1. Consider a massive spin-1/2 particle with four-momentum \( p^\mu = (E; \vec{p}) \) and helicity \( \lambda \) (where \( 2\lambda = \pm 1 \)). The spin four-vector is defined as:

\[
s^\mu = \frac{1}{m} \left( |\vec{p}|; \frac{E\vec{p}}{|\vec{p}|} \right).
\]

Verify that \( s \cdot p = 0 \) and \( s \cdot s = -1 \).

By explicit computation,

\[
s \cdot p = \frac{1}{m} (E|\vec{p}| - E|\vec{p}|) = 0,
\]
\[
s \cdot s = \frac{1}{m^2} (|\vec{p}|^2 - E^2) = -1,
\]

after using \( \vec{p} \cdot \vec{p} = |\vec{p}|^2 \) and the mass-shell condition \( E^2 = |\vec{p}|^2 + m^2 \).

(a) Show that both the particle and antiparticle helicity spinors \( u(p, \lambda) \) and \( v(p, \lambda) \) are eigenstates of \( \gamma_5 \) with eigenvalue equal to \( 2\lambda \).

I shall provide two different methods for solving part (a) of this problem.

**Method 1:**

We shall employ the low-energy (Dirac) representation of the gamma matrices, where

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},
\]

when written in block matrix form, where \( I \) is the \( 2 \times 2 \) identity matrix. In this representation, the \( u \) and \( v \) helicity spinors are given by

\[
u(p, \lambda) = \sqrt{E + m} \begin{pmatrix} \chi^{(\lambda)} \\ \frac{2\lambda|\vec{p}|}{E + m} \chi^{(\lambda)} \end{pmatrix}, \quad v(p, \lambda) = \sqrt{E + m} \begin{pmatrix} |\vec{p}| \chi^{(-\lambda)} \\ \frac{E + m}{-2\lambda \chi^{(-\lambda)}} \chi^{(-\lambda)} \end{pmatrix},
\]

after making use of

\[
\frac{\vec{\sigma} \cdot \vec{p}}{2|\vec{p}|} \chi^{(\lambda)} = \lambda \chi^{(\lambda)}.
\]
It follows that

\[ \not s = \gamma^\mu s_\mu = \gamma^0 s_0 - \vec{\gamma} \cdot \vec{s} = \begin{pmatrix} s_0 & -\vec{\sigma} \cdot \vec{s} \\ \vec{\sigma} \cdot \vec{s} & -s_0 \end{pmatrix}, \tag{2} \]

\[ \gamma_5 \not s \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} s_0 & -\vec{\sigma} \cdot \vec{s} \\ \vec{s} \cdot \vec{\sigma} & -s_0 \end{pmatrix} = \begin{pmatrix} \vec{\sigma} \cdot \vec{s} & -s_0 \\ s_0 & -\vec{\sigma} \cdot \vec{s} \end{pmatrix}. \tag{3} \]

Using \( s_0 = |\vec{p}|/m \) and \( \vec{s} = E\vec{p}/(m|\vec{p}|) \) in eq. (3) yields,

\[ \gamma_5 \not s = \frac{1}{m} \begin{pmatrix} \frac{E}{|\vec{p}|} \vec{\sigma} \cdot \vec{p} & -|\vec{p}| \\ |\vec{p}| & -\frac{E}{|\vec{p}|} \vec{\sigma} \cdot \vec{p} \end{pmatrix}. \]

Hence,

\[ \gamma_5 \not s u(\vec{p}, \lambda) = \frac{1}{m} \sqrt{E + m} \begin{pmatrix} \frac{E}{|\vec{p}|} \vec{\sigma} \cdot \vec{p} \chi^{(\lambda)} - \frac{2\lambda|\vec{p}|^2}{E + m} \chi^{(\lambda)} \\ |\vec{p}| \chi^{(\lambda)} - \frac{2\lambda E}{E + m} \vec{\sigma} \cdot \vec{p} \chi^{(\lambda)} \end{pmatrix}. \]

The above result can be simplified by making use of eq. (1). The end result is

\[ \gamma_5 \not s u(\vec{p}, \lambda) = \frac{1}{m} \sqrt{E + m} \begin{pmatrix} 2\lambda \left( E - \frac{|\vec{p}|^2}{E + m} \right) \chi^{(\lambda)} \\ |\vec{p}| \left( 1 - \frac{E}{E + m} \right) \chi^{(\lambda)} \end{pmatrix} \]

\[ = \frac{1}{m} \sqrt{E + m} \begin{pmatrix} 2\lambda m \chi^{(\lambda)} \\ m|\vec{p}| \chi^{(\lambda)} \end{pmatrix} \]

\[ = \frac{1}{m} \sqrt{E + m} \begin{pmatrix} \chi^{(\lambda)} \\ 2\lambda |\vec{p}| \chi^{(\lambda)} \end{pmatrix} \]

\[ = 2\lambda \left( E - \frac{|\vec{p}|^2}{E + m} \right) \chi^{(\lambda)} \]

\[ = 2\lambda u(\vec{p}, \lambda), \]

after using \( E^2 = |\vec{p}|^2 + m^2 \) and \((2\lambda)^2 = 1\). The latter is a consequence of the fact that the possible values of the helicity are \( \lambda = \pm \frac{1}{2} \). Thus, we conclude that

\[ \boxed{\gamma_5 \not s u(\vec{p}, \lambda) = 2\lambda u(\vec{p}, \lambda)} \tag{4} \]
Similarly,

\[
\gamma_5 \hat{s} v(\vec{p}, \lambda) = \frac{1}{m} \sqrt{E + m} \left( \frac{E}{E + m} \vec{\sigma} \cdot \vec{p} \chi^{(-\lambda)} + 2\lambda |\vec{p}| \chi^{(-\lambda)} \right)
\]

\[
= \frac{1}{m} \sqrt{E + m} \left( 2\lambda |\vec{p}| \left( 1 - \frac{E}{E + m} \right) \chi^{(-\lambda)} \right)
\]

\[
= \frac{1}{m} \sqrt{E + m} \left( 2\lambda m |\vec{p}| \frac{\chi^{(-\lambda)}}{E + m} \right)
\]

\[
= 2\lambda |\vec{p}| \left( \frac{\gamma^0 |\vec{p}|}{E + m} \chi^{(-\lambda)} \right)
\]

\[
= 2\lambda v(\vec{p}, \lambda),
\]

Thus, we conclude that

\[
\gamma_5 \hat{s} v(\vec{p}, \lambda) = 2\lambda v(\vec{p}, \lambda)
\] (5)

**Method 2:**

This method is independent of the choice of representation for the gamma matrices. First, we note that

\[
\gamma^0 s_0 - \vec{\gamma} \cdot \vec{s} = \frac{1}{m} \left( |\vec{p}| \gamma^0 - \frac{E}{|\vec{p}|} \vec{\gamma} \cdot \vec{p} \right)
\]

\[
= \frac{E}{m|\vec{p}|} \left( \gamma^0 \frac{|\vec{p}|^2}{E} - \vec{\gamma} \cdot \vec{p} \right) = \frac{E}{m|\vec{p}|} \left( \vec{p} - \frac{m^2}{E} \gamma^0 \right),
\]

after using \(|\vec{p}|^2 = E^2 - m^2\) and recognizing that \(\vec{p} = p_\mu \gamma^\mu = E \gamma^0 - \vec{\gamma} \cdot \vec{p}\). Hence, it follows that

\[
\hat{s} u(p, \lambda) = \frac{E}{m|\vec{p}|} \left( \vec{p} - \frac{m^2}{E} \gamma^0 \right) u(\vec{p}, \lambda) = \frac{1}{|\vec{p}|} \left( E - m \gamma^0 \right) u(\vec{p}, \lambda).
\] (6)

In the last step above, the Dirac equation has been employed, \(\hat{p} u(p, \lambda) = m u(\vec{p}, \lambda)\).
To make further progress, recall that $\Sigma^{\mu\nu} = \frac{1}{2} i [\gamma^\mu, \gamma^\nu]$. It then follows that

$$\Sigma^i = \frac{1}{2} i \epsilon^{ijk} \Sigma_{jk} = \frac{1}{2} i \epsilon^{ijk} \gamma^j \gamma^k = \gamma_5 \gamma^i,$$

where $\gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3$. In particular,

$$\vec{\Sigma} \cdot \vec{\rho} = \gamma_5 \gamma^0 \vec{\gamma} \cdot \vec{\rho} = \gamma_5 \gamma^0 \left(E \gamma^0 - \frac{\vec{p}}{m} \right) = \gamma_5 \left(E - \gamma^0 \frac{\vec{p}}{m} \right),$$

after using $(\gamma^0)^2 = I$. Multiplying eq. (8) on the left by $\gamma_5$ and using $(\gamma_5)^2 = I$, it follows that

$$\gamma_5 \vec{\Sigma} \cdot \vec{p} = E - \gamma^0 \frac{\vec{p}}{m}.$$  

Using the Dirac equation again,

$$\gamma_5 \vec{\Sigma} \cdot \vec{p} u(\vec{p}, \lambda) = \left(E - m \gamma^0\right) u(\vec{p}, \lambda).$$

Comparing with eq. (6), it follows that

$$\gamma_5 \slashed{\Sigma} u(p, \lambda) = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|} u(\vec{p}, \lambda).$$

Recalling that helicity spinors satisfy

$$\frac{\vec{\Sigma} \cdot \vec{p}}{2|\vec{p}|} u(\vec{p}, \lambda) = \lambda u(\vec{p}, \lambda),$$

we can conclude that

$$\gamma_5 \slashed{\Sigma} u(\vec{p}, \lambda) = 2\lambda u(\vec{p}, \lambda).$$

A similar analysis can be performed by applying $\gamma_5 \slashed{\Sigma}$ to $v(\vec{p}, \lambda)$. In this case, the relevant Dirac equation now reads, $\gamma_5 \gamma^0 \gamma^0 \vec{p} v(\vec{p}, \lambda) = -m v(\vec{p}, \lambda)$. Following the derivation above,

$$\gamma_5 \slashed{\Sigma} v(\vec{p}, \lambda) = \left(\gamma_5 \gamma^0 \vec{p} \right) v(\vec{p}, \lambda) = -\frac{1}{|\vec{p}|} \left(E - \gamma_0 \frac{\vec{p}}{m} \right) v(\vec{p}, \lambda) = -\frac{1}{|\vec{p}|} \gamma_5 \vec{\Sigma} \cdot \vec{p} v(\vec{p}, \lambda).$$

Recall that the antiparticle helicity spinor satisfies

$$\frac{\vec{\Sigma} \cdot \vec{p}}{2|\vec{p}|} v(\vec{p}, \lambda) = -\lambda v(\vec{p}, \lambda).$$

In particular, notice the minus sign above [in contrast to eq. (9)]. Hence, we can conclude that

$$\gamma_5 \slashed{\Sigma} v(\vec{p}, \lambda) = 2\lambda v(\vec{p}, \lambda).$$

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1Note that $\gamma_5 \gamma^0 \gamma^i = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^0 \gamma^i = -i \gamma^1 \gamma^2 \gamma^3 \gamma^i$, after employing $(\gamma^0)^2 = I$ and $\{\gamma^0, \gamma^i\} = 0$. Using $(\gamma^i)^2 = -I$, one can easily verify that $\frac{1}{6} \epsilon^{ijk} \gamma^j \gamma^k = -\gamma^1 \gamma^2 \gamma^3$, thus justifying eq. (7). As a check, we multiply both sides of the latter equation by $\gamma^i$ and sum over $i$ to obtain $\frac{1}{6} \epsilon^{ijk} \gamma^j \gamma^k \gamma^i = \gamma^1 \gamma^2 \gamma^3$. 


(b) Using the results of part (a), derive the following formulae:

\[ u(p, \lambda) \bar{u}(p, \lambda) = \frac{1}{2} (1 + 2\lambda \gamma_5) (\not{p} + m), \]

\[ v(p, \lambda) \bar{v}(p, \lambda) = \frac{1}{2} (1 + 2\lambda \gamma_5) (\not{p} - m). \]

These are called the *helicity spinor projection operators*. Check the above formulae by evaluating both sides of the equations in the rest frame.

Using eqs. (4) and (5), it follows that

\[ \frac{1}{2} (1 + 2\lambda' \gamma_5) u(\vec{p}, \lambda) = \frac{1}{2} (1 + 4\lambda \lambda') u(\vec{p}, \lambda), \quad (10) \]

\[ \frac{1}{2} (1 + 2\lambda' \gamma_5) v(\vec{p}, \lambda) = \frac{1}{2} (1 + 4\lambda \lambda') v(\vec{p}, \lambda), \quad (11) \]

where \( \lambda, \lambda' = \pm \frac{1}{2} \). In particular, for these values of \( \lambda, \lambda' \), we have \( \frac{1}{2} (1 + 4\lambda \lambda') = \delta_{\lambda \lambda'} \). We can therefore rewrite eqs. (10) and (11) as follows,

\[ \frac{1}{2} (1 + 2\lambda' \gamma_5) u(\vec{p}, \lambda) = \delta_{\lambda \lambda'} u(\vec{p}, \lambda), \quad (12) \]

\[ \frac{1}{2} (1 + 2\lambda' \gamma_5) v(\vec{p}, \lambda) = \delta_{\lambda \lambda'} v(\vec{p}, \lambda), \quad (13) \]

To make further progress, consider the helicity projection operators,

\[ \sum_\lambda u(\vec{p}, \lambda) \bar{u}(\vec{p}, \lambda) = \not{p} + m, \quad (14) \]

\[ \sum_\lambda v(\vec{p}, \lambda) \bar{v}(\vec{p}, \lambda) = \not{p} - m. \quad (15) \]

Multiplying eqs. (14) and (15) each by the factor \( \frac{1}{2} (1 + 2\lambda' \gamma_5) \) and using eqs. (12) and (13), we see that the end result is to project out the \( \lambda = \lambda' \) piece of the sum. It then follows that

\[ u(p, \lambda) \bar{u}(p, \lambda) = \frac{1}{2} (1 + 2\lambda \gamma_5) (\not{p} + m), \quad (16) \]

\[ v(p, \lambda) \bar{v}(p, \lambda) = \frac{1}{2} (1 + 2\lambda \gamma_5) (\not{p} - m). \quad (17) \]

(c) Show that in the high energy limit, \( E \gg m, \ s^\mu = p^\mu / m + \mathcal{O}(m/E) \). Using this result and the result of part (a), show that in the massless limit, \( u(p, \lambda) \) and \( v(p, \lambda) \) are also eigenstates of \( \gamma_5 \). What are the corresponding eigenvalues?

Consider the high energy limit where \( E \gg m \). Then,

\[ s^\mu = \frac{1}{m} \left( \left| \vec{p} \right| ; \frac{E \vec{p}}{| \vec{p} |} \right) = \frac{E}{m | \vec{p} |} \left( \left| \vec{p} \right|^2 / E ; \vec{p} \right) = \frac{E}{m (E^2 - m^2)^{1/2}} \left( E - m^2 / E ; \vec{p} \right) \]

\[ = \frac{1}{m} \left( 1 - \frac{m^2}{E^2} \right)^{-1/2} \left\{ (E ; \vec{p}) - \frac{m^2}{E}(1 ; \vec{0}) \right\}, \quad (18) \]
after using $|\vec{p}|^2 = E^2 - m^2$. Note that no approximations were used in deriving eq. (18). We can now apply the condition that $E \gg m$, in which case

$$\left(1 - \frac{m^2}{E^2}\right)^{-1/2} \approx 1 + \frac{m^2}{2E}.$$ 

Inserting this result back into eq. (18) and identifying the four-vector $p^\mu = (E, \vec{p})$, it follows that

$$s^\mu = \left\{ \frac{p^\mu}{m} - \frac{m}{E}(1; \vec{0}) \right\} \left[ 1 + \mathcal{O}\left(\frac{m^2}{E^2}\right) \right].$$

Dropping the $\mathcal{O}(m^2/E^2)$ terms yields,

$$s^\mu = \frac{p^\mu}{m} + \mathcal{O}\left(\frac{m}{E}\right). \hspace{1cm} (19)$$

Recall eqs. (4) and (5), which we rewrite below for convenience,

$$\gamma_5 u(\vec{p}, \lambda) = 2\lambda u(\vec{p}, \lambda),$$
$$\gamma_5 v(\vec{p}, \lambda) = 2\lambda v(\vec{p}, \lambda).$$

Applying the approximation given by eq. (19) and using the Dirac equations,

$$\not{\partial} u(\vec{p}, \lambda) = m u(\vec{p}, \lambda), \hspace{1cm} \not{\partial} v(\vec{p}, \lambda) = -m v(\vec{p}, \lambda),$$

it follows that

$$\gamma_5 u(\vec{p}, \lambda) = 2\lambda u(\vec{p}, \lambda), \hspace{1cm} \gamma_5 v(\vec{p}, \lambda) = -2\lambda v(\vec{p}, \lambda). \hspace{1cm} (20)$$

Note that these equations are only approximate in the high energy limit since terms of $\mathcal{O}(m/E)$ have been dropped. But, in the massless limit ($m \to 0$), eq. (20) becomes exact.

(d) Following the limiting procedure of part (c), deduce the helicity spinor projection operators [see part (b)] for the case of massless spin-1/2 particles.

By multiplying out the terms on the right hand side of eqs. (16) and (17), we obtain,

$$u(p, \lambda)\overline{\pi}(p, \lambda) = \frac{1}{2} \left( \not{\dot{p}} + m + 2\lambda \gamma_5 \not{\dot{p}} + 2\lambda m \gamma_5 \not{\dot{p}} \right), \hspace{1cm} (21)$$
$$v(p, \lambda)\overline{\pi}(p, \lambda) = \frac{1}{2} \left( \not{\dot{p}} - m + 2\lambda \gamma_5 \not{\dot{p}} - 2\lambda m \gamma_5 \not{\dot{p}} \right). \hspace{1cm} (22)$$

To obtain the massless helicity projection operators, we first put $\not{s} = \not{\dot{p}}/m$ and then take the limit as $m \to 0$. For the particle helicity spinor,

$$u(p, \lambda)\overline{\pi}(p, \lambda) = \frac{1}{2} \left( \not{\dot{p}} + m + 2\lambda \gamma_5 \not{\dot{p}}/m + 2\lambda m \gamma_5 \not{\dot{p}} \right) = \frac{1}{2} \left( 1 + 2\lambda \gamma_5 \right) \not{\dot{p}} + \mathcal{O}(m),$$

after noting that $\not{\dot{p}}^2 = p^2 = m^2$. Taking the $m \to 0$ limit yields

$$u(p, \lambda)\overline{\pi}(p, \lambda) = \frac{1}{2} \left( 1 + 2\lambda \gamma_5 \right) \not{\dot{p}}, \hspace{1cm} \text{for } m = 0.$$
For the antiparticle helicity spinor.
\[ v(p, \lambda) \bar{\Psi}(p, \lambda) = \frac{1}{2} (\gamma \cdot p - m + 2\gamma_5 p \cdot \gamma_5 / m - 2\gamma_5 \gamma_5) + \mathcal{O}(m). \]
Taking the \( m \to 0 \) limit yields
\[ v(p, \lambda) \bar{\Psi}(p, \lambda) = \frac{1}{2} (1 - 2\gamma_5) \gamma_5, \quad \text{for } m = 0. \]

2. For a four-component Dirac field, the transformations
\[ \Psi(x) \to \Psi'(x) = \exp(i\alpha \gamma_5) \Psi(x), \quad \Psi^\dagger(x) \to \Psi'^\dagger(x) = \Psi^\dagger(x) \exp(-i\alpha \gamma_5), \quad (23) \]
where \( \alpha \) is an arbitrary real parameter, are called chiral phase transformations.

(a) Show that the Dirac Lagrangian density,
\[ \mathcal{L} = \bar{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x), \quad (24) \]
is only invariant under chiral phase transformations in the zero-mass limit, \( m = 0 \). Using Noether’s theorem, prove that the corresponding conserved current (in the \( m = 0 \) limit) is the axial vector current \( J_\mu^A(x) \equiv \bar{\Psi}(x) \gamma^\mu \gamma_5 \Psi(x) \).

Note that under the chiral transformations given by eq. (23),
\[ \bar{\Psi}'(x) = \bar{\Psi}'^\dagger(x) \gamma^0 = \bar{\Psi}^\dagger(x) e^{-i\alpha \gamma_5} \gamma^0 = \bar{\Psi}^\dagger(x) \gamma^0 \gamma^0 e^{-i\alpha \gamma_5} \gamma^0 = \bar{\Psi}(x) \gamma^0 e^{-i\alpha \gamma_5} \gamma^0, \]
after using \( \bar{\Psi} = \Psi^\dagger \gamma^0 \) and \((\gamma^0)^2 = I\) (where \( I \) is the \( 4 \times 4 \) identity matrix). Next, we can express the exponential in terms of its Taylor series, so that
\[ \gamma^0 e^{-i\alpha \gamma_5} \gamma^0 = \gamma^0 \left( \sum_{n=0}^{\infty} \frac{(-i\alpha \gamma_5)^n}{n!} \right) \gamma^0. \quad (25) \]
Consider \( \gamma^0 \gamma_5 \gamma^0 \). If \( n \) is even then \((\gamma_5)^n = [(\gamma_5)^2]^{n/2} = 1\) since \( n/2 \) is an integer and \((\gamma_5)^2 = I\). It follows that \((\gamma_5)^n = I\) for even \( n \). If \( n \) is odd then \((\gamma_5)^n = \gamma_5 (\gamma_5)^{n-1} = \gamma_5 I = \gamma_5\) since in this case \( n - 1 \) is even. Thus, we can rewrite eq. (25) as,
\[ \gamma^0 e^{-i\alpha \gamma_5} \gamma^0 = \gamma^0 \left( \sum_{n=0}^{\infty} \frac{(-i\alpha \gamma_5)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i\alpha \gamma_5)^{2n+1}}{(2n+1)!} \right) \gamma^0. \]
\[ = I \sum_{n=0}^{\infty} \frac{(-i\alpha)^{2n}}{(2n)!} + \gamma_5 \gamma^0 \sum_{n=0}^{\infty} \frac{(-i\alpha)^{2n+1}}{(2n+1)!} \gamma^0 \gamma_5 \]
\[ = I \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n}}{(2n)!} - \gamma_5 \sum_{n=0}^{\infty} \frac{(-i\alpha)^{2n+1}}{(2n+1)!} \]
\[ = \sum_{n=0}^{\infty} \frac{(i\alpha \gamma_5)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\alpha \gamma_5)^{2n+1}}{(2n+1)!} = e^{i\alpha \gamma_5}, \]
where we have used the anticommutation relation,

$$\{\gamma_5, \gamma^\mu\} = 0,$$

(26)
to conclude that $\gamma^0 \gamma_5 \gamma^0 = -\gamma^0 \gamma_5 = -\gamma_5$. Consequently, eq. (25) yields

$$\overline{\Psi}(x) = \overline{\Psi}(x)e^{i\alpha\gamma_5}.$$

It then follows that

$$\overline{\Psi'} \Psi' = \overline{\Psi}e^{2i\alpha\gamma_5}\Psi.$$

(27)

Next, we consider

$$\overline{\Psi'} \gamma^\mu \partial_\mu \Psi' = \overline{\Psi}e^{i\alpha\gamma_5} \gamma^\mu e^{i\alpha\gamma_5} \partial_\mu \Psi = \overline{\Psi}e^{i\alpha\gamma_5} e^{-i\alpha\gamma_5} \gamma^\mu \partial_\mu \Psi = \overline{\Psi} \gamma^\mu \partial_\mu \Psi.$$

(28)

Note that $\alpha$ is independent of $x$, so that one can pass the $\partial_\mu$ past the $e^{i\alpha\gamma_5}$ to obtain the above result. Using eq. (26), a computation analogous to the one presented above yields

$$\gamma^\mu e^{i\alpha\gamma_5} = \gamma^\mu \left( \sum_{n=0}^{\infty} \frac{(i\alpha\gamma_5)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\alpha\gamma_5)^{2n+1}}{(2n+1)!} \right)$$

$$= \gamma^\mu \left( \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n}}{(2n)!} + \gamma_5 \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n+1}}{(2n+1)!} \right)$$

$$= \left( \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n}}{(2n)!} - \gamma_5 \sum_{n=0}^{\infty} \frac{(i\alpha)^{2n+1}}{(2n+1)!} \right) \gamma^\mu$$

$$= \left( \sum_{n=0}^{\infty} \frac{(-i\alpha\gamma_5)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-i\alpha\gamma_5)^{2n+1}}{(2n+1)!} \right) \gamma^\mu = e^{-i\alpha\gamma_5} \gamma^\mu.$$

Hence, eq. (28) yields

$$\overline{\Psi} \gamma^\mu \partial_\mu \Psi' = \overline{\Psi}e^{i\alpha\gamma_5} \gamma^\mu e^{i\alpha\gamma_5} \partial_\mu \Psi = \overline{\Psi}e^{i\alpha\gamma_5} e^{-i\alpha\gamma_5} \gamma^\mu \partial_\mu \Psi = \overline{\Psi} \gamma^\mu \partial_\mu \Psi.$$

(29)

Combining the results of eqs. (27) and (29), it follows that under a chiral phase transformation given by eq. (23), the Lagrangian density transforms as

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \rightarrow \overline{\Psi}(i\gamma^\mu \partial_\mu - me^{2i\alpha\gamma_5})\Psi.$$

If $m \neq 0$, then $\mathcal{L}$ is clearly not invariant under $\Psi \rightarrow \exp(i\alpha\gamma_5)\Psi$. However, $\mathcal{L}$ is invariant if $m = 0$. In the case of $m = 0$, the invariance under chiral phase transformations implies the existence of a conserved Noether current. Since the chiral phase transformation is independent of $x$ (it is a global transformation), the Noether current is given by

$$\alpha J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \delta \Psi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \overline{\Psi})} \delta \overline{\Psi}(x),$$

(30)
where the variations, \( \delta \Psi(x) \equiv \Psi'(x) - \Psi(x) \) and \( \delta \overline{\Psi}(x) \equiv \overline{\Psi}'(x) - \overline{\Psi}(x) \), are evaluated to first order in \( \alpha \),

\[
\delta \Psi(x) = i \alpha \gamma_5 \Psi(x), \quad \delta \overline{\Psi}(x) = \overline{\Psi}(x) i \alpha \gamma_5 .
\]

Inserting the massless Lagrangian density, \( \mathcal{L} = i \overline{\Psi} \gamma^\mu \partial_\mu \Psi \), into eq. (30), the end result is

\[
J^\mu = -\Psi \gamma^\mu \gamma_5 \Psi .
\]

The overall constant is arbitrary, since it just defines the scale of the conserved charge. The standard practice is to drop the overall minus sign and define the conserved current by

\[
J_A^\mu = \overline{\Psi} \gamma^\mu \gamma_5 \Psi .
\]

The subscript \( A \) (for “axial”) is often used here and the corresponding conserved current is called the axial current.

To verify that the axial current is conserved if \( m = 0 \), note that the field equations corresponding to the massless Lagrangian density, \( \mathcal{L} = i \overline{\Psi} \gamma^\mu \partial_\mu \Psi \), are given by

\[
\gamma^\mu (\partial_\mu \Psi) = 0 , \quad (\partial_\mu \overline{\Psi}) \gamma^\mu = 0 .
\] (31)

Using the field equations in the computation of the divergence of the axial current,

\[
\partial_\mu J_A^\mu = \partial_\mu (\overline{\Psi} \gamma^\mu \gamma_5 \Psi) = (\partial_\mu \overline{\Psi}) \gamma^\mu \gamma_5 \Psi + \overline{\Psi} \gamma^\mu \gamma_5 (\partial_\mu \Psi) = (\partial_\mu \overline{\Psi}) \gamma^\mu \gamma_5 \Psi - \overline{\Psi} \gamma_5 \gamma^\mu (\partial_\mu \Psi) = 0 ,
\]

where we have used eq. (26) to push the \( \gamma_5 \) past the \( \gamma^\mu \), and then employed the field equations given by eq. (31). Thus, indeed \( \partial_\mu J_A^\mu = 0 \), and we conclude that in the case of \( m = 0 \), the axial current is conserved, as expected from Noether’s theorem.

(b) Introduce the left-handed and right-handed fields:

\[
\Psi_L(x) \equiv \frac{1}{2} (1 - \gamma_5) \Psi(x) , \quad \Psi_R(x) \equiv \frac{1}{2} (1 + \gamma_5) \Psi(x) .
\]

Noting that \( \Psi(x) = \Psi_L(x) + \Psi_R(x) \), rewrite the Dirac Lagrangian in terms of the two independent fields \( \Psi_L(x) \) and \( \Psi_R(x) \). Use the hint below to simplify this Lagrangian (by removing any terms that vanish). Starting from the resulting Lagrangian, deduce the (Lagrange) field equations for the left and right-handed fields in the case of non-vanishing mass \( m \), and show that the two field equations decouple in the limit of \( m = 0 \).

HINT: Show that \( \overline{\Psi}_L = \Psi_L^\dagger \gamma^0 = \frac{1}{2} \overline{\Psi}(1 + \gamma_5) \), etc. Then, prove that \( \overline{\Psi}_L \gamma^\mu \Psi_R = 0 \) and \( \overline{\Psi}_L \Psi_L = 0 \), and use these and similar results to simplify your Lagrangian.

In the notation used in this problem, \( 1 \pm \gamma_5 \) above really means \( I \pm \gamma_5 \). Henceforth we shall write the \( 4 \times 4 \) identity matrix using the symbol \( 1 \). It should be clear from the context whether \( 1 \) is the identity matrix or a number.

First, we note that

\[
\overline{\Psi}_L = \Psi_L^\dagger \gamma^0 = \left[ \frac{1}{2} (1 - \gamma_5) \Psi \right]^\dagger \gamma^0 = \frac{1}{2} \Psi^\dagger (1 - \gamma_5^\dagger) \gamma^0 = \frac{1}{2} \overline{\Psi} \gamma^0 (1 - \gamma_5^\dagger) \gamma^0 = \frac{1}{2} \overline{\Psi} (1 + \gamma_5) ,
\] (32)
where we have used \((\gamma^0)^2 = 1\) and
\[
\gamma^0 \gamma^5 \gamma^0 = -\gamma^5 .
\] (33)

It follows that
\[
\overline{\Psi}_L \gamma^\mu \Psi_R = \frac{1}{4} \overline{\Psi}_R (1 + \gamma_5) \gamma^\mu (1 + \gamma_5) \Psi = \frac{1}{4} \overline{\Psi} \gamma^\mu (1 - \gamma_5) (1 + \gamma_5) \Psi = 0 ,
\]
after using eq. (26) and noting that \((1 - \gamma_5) (1 + \gamma_5) = 1 - (\gamma_5)^2 = 0\). Hence, \(\overline{\Psi}_L \gamma^\mu \Psi_R = 0\).

A similar computation yields \(\overline{\Psi}_R \gamma^\mu \Psi_L = 0\). Hence, we conclude that
\[
\Psi_L \gamma^\mu \Psi_R = 0.
\]

An analogous computation yields,
\[
\overline{\Psi} \Psi = (\overline{\Psi}_L + \overline{\Psi}_R) (\Psi_L + \Psi_R) = \overline{\Psi}_L \Psi_R + \overline{\Psi}_R \Psi_L .
\]

In obtaining this result, we made use of the fact that
\[
\overline{\Psi}_L \Psi_L = \frac{1}{4} \overline{\Psi} (1 + \gamma_5) (1 - \gamma_5) \Psi = 0 ,
\]
and similarly \(\overline{\Psi}_R \Psi_R = 0\).

Thus, in terms of the independent degrees of freedom, \(\Psi_L\) and \(\Psi_R\), the Dirac Lagrangian can be written as
\[
\mathcal{L} = i \overline{\Psi}_L \gamma^\mu \partial^\mu \Psi_L + i \overline{\Psi}_R \gamma^\mu \partial^\mu \Psi_R - m (\overline{\Psi}_L \Psi_R + \overline{\Psi}_R \Psi_L ) .
\] (34)

In obtaining the Lagrange field equations, we treat the variables \(\Psi_L, \Psi_R, \overline{\Psi}_L\) and \(\overline{\Psi}_R\) independently,
\[
\frac{\partial \mathcal{L}}{\partial \overline{\Psi}_L} = -m \overline{\Psi}_R ,
\]
\[
\frac{\partial \mathcal{L}}{\partial \overline{\Psi}_R} = -m \overline{\Psi}_L ,
\]
\[
\frac{\partial \mathcal{L}}{\partial (\partial^\mu \Psi_L)} = i \overline{\Psi}_L \gamma^\mu ,
\]
\[
\frac{\partial \mathcal{L}}{\partial (\partial^\mu \Psi_R)} = i \overline{\Psi}_R \gamma^\mu .
\]
The resulting field equations are
\[
i (\partial^\mu \overline{\Psi}_L) \gamma^\mu + m \overline{\Psi}_R = 0 ,
\]
\[
i (\partial^\mu \overline{\Psi}_R) \gamma^\mu + m \overline{\Psi}_L = 0 .
\]

\(^2\)Recall that in class, we defined the matrix \(A\) via the relation \(\Psi \equiv \Psi^\dagger A\), where \(A \gamma^\mu A^{-1} = \gamma^\mu\). Using this latter relation and the definition of \(\gamma_5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3\), it follows that
\[
A \gamma_5 A^{-1} = -i (A \gamma^0 A^{-1}) (A \gamma^1 A^{-1}) (A \gamma^2 A^{-1}) (A \gamma^3 A^{-1}) = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i (\gamma^3 \gamma^2 \gamma^1 \gamma^0) = i (\gamma^0 \gamma^1 \gamma^2 \gamma^3) = i (-i \gamma_5) = -\gamma_5 ,
\]
where we have used the anticommutation relations, \(\{ \gamma^\mu, \gamma^\nu \} = 2 \delta^\mu^\nu\) to reorder the gamma matrices into to rewrite the product in terms of \(\gamma_5\). In all practical gamma matrix conventions, we may choose \(A = \gamma^0\). Noting that \((\gamma^0)^2 = 1\), it follows that \(A^{-1} = \gamma^0\). Hence, \(\gamma^0 \gamma_5 \gamma^0 = -\gamma^5\), and eq. (33) immediately follows.
Similarly, we can compute the field equations of the conjugate fields (or equivalently, take the Dirac conjugate of the above equations),
\[ i\gamma^\mu \partial_\mu \Psi_L - m\Psi_R = 0, \quad i\gamma^\mu \partial_\mu \Psi_R - m\Psi_L = 0. \]

When \( m = 0 \), we see that the equations for the left-handed and right-handed fields decouple,
\[ i\gamma^\mu \partial_\mu \Psi_L = i\gamma^\mu \partial_\mu \Psi_R = 0. \]

(c) Compare the results of part (b) to the Dirac equation in two-component notation. Discuss the relation between the two-component and four-component treatments.

In class, we showed that the Lagrangian density for a Dirac equation in two-component spinor notation is given by
\[ \mathcal{L} = i\chi \bar{\sigma}^\mu \partial_\mu \chi + i\eta \bar{\sigma}^\mu \partial_\mu \eta - m(\chi \eta + \bar{\chi} \bar{\eta}). \]  

The corresponding field equations are
\[ i\bar{\sigma}^\mu \partial_\mu \chi = m\eta, \quad i\bar{\sigma}^\mu \partial_\mu \eta = m\bar{\chi}, \]
\[ i\sigma^\mu \partial_\mu \bar{\eta} = m\chi, \quad i\sigma^\mu \partial_\mu \bar{\chi} = m\eta. \]

The four component Dirac spinor is identified by
\[ \Psi = \begin{pmatrix} \chi_\alpha \\ \eta^\alpha \end{pmatrix}. \]

Using the chiral (or “high-energy”) representation of the gamma matrices,
\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \]

it follows that
\[ \Psi_L = \frac{1}{2}(1 - \gamma_5)\Psi = \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \quad \Psi_R = \frac{1}{2}(1 + \gamma_5)\Psi = \begin{pmatrix} 0 \\ \eta \end{pmatrix}. \]

That is \( \Psi_L \) is identified with the two-component spinor \( \chi \) and \( \Psi_R \) is identified with the two-component spinor \( \eta \). Likewise,
\[ \bar{\Psi} = \begin{pmatrix} \eta^\alpha \\ \bar{\chi}_\alpha \end{pmatrix}. \]

In light of eq. (32), it follows that
\[ \bar{\Psi}_L = (0 \ \bar{\chi}), \quad \bar{\Psi}_R = (\eta \ 0). \]
The terms of the Lagrangian density, eq. (34), can be expressed in terms of the two-component spinor fields as follows,
\[ \Psi_L^\gamma \partial_\mu \Psi_L = \chi \sigma^\mu \partial_\mu \chi, \quad \Psi_R^\gamma \partial_\mu \Psi_R = \eta \sigma^\mu \partial_\mu \eta, \]
\[ \Psi_L \Psi_R = \chi \eta, \quad \Psi_R \Psi_L = \eta \chi. \]
Using the two-component spinor identities, \( \eta \chi = \chi \eta \) and
\[ \eta \sigma^\mu \partial_\mu \eta = - (\partial_\mu \eta) \sigma^\mu \eta - \partial_\mu (\eta \sigma^\mu \eta), \]
we see that
\[ i \Psi_L^\gamma \partial_\mu \Psi_L + i \Psi_R^\gamma \partial_\mu \Psi_R = i \chi \sigma^\mu \partial_\mu \chi + i \eta \sigma^\mu \partial_\mu \eta + \text{total divergence}. \]
Since a total divergence can be dropped from the Lagrangian density (since it does not contribute to the action), we see that the Lagrangian density expressed in terms of four-component spinors \( \Psi_L \) and \( \Psi_R \) in eq. (34) is equivalent to the two-component Lagrangian density given in eq. (35).

3. In class, we wrote down an expression for the momentum operator \( P^\mu \) in the two cases of a non-interacting scalar and Dirac field theory, respectively. We then inserted the mode expansions for the corresponding quantum fields and obtained \( P^\mu \) as a sum over modes.

(a) Fill in the steps in the case of Dirac field theory; i.e., derive the expression for \( P^\mu \) as a sum over modes \( \{ \vec{p}, s \} \), where \( s \) is the spin quantum number.

In the case of Dirac field theory, we obtained the following expressions in class,
\[ P^0 = H = \int d^3 x \Psi^\dagger(x) \left( -i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} + m \gamma^0 \right) \Psi(x), \tag{36} \]
and
\[ \vec{P} = -i \int d^3 x \Psi^\dagger(x) \vec{\nabla} \Psi(x). \tag{37} \]
We can simplify the expression for \( P^0 \) by using the Dirac equation,
\[ (i \gamma^\mu \partial_\mu - m) \Psi(x) = 0. \tag{38} \]
Noting that \( \gamma^\mu \partial_\mu = \gamma^0 (\partial / \partial t) + i \vec{\gamma} \cdot \vec{\nabla} \) we can express eq. (38) as
\[ (-i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} + m \gamma^0) \Psi(x) = i \frac{\partial \Psi}{\partial t}, \]
after using \( (\gamma^0)^2 = I \). Hence, we can rewrite eq. (36) as
\[ P^0 = i \int d^3 x \Psi^\dagger(x) \frac{\partial \Psi}{\partial t}. \tag{39} \]
We shall perform the mode expansions in a cubical box of finite volume \( V \). Then,

\[
\Psi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{\lambda} \frac{1}{2E_p} [u^{(\lambda)}(\vec{p}) a_{\vec{p},\lambda} e^{-ip \cdot x} + v^{(\lambda)}(\vec{p}) b^\dagger_{\vec{p},\lambda} e^{ip \cdot x}]
\]  

(40)

\[
\Psi^\dagger(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{\lambda} \frac{1}{2E_p} [u^{(\lambda)\dagger}(\vec{p}) a_{\vec{p},\lambda}^\dagger e^{ip \cdot x} + v^{(\lambda)\dagger}(\vec{p}) b_{\vec{p},\lambda} e^{-ip \cdot x}]
\]  

(41)

where \( E_p = (|\vec{p}|^2 + m^2)^{1/2} \). Noting that \( p \cdot x = E_p t - \vec{p} \cdot \vec{x} \), it follows that

\[
\tilde{\nabla} \Psi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_{\lambda} \frac{i\vec{p}}{2E_p} [u^{(\lambda)}(\vec{p}) a_{\vec{p},\lambda} e^{-ip \cdot x} - v^{(\lambda)}(\vec{p}) b^\dagger_{\vec{p},\lambda} e^{ip \cdot x}].
\]  

(42)

Inserting eqs. (40) and (41) into eq. (39) yields

\[
P^0 = \frac{1}{V} \sum_{\vec{p},\vec{p}'} \sum_{\lambda,\lambda'} \int d^3x \frac{E_{p'}}{(2E_p)^{1/2}(2E_{p'})^{1/2}} [u^{(\lambda)}(\vec{p}) a_{\vec{p},\lambda} e^{ip \cdot x} + v^{(\lambda)}(\vec{p}) b^\dagger_{\vec{p},\lambda} e^{-ip \cdot x}]
\]

\[
\times [u^{(\lambda')}(\vec{p}') a_{\vec{p}',\lambda'} e^{-ip' \cdot x} - v^{(\lambda')}(\vec{p}') b^\dagger_{\vec{p}',\lambda'} e^{ip' \cdot x}].
\]  

(43)

One can perform the integral over \( \vec{x} \) by employing

\[
\frac{1}{V} \int_V e^{i(\vec{p} - \vec{p}') \cdot \vec{x}} d^3x = \delta_{\vec{p},\vec{p}'}.
\]

Then, eq. (43) simplifies to

\[
P^0 = \frac{1}{2} \sum_{\vec{p}} \sum_{\lambda,\lambda'} \left\{ a_{\vec{p},\lambda}^\dagger a_{\vec{p}',\lambda'} u^{(\lambda)}(\vec{p}) u^{(\lambda')}(\vec{p}') - b_{\vec{p},\lambda}^\dagger b_{\vec{p}',\lambda'} v^{(\lambda)}(\vec{p}) v^{(\lambda')}(\vec{p})
\]

\[
+ b_{\vec{p},\lambda} a_{\vec{p}',\lambda'} e^{-2iE_p v^{(\lambda)}(\vec{p}) u^{(\lambda')}(\vec{p})} - a_{\vec{p},\lambda}^\dagger b_{\vec{p}',\lambda'} e^{2iE_p v^{(\lambda)}(\vec{p}) u^{(\lambda')}(\vec{p})} \right\}.
\]  

(44)

We can simplify the above expression by employing the following identities,

\[
u^{(\lambda)}(\vec{p}) u^{(\lambda')}(\vec{p}) = v^{(\lambda)\dagger}(\vec{p}) v^{(\lambda')}(\vec{p}) = 2E_p \delta_{\lambda\lambda'},
\]

\[
u^{(\lambda)\dagger}(\vec{p}) v^{(\lambda)}(\vec{p}) = v^{(\lambda)\dagger}(\vec{p}) u^{(\lambda)}(\vec{p}) = 0.
\]

Then, eq. (43) further simplifies to

\[
P^0 = \sum_{\vec{p},\lambda} E_p (a_{\vec{p},\lambda}^\dagger a_{\vec{p},\lambda} - b_{\vec{p},\lambda}^\dagger b_{\vec{p},\lambda}).
\]  

(45)

In obtaining eq. (45), we have not yet applied the normal-ordering prescription. In the case of fermion creation and annihilation operators, normal ordering is performed by applying the anticommutation relations,

\[
\{b_{\vec{p},\lambda}, b_{\vec{p}',\lambda'}^\dagger\} = \delta_{\vec{p},\vec{p}'} \delta_{\lambda\lambda'},
\]  

(46)
and dropping the infinite c-number constant (which corresponds to the vacuum energy). The end result is

\[
P^0 = \sum_{\vec{p}, \lambda} E_p (a^{\dagger}_{\vec{p}, \lambda} a_{\vec{p}, \lambda} + b^{\dagger}_{\vec{p}, \lambda} b_{\vec{p}, \lambda})
\]

The computation of \(\vec{P}\) is similar.

\[
\vec{P} = \frac{1}{V} \sum_{\vec{p}, \vec{p}', \lambda, \lambda'} \int d^3x \frac{\vec{p}'}{(2E_p)^{1/2}(2E_{p'})^{1/2}} \left[ u^{(\lambda)}(\vec{p}) e^{ip \cdot x} + v^{(\lambda)}(\vec{p}) e^{-ip \cdot x} \right]
\]

\[
\times \left[ u^{(\lambda')}(\vec{p}') e^{-ip' \cdot x} - v^{(\lambda')}(\vec{p}') e^{ip' \cdot x} \right].
\]

\[
= \sum_{\vec{p}, \lambda, \lambda'} \frac{\vec{p}}{2E_p} \left\{ a^{\dagger}_{\vec{p}, \lambda} a_{\vec{p}', \lambda'} u^{(\lambda)}(\vec{p}) u^{(\lambda')}(\vec{p}) - b_{\vec{p}, \lambda} b^{\dagger}_{\vec{p}', \lambda'} v^{(\lambda)}(\vec{p}) v^{(\lambda')}(\vec{p}) \right. 
\]

\[
- b_{\vec{p}, \lambda} a_{-\vec{p}', \lambda'} e^{-2iE_p} v^{(\lambda)}(\vec{p}) u^{(\lambda')}(\vec{p}) e^{-2iE_{p'}} u^{(\lambda')}(\vec{p}) v^{(\lambda)}(\vec{p}) 
\]

\[
+ a^{\dagger}_{\vec{p}, \lambda} b^{\dagger}_{-\vec{p}', \lambda'} e^{2iE_p} u^{(\lambda)}(\vec{p}) v^{(\lambda')}(\vec{p}) e^{2iE_{p'}} v^{(\lambda)}(\vec{p}) u^{(\lambda')}(\vec{p}) \right\}.
\]

\[
= \sum_{\vec{p}, \lambda} \vec{p} (a^{\dagger}_{\vec{p}, \lambda} a_{\vec{p}, \lambda} - b_{\vec{p}, \lambda} b^{\dagger}_{\vec{p}, \lambda}).
\]

We can again employ eq. (46). This time, there is no need to normal order, since the zero point momentum vanishes due to the exact cancellation of positive and negative three-momenta,

\[
\sum_{\vec{p}} \vec{p} = 0.
\]

The end result is

\[
\vec{P} = \sum_{\vec{p}, \lambda} \vec{p} (a^{\dagger}_{\vec{p}, \lambda} a_{\vec{p}, \lambda} + b^{\dagger}_{\vec{p}, \lambda} b_{\vec{p}, \lambda})
\]

(b) Prove that \(P^\mu |0\rangle = 0\), where \(|0\rangle\) is the vacuum state. Interpret the result.

The vacuum state is defined by the property that

\[
a_{\vec{p}, \lambda} |0\rangle = b_{\vec{p}, \lambda} |0\rangle = 0.
\]

It then immediately follows from eqs. (47) and (48) that

\[
P^\mu |0\rangle = 0.
\]

The interpretation of this result is straightforward. \(P^\mu |0\rangle = 0\) means that the vacuum state is an eigenstate of the four-momentum operator with zero eigenvalue. That is, the four-momentum of the vacuum is zero. Thus, the vacuum state is Lorentz invariant since if \(p^\mu = 0\) in one reference frame, then \(p^\mu = 0\) in all reference frames.
4. In Dirac field theory governed by the Lagrangian \( \mathcal{L} = \overline{\Psi}(i\partial - m)\Psi \), the conserved angular momentum tensor operator is given by:

\[
J^{\mu\nu} = \int d^3x \psi^\dagger(x) \left[ i(x^\mu \partial^\nu - x^\nu \partial^\mu) + \frac{1}{2} \Sigma^{\mu\nu} \right] \psi(x) ,
\]

where \( \Sigma^{\mu\nu} \equiv \frac{1}{2} i(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \) and \( \Psi(x) \) is the Dirac (four-component) field operator. As usual, we define the corresponding vector angular momentum and boost operators by

\[
\vec{J} = \frac{1}{2} \epsilon^{ijk} J_{jk} \quad \text{and} \quad K^i = J^{0i} .
\]

First we note that \( J^i = \frac{1}{2} \epsilon^{ijk} J_{jk} \) and \( K^i = J^{0i} \), eq. (49) yield

\[
\vec{\mathcal{J}} = \int d^3x \Psi^\dagger(x)(-i\vec{\sigma} \times \vec{\nabla} + \frac{1}{2} \vec{\Sigma}) \Psi(x) ,
\]

where \( \Sigma^i = \frac{1}{2} \epsilon^{ijk} \Sigma_{jk} = \frac{1}{4} i \epsilon^{ijk} [\gamma_i, \gamma_j] \), and

\[
K^i = \int d^3x \Psi^\dagger(x) \left[ i(x^0 \partial^i - x^i \partial^0) + \frac{1}{2} \Sigma^{0i} \right] \Psi(x) .
\]

Note that in both \( \vec{\mathcal{J}} \) and \( K^i \) above, we interpret the two terms in the corresponding integrands as the “orbital” part and the “spin” part, respectively.

(a) Prove that \( \vec{\mathcal{J}} \) is an hermitian operator.

Using \( A \gamma^\mu A^{-1} = \gamma^\mu \) [cf. footnote 1], it follows that

\[
A \Sigma^{\mu\nu} A^{-1} = \frac{1}{2} i A \left( \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right) A^{-1} = \frac{1}{2} i (A \gamma^\mu A^{-1} A \gamma^\nu A^{-1} - A \gamma^\nu A^{-1} A \gamma^\mu A^{-1})
\]

\[
= \frac{1}{2} i \left( \gamma^\mu \gamma^\nu \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \gamma^\mu \gamma^\nu \right) = \frac{1}{2} i [(\gamma^\mu \gamma^\nu) \gamma^\mu \gamma^\nu] = \Sigma^{\mu\nu} ,
\]

Since we may take \( A = \gamma^0 \), we obtain

\[
\Sigma^{ij} = \gamma^0 \Sigma^{ij} \gamma^0 = \frac{1}{2} i \gamma^0 (\gamma^i \gamma^j - \gamma^j \gamma^i) \gamma^0 = \frac{1}{2} i (\gamma^i \gamma^j - \gamma^j \gamma^i) = \Sigma^{ij} ,
\]

after using the anticommutation relations, \( \{ \gamma^0 , \gamma^i \} = 0 \) and \( (\gamma^0)^2 = I \). Hence, we conclude that \( \Sigma^\dagger = \Sigma \). More explicitly, in both the Dirac and the chiral representations of the gamma matrices,

\[
\Sigma = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} ,
\]

which is manifestly hermitian. It then follows that

\[
(\Psi^\dagger \Sigma \Psi)^\dagger = \Psi^\dagger \Sigma \Psi .
\]

Using eq. (50)

\[
\left( -i \epsilon^{ijk} \int d^3x \Psi^\dagger(x)x_j \partial_k \Psi(x) \right)^\dagger = i \epsilon^{ijk} \int d^3x \left[ \partial_k \Psi^\dagger(x) \right] x_j \Psi(x)
\]

\[
= -i \epsilon^{ijk} \int d^3x \Psi^\dagger(x) \partial_k \left[ x_j \Psi(x) \right] = -i \epsilon^{ijk} \int d^3x \Psi^\dagger(x)x_j \partial_k \Psi(x) ,
\]

(53)
after integrating by parts (and dropping the surface terms at infinity) and noting that
\[ \epsilon^{ijk} \partial_k [x^j \Psi(x)] = \epsilon^{ijk} g_{kj} \Psi(x) + \epsilon^{ijk} x^j \partial_k \Psi(x) = \epsilon^{ijk} x^j \partial_k \Psi(x). \]

In the penultimate step above, we used \( \partial_{\mu} x_{\nu} = g_{\mu \nu} \). Since \( g_{kj} = g_{jk} \) and \( \epsilon^{ijk} \) is antisymmetric under the interchange of \( j \) and \( k \), it follows that \( \epsilon^{ijk} g_{kj} = 0 \).

Using eqs. (52) and (53), we conclude that \( J^i \dagger = J^i \).

(b) Derive the following form for \( \vec{K} \):
\[ \vec{K} = x^0 \vec{P} - i \frac{1}{2} \int d^3 x \bar{x} \Psi^\dagger(x) \partial_0 \Psi(x), \quad (54) \]
where \( \vec{P} \) is the three-vector momentum operator of Dirac field theory and
\[ \Psi^\dagger(x) \partial_0 \Psi(x) \equiv \Psi^\dagger(x) \frac{\partial \Psi(x)}{\partial t} - \frac{\partial \Psi^\dagger(x)}{\partial t} \Psi(x). \]

First, we note that \( \Sigma^0i = \frac{1}{2} i (\gamma^0 \gamma^i - \gamma^i \gamma^0) = i \gamma^0 \gamma^i \) after using the anticommutation relations, \( \{ \gamma^0, \gamma^i \} = 0 \). Next, consider the following identity,
\[ \frac{1}{2} i \int d^3 x \partial_k [x^j \Psi^\dagger(x) \gamma^k \Psi(x)] = 0, \quad (55) \]
which is valid under the assumption that the field \( \Psi(x) \) vanishes at spatial infinity. Evaluating the derivative using Leibniz’s rule,
\[ \frac{1}{2} i \int d^3 x \partial_k [x^j \Psi^\dagger(x) \gamma^k \Psi(x)] = 0. \]
Noting that \( \frac{1}{2} i \bar{\Psi} \gamma^i \Psi = \frac{1}{2} i \Psi^\dagger \gamma^0 \gamma^i \Psi = \frac{1}{2} \Psi^\dagger \Sigma^0i \Psi \), eq. (55) yields,
\[ \frac{1}{2} i \int d^3 x \Psi^\dagger(x) \Sigma^0i \Psi(x) = -\frac{1}{2} i \int d^3 x [(\partial_k \bar{\Psi}) \gamma^k \Psi + \bar{\Psi} \gamma^k (\partial_k \Psi)] x^i. \quad (56) \]
Further simplification can be achieved by noting that \( \Psi(x) \) satisfies the free-field Dirac equation, \((i \gamma^\mu \partial_\mu - m) \Psi = 0 \), which can be written as
\[ i \gamma^0 \partial_0 \Psi + i \gamma^j \partial_j \Psi = m \Psi. \quad (57) \]
Taking the adjoint of the above equation and multiplying on the right by \( \gamma^0 \), it follows that
\[ i \partial_0 \bar{\Psi} \gamma^0 + i \partial_j \bar{\Psi} \gamma^j = -m \bar{\Psi}, \quad (58) \]
where we have used \( \bar{\Psi} = \Psi^\dagger \gamma^0 \) and \( \gamma^0 \gamma^\mu \gamma^0 = \gamma^\mu \dagger \). Multiply eq. (57) from the left by \( \bar{\Psi} \) and multiply eq. (58) from the right by \( \Psi \). Adding the two resulting equations yields,
\[ i (\partial_0 \bar{\Psi}) \gamma^0 \Psi + i \bar{\Psi} \gamma^j (\partial_j \Psi) = -i \bar{\Psi} \gamma^0 (\partial_0 \Psi) - i (\partial_0 \bar{\Psi}) \gamma^0 \Psi. \quad (59) \]
Inserting the result of eq. (59) back into eq. (56), and using \( \Psi = \Psi \gamma^0 \) and \( (\gamma^0)^2 = I \), the end result is

\[
\frac{1}{2} \int d^3 x \, \Psi^\dagger(x)\Sigma_{0i} \Psi(x) = \frac{1}{2} i \int d^3 x \, [\Psi^\dagger(\partial_0 \Psi) + (\partial_0 \Psi^\dagger)\Psi] x^i. \tag{60}
\]

Finally, we can use eq. (60) to rewrite the expression for \( K^i \) given in eq. (51). After simplifying the resulting expression, we obtain

\[
K^i = i \int d^3 x \, x^0 \Psi^\dagger(x) \partial^i \Psi(x) - \frac{1}{2} i \int d^3 x \, \Psi^\dagger(x) \overset{\leftrightarrow}{\partial_0} \Psi(x). \tag{61}
\]

Using eq. (37), we recognize

\[
P^i = i \int d^3 x \, \Psi^\dagger(x) \partial^i \Psi(x).
\]

Hence, we end up with

\[
\vec{K} = x^0 \vec{P} - \frac{i}{2} \int d^3 x \, \vec{x} \Psi^\dagger(x) \overset{\leftrightarrow}{\partial_0} \Psi(x), \tag{61}
\]

as requested.

(c) Prove that \( \vec{K} \) is an hermitian operator.

First, we note that

\[
\vec{P}^\dagger = \int d^3 x \, [\Psi^\dagger(x)(-i \vec{\nabla} \Psi(x))]^\dagger = \int d^3 x \, [(i \vec{\nabla} \Psi^\dagger(x)) \Psi(x)]
\]

\[
= -i \int d^3 x \, \Psi^\dagger(x) \vec{\nabla} \Psi(x) = \vec{P}, \tag{62}
\]

after integration by parts (and dropping the surface terms at infinity). Next, we compute

\[
(-i \psi^\dagger \partial_0 \psi)^\dagger = i[\Psi^\dagger(\partial_0 \Psi) - (\partial_0 \Psi^\dagger)\Psi]^\dagger = i[(\partial_0 \Psi)^\dagger \Psi - \Psi^\dagger(\partial_0 \Psi)] = -i \psi^\dagger \partial_0 \psi. \tag{63}
\]

Using eqs. (62) and (63), it follows from eq. (61) that \( \vec{K}^\dagger = \vec{K} \).

**REMARK:**

Consider the original form of \( K^i \) given by eq. (51), we note that \( \Sigma^{0i} \) \( \dagger \) \( = -\Sigma^{0i} \) \( \dagger \), which implies that

\[
(\Psi^\dagger \Sigma^{0i} \Psi)^\dagger = -\Psi^\dagger \Sigma^{0i} \Psi.
\]

Thus, in order to satisfy \( \vec{K}^\dagger = \vec{K} \), there must be a cancellation between the contribution of the “orbital” part and the “spin” part of \( K^i \) that exactly cancels out the antihermitean terms in \( K^i \).
(d) Verify that $d\vec{K}/dt = 0$.

To prove that $d\vec{K}/dt = 0$, we must first prove that $d\vec{P}/dt = 0$. Starting from eq. (37),

$$dP^i/dt = i\frac{d}{dt} \int d^3x \Psi^\dagger \partial^i \Psi(x) = i \int d^3x \left[ (\partial_0 \Psi^\dagger) \partial^i \Psi + \Psi^\dagger \partial_0 \partial^i \Psi \right]$$

$$= i \int d^3x \left[ (\partial_0 \Psi^\dagger) \gamma^0 \partial^i \Psi + \Psi^\dagger \gamma^0 \partial_0 \partial^i \Psi \right],$$

(64)

after using $\Psi^\dagger = \overline{\Psi} \gamma^0$ and $\gamma^0 \partial_0 \partial^i \Psi = \partial^i \gamma^0 \partial_0 \Psi$ (since the partial derivatives commute when acting on smooth functions). We can use the Dirac equation [cf. eqs. (57) and (58)] to eliminate $i\gamma^0 \partial_0 \Psi$ and $i\partial_0 \overline{\Psi} \gamma^0$ in eq. (64). The resulting terms proportional to $m$ exactly cancel and we end up with,

$$\frac{dP^i}{dt} = -i \int d^3x \left[ (\partial_0 \overline{\Psi}) \gamma^0 \partial^i \Psi + \overline{\Psi} \gamma^0 \partial_0 \partial^i \Psi \right] = -i \int d^3x \partial_0 (\overline{\Psi} \gamma^0 \partial^i \Psi) = 0,$$

under the assumption that the surface terms at infinity vanish.

Next, we compute $d\vec{K}/dt = 0$ starting from the expression obtained in eq. (61). Using $x^0 = t$ and $d\vec{P}/dt = 0$, we obtain

$$\frac{dK^i}{dt} = P^i - \frac{1}{2} i \int d^3x x^i \frac{\partial}{\partial t} \left( \Psi^\dagger \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^\dagger}{\partial t} \Psi \right)$$

$$= P^i - \frac{1}{2} i \int d^3x x^i \left( \Psi^\dagger \frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi^\dagger}{\partial t^2} \Psi \right),$$

(65)

after noting that two additional terms cancel exactly. Since $\Psi$ satisfies the Dirac equation, we know that it also satisfies the Klein-Gordon equation,

$$\frac{\partial^2 \Psi}{\partial t^2} = (\overline{\nabla}^2 - m^2) \Psi.$$

Hence, we can rewrite eq. (65) as

$$\frac{dK^i}{dt} = P^i - \frac{1}{2} i \int d^3x x^i \left( \Psi^\dagger \overline{\nabla}^2 \Psi - (\overline{\nabla}^2 \Psi^\dagger) \Psi \right),$$

$$= P^i - \frac{1}{2} i \int d^3x x^i \partial_j [\Psi^\dagger (\partial^i \Psi) - (\partial^i \Psi^\dagger) \Psi],$$

(66)

where we have written $\overline{\nabla}^2 = -\partial_j \partial^j$. Integrating by parts and dropping the surface term at infinity yields,

$$\frac{dK^i}{dt} = P^i - \frac{1}{2} i \int d^3x \left[ \Psi^\dagger (\partial^i \Psi) - (\partial^i \Psi^\dagger) \Psi \right] = P^i - i \int d^3x \Psi^\dagger \partial^i \Psi = P^i - P^i = 0,$$

after identifying the integral expression for $P^i$ given in eq. (37).
5. This problem concerns the discrete symmetries $P$, $C$ and $T$.

(a) Let $\Phi(x)$ be a complex-valued scalar field previously considered in problem 2 of Problem Set #2. The unitary operators $P$, $C$ and an antiunitary operator $T$ act on the quantized free complex scalar field as follows,

$$P\Phi(t; \vec{x})P^{-1} = \Phi(t; -\vec{x}),$$  

(67)

$$C\Phi(t; \vec{x})C^{-1} = \Phi^\dagger(t; \vec{x}),$$  

(68)

$$T\Phi(t; \vec{x})T^{-1} = \Phi(-t; \vec{x}),$$  

(69)

where $\Phi^\dagger$ is the hermitian conjugate of the field operator $\Phi$. Determine the action of $P$, $C$ and $T$ on the annihilation operators $a(\vec{k})$ and $b(\vec{k})$ for the charged scalar particles and antiparticles, respectively.

Consider the mode expansion of a free complex scalar field given in problem 2 of Problem Set #2 (where we follow the conventions used in the solution given in Solution Set #2),

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} \left[ a(\vec{k})e^{-ik\cdot x} + b^\dagger(\vec{k})e^{ik\cdot x} \right].$$  

(70)

where $k = (E_k; \vec{k})$ and $\omega_k = (|\vec{k}|^2 + m^2)^{1/2}$.

First, consider the parity transformation. Eq. (67) implies that

$$Pa(\vec{k})P^{-1} = a(-\vec{k}), \quad Pb(\vec{k})P^{-1} = b(-\vec{k}),$$  

(71)

$$Pa^\dagger(\vec{k})P^{-1} = a^\dagger(-\vec{k}), \quad Pb^\dagger(\vec{k})P^{-1} = b^\dagger(-\vec{k}).$$  

(72)

To verify this assertion, let us first define the four-vectors,

$$\vec{k} \equiv (\omega_k; -\vec{k}), \quad \vec{x} = (x_0; -\vec{x}).$$  

(73)

Then, eq. (70) implies that

$$\Phi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} \left[ a(-\vec{k})e^{-ik\cdot x} + b^\dagger(-\vec{k})e^{ik\cdot x} \right].$$  

(74)

Changing the integration variable, $\vec{k} \rightarrow -\vec{k}$ and noting that $k\cdot x = \vec{k} \cdot \vec{x}$, it follows that

$$\Phi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{2E_k} \left[ a(-\vec{k})e^{-ik\cdot x} + b^\dagger(-\vec{k})e^{ik\cdot x} \right] = P\Phi(x)P^{-1},$$  

after using eqs. (71) and (72). Indeed, eq. (74) is equivalent to eq. (67).

Second, consider the time reversal transformation. Eq. (69) implies that

$$Ta(\vec{k})T^{-1} = a(-\vec{k}), \quad Tb(\vec{k})T^{-1} = b(-\vec{k}),$$  

(75)

$$Ta^\dagger(\vec{k})T^{-1} = a^\dagger(-\vec{k}), \quad Tb^\dagger(\vec{k})T^{-1} = b^\dagger(-\vec{k}).$$  

(76)

In contrast to fermion field theory, there is no extra minus sign in $Pb(\vec{k})P^{-1}$. Thus a boson and its antiparticle have the same parity quantum number.
The subtlety of this analysis is that $T$ is an antiunitary operator, which implies that
\[ e^{-ik \cdot x} T^{-1} = T^{-1} e^{ik \cdot x}. \]  
(77)
Hence, following a calculation similar to the one given above in the case of parity,
\[
\Phi(-\bar{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{2E_k} \left[ a(\bar{k}) e^{ik \cdot \bar{x}} + b^\dagger(\bar{k}) e^{-ik \cdot \bar{x}} \right]
= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{2E_k} \left[ a(-\bar{k}) e^{ik \cdot x} + b^\dagger(-\bar{k}) e^{-ik \cdot x} \right]
= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{2E_k} \left[ T a(\bar{k}) T^{-1} e^{ik \cdot x} + T b^\dagger(\bar{k}) T^{-1} e^{-ik \cdot x} \right]
= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{2E_k} T \left[ a(\bar{k}) e^{ik \cdot x} + b^\dagger(\bar{k}) e^{-ik \cdot x} \right] T
= T \Phi(x) T^{-1},
\]  
(78)
after employing eq. (77) at the penultimate step. Indeed, eq. (78) is equivalent to eq. (69).

Finally, we consider the charge conjugation transformation. I claim that
\[
Ca(\bar{k}) C^{-1} = b(\bar{k}), \quad Cb(\bar{k}) C^{-1} = a(\bar{k}),
\]  
(79)
\[
Ca^\dagger(\bar{k}) C^{-1} = b^\dagger(\bar{k}), \quad Cb^\dagger(\bar{k}) C^{-1} = a^\dagger(\bar{k}).
\]  
(80)
To verify this assertion, we note that
\[
\Phi^\dagger(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{2E_k} \left[ b(\bar{k}) e^{-ik \cdot x} + a^\dagger(\bar{k}) e^{ik \cdot x} \right]
= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{2E_k} \left[ Ca(\bar{k}) C^{-1} e^{-ik \cdot x} + C b^\dagger(\bar{k}) C^{-1} e^{ik \cdot x} \right] = C \Phi(x) C^{-1},
\]
which is equivalent to eq. (68).

(b) Consider the conserved current operator $J^\mu$, 
\[
J^\mu = i : \Phi^\dagger (\partial^\mu \Phi) - (\partial^\mu \Phi^\dagger) \Phi : 
\]
where $:\mathcal{O}:$ indicates that the operator $\mathcal{O}$ is normal ordered. Determine the transformation properties of $J^\mu$ under $P$, $T$ and $C$. Why is the normal ordering in the definition of the operator $J^\mu$ necessary?

We now consider the properties of $J^\mu$ under $P$, $C$, and $T$. Using eq. (67),
\[
\begin{align*}
P J^\mu (t ; \bar{x}) P^{-1} &= \begin{cases} 
J^0 (t ; -\bar{x}) , & \text{for } \mu = 0 , \\
-J^i (t ; -\bar{x}) , & \text{for } \mu = i ,
\end{cases}
\end{align*}
\]
and \(i = 1, 2\) or \(3\). The above result is obtained after noting that under a parity transformation, \(x^\mu \rightarrow \bar{x}^\mu\) [cf. eq. (73)], so that

\[
\partial_\mu \rightarrow \bar{\partial}_\mu \equiv \frac{\partial}{\partial \bar{x}^\mu} = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \frac{\partial}{\partial x^\nu} = \begin{cases} 
\partial_\nu, & \text{for } \mu = 0, \\
-\partial_\nu, & \text{for } \mu = i.
\end{cases}
\]

In light of eq. (69), a similar computation yields,

\[
TJ^\mu(t; \bar{x})T^{-1} = \begin{cases} 
J^0(-t; \bar{x}), & \text{for } \mu = 0, \\
-J^i(-t; \bar{x}), & \text{for } \mu = i,
\end{cases}
\]

In this case, two minus signs compensate each other. The first minus sign arises because under a time reversal transformation, \(x^\mu \rightarrow -\bar{x}^\mu\), which yields

\[
\partial_\mu \rightarrow \begin{cases} 
-\partial_\nu, & \text{for } \mu = 0, \\
\partial_\nu, & \text{for } \mu = i.
\end{cases}
\]

The second minus sign arises due to the fact that \(T\) is an antiunitary operator, which yields

\[
TJ^\mu T^{-1} = -iT:\left[\Phi^\dagger(\partial^\mu \Phi) - (\partial^\mu \Phi^\dagger)\Phi\right]:T^{-1},
\]

after \(T\) passes through the \(i\).

Finally, we consider the charge conjugation transformation. Using eq. (68),

\[
CJ^\mu C^{-1} = i:\left[\Phi(\partial^\mu \Phi^\dagger) - (\partial^\mu \Phi^\dagger)\Phi\right]:= -i:\left[\Phi^\dagger(\partial^\mu \Phi) - (\partial^\mu \Phi^\dagger)\Phi\right]:= -J^\mu. \quad (81)
\]

In the penultimate step above, the order of the field operators was modified. Since the normal ordering prescription dictates the proper ordering of the creation and annihilation operators, it follows that the order of the operators appearing inside the normal ordering symbol is immaterial. Eq. (81) implies that the current is odd under charge conjugation, as expected.

Note that the properties of \(J^\mu\) under parity and time-reversal do not depend on the presence of the normal ordering prescription. In the case of charge conjugation, the normal ordering prescription is necessary only in the case where the operators within the normal ordering symbols do not commute. It is easy to check that the normal ordering prescription is required only for the case of \(J^0\). In this case,

\[
J^0 = i:\left[\Phi^\dagger(\partial^0 \Phi) - (\partial^0 \Phi^\dagger)\Phi\right]:= i:\left[\Phi^\dagger \Pi^\dagger - \Pi \Phi\right];
\]

where \(\Pi\) and \(\Pi^\dagger\) are the canonical momentum fields. Since

\[
[\Phi(\bar{x}, t), \Pi(\bar{x}', t)] = [\Phi^\dagger(\bar{x}, t), \Pi^\dagger(\bar{x}', t)] = i\delta^3(\bar{x} - \bar{x}'),
\]

it is clear that \(\Phi(\bar{x}, t)\) and \(\Pi(\bar{x}, t)\) [and likewise, \(\Phi^\dagger(\bar{x}, t)\) and \(\Pi^\dagger(\bar{x}, t)\)] do not commute. This is not surprising, since the reason for the normal ordering prescription in the definition of the current is to remove the infinite vacuum charge. In the case of \(J^i\), the corresponding field operators commute and the normal ordering prescription is not required.
Consider a theory of a charged scalar field $\Phi(x)$ and a Dirac fermion field $\Psi(x)$. Show that any Lorentz invariant hermitian local operator$^4$ built from products of $\Phi(x)$, $\Psi(x)$ and their conjugates has $CPT = +1$.

Under CPT, the results of parts (a) and (b) imply that

$$CPT \Phi(x)(CPT)^{-1} = \Phi^\dagger(-x),$$

$$CPT J^\mu(x)(CPT)^{-1} = -J^\mu(-x),$$

where $-x \equiv (-t; -\vec{x})$. If we build any local evaluated hermitian Lorentz invariant operator, $\mathcal{O}(x)$, out of scalar fields and derivatives, then

$$CPT \mathcal{O}(x)(CPT)^{-1} = \mathcal{O}(-x).$$

(82)

Examples of such operators would be $|\Phi(x)|^2$, $\partial_\mu \Phi^\dagger(x) \partial^\mu \Phi(x)$, $J_\mu(x) J^\mu(x)$, etc. An equivalent way to write eq. (82) is

$$CPT \mathcal{O}(x)(CPT)^{-1} = \eta_{CPT} \mathcal{O}(-x),$$

where $\eta_{CPT} = +1$. We then say colloquially that the operator $\mathcal{O}(x)$ has $CPT = +1$.

In class, we showed that the bilinear covariants, constructed out of fermion fields and their Dirac adjoints, had the following transformation properties,

$$CPT S(x)(CPT)^{-1} = S(-x),$$

$$CPT V^\mu(x)(CPT)^{-1} = -V^\mu(-x),$$

$$CPT F^{\mu\nu}(x)(CPT)^{-1} = F^{\mu\nu}(-x),$$

where $S(x)$ was a scalar or pseudoscalar operator, $V^\mu(x)$ was a vector or an axial vector operator and $F^{\mu\nu}$ was either an antisymmetric second rank tensor or antisymmetric second rank pseudotensor operator. Combining any of the above operators to make a Lorentz invariant operator $\mathcal{O}(x)$ again yields eq. (82).

Any such operator $\mathcal{O}(x)$ is a candidate for a term in a Lagrangian density. Integrating the Lagrangian density over all of spacetime yields the action. Since

$$\int d^4 x \mathcal{O}(-x) = \int d^4 x \mathcal{O}(x),$$

after a change of the integration variable, $x \rightarrow -x$, it follows that all such terms in the action are invariant under a CPT transformation. This is a statement of the CPT theorem of quantum field theory.

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$^4$By definition, a local operator is a product of fields or derivatives of fields in which each field is evaluated at the same spacetime point.