to express S in terms of field operators in a more direct way. The most standard way of doing this is called **Lehmann-Symanzik-Zimmermann** (or **LSZ**) **formalism** [Leh 55, Leh 57].

For simplicity of description, we consider only a real scalar field $\phi(x)$. The Klein-Gordon operator $\Box + m_r^{\,2}$ is denoted by K. We intoduce a complete set of orthonormal positive-frequency solutions, $f_k(x)$, to the Klein-Gordon equation:

$$Kf_k = 0, \tag{1.4.4 – 1}$$

$$i\int d\mathbf{x}\,[f_k^*(x)\partial_0 f_l(x) - \partial_0 f_k^*(x)\cdot f_l(x)] = \delta_{kl}, \tag{1.4.4 – 2}$$

$$\sum_k f_k(x)f_k^*(y) = \Delta^{(+)}(x-y; m_r^{\,2}). \tag{1.4.4 – 3}$$

We then define

$$\phi_k(x^0) \equiv i \int d\boldsymbol{x}\, [\phi(x)\partial_0 f_k(x) - \partial_0\phi(x)\cdot f_k(x)], \tag{1.4.4 – 4}$$

and introduce the asymptotic-field creation operators ${\phi^{\text{in}}}_k^\dagger$ and ${\phi^{\text{out}}}_k^\dagger$ as the weak $x^0 \to \mp\infty$ limits of $\phi_k(x^0)$; of course, ${\phi^{\text{as}}}_k^\dagger$ is independent of time.

The **time-ordered product** of field operators is defined by

$$\mathrm{T}\phi(x_1)\cdots\phi(x_n) \equiv \sum_\sigma \left[\prod_{j=1}^{n-1}\theta(x_{\sigma(j)} - x_{\sigma(j+1)})\right]\phi(x_{\sigma(1)})\cdots\phi(x_{\sigma(n)}), \tag{1.4.4 – 5}$$

where σ stands for a permutation of $\{1, 2, \cdots, n\}$ and the summation runs over all permutations. The vacuum expectation value of Eq.(5) is called a **τ-function**:

$$\tau(x_1, \cdots, x_n) \equiv \langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)|0\rangle. \tag{1.4.4 – 6}$$

We first prove that

$$\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)\cdot {\phi^{\text{in}}}_k^\dagger|0\rangle = i\int d^4y\, K^y\,\tau(x_1, \cdots, x_n, y)\cdot f_k(y). \tag{1.4.4 – 7}$$

We substitute

$$ {\phi^{\text{in}}}_k^\dagger = i \lim_{y^0\to-\infty}\int d\boldsymbol{y}[\phi(y)\partial_0 f_k(y) - \partial_0\phi(y)\cdot f_k(y)] \tag{1.4.4 – 8}$$

into the lhs. of Eq.(7). Since $x_j^0 > y^0$ for any j, $\phi(y)$ can be included under the T symbol. Then, by using

$$\langle 0|{\phi^{\text{out}}}_k^\dagger \mathrm{T}\phi(x_1)\cdots\phi(x_n)|0\rangle = 0, \tag{1.4.4 – 9}$$

we can rewrite the integral as follows:

$$\lim_{y^0\to-\infty}\int d\boldsymbol{y}[\cdots] = -\int_{-\infty}^{+\infty} d^4y\,\partial_0^y[\cdots]. \tag{1.4.4 – 10}$$

We thus obtain

$$\begin{aligned}
&\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)\cdot{\phi^{\text{in}}}_k^\dagger|0\rangle \\
&\quad = -i\int d^4y\,[\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)\phi(y)|0\rangle(\partial_0)^2 f_k(y) \\
&\qquad\qquad - (\partial_0^y)^2\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)\phi(y)|0\rangle\cdot f_k(y)],
\end{aligned} \tag{1.4.4 – 11}$$

from which Eq.(7) immediately follows by using Eq.(1) and by integrating by parts.

If no indices coincide with each other, the above reasoning can be easily generalized to proving the formula

$$\begin{aligned}
&\langle 0| \prod_{j=1}^{s} \phi^{\text{out}}{}_{l_j} \mathrm{T}\phi(x_1)\cdots\phi(x_n)\cdot\prod_{i=1}^{r}\phi^{\text{in}}{}_{k_i}^{\dagger}|0\rangle \\
&= i\int d^4y\, K^y \langle 0|\prod_{j=1}^{s}\phi^{\text{out}}{}_{l_j}\mathrm{T}\phi(x_1)\cdots\phi(x_n)\phi(y)\prod_{i=1}^{r-1}\phi^{\text{in}}{}_{k_i}^{\dagger}|0\rangle\cdot f_{k_r}(y) \\
&= i\int d^4y\, K^y \langle 0|\prod_{j=1}^{s-1}\phi^{\text{out}}{}_{l_j}\mathrm{T}\phi(y)\phi(x_1)\cdots\phi(x_n)\prod_{i=1}^{r}\phi^{\text{in}}{}_{k_i}^{\dagger}|0\rangle\cdot f_{l_s}^{*}(y).
\end{aligned} \tag{1.4.4 - 12}$$

This formula is called **reduction formula**. By using Eq.(12) repeatedly, we can express

$$\langle 0|\prod_j \phi^{\text{in}}{}_{l_j}\cdot S\prod_i\phi^{\text{in}}{}_{k_i}^{\dagger}|0\rangle = \langle 0|\prod_j\phi^{\text{out}}{}_{l_j}\cdot\prod_i\phi^{\text{in}}{}_{k_i}^{\dagger}|0\rangle \tag{1.4.4 - 13}$$

in terms of τ-functions.

Taking account of the possible coincidence of indices, we can write the result in a simple operator form:

$$S =: \exp\left(\int d^4y\,\phi^{\text{in}}(y)K^y\frac{\delta}{\delta J(y)}\right) : \langle 0|\mathrm{T}\exp i\int d^4x\, J(x)\phi(x)|0\rangle\,|_{J=0}\,, \tag{1.4.4 - 14}$$

where the pair of double dots stands for normal product and $J(x)$ is a c-number source function.

For further discussions on asymptotic fields and asymptotic states, especially in the indefinite-metric cases, see Appendix A.4.

1.5 SPONTANEOUS BREAKDOWN OF SYMMETRY

Spontaneous breakdown of symmetry is the breakdown of symmetry at the level of the representation of field operators. It is a very interesting phenomenon which cannot occur in a system of finite degrees of freedom. It was originally found in solid-state physics, and then by analogy it was brought into quantum field theory by Nambu and Jona-Lasinio [Nam 61]. The central subject in the spontaneous breakdown of symmetry is the Goldstone theorem claiming the existence of a massless mode [Gol 61].

1.5.1 Goldstone theorem

A symmetry generator Q, of course, commutes with P_μ, but the vacuum $|0\rangle$ need not be a simultaneous eigenstate of Q and P_μ in quantum field theory. That is, we may have

$$Q|0\rangle \neq |0\rangle\langle 0|Q|0\rangle. \tag{1.5.1 - 1}$$

Although Eq.(1) most clearly characterizes the spontaneous breakdown of the symmetry generated by Q, it is problematic to write Eq.(1) explicitly because $Q|0\rangle$ does not exist at least in the positive-metric case. Indeed, if $Q|0\rangle$ were well-defined, we would have

$$\begin{aligned}\langle 0|Q\cdot Q|0\rangle &= \int dx \int dy\, \langle 0|J^0(x)J^0(y)|0\rangle \\ &= \int dx \int dy\, f(x-y) = \infty \end{aligned} \tag{1.5.1 - 2}$$

because of translational invariance and metric positivity.

Instead of Eq.(1), the spontaneous breakdown of symmetry is characterized by the **Goldstone commutator**

$$\langle 0|[iQ, \chi(x)]_\mp |0\rangle \neq 0 \tag{1.5.1 - 3}$$

for some operator $\chi(x)$. Indeed, if $Q|0\rangle = |0\rangle\langle 0|Q|0\rangle$, the lhs. of Eq.(3) would vanish; that is, if $Q|0\rangle$ is sensible, Eq.(3) implies Eq.(1).

The simplest example of Eq.(3) is found in the free massless scalar field theory. This theory is invariant under the transformation $\phi \rightarrow \phi' = \phi + \alpha$, α

being a c-number constant. The corresponding generator is given by

$$Q = \int d\boldsymbol{x}\, \partial_0 \phi(x). \tag{1.5.1 - 4}$$

Since

$$[iQ, \phi(x)] = 1, \tag{1.5.1 - 5}$$

this symmetry is spontaneously broken. Here, the fact that $\phi(x)$ is massless is crucial.

In general, the following theorem, which is called **Goldstone theorem,** holds: *If Eq.(3) holds, then $\chi(x)$ contains a massless discrete spectrum.* Here, as remarked in Sec.1.2.4, Eq.(3) should be understood as

$$\int d\boldsymbol{x}\, \langle 0|[J^0(x), \chi(y)]_{\mp}|0\rangle = c \neq 0, \tag{1.5.1 - 6}$$

where c is a constant because of translational invariance. Proof goes as follows [Gol 62]. (Proof can be made more rigorously [Kas 66, Eza 67, Ree 68].)

Because of translational invariance, we can write

$$\langle 0|[J^0(x), \chi(y)]_{\mp}|0\rangle = \int d^4p\, f(p) e^{-ip(x-y)}. \tag{1.5.1 - 7}$$

Combining Eq.(7) with Eq.(6), we have

$$c = (2\pi)^3 \int d^4p\, f(p)\delta(\boldsymbol{p}) e^{-ip(x-y)}, \tag{1.5.1 - 8}$$

whose Fourier transform is

$$c\delta^4(p) = (2\pi)^3 f(p)\delta(\boldsymbol{p}). \tag{1.5.1 - 9}$$

Since $c \neq 0$, $f(p)$ must become proportional to $\delta(p_0)$ when $\boldsymbol{p} = 0$, that is, it contains a *one-dimensional* δ-like singularity. This fact implies that $f(p)$ has a massless discrete spectrum in its spectral representation.

This massless spectrum is called **Goldstone mode**. In the covariant field theory, it must represent a particle, which is called **Goldstone boson** or **Nambu-Goldstone (NG) boson** (or **NG fermion** if χ is fermionic). For the internal symmetry, Q is Lorentz invariant, whence χ must be spinless, that is, the NG boson is spinless.

It is quite instructive to provide an alternative proof of the Goldstone theorem by means of asymptotic fields [Ume 65, Sen 67]. Let $\{\varphi^{\text{as}}{}_A(x)\}$ be the totality of independent asymptotic fields. Since $\varphi^{\text{as}}{}_A$ satisfies a free-field equation, $[iQ, \varphi^{\text{as}}{}_A]_{\mp}$ also does, whence it must be linear with respect to asymptotic fields; that is,

$$[iQ, \varphi^{\text{as}}{}_A(x)]_{\mp} = \alpha_A{}^B \varphi^{\text{as}}{}_B(x) + \beta_A, \qquad (1.5.1-10)$$

where $\alpha_A{}^B$ is a differential operator with c-number constant coefficients, β_A being a c-number constant. Of course, $\alpha_A{}^B = 0$ if the quantum numbers of $\varphi^{\text{as}}{}_A$ and $\varphi^{\text{as}}{}_B$ are not common.

Since

$$\langle 0|\varphi^{\text{as}}{}_A(x)|0\rangle = 0 \qquad (1.5.1-11)$$

for any $\varphi^{\text{as}}{}_A$, Eq.(10) yields

$$\langle 0|[iQ, \varphi^{\text{as}}{}_A(x)]_{\mp}|0\rangle = \beta_A. \qquad (1.5.1-12)$$

Comparing Eq.(12) with Eq.(3), we find that a necessary and sufficient condition for the spontaneous breakdown of symmetry is that we do not have $\beta_A = 0$ for all A's. Let $\beta_A = c \neq 0$ for some A, and $\chi^{\text{as}}(x)$ be the corresponding asymptotic field:

$$\langle 0|[iQ, \chi^{\text{as}}(x)]_{\mp}|0\rangle = c \neq 0. \qquad (1.5.1-13)$$

Since χ^{as} is a free field, it satisfies a linear differential equation $\tilde{K}\chi^{\text{as}} = 0$. Then Eq.(13) implies that $\tilde{K}c = 0$, that is, the differential operator $\tilde{K}$ cannot have a constant term. Thus χ^{as} must contain a massless mode.

Since any (anti)commutator of asymptotic fields is a c-number (see Appendix A.4), Eq.(10) shows that Q consists of quadratic terms and linear ones of asymptotic fields. From the above result, we see that the existence of a linear term in Q characterizes the spontaneous breakdown of symmetry.

Mathematical discussions on the spontaneous breakdown of symmetry in the indefinite-metric cases are given in Appendix A.3.

1.5.2 Goldstone model

Whether or not the spontaneous breakdown of symmetry occurs is often dependent on the values of the parameters involved in the theory.

We consider a complex scalar field having a ϕ^4 interaction. Its Lagrangian density is given by

$$\mathcal{L}_G \equiv \partial^\mu \phi^\dagger \cdot \partial_\mu \phi - u\phi^\dagger\phi - (\lambda/4)(\phi^\dagger\phi)^2. \qquad (1.5.2-1)$$

Here we assume $\lambda > 0$ because if $\lambda < 0$ there is no vacuum, but we do not prescribe the sign of the mass term. The field equation which follows from Eq.(1) is

$$(\Box + u)\phi + (\lambda/2)(\phi^\dagger\phi)\phi = 0. \qquad (1.5.2-2)$$

If ϕ is expanded in powers of its asymptotic field, the zeroth order term is

$$\langle 0|\phi(x)|0\rangle \equiv v/\sqrt{2}. \qquad (1.5.2-3)$$

Hence, in the zeroth-order approximation, Eq.(2) yields

$$(4u + \lambda|v|^2)v = 0. \qquad (1.5.2-4)$$

Since $\lambda > 0$, the solution to Eq.(4) is $v = 0$ only, if $u > 0$. This solution is consistent with the phase invariance of $\mathcal{L}_G$, that is, the phase symmetry is not spontaneously broken. On the other hand, if $u < 0$, then Eq.(4) has infinitely many solutions

$$|v| = \sqrt{-4u/\lambda} \qquad (1.5.2-5)$$

in addition to $v = 0$. The corresponding potential energy becomes

$$\begin{aligned} E_0 &\equiv u|v/\sqrt{2}|^2 + (\lambda/4)|v/\sqrt{2}|^4 \\ &= -u^2/\lambda < 0. \end{aligned} \qquad (1.5.2-6)$$

That is, it is lower than the energy corresponding to $v = 0$. Thus the vacuum should realize Eq.(3) with Eq.(5). Since

$$[iQ, \phi(x)] = -i\phi(x) \qquad (1.5.2-7)$$

for the phase-transformation generator Q [see Eq.(1.3.3-8)], Eq.(3) with Eq.(5) shows that this symmetry is spontaneously broken. The theory given by $\mathcal{L}_G$ with $u < 0$ is called the **Goldstone model** [Gol 61], which is a very instructive example of spontaneous breakdown.

Although v is, in general, complex, we can make it real positive by redefining ϕ by the phase factor of v^{-1}. Hence, without loss of generality, we may assume $v > 0$. It is convenient to write

$$\sqrt{2}\phi(x) = v + \varphi(x) + i\chi(x), \tag{1.5.2 - 8}$$

where both real scalar fields $\varphi(x)$ and $\chi(x)$ have a vanishing vacuum expectation value. Substituting Eq.(8) into Eq.(1), we have

$$\begin{aligned}\mathcal{L}_{\mathrm{G}} = \frac{1}{2}(\partial^\mu\varphi\cdot\partial_\mu\varphi - M^2\varphi^2) + \frac{1}{2}\partial^\mu\chi\cdot\partial_\mu\chi \\ - \frac{1}{2}gM\varphi(\varphi^2+\chi^2) - \frac{1}{8}g^2(\varphi^2+\chi^2)^2 - E_0,\end{aligned} \tag{1.5.2 - 9}$$

where

$$M^2 \equiv -2u > 0, \tag{1.5.2 - 10}$$

$$g \equiv \sqrt{\lambda/2} > 0, \tag{1.5.2 - 11}$$

whence

$$v = M/g. \tag{1.5.2 - 12}$$

The above zeroth-order approximation will be good if g is very small. In this case, φ is a massive scalar field having a mass M, while χ is massless. Of course, the mass of φ receives quantum correction, but χ remains exactly massless because of the Goldstone theorem. The asymptotic field, χ^{as}, of χ satisfies

$$[\chi^{\mathrm{as}}(x), \chi^{\mathrm{as}}(y)] = iD(x-y). \tag{1.5.2 - 13}$$

The generator Q can be expressed as

$$Q = Z^{-1/2}v\int d\boldsymbol{x}\,\partial_0\chi^{\mathrm{as}}(x), \tag{1.5.2 - 14}$$

where Z denotes the renormalization constant of χ.

In the above, the representation of the field operator has been constructed on the basis of a particular choice of the phase of v. For various choices of it , we have physically equivalent but unitarily inequivalent representations. There is no need for considering all irreducible representations simultaneously so as to recover the invariance under the phase transformation.

1.5.3 Nambu-Jona-Lasinio model

In the Goldstone model, the NG boson $\chi(x)$ is a part of a primary field $\phi(x)$. But, *in general, $\chi(x)$ may be a composite field.* Such a simple example is the **Nambu-Jona-Lasinio model** [Nam 61].

The Lagrangian density of this model is given by

$$\mathcal{L}_{\mathrm{NJL}} \equiv i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + g[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]. \qquad (1.5.3-1)$$

It is invariant under the chiral transformation

$$\psi \rightarrow \psi' = \exp(-i\theta\gamma_5)\psi, \qquad \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}\exp(-i\theta\gamma_5), \qquad (1.5.3-2)$$

as is easily confirmed by using ${\gamma_5}^2 = 1$ and $\exp(-i\theta\gamma_5) = \cos\theta - i\gamma_5\sin\theta$. If this invariance is not broken, ψ must be massless. But, according to the analysis based on the self-consistent self-energy equation (though it suffers, unfortunately, from serious divergence difficulty), it is possible for ψ to acquire a non-zero mass if the coupling constant g is negative and sufficiently large.

The chiral transformation, Eq.(2), is generated by

$$Q_5 \equiv \int dx\, \psi^{\dagger}\gamma_5\psi. \qquad (1.5.3-3)$$

Indeed, the canonical anticommutation relation yields

$$[iQ_5, \psi] = -i\gamma_5\psi. \qquad (1.5.3-4)$$

Hence, for the **pseudoscalar density**

$$\rho \equiv -i\bar{\psi}\gamma_5\psi, \qquad (1.5.3-5)$$

we have

$$[iQ_5, \rho] = -2\bar{\psi}\psi. \qquad (1.5.3-6)$$

Therefore, if

$$\langle 0|\bar{\psi}(x)\psi(x)|0\rangle \neq 0, \qquad (1.5.3-7)$$

then the chiral invariance is spontaneously broken, and $\rho(x)$ contains the Goldstone mode. That is, the NG boson is a composite particle of ψ and $\bar{\psi}$.

Addendum 1.A DIRAC METHOD OF QUANTIZATION

We describe the **Dirac method of quantization** for a singular system and show that it is independent of the choice of canonical variables.

For simplicity of description, we consider a system of finite degrees of bosonic freedom only. Canonical variables q_i $(i = 1, \ldots, n)$ are functions of time τ, and the Lagrangian L is a function of q_i and $\dot{q}_i \equiv dq_i/d\tau$. One wishes to define the canonical conjugates p_i by $\partial L/\partial \dot{q}_i$, but, in general, the simultaneous equations

$$p_i = \partial L/\partial \dot{q}_i \tag{1.A-1}$$

are solvable with respect to $\dot{q}_i$ *only partially*. That is, Eq.(1) may yield some relations involving no $\dot{q}_i$. Such relations are called **constraints**, and the system having constraints is called a **singular system**. Dirac [Dir 64] presented a general method for quantizing such a singular system.

As mentioned above, we suppose that Eq.(1) implies the existence of indepedent constraints

$$\phi_\alpha = 0, \tag{1.A-2}$$

where ϕ_α $(\alpha = 1, \ldots, r \leq n)$ are functions of q_i and p_i. We consider a variation of the Hamiltonian $p_i\dot{q}_i - L$:

$$\delta(p_i\dot{q}_i - L) = \dot{q}_i\delta p_i - \frac{\partial L}{\partial q_i}\delta q_i + (p_i - \frac{\partial L}{\partial \dot{q}_i})\delta \dot{q}_i. \tag{1.A-3}$$

If Eq.(1) is used, Eq.(3) shows that the Hamiltonian is expressible in terms of q_i and p_i. Of course, this statement is valid under the validity of the constraints, Eq.(2). Hence we should consider a generalized Hamiltonian

$$H \equiv p_i\dot{q}_i - L + v_\alpha\phi_\alpha, \tag{1.A-4}$$

where v_α $(\alpha = 1, \ldots, r)$ are undetermined functions.

The time development of a quantity χ is given by

$$\dot{\chi} = (\chi,\ H)_{\mathrm{P}}, \tag{1.A-5}$$

where $(A,\ B)_{\mathrm{P}}$ is the **Poisson bracket** defined by

$$(A,\ B)_{\mathrm{P}} \equiv \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}. \tag{1.A-6}$$

Since the constraints must hold for any τ, consistency requires to have

$$(\phi_\alpha,\ H)_{\mathrm{P}} = 0, \tag{1.A-7}$$

Some of Eq.(7) may be satisfied by choosing v_α appropriately, but the remainder may be new conditions, which must be regarded as constraints. We should therefore consider Eq.(7) for these new constraints. Repeating this procedure until we no longer obtain new constraints, we obtain a set of independent constraints Eq.(2) with $\alpha = 1, \ldots, s$ $(r \le s \le 2n)$.

If $(\phi_\alpha,\ \chi)_{\mathrm{P}}$ $(\alpha = 1, \ldots, s)$ are all written as linear combinations of constraints, χ is called of **first class**. Otherwise, χ is of **second class**. Let $\mathcal{A}$ be the $s \times s$ matrix formed by $(\phi_\alpha,\ \phi_\beta)_{\mathrm{P}}$. If $\det\mathcal{A} = 0$, certain combinations ϕ' of ϕ_α are first-class constraints. When quantized, these first-class constraints are interpreted as conditions restricting state vectors in such forms as $\phi'|f\rangle = 0$, because there exist some quantities χ such that $(\phi',\ \chi)_{\mathrm{P}} \neq 0$. When applied to quantum field theory, however, such conditions are generally inconsistent with the existence of the vacuum (see Sec.2.2.3). We therefore do not wish to have any first-class constraints.

Since the first-class constraints are nothing but the generators of gauge transformations (i.e., canonical transformations which leave H and all constraints invariant), they become of second class by adding gauge-fixing constraints. Indeed, if we start with the Lagrangian modified by gauge fixing in such a way that no gauge invariance remains, then we encounter no first-class constraints. In the following, therefore, we assume that all constraints are of second class.

In this case, the matrix $\mathcal{A}$ is invertible, whence s is even because $\mathcal{A}$ is antisymmetric. We then define the **Dirac bracket** of two quantities A and B by

$$(A,\ B)_{\mathrm{D}} \equiv (A,\ B)_{\mathrm{P}} - (A,\ \phi_\alpha)_{\mathrm{P}}(\mathcal{A}^{-1})_{\alpha\beta}(\phi_\beta,\ B)_{\mathrm{P}}. \tag{1.A-8}$$

The Dirac bracket has the same properties as those (antisymmetry, Leibniz rule, and Jacobi identity) of the Poisson bracket. Furthermore, *the Dirac bracket between ϕ_α and any quantity χ vanishes*:

$$(\phi_\alpha,\ \chi)_{\mathrm{D}} = 0. \tag{1.A-9}$$

Quantization is carried out by replacing the Dirac bracket by $-i$ times a commutator:

$$(A,\ B)_{\mathrm{D}} \quad \rightarrow \quad -i[A,\ B]. \tag{1.A-10}$$

Here, because of Eq.(9), *we can regard Eq.(2) as operator identities.* We no longer need to worry about the existence of constraints. The Dirac method of quantization is thus quite useful for quantizing a singular system.

Now, there arises a question: Is the Dirac bracket independent of the choice of canonical variables? It is well known that the Poisson bracket is invariant under the canonical transformation. In the following, we show that the Dirac bracket is also invariant under the canonical transformation [Kug unp.c].

The Dirac bracket can easily be expressed in terms of the **Lagrange bracket**, which is defined as follows. Let z_j $(j = 1, \ldots, 2n)$ be functions of q_i and p_i such that q_i and p_i are expressible in terms of z_j. Then the Lagrange bracket between z_j and z_k is

$$(z_j,\ z_k)_\mathrm{L} \equiv \frac{\partial q_i}{\partial z_j}\frac{\partial p_i}{\partial z_k} - \frac{\partial q_i}{\partial z_k}\frac{\partial p_i}{\partial z_j}. \qquad (1.A-11)$$

The $2n \times 2n$ matrix formed by $(z_j,\ z_k)_\mathrm{L}$ is the inverse matrix of the one formed by the corresponding Poisson brackets:

$$(z_j,\ z_k)_\mathrm{L}(z_j,\ z_\ell)_\mathrm{P} = \delta_{k\ell}. \qquad (1.A-12)$$

Thus, as is well known, the Lagrange bracket is invariant under the canonical transformation.

We choose $z_j = \phi_\alpha$ for $j = 2n - s + \alpha$ and $z_j = z_a$ for $j = a = 1, 2, \ldots, 2n - s$. Then, with the aid of Eqs.(9), (8) and (12), we have

$$\begin{aligned}(z_a,\ z_b)_\mathrm{L}(z_a,\ z_c)_\mathrm{D} &= (z_a,\ z_b)_\mathrm{L}(z_a,\ z_c)_\mathrm{D} + (\phi_\alpha,\ z_b)_\mathrm{L}(\phi_\alpha,\ z_c)_\mathrm{D} \\ &= (z_j,\ z_b)_\mathrm{L}(z_j,\ z_c)_\mathrm{D} \\ &= \delta_{bc}. \end{aligned} \qquad (1.A-13)$$

Thus the $(2n - s) \times (2n - s)$ matrix formed by the Dirac brackets $(z_a,\ z_c)_\mathrm{D}$ is the inverse matrix of the restricted one of the Lagrange brackets $(z_a,\ z_b)_\mathrm{L}$. It follows from this fact that *the Dirac bracket is invariant under the canonical transformation.*

Finally, we present two important examples of the invariance of the Dirac brackets [Kug unp.c]. The action is not altered by adding total divergence. In the present formulation, this amounts to considering a change of L by $\dot{F}$, where F is a function of q_i only (i.e., involving no $\dot{q}_i$). In this case, the canonical variables

p_i change by $\partial F/\partial \dot{q}_i$. Thus the Poisson brackets remain invariant, and so are the Dirac brackets.

The second example is the situation in which multipliers are present as discussed in Sec.1.2.3. Corresponding to Eq.(1.2.3-1), we consider the Lagrangian

$$L = L_0(q, \dot{q}) + F_\alpha(q, \dot{q})\, b_\alpha + G(b). \tag{1.A – 14}$$

Here L_0 involves no $\dot{q}_\alpha$,[1] F_α is linear in $\dot{q}_A$ and $\det\partial F_\alpha/\partial \dot{q}_\beta \neq 0$. In the Dirac quantization, not only q_a but also b_α are taken as canonical variables. The canonical conjugates of q_α and b_α are

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial F_\beta}{\partial \dot{q}_\alpha}\, b_\beta, \tag{1.A – 15}$$

$$\pi_\alpha = \frac{\partial L}{\partial \dot{b}_\alpha} = 0. \tag{1.A – 16}$$

They are constraints because $\partial F_\beta/\partial \dot{q}_\alpha$ involves no $\dot{q}_A$. Hence we write

$$\phi_{1\alpha} \equiv p_\alpha - \frac{\partial F_\beta}{\partial \dot{q}_\alpha}\, b_\beta, \qquad \phi_{2\alpha} \equiv \pi_\alpha. \tag{1.A – 17}$$

Then, since

$$(\phi_{2\alpha},\ \phi_{2\beta})_{\mathrm{P}} = 0, \tag{1.A – 18}$$

$$(\phi_{1\alpha},\ \phi_{2\beta})_{\mathrm{P}} = -\partial F_\beta/\partial \dot{q}_\alpha, \tag{1.A – 19}$$

we have

$$\det \mathcal{A} = (\det \partial F_\beta/\partial \dot{q}_\alpha)^2 \neq 0, \tag{1.A – 20}$$

that is, $\mathcal{A}$ is invertible. Thus the constraints $\phi_{1\alpha}$ and $\phi_{2\alpha}$ are of second class.

Because $(\partial F_\beta/\partial \dot{q}_\alpha)$ is invertible, $\{q_A, b_\alpha, p_B, \pi_\beta\}$ is expressible in terms of $\{q_A, p_B, \phi_{1\alpha}, \phi_{2\beta}\}$, which we adopt as z_j. Then Eq.(13) implies that

$$(q_A,\ q_B)_{\mathrm{D}} = (p_A,\ p_B)_{\mathrm{D}} = 0, \tag{1.A – 21}$$

$$(q_A,\ p_B)_{\mathrm{D}} = \delta_{AB}. \tag{1.A – 22}$$

We thus obtain the canonical commutation relations which are exactly the same as those in which canonical variables are q_A only but not b_α. The validity of this result is crucially due to the fact that L involves no $\dot{b}_\alpha$, that is, it is not correct to regard a primary variable b_α as non-canonical if L involves $\dot{b}_\alpha$.

[1] $\{q_\alpha\}$ is a subset of $\{q_A\}$ and has a one-to-one correspondence with $\{b_\alpha\}$.