DUE: TUESDAY NOVEMBER 29, 2016

1. Consider the following Lagrangian density involving two real scalar fields ϕ and χ,

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}(\partial_\mu \chi)^2 - \tfrac{1}{2}m_\phi^2 \phi^2 - \tfrac{1}{2}m_\chi^2 \chi^2 - \mu\phi\chi\chi\,, \tag{1}$$

where μ is a parameter with units of mass (in units of $\hbar = c = 1$). Due to the term in eq. (1) proportional to μ, a ϕ particle can decay into two χ particles provided that $m_\phi > 2m_\chi$. Assuming that this latter condition is satisfied, calculate the lifetime of the ϕ particle to lowest order in μ.

2. Consider the decay: $\mu^- \to e^- \nu_\mu \bar{\nu}_e$. The lifetime ($\tau$) of the muon in its rest frame can be computed from the following formula:

$$\tau^{-1} = \frac{1}{2M} \int |\mathcal{M}|^2 \, (2\pi)^4 \, \delta^4(p - p_1 - p_2 - p_3) \, \mathrm{dLips}\,, \tag{2}$$

where M is the muon mass, p is the muon four-momentum, p_1 is the electron four-momentum and p_2 and p_3 are the neutrino four-momenta. The notation dLips is short-hand for Lorentz invariant phase space:

$$\mathrm{dLips} \equiv \prod_j \frac{d^3 p_j}{(2\pi)^3 (2E_j)}\,. \tag{3}$$

The theory of weak interactions yields the following leading-order prediction for square of the invariant matrix element (after averaging over initial spins and summing over final spins):

$$|\mathcal{M}|^2 = 64\, G_F^2 \, (p \cdot p_2)\, (p_1 \cdot p_3)\,, \tag{4}$$

where $G_F = 1.16638 \times 10^{-5}$ GeV^{-2} is Fermi's constant. Note that the dot products in eq. (4) involve four-vectors.

(a) Insert eq. (4) into eqs. (2) and (3) and carry out all integrations. Work in the approximation that the electron and neutrino masses are zero. Here are some tricks to make the calculation simple. First, we shall integrate over the neutrino four-momenta. Define $q = p - p_1$ and consider:

$$I^{\mu\nu}(q) \equiv \int \frac{d^3 p_2}{2E_2} \frac{d^3 p_3}{2E_3} \, p_2^\mu \, p_3^\nu \, \delta^4(q - p_2 - p_3)\,. \tag{5}$$

Since d^3p/E is Lorentz invariant, it follows that $I^{\mu\nu}$ transforms like a second-rank tensor. Clearly $I^{\mu\nu}$ is a function of the four-vector q. Noting that $g^{\mu\nu}$ and $q^\mu q^\nu$ are the only second-rank tensors that can be constructed using the four-vector q, it follows that

$$I^{\mu\nu}(q) = A(q^2)\, g^{\mu\nu} + B(q^2)\, q^\mu q^\nu\,, \tag{6}$$

where A and B are functions of the Lorentz scalar quantity q^2.

Thus, we have reduced the problem of computing $I^{\mu\nu}$ to the problem of evaluating the scalar functions A and B. One can evaluate A and B by the following trick. First multiply both sides of eqs. (5) and (6) by $g_{\mu\nu}$ (summing over μ and ν). The δ-function in eq. (5) requires that $q = p_2 + p_3$. From this relation and using the fact that the neutrinos are massless (*i.e.*, $p_2^2 = p_3^2 = 0$), we can evaluate $p_2 \cdot p_3$ in terms of q^2. The result is that one can express a linear combination of A and B in terms of the integral:

$$I \equiv \int \frac{d^3p_2}{2E_2} \frac{d^3p_3}{2E_3} \, \delta^4(q - p_2 - p_3) \,. \tag{7}$$

Next, multiply both sides of eqs. (5) and (6) by $q_\mu q_\nu$ (summing over μ and ν). This yields a second equation for A and B in terms of the integral I. We now have two equations for A and B in terms of I. Thus, once we know the value of I, we can trivially evaluate A and B. To evaluate I, first note that it is a Lorentz invariant, so we can choose any frame of reference to perform the calculation. Choose the center-of-mass frame of the two neutrinos (where $\vec{\boldsymbol{p}}_2 + \vec{\boldsymbol{p}}_3 = 0$). In particular, note that in this frame (due to the fact that the neutrinos are both massless), $E_2 = E_3$. The integration over $\vec{\boldsymbol{p}}_2$ is immediate by using three of the four δ-functions. Converting to spherical co-ordinates, the remaining integrals can be done almost by inspection. You should find that I is a simple constant.

Now that $I^{\mu\nu}$ has been computed, we are left with:

$$\tau^{-1} = \frac{1}{2M}(2\pi)^{-5}\, 64 G_F^2 \int \frac{d^3p_1}{2E_1}\, p^\mu p_1^\nu \, I_{\mu\nu}(q) \,. \tag{8}$$

Inserting the result for $I_{\mu\nu}$ obtained from the above calculation, and recalling that $q = p - p_1$, we can complete the computation by performing the last set of integrals in eq. (8). Again, we are free to choose an arbitrary frame. This time, the simplest choice is the muon rest frame, where $p = (M; \vec{\boldsymbol{0}})$. Since the electron is massless, we can take $p_1 = E_1(1; 0, 0, 1)$; that is, we choose the z-axis to be the electron direction in the muon rest frame. Converting again to spherical coordinates, the remaining challenge is to determine the range of integration for the "radial" variable $|\vec{\boldsymbol{p}}_1| \equiv E_1$. Clearly, $0 \leq E_1 \leq E_{\rm max}$. The maximum value of E_1 is determined from the inequality $q^2 \geq 0$ (since $q = p_2 + p_3$ is the sum of two light-like four vectors); hence, $(p - p_1)^2 \geq 0$. Inserting the explicit forms for the four-vectors given above, we easily obtain $E_{\rm max}$. Complete the evaluation of the remaining integrals, and obtain a formula for τ^{-1} in terms of G_F and the muon mass.

(b) Using $M = 105.66$ MeV, evaluate the muon lifetime (in seconds). To accomplish this last step, you will have to relate the energy unit of electron volts (eV) to the time unit of seconds. Effectively, this requires that you restore the appropriate factors of $\hbar$ and c (which have been set to one throughout the above calculation). Compare your result with the muon lifetime listed in the *2016 Review of Particle Physics* [see `http://www-pdg.lbl.gov/2016/tables/rpp2016-sum-leptons.pdf`] compiled by the Particle Data Group.

3. The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed to lowest order in the electromagnetic coupling e without quantizing the electromagnetic field. Instead, we shall treat the electromagnetic field as a given classical potential $A^\mu(x)$. The interaction Hamiltonian is

$$H_{\rm int} = -\int d^3x\, e\overline{\Psi}(x)\gamma^\mu\Psi(x)A_\mu(x)\,,$$

where $\Psi(x)$ is the quantized four-component Dirac field and the charge of the electron is $-e$.

(a) Show that the T-matrix element for the electron scattering off a localized classical potential, to lowest order in e, is given by

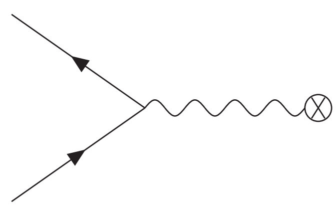
$$\langle p'|iT|p\rangle = ie\overline{u}(p')\gamma^\mu u(p)\,\widetilde{A}_\mu(p'-p)\,,$$

where $\widetilde{A}_\mu(q)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

(b) If $A^\mu(x)$ is time-independent, its Fourier transform contains a delta function of energy. It is then natural to define,

$$\langle p'|iT|p\rangle \equiv 2\pi i\delta(E_f - E_i)\mathcal{M}\,, \tag{9}$$

where E_i and E_f are the initial and final energies of the electron, and to adopt a new Feynman rule for computing $\mathcal{M}$,



$$= ie\gamma^\mu\widetilde{A}_\mu(\vec{\boldsymbol{q}})$$

where $\widetilde{A}_\mu(\vec{\boldsymbol{q}})$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given the definition of $\mathcal{M}$ given in eq. (9), show that the cross section for scattering off a time-independent, localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\,|\mathcal{M}|^2(2\pi)\delta(E_f - E_i)\,, \tag{10}$$

where v_i is the magnitude of the initial velocity of the electron. Integrate over the final state three momentum p_f to find a simple expression for $d\sigma/d\Omega$.

(c) Specialize to the case of electron scattering from a Coulomb potential,

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}}) = 0\,, \qquad A^0(\vec{\boldsymbol{x}}) = \frac{Ze}{4\pi r}\,, \qquad \text{where } r \equiv |\vec{\boldsymbol{x}}|,$$

Using eq. (10) and working in the nonrelativistic limit, average over the initial electron spins and sum over the final electron spins. Derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4m^2v^4\sin^4(\theta/2)}\,,$$

where $\alpha \equiv e^2/(4\pi)$ and v is the magnitude of the electron velocity.

4. Spin-1 helicity wave functions (sometimes called polarization vectors) are four-vectors denoted by $\epsilon^\mu(k,\lambda)$, where $k^\mu = (E_k\,;\,\vec{\boldsymbol{k}})$ is the four-momentum of the particle. The helicity λ can take on three possible values ($\lambda = -1, 0, 1$) if the particle is massive, and two possible values ($\lambda = \pm 1$) if the particle is massless. Suppose $\vec{\boldsymbol{k}}$ points in a direction specified by polar and azimuthal angles θ and ϕ with respect to the z-axis. Then the transverse ($\lambda = \pm 1$) polarization vectors are given by:

$$\epsilon^\mu(k,\pm 1) = \sqrt{\tfrac{1}{2}}\,e^{\pm i\phi}\left(0\,;\,\mp\cos\theta\cos\phi + i\sin\phi, -i\cos\phi \mp \cos\theta\sin\phi, \pm\sin\theta\right)\,.$$

The above result holds for both massless and massive spin-one particles. If $m \neq 0$, one also needs the polarization four-vector of the longitudinal ($\lambda = 0$) state:

$$\epsilon^\mu(k,0) = \frac{1}{m}\left(|\vec{\boldsymbol{k}}|\,;\,E_k\frac{\vec{\boldsymbol{k}}}{|\vec{\boldsymbol{k}}|}\right)\,.$$

(a) Show that the spin-one polarization vectors satisfy:

$$k\cdot\epsilon(k,\lambda) = 0\,,$$

$$\epsilon(k,\lambda)\cdot\epsilon(k,\lambda')^* = -\delta_{\lambda\lambda'}\,.$$

(b) If $m \neq 0$, prove that the polarization sum is given by:

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda)^* = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\,.$$

(c) For photons, $m = 0$, and the polarization sum is taken only over transverse polarizations, $\lambda = \pm 1$. Define a fixed four vector, $n^\mu \equiv (1;0,0,0)$. Show that the photon polarization sum is given by:

$$\sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda)^* = -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n} - \frac{k_\mu k_\nu}{(k\cdot n)^2}\,.$$

You are not required to hand in problem 5 below, and it will not be graded. However, the results below should be a part of your mathematical toolkit. In particular, these results play an important role in scattering theory and the theory of dispersion. Thus, I include it here for future reference.

5. Prove the following identity:

$$\frac{1}{x \pm i\epsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x)\,,$$

where $\epsilon > 0$ is an infinitesimal quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin, and then integrated over a range of x containing the origin. That is, prove that:

$$\int_{-\infty}^{+\infty} \frac{f(x)dx}{x \pm i\epsilon} = \mathrm{P}\int_{-\infty}^{+\infty} \frac{f(x)dx}{x} \mp i\pi f(0)\,,$$

where the principal value integral is defined as:

$$\mathrm{P}\int_{-\infty}^{+\infty} \frac{f(x)dx}{x} \equiv \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta} \frac{f(x)dx}{x} + \int_{+\delta}^{+\infty} \frac{f(x)dx}{x}\right\}\,,$$

assuming $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x| \to \infty$.