# Warming Up to Finite-Temperature Field Theory

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# Overview

- Motivations
- Quantum statistical mechanics (quick!)
- Path integral representation of partition function in quantum mechanics
  - Quantum harmonic oscillator
- Path integral representation of partition function in quantum field theory
  - Partition function for free scalar field and its free energy density
  - Partition function for interacting scalar fields and free energy density

# Motivations

- QFT formulated at zero-temperature
  - What are the effects of heating up QFTs?
- Initial motivations found in condensed matter physics
- Later applications to quantum chromodynamics, cosmology, and astrophysics
  - Quark-gluon plasmas (QGPs), heavy-ion collisions, baryogenesis, inflation and phase transitions in the early universe, stellar emission of neutrinos, etc.
- Advantages over kinetic/many-body theory
  - Path integral
  - Non-Abelian gauge interactions
  - Lorentz invariant

# Quantum Statistical Mechanics

- For quantum system described by Hamiltonian $H$ and set of conserved number operators $\hat{N}_i$ that commute with $H$, the density matrix:

$$\hat{\rho} = \exp[-\beta(H - \mu_i \hat{N}_i)]$$

(natural units used throughout $\hbar = c = k_B = 1$)

- Can calculate expectation values of observables by

$$\langle \hat{A} \rangle = \frac{Tr[\hat{A}\hat{\rho}]}{Tr[\hat{\rho}]}$$

- The grand canonical partition function can be used to calculate thermodynamic properties of a system:

$$Z = Tr[\hat{\rho}]$$

- Pressure, entropy, particle number and free energy: $P = \frac{\partial(T \ln Z)}{\partial V}$, $N_i = \frac{\partial(T \ln Z)}{\partial \mu_i}$, $S = \frac{\partial(T \ln Z)}{\partial T}$, $F = -T \ln Z$

# Partition Function: Harmonic Oscillator

- For the (bosonic) quantum harmonic oscillator, the number operator is $\hat{N} = a^\dagger a$ and the Hamiltonian is $H = \omega(\hat{N} + \frac{1}{2})$
- The partition function (with $\mu = 0$) can be written as

$$Z = \exp[-\frac{\beta\omega}{2}]Tr[\exp[-\beta(\omega)\hat{N}]] = \exp[-\frac{\beta\omega}{2}]\sum_{n=0}^{\infty}\langle n|\exp[-\beta(\omega)\hat{N}]|n\rangle$$

$$= \exp[-\frac{\beta\omega}{2}]\sum_{n=0}^{\infty}\exp[-\beta(\omega)n] = \exp[-\frac{\beta\omega}{2}](1-\exp[-\beta\omega])^{-1}$$

$$= [2\sinh(\frac{\omega\beta}{2})]^{-1}$$

# Path Integral for Partition Function in QM

- There are few systems for which the partition function can be calculated exactly.
- A more useful representation is that of the path integral
- Start here, translate results to QFTs
- The partition function can be written as

$$Z = Tr[\exp(-\beta H)] = \int dx \langle x|\exp(-\beta H)$$

  Defining $\beta = \epsilon N$ the exponential is split into a product of N pieces

$$Z = \int dx \langle x|\exp(-\epsilon H) \cdots \exp(-\epsilon H)|x\rangle$$

- Inserting $\mathbb{1} = \int \frac{dp_i}{2\pi} |p_i\rangle\langle p_i|$ on the left of each exponential and $\mathbb{1} = \int dx|x_i\rangle\langle x_i|$ on the right ($i$ increasing from right to left in both cases with $i = 1, \ldots, N$)

# Path Integral for Partition Function in QM

the partition function becomes

$$Z = \int \frac{dx dx_i dp_i}{2\pi} \langle x|p_N\rangle\langle p_N|\exp(-\epsilon H)|x_N\rangle\langle x_N|p_{N-1}\rangle$$
$$\times\langle p_{N-1}|\exp(-\epsilon H)|x_{N-1}\rangle\langle x_{N-1}|p_{N-2}\rangle\cdots\langle x_2|p_1\rangle\langle p_1|\exp(-\epsilon H)|x_1\rangle\langle x_1|x\rangle \tag{1}$$

- On the very left of equation (1) above with $\langle x|$ playing role of $\langle x_{i+1}|$ we have:

$$\langle x_{i+1}|p_i\rangle\langle p_i|\exp(-\epsilon H)|x_i\rangle = \exp(ip_i x_{i+1})\langle p_i|\exp(-\epsilon H)|x_i\rangle$$
$$= \exp(ip_i x_{i+1})\exp(ip_i x_i)\exp[-\epsilon(\frac{p_i^2}{2m} + V(x_i) + \mathcal{O}(\epsilon))]$$
$$= \exp[-\epsilon(\frac{p_i^2}{2m} - ip_i\frac{x_{i+1} - x_i}{\epsilon} + V(x_i) + \mathcal{O}(\epsilon))]$$

- Using the delta function at the very right of equation (1), the integral can be carried out over $x$ yielding the partition function

# Path Integral for Partition Function in QM

$$Z = \lim_{N\to\infty}$$

$$\int \Big[\prod_{i=1}^{N} \frac{dx_i dp_i}{2\pi}\Big] \exp\Big[-\sum_{j=1}^{N} \epsilon(\frac{p_j^2}{2m} - ip_j \frac{x_{j+1}-x_j}{\epsilon} + V(x_j) + \mathcal{O}(\epsilon))\Big]\Bigg|_{x_{N+1}=x_1} \quad (2$$

- The momentum integral is Gaussian and can be easily evaluated to be

$$\int_{-\infty}^{\infty} \frac{dp_i}{2\pi} \exp[-\epsilon(\frac{p_i^2}{2m} - ip_i \frac{x_{i+1}-x_i}{\epsilon})]$$

$$= \sqrt{(\frac{m}{2\pi\epsilon})} \exp(-\frac{m(x_{i+1}-x_i)^2}{2\epsilon}) \quad (3)$$

# Path Integral for Partition Function in QM

Thus the partition function becomes

$$Z = \lim_{N\to\infty} \int \Big[\prod_{i=1}^{\infty} \frac{dx_i}{\sqrt{\frac{2\pi\epsilon}{m}}}\Big] \exp\Big[-\sum_{j=1}^{\infty} \epsilon(\frac{m}{2}(\frac{x_{j+1}-x_j}{\epsilon})^2 + V(x_j))\Big]\Bigg|_{x_{N+1}=x_1}$$

- In the continuum limit, defining $(\frac{m}{2\pi\epsilon})^{N/2} = C$, where $C$ is infinite, the above equation becomes

$$Z = C\int_{x(\beta)=x(0)} \mathcal{D}[x] \exp\Big[-\int_0^\beta (\frac{m}{2}(\frac{dx}{d\tau})^2 + V(x))\Big]$$

- Compare to zero-temperature QM counterpart:

$$Z_0 = C\int \mathcal{D}[x] \exp(i\int dt\mathcal{L}_0)$$

- Finite-temp partition function can be achieved in four steps:

- Wick rotation using $\tau = it$
- Introduce "finite-temperature" Lagrangian: $L = \mathcal{L}_0(\tau = it) = \frac{m}{2}(\frac{dx}{d\tau})^2 + V(x)$
- Restrict integration by $\int dt \to \int_0^\beta d\tau$
- Impose periodicity on $x(\tau)$ by $x(\beta) = x(0)$

* This process is the imaginary time formalism of doing field theory
* Can be used also in quantum field theory and can be used for fermions and bosons

# Path Integral for QHO

- To go to momentum space, write $x(\tau)$ as Fourier series: $x(\tau) = T\sum_{n=-\infty}^{\infty} x_n \exp(i\omega_n\tau)$
- Periodicity requires $e^{i\omega_n\beta} = 1 \Rightarrow \omega_n = 2\pi nT$
- Impose reality on the $x(\tau) \Rightarrow x^*(\tau) = x(\tau)$ making the $x_n^* = x_{-n}$ and can now write $x_n = a_n + ib_n \Rightarrow x_n^* = a_n - ib_n = x_{-n} = a_{-n} + ib_{-n} \Rightarrow a_n = a_{-n},\ b_n = -b_{-n}$
- Letting $b_0 = 0$ and putting this all together,

$$x(\tau) = T\Big[a_0 + \sum_{n=1}^{\infty}[(a_n + ib_n)e^{i\omega_n\tau} + (a_n - ib_n)e^{-i\omega_n\tau}]\Big] \quad (4)$$

- Fourier representation allows quadratics of the form

$$\int_0^\beta d\tau x(\tau)y(\tau) = T^2\sum_{m,n} x_n y_m \int_0^\beta d\tau e^{i(\omega_n+\omega_m})$$

$$= \sum_{m,n} x_n y_m \delta_{n,-m} = \sum_n x_n y_{-n}$$

- Next, using the four steps above for the action of the quantum harmonic oscillator:

$$-\int_0^\beta d\tau \frac{m}{2}[\frac{dx}{d\tau}\frac{dx}{d\tau} + \omega^2 x(\tau)x(\tau)]$$

$$= -\frac{mT}{2}\sum_{n=-\infty}^{\infty} x_n(-\omega_n\omega_{-n} + \omega^2)x_{-n}$$

and using $\omega_{-n} = -\omega_n$ the action becomes:

$$S = -\frac{mT}{2}\omega^2 a_0^2 - mT\sum_{n=1}^{\infty}(\omega_n^2 + \omega^2)(a^2 + b^2)$$

- Turning our attention to the product of integrals $\mathcal{D}[x]$ this can be rewritten by changing variables to $a_n$, $b_n$ to obtain

$$\mathcal{D}[x] = |det(\frac{\delta x(\tau)}{\delta x_n})|da_0[\prod_{n\geq 1} da_n db_n]$$

- Defining a new constant $C' = C|det(\frac{\delta x(\tau)}{\delta x_n})|$ the harmonic oscillator partition function is

$$Z = C' \int_{-\infty}^{\infty} da_0 \int_{-\infty}^{\infty} \Big[ \prod_{n\geq 1} da_n db_n \Big] \times \exp\Big[ -\frac{mT}{2}\omega^2 a_0^2 - mT \sum_{n\geq 1} (\omega_n^2 + \omega^2)(a_n^2 + b_n^2) \Big] \quad (5)$$

- Solving the Gaussian integrals above (and solving for $C'$) yields

$$Z = \frac{1}{\omega\beta} \prod_{n=1}^{\infty} \frac{\omega_n^2}{\omega_n^2 + \omega^2}$$

## Remarks

– The partition function as well as other extensive quantities like volume, temperature, and pressure are all finite, but with the path integral, this is not immediately clear.

- In quantum field theory, however, these divergences may remain.
- When calculating most physically relevant variables, this infinite constant falls out.

# Free Scalar Fields

- For free scalar fields, we can follow essentially the same procedure taking $x(\tau) \to \phi(\tau, \vec{x})$

- Why can we do this?

  – The Lagrangian for a scalar field, given by $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) = (\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)(\partial_i\phi) - V(\phi)$ which is a collection of near-independent harmonic oscillators (with m=1).

  – Interact through spatial derivatives, approximately given

$$\partial_i\phi \approx \frac{\phi(t, \vec{x} + \epsilon\vec{e_i}) - \phi(t, \vec{x})}{\epsilon}$$

  and since the Hamiltonian depends only on terms quadratic in canonical momentum, this coupling cannot change the previous derivation fundamentally

- Taking derivation from previous section

$$Z = \int_{\phi(\beta,\vec{x})=\phi(0,\vec{x})} \prod_x C\mathcal{D}[\phi] \exp[-\int_0^\beta d\tau \int d^3L]$$

where

$$L = -\mathcal{L}(t = i\tau) = \frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 + \sum_{i=1}^{3} \frac{1}{2}(\frac{\partial\phi}{\partial x_i})^2 + V(\phi)$$

- As with standard QM case, expand the fields in a Fourier series:

$$\phi(\tau, \vec{x}) = T \sum_{n=-\infty}^{\infty} \tilde{\phi}(\omega_n, \vec{x}) e^{i\omega_n\tau}$$

- In this case, it is convenient to let the directions of the spatial coordinates be momentarily finite. Denoting these directions as $L_i$ and using

$$f(x_i) = \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \tilde{f}(n_i) e^{ik_i x_i}$$

where $k_i = \frac{2\pi n_i}{L_i}$

- Taking the volume to infinity gives

$$\frac{1}{L_i}\sum_{n_i} = \frac{1}{2\pi}\sum_{n_i}\Delta k_i \to \int \frac{dk_i}{2\pi}$$

where $\Delta k_i = \frac{2\pi}{L_i}$ and

$$\phi(\tau,\vec{x}) = \frac{T}{V}\sum_{\omega_n}\sum_{\vec{k}}\tilde{\phi}(\tau,\vec{k})\exp(i\omega_n\tau - i\vec{k}\cdot\vec{x})$$

- For brevity, will quote result (follow similar steps as previous section)

$$Z = \exp(-\int_0^\beta d\tau \int L) = \prod_{\vec{k}}\Big[\exp[-\frac{T}{2V}\sum_{\omega_n}(\omega_n^2 + \vec{k}^2 + m^2)|\tilde{\phi}(\omega_n,\vec{k}|)^2]\Big] \quad (6)$$

- Looks like product of harmonic oscillator partition functions

- Comparing to the quantum harmonic oscillator $\omega^2 \to \vec{k}^2 + m^2 = E_k^2$, $m \to \frac{1}{V}$, $|x_n|^2 \to |\tilde{\phi}(\omega_n, \vec{k})|^2$

- Continuing with the analogies to QHO,

$$Z = \prod_{\vec{k}} \Big[ T \prod_{n=-\infty}^{\infty} (\omega_n^2 + E_k^2)^{-1/2} \prod_{n \neq 0} (\omega_n^2)^{1/2} \Big]$$

$$= \exp\Big[ \sum_{\vec{k}} \ln(T) + \frac{1}{2} \sum_{n \neq 0} \ln(\omega_n^2) - \frac{1}{2} \sum_{n} \ln(\omega_n^2 + E_k^2) \Big] = \exp(-\frac{F}{T})$$

- In order to obtain the free energy density, take limit at volume goes to infinity to find

$$\lim_{V \to \infty} \frac{F}{V} = \int \frac{d^3 k}{(2\pi)^3} \Big[ T \sum_{\omega_n} \frac{1}{2} \ln(\omega_n^2 + E_k^2) - T \sum_{\omega_{n \neq 0}} \frac{1}{2} \ln(\omega_n^2) - T \ln(T) \Big]$$

# Interacting Theory

- As in zero temperature QFT, free theory is only exactly solvable theory
- But because of the way the thermal theory is built, we can use perturbative techniques in much the same way to approximate interactions
- In much the same way as was done in this course, we can write the partition function as

$$Z = \int \mathcal{D}[\phi] exp[-S_0 + S_I]$$

  where $S_0$ is the action of the free scalar field theory and $S_I$ is the action of the interacting theory
- This can be written in the form

$$Z = C \int \mathcal{D}[\phi] e^{-S_0} [1 - S_I + \frac{1}{2!} S_I^2 - \frac{1}{3!} S_I^3]$$

and using

$$\langle S_I^n \rangle = \frac{\int \mathcal{D}[\phi] S_I^n e^{-S_0}}{\int \mathcal{D}[\phi] e^{-S_0}}$$

$$\Rightarrow \; Z = Z_0(1 - \langle S_I \rangle + \frac{1}{2!}\langle S_I^2 \rangle + \cdots)$$

- Taking the logarithm, it becomes obvious that the first term in the above equation corresponds to a term proportional to the free energy density and the terms following correspond to the corrections due to interactions

# Feynman Rules for $\phi^4$-theory

- Taking the logarithm of the the partition function for interacting fields

$$\ln Z = \ln Z_0 + \ln(1 - \langle S_I \rangle + \frac{1}{2!}\langle S_I^2 \rangle + \cdots)$$

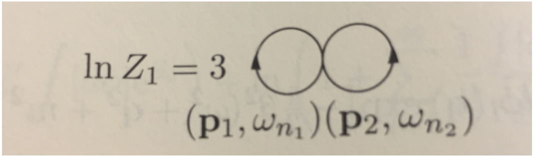
- The second logarithm corresponds to the interacting fields, $\ln Z_I = \ln(1 - \langle S_I \rangle + \frac{1}{2!}\langle S_I^2 \rangle + \cdots)$ and can be expanded using $ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ to find the first order correction:

$$\ln Z_1 \approx -\langle S_I \rangle_0 = \frac{\lambda}{4}\Big[\frac{\int_0^\beta d\tau \int d^3x \int \mathcal{D}[\phi] \exp(-S_0)\phi^4}{\int \mathcal{D}[\phi] \exp(-S_0)}\Big]$$

- Skipping the steps of evaluating the above formula, which involve Fourier expanding the fields, the first order correction for a $\phi^4$-theory is

$$\ln Z_1 = -\frac{3}{4}\lambda\beta V\Big[T\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_n^2 + \vec{k}^2 + m^2}\Big]^2$$

- As in zero-temperature QFT, this has a diagrammatic representation




- "Finite-Temperature Feynman Rules" for $\phi^4$-theory

  - Draw all connected diagrams
  - Determine the combinatoric factor for each diagram
  - Include a factor of $T \sum_n \int [\frac{d^3p}{(2\pi)^3}] \mathcal{D}_0(\omega_n, \vec{p})$
  - Include a factor of $-\lambda$ for each vertex
  - Include a factor of $(2\pi)^3 \delta(\vec{p_{in}} - \vec{p_{out}}) \beta \delta_{\omega_{in}, \omega_{out}}$ for each vertex, corresponding to energy-momentum conservation. There will be one factor of $\beta (2\pi)^3 \delta \vec{0} = \beta V$ left over.

# Summary/Next Steps

- Calculated the partition function of the QHO using a path integral
- Developed the general tools necessary for calculating observable quantities
- Were able to extend this to a simple scalar QFT
- Somethings missing: chemical potential, physical behavior discussions
- Extend to more concepts in QFTs: gauge theories, symmetry breaking, lattice-QCD/QGP calculations...

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