DUE: TUESDAY OCTOBER 11, 2016

1. It is often possible to derive a field theory as a limit of a discrete system. Perhaps the simplest example is a one-dimensional infinite system of point masses, m, separated by springs of spring constant k and equilibrium length a. Let η_i be the displacement from equilibrium of the ith point mass. Derive the exact Lagrangian and Lagrange equations for this system. Then consider the limit: m, $a \to 0$, $k \to \infty$, with $\mu \equiv m/a$ and $Y \equiv ka$ held fixed. Replacing η_i with a smooth function $\eta(x,t)$, show that in this limit the Lagrangian may be written in the following form:

$$L = \int dx \, \tfrac{1}{2} \left[\mu \left(\frac{\partial \eta}{\partial t} \right)^2 - Y \left(\frac{\partial \eta}{\partial x} \right)^2 \right] .$$

Write down the corresponding (partial differential) Lagrange equations.

2. Consider a set of N real scalar fields $\phi_r(x)$ $(r = 1, 2, \ldots, N)$ and the corresponding conjugate fields $\pi_r(x)$. To quantize the scalar field theory, one imposes the equal-time commutation relations:

$$[\phi_r(\vec{\boldsymbol{x}}, t), \pi_s(\vec{\boldsymbol{y}}, t)] = i \, \delta_{rs} \, \delta^3(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}) \, ,$$
$$[\phi_r(\vec{\boldsymbol{x}}, t), \phi_s(\vec{\boldsymbol{y}}, t)] = [\pi_r(\vec{\boldsymbol{x}}, t), \pi_s(\vec{\boldsymbol{y}}, t)] = 0 \, .$$

(a) Show that the momentum operator of the fields,

$$P^j \equiv \int d^3\boldsymbol{x} \, \pi_r(x) \frac{\partial \phi_r(x)}{\partial x_j}$$

(implicit sum over r implied), satisfies the equations

$$[P^j, \phi_r(x)] = -i \frac{\partial \phi_r(x)}{\partial x_j} \, , \qquad [P^j, \pi_r(x)] = -i \frac{\partial \pi_r(x)}{\partial x_j} \, .$$

Hence, show that any operator $F(\phi_r(x), \pi_r(x))$, which can be expanded in a power series in the field operators $\phi_r(x)$ and $\pi_r(x)$, satisfies

$$[P^j, F(x)] = -i \frac{\partial F(x)}{\partial x_j} \, .$$

(b) Starting from Hamilton's field equations, deduce the Heisenberg equations of motion for the operator $F(x)$ of part (a):

$$[H, F(x)] = -i\frac{\partial F(x)}{\partial t}\,,$$

where H is the Hamiltonian. Combine this result with the one from part (a) by identifying $H = P^0$ to obtain a covariant form for the Heisenberg equations of motion.

(c) Under a translation of coordinates, $x_\mu \to x'_\mu \equiv x_\mu + a_\mu$ (where a_μ is a fixed four-vector), a scalar field remains invariant:

$$\phi'(x') = \phi(x)\,.$$

This may be rewritten as $\phi'(x) = \phi(x - a)$. Show that the corresponding unitary transformation

$$\phi(x) \to \phi'(x) = U\,\phi(x)\,U^{-1}$$

is given by the unitary operator $U = \exp(-ia_\mu P^\mu)$, where $P^\mu = (H\,;\,P^j)$ and P^j is the momentum operator of the field introduced in part (a).

3. Show that the Lagrangian density:

$$\mathcal{L} = \tfrac{1}{2}[\partial_\alpha V^\alpha(x)][\partial_\beta V^\beta(x)] - \tfrac{1}{2}[\partial_\alpha V_\beta(x)][\partial^\alpha V^\beta(x)] + \tfrac{1}{2}m^2\,V_\alpha(x)V^\alpha(x)$$

for the real vector field $V^\alpha(x)$ leads to the field equations

$$\left[g_{\alpha\beta}(\Box + m^2) - \partial_\alpha\partial_\beta\right]V^\beta(x) = 0\,,$$

and prove that $V^\alpha(x)$ automatically satisfies the Lorentz condition: $\partial_\alpha V^\alpha(x) = 0$ (assuming $m \neq 0$). [*HINT:* the latter follows from the field equations.]

4. Consider the free massless Klein-Gordon action:

$$S_0 = \tfrac{1}{2}\int d^4x\,\partial^\mu\phi(x)\partial_\mu\phi(x)\,.$$

The *dilatation* transformation is a space-time transformation defined by

$$x \to x' \equiv e^\alpha x\,,$$

where x is the position four-vector and α is a real parameter. Under the dilatation transformation, the scalar field $\phi(x)$ transforms as:

$$\phi(x) \to \phi'(x') = \phi(x)\,\exp(-d_\phi\alpha)\,,$$

for some appropriate choice of the constant d_ϕ.

(a) Show that the dilatation transformation is a symmetry of the free massless Klein-Gordon action for a unique choice of d_ϕ.

(b) Find the Noether current associated to the dilatation symmetry. Verify that it is conserved when $\phi(x)$ satisfies its Lagrange field equations.

(c) Suppose we add an interaction term to S_0:

$$S = \tfrac{1}{2}\int d^4x \left\{\partial^\mu\phi(x)\partial_\mu\phi(x) - \lambda\,[\phi(x)]^4\right\} .$$

What is the mass dimension of the coupling λ? Is the dilatation transformation a symmetry of the modified action S?

(d) Show that the dilatation transformation is *not* a symmetry of the free *massive* Klein-Gordon action:

$$S = \tfrac{1}{2}\int d^4x \left\{\partial^\mu\phi(x)\partial_\mu\phi(x) - m^2\,[\phi(x)]^2\right\} .$$

(e) Based on the results of this problem, can you guess a rule for determining by inspection when the dilatation symmetry is present or absent?

5. Consider a single real scalar field $\phi(x)$, which is a function of the four-vector x. The vacuum expectation value of the product of free scalar fields is defined by

$$D(x,y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle .$$

(a) Show that $D(x,y)$ is translationally invariant, which means that $D(x,y)$ is a function only of $x-y$. As a result of this observation, we are free to set $y=0$ with no loss of generality.

HINT: You may assume that the vacuum is translationally invariant, i.e., $P^\mu|0\rangle = 0$.

(b) Employing the expansion of a free scalar field in terms of creation and annihilation operators, derive the following integral expression for $D(x)\equiv D(x,0)$,

$$D(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot x} ,$$

where $p\cdot x \equiv g_{\mu\nu}p^\mu x^\nu$, with $p^\mu = (E_{\mathbf{p}}\,;\,\vec{\boldsymbol{p}})$ and $E_{\mathbf{p}} = \sqrt{|\vec{\boldsymbol{p}}|^2+m^2}$.

(c) Show that $D(x)$ satisfies the Klein-Gordon equation, *i.e.*, $(\Box + m^2)D(x) = 0$.

HINT: First, prove the identity:

$$\frac{1}{2E_p}\delta(p_0 - E_{\mathbf{p}}) = \delta(p^2-m^2)\Theta(p_0) ,$$

where $p^2 = p_0^2 - |\vec{\boldsymbol{p}}|^2$ is the square of the momentum four-vector and $\Theta(x)$ is the step function,

$$\Theta(x) \equiv \begin{cases} 1 & x > 0\,, \\ 0 & x < 0\,. \end{cases}$$

Using the above identity, convert $D(x)$ above from a three-dimensional integral to a four-dimensional integral over $d^4p \equiv d^3\boldsymbol{p}\, dp_0$, where $-\infty < p_0 < \infty$. In this form, it is easy to prove that $D(x)$ satisfies the Klein-Gordon equation.

(d) Evaluate $D(x)$ explicitly in terms of Bessel functions.

(e) Determine the precise form of the singularities of $D(x)$ on the light cone [*i.e.*, for $x^2 \equiv t^2 - |\vec{\boldsymbol{x}}|^2 = 0$]. *NOTE:* these will involve terms such as $\delta(x^2)$ and $1/x^2$.

HINTS for parts (d) and (e): There are a number of ways to compute the integral of this problem. Here is one way. First write $d^3\boldsymbol{p} = p^2\, dp\, d\Omega$ and integrate over angles. To compute the remaining integral over p requires a trick, since this integral is not formally convergent. Technically, $D(x)$ is not a function; instead it is a distribution (like the delta function). But, one can easily write the integral as the derivative (with respect to the radial distance $r \equiv |\vec{\boldsymbol{x}}|$) of another integral that is well defined. Then, consult the integral tables (such as Gradshteyn and Ryzhik) and convince yourself that:

$$\int_0^\infty \frac{\sin(t\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}} \cos pr\, dp = \frac{\pi}{2}\,\epsilon(t)\Theta(t^2-r^2)J_0(m\sqrt{t^2-r^2})\,,$$

$$\int_0^\infty \frac{\cos(t\sqrt{p^2+m^2})}{\sqrt{p^2+m^2}} \cos pr\, dp = -\frac{\pi}{2}\,\Theta(t^2-r^2)N_0(m\sqrt{t^2-r^2}) + \Theta(r^2-t^2)K_0(m\sqrt{r^2-t^2})\,,$$

where J_0 and N_0 are Bessel functions of the first and second kind, and K_0 is one of the modified Bessel functions. The step function $\Theta(x)$ was defined in part (b), and the sign function is

$$\epsilon(x) \equiv \frac{x}{|x|} = \begin{cases} 1 & x > 0\,, \\ -1 & x < 0\,. \end{cases}$$

Note that the delta function is related to the step function:

$$\delta(x) = \frac{d\Theta(x)}{dx}\,.$$

Using the integrals quoted above, complete the evaluation of $D(x)$. Finally, employing small argument expansions of the Bessel functions, determine the leading behavior of $D(x)$ near the light cone (you may discard any term that vanishes as $x^2 \to 0$).