DUE: THURSDAY NOVEMBER 10, 2016

1. Consider a massive spin-1/2 particle with four-momentum $p^\mu = (E; \vec{\boldsymbol{p}})$ and helicity λ (where $2\lambda = \pm 1$). The spin four-vector is defined as:

$$s^\mu = \frac{1}{m}\left(|\vec{\boldsymbol{p}}|\,;\, E\frac{\vec{\boldsymbol{p}}}{|\vec{\boldsymbol{p}}|}\right) .$$

Verify that $s \cdot p = 0$ and $s \cdot s = -1$.

(a) Show that both the particle and antiparticle helicity spinors $u(p, \lambda)$ and $v(p, \lambda)$ are eigenstates of $\gamma_5 \not{s}$ with eigenvalue equal to 2λ.

(b) Using the results of part (a), derive the following formulae:

$$u(p,\lambda)\bar{u}(p,\lambda) = \tfrac{1}{2}(1 + 2\lambda\gamma_5\not{s})(\not{p} + m) ,$$
$$v(p,\lambda)\bar{v}(p,\lambda) = \tfrac{1}{2}(1 + 2\lambda\gamma_5\not{s})(\not{p} - m) .$$

These are called the *helicity spinor projection operators*. Check the above formulae by evaluating both sides of the equations in the rest frame.

HINT: Starting, *e.g.*, with $\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) = \not{p} + m$, multiply both sides of this equation by a suitable operator that projects out one term in the sum over λ. Using the results of part (a), it should be easy to discover the required operator.

(c) Show that in the high energy limit, $E \gg m$, $s^\mu = p^\mu/m + \mathcal{O}(m/E)$. Using this result and the result of part (a), show that in the massless limit, $u(p,\lambda)$ and $v(p,\lambda)$ are also eigenstates of γ_5. What are the corresponding eigenvalues?

(d) Following the limiting procedure of part (c), deduce the helicity spinor projection operators [see part (b)] for the case of massless spin-1/2 particles.

2. For a four-component Dirac field, the transformations

$$\Psi(x) \to \Psi'(x) = \exp(i\alpha\gamma_5)\Psi(x) , \qquad \Psi^\dagger(x) \to \Psi'^\dagger(x) = \Psi^\dagger(x)\exp(-i\alpha\gamma_5) ,$$

where α is an arbitrary real parameter, are called chiral phase transformations.

(a) Show that the Dirac Lagrangian density, $\mathcal{L} = \overline{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x)$, is only invariant under chiral phase transformations in the zero-mass limit, $m = 0$. Using Noether's theorem, prove that the corresponding conserved current (in the $m = 0$ limit) is the axial vector current $J^\mu_A(x) \equiv \overline{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)$.

(b) Introduce the left-handed and right-handed fields:

$$\Psi_L(x) \equiv \tfrac{1}{2}(1-\gamma_5)\Psi(x)\,, \qquad\qquad \Psi_R(x) \equiv \tfrac{1}{2}(1+\gamma_5)\Psi(x)\,.$$

Noting that $\Psi(x) = \Psi_L(x) + \Psi_R(x)$, rewrite the Dirac Lagrangian in terms of the two independent fields $\Psi_L(x)$ and $\Psi_R(x)$. Use the hint below to simplify this Lagrangian (by removing any terms that vanish). Starting from the resulting Lagrangian, deduce the (Lagrange) field equations for the left and right-handed fields in the case of non-vanishing mass m, and show that the two field equations decouple in the limit of $m = 0$.

HINT: Show that $\overline{\Psi}_L \equiv \Psi_L^\dagger \gamma^0 = \frac{1}{2}\overline{\Psi}(1+\gamma_5)$, *etc.* Then, prove that $\overline{\Psi}_L \gamma^\mu \Psi_R = 0$ and $\overline{\Psi}_L \Psi_L = 0$, and use these and similar results to simplify your Lagrangian.

(c) Compare the results of part (b) to the Dirac equation in two-component notation. Discuss the relation between the two-component and four-component treatments.

3. In class, we wrote down an expression for the momentum operator P^μ in the two cases of a non-interacting scalar and Dirac field theory, respectively. We then inserted the mode expansions for the corresponding quantum fields and obtained P^μ as a sum over modes.

(a) Fill in the steps in the case of Dirac field theory; *i.e.*, derive the expression for P^μ as a sum over modes $\{\vec{\boldsymbol{p}}, s\}$, where s is the spin quantum number.

(b) Prove that $P^\mu \left|0\right\rangle = 0$, where $\left|0\right\rangle$ is the vacuum state. Interpret the result.

4. In Dirac field theory governed by the Lagrangian $\mathscr{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi$, the conserved angular momentum tensor operator is given by:

$$J^{\mu\nu} = \int d^3x\, \psi^\dagger(x)\left[i(x^\mu\partial^\nu - x^\nu\partial^\mu) + \tfrac{1}{2}\Sigma^{\mu\nu}\right]\psi(x)\,,$$

where $\Sigma^{\mu\nu} \equiv \frac{1}{2}i(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ and $\Psi(x)$ is the Dirac (four-component) field operator. As usual, we define the corresponding vector angular momentum and boost operators by $J^i = \frac{1}{2}\epsilon^{ijk}J_{jk}$ and $K^i = J^{0i}$.

(a) Prove that $\vec{\boldsymbol{J}}$ is an hermitian operator.

(b) Derive the following form for $\vec{\boldsymbol{K}}$:

$$\vec{\boldsymbol{K}} = x^0\vec{\boldsymbol{P}} - \frac{i}{2}\int d^3x\, \vec{\boldsymbol{x}}\, \Psi^\dagger(x)\overleftrightarrow{\partial_0}\,\Psi(x)\,, \tag{1}$$

where $\vec{\boldsymbol{P}}$ is the three-vector momentum operator of Dirac field theory and

$$\Psi^\dagger(x)\overleftrightarrow{\partial_0}\,\Psi(x) \equiv \Psi^\dagger(x)\frac{\partial\Psi(x)}{\partial t} - \frac{\partial\Psi^\dagger(x)}{\partial t}\Psi(x)\,.$$

HINT: To derive eq. (1), it is very convenient to make use of the following identity

$$\frac{i}{2}\int d^3x\,\partial_k\,[x^i\,\overline{\Psi}(x)\gamma^k\Psi(x)] = 0\,,$$

which is valid under the assumption that the field $\Psi(x)$ vanishes at spatial infinity. Further simplification can be achieved by noting that $\Psi(x)$ satisfies the free-field Dirac equation.

(c) Prove that $\vec{\boldsymbol{K}}$ is an hermitian operator.

(d) Verify that $d\vec{\boldsymbol{K}}/dt = 0$.

5. This problem concerns the discrete symmetries P, C and T.

(a) Let $\Phi(x)$ be a complex-valued scalar field previously considered in problem 2 of Problem Set #2. The unitary operators P, C and an antiunitary operator T act on the quantized free complex scalar field as follows,

$$\begin{aligned}
P\Phi(t\,;\,\vec{\boldsymbol{x}})P^{-1} &= \Phi(t\,;\,-\vec{\boldsymbol{x}})\,,\\
C\Phi(t\,;\,\vec{\boldsymbol{x}})C^{-1} &= \Phi^*(t\,;\,\vec{\boldsymbol{x}})\,,\\
T\Phi(t\,;\,\vec{\boldsymbol{x}})T^{-1} &= \Phi(-t\,;\,\vec{\boldsymbol{x}})\,,
\end{aligned}$$

where Φ^* is the complex conjugate of the field Φ. Determine the action of P, C and T on the annihilation operators $a_{\vec{\boldsymbol{p}}}$ and $b_{\vec{\boldsymbol{p}}}$ for the charged scalar particles and antiparticles, respectively.

(b) Consider the conserved current operator J^μ,

$$J^\mu = i\,\boldsymbol{:}\big[\Phi^*(\partial^\mu\Phi) - (\partial^\mu\Phi^*)\Phi\big]\boldsymbol{:}$$

where $\boldsymbol{:}\mathcal{O}\boldsymbol{:}$ indicates that the operator $\mathcal{O}$ is normal ordered. Determine the transformation properties of J^μ under P, C and T. Why is the normal ordering in the definition of the operator J^μ necessary?

(c) Consider a theory of a charged scalar field $\Phi(x)$ and a Dirac fermion field $\Psi(x)$. Show that any Lorentz invariant hermitian local operator* built from products of $\Phi(x)$, $\Psi(x)$ and their conjugates has $CPT = +1$.

*By definition, a local operator is a product of fields in which each field is evaluated at the same spacetime point.