

In the first chapter, we focused on quantum field theories of free fermions. In order to construct renormalizable interacting quantum field theories, we must introduce additional fields. The requirement of renormalizability imposes two constraints. First, the terms of the interaction Lagrangian must be no higher than mass dimension-four. Thus, no (perturbatively) renormalizable interacting theory that consists only of spin-1/2 fields exists, since the simplest interaction term involving fermions is a dimension-six four-fermion interaction. Renormalizable interacting theories consisting of scalars, fermions and spin-one bosons can be constructed. The vector bosons must either be abelian vector fields or non-abelian gauge fields. This exhausts all possible renormalizable field theories.

The Standard Model is a spontaneously broken non-abelian gauge theory containing elementary scalars, fermions and spin-one gauge bosons. Typically, one refers to the spin-0 and spin-1/2 fields (which are either neutral or charged with respect to the underlying gauge group) as *matter fields*, whereas the spin-1 gauge bosons are called *gauge fields*. In this chapter we review the ingredients for constructing non-abelian (Yang-Mills) gauge theories and their breaking via the dynamics of self-interacting scalar fields. The Standard Model of fundamental particles and interactions is then exhibited, and some of its properties are described.

4.1 Abelian Gauge field theory

The first (and simplest) known gauge theory is quantum electrodynamics (QED). This was a very successful theory that described the interactions of electrons, positrons and photons. The Lagrangian of QED is given by:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\xi^\dagger \bar{\sigma}^\mu D_\mu \xi + i\eta^\dagger \bar{\sigma}^\mu D_\mu \eta - m(\xi\eta + \xi^\dagger \eta^\dagger), \quad (4.1.1)$$

where the electromagnetic field strength tensor $F_{\mu\nu}$ is defined in terms of the gauge field A_μ as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (4.1.2)$$

and the covariant derivative D_μ is defined as:

$$D_\mu \psi(x) \equiv (\partial_\mu + ieq_\psi A_\mu)\psi(x), \quad (4.1.3)$$

where $\psi(x) = \xi(x)$ or $\eta(x)$ with $q_\xi = -1$ and $q_\eta = +1$. Note that we have written eq. (4.1.1) in terms of the two-component charged fermion fields ξ and η . The identification of these fields with the electron and positron (with corresponding electric charges q_ψ in units of $e > 0$) has been given in Section E.¹

The QED Lagrangian consists of a sum of the kinetic energy term for the gauge (photon) field, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and the Dirac Lagrangian with the ordinary derivative ∂_μ replaced by a covariant derivative D_μ . This Lagrangian is invariant under a local U(1) gauge transformation:

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\Lambda(x), \quad (4.1.4)$$

$$\xi(x) \rightarrow \exp[ie\Lambda(x)]\xi(x), \quad (4.1.5)$$

$$\eta(x) \rightarrow \exp[-ie\Lambda(x)]\eta(x). \quad (4.1.6)$$

The Feynman rule for electrons interacting with photons is obtained by taking $G_\psi = eq_\psi$ for $\psi = \xi$ and η in eq. (2.6.1). That is, we employ $G_\xi = -G_\eta = -e$ in the two-component Feynman rules displayed in Fig. 2.6. One can easily check that the corresponding four-component Feynman rule of Fig. 3.1 yields the well-known rule for the $e^+e^-\gamma$ vertex of QED.

One can extend the theory above by including charged scalars among the possible matter fields. For example, a complex scalar field $\Phi(x)$ of definite U(1) charge q_Φ will transform under the local U(1) gauge transformation:

$$\Phi(x) \rightarrow \exp[-ieq_\Phi\Lambda(x)]\Phi(x). \quad (4.1.7)$$

A gauge invariant Lagrangian involving the scalar fields can be obtained from the free-field scalar Lagrangian of eq. (3.0.3), by replacing $\partial_\mu\Phi(x)$ with $D_\mu\Phi(x) \equiv (\partial_\mu + ieq_\Phi)\Phi(x)$. One may also add gauge-invariant Yukawa interactions of the form

$$\mathcal{L}_Y = -y_{ijk}\Phi_i\psi_j\psi_k + \text{h.c.}, \quad (4.1.8)$$

where Φ_i consist of either neutral or charged scalar fields and ψ_i consist of neutral Majorana (χ) or charged Dirac pairs (ξ and η) of two-component fermion fields. The (complex) Yukawa couplings y_{ijk} vanish unless the condition $q_{\Phi_i} + q_{\psi_j} + q_{\psi_k} = 0$ is satisfied, as required by gauge invariance.

4.2 Non-abelian gauge groups and their Lie algebras

Abelian gauge field theory can be generalized by replacing the abelian U(1) gauge group of QED with a non-abelian gauge group G. We again consider possible matter fields—multiplets of scalar fields $\Phi_i(x)$ and two-component fermion fields $\psi_i(x)$ [or

¹ It is a simple matter to rewrite eq. (4.1.1) in terms of the four-component spinor electron field. Simply replace $\partial_\mu\Psi(x)$ with $D_\mu\Psi(x)$ in the Dirac Lagrangian [eq. (3.2.60)] with $q_\Psi = -1$ and add the kinetic energy term for the gauge fields.

equivalently, four component fermions $\Psi_i(x)$] that are either neutral or charged with respect to G .

The symmetry group G can be expressed in general as a direct product of a finite number of simple compact Lie groups and $U(1)$. A direct product of simple Lie groups [*i.e.*, with no $U(1)$ factors] is called a semi-simple Lie group. The list of all possible simple Lie groups are known and consist of $SU(n)$, $SO(n)$, $Sp(n)$ and five exceptional groups (G_2 , F_4 , E_6 , E_7 and E_8). Given some matter field (either scalar or fermion), which we generically designate by $\phi_i(x)$, the gauge transformation under which the Lagrangian is invariant, is given by:

$$\phi_i(x) \rightarrow U_i^j(g)\phi_j(x), \quad i, j = 1, 2, \dots, d(R), \quad (4.2.1)$$

where g is an element of G (that is, g is a specific gauge transformation) and $U(g)$ is a (possibly reducible) $d(R)$ -dimensional unitary representation R of the group G . One is always free to redefine the fields via $\phi(x) \rightarrow V\phi(x)$, where V is any fixed unitary matrix (independent of the choice of g). The gauge transformation law for the redefined $\phi(x)$ now has $U(g)$ replaced by $V^{-1}U(g)V$. If $U(g)$ is a reducible representation, then it is possible to find a V such that the $U(g)$ for *all* group elements g assume a block diagonal form. Otherwise, the representation $U(g)$ is irreducible. The matter fields of the gauge theory generally form a reducible representation, which can subsequently be decomposed into their irreducible pieces. Irreducible representations imply that the corresponding multiplets transform only among themselves, and thus we can focus on these pieces separately without loss of generality.

The *local* gauge transformation $U(g)$ is also a function of space-time position, x^μ . Explicitly, any group element that is continuously connected to the identity takes the following form:

$$U(g(x)) = \exp[-ig_a\Lambda^a(x)T^a], \quad (4.2.2)$$

where there is an implicit sum over the repeated index $a = 1, 2, \dots, d_G$. The T^a are a set of d_G linearly independent hermitian matrices² called generators of the Lie group, and the corresponding $\Lambda^a(x)$ are arbitrary x -dependent functions. The constants g_a (which is analogous to e of the abelian theory) are called the gauge couplings. There is a separate coupling g_a for each simple group or $U(1)$ factor of the gauge group G . Thus, the generators T^a separate out into distinct classes, each of which is associated with a simple group or one of the $U(1)$ factors contained in the direct product that defines G . In particular, $g_a = g_b$ if T^a and T^b are in the same class. If G is simple, then $g_a = g$ for all a .

Lie group theory teaches us that the number of linearly independent generators, d_G , depends only on the abstract definition of G (and not on the choice of representation). Thus, d_G is also called the dimension of the Lie group G . Moreover, the

² The condition of linear independence means that $c^a T^a = 0$ (implicit sum over a) implies that $c^a = 0$ for all a .

commutator of two generators is a linear combination of generators:

$$[T^a, T^b] = if^{abc}T^c, \quad (4.2.3)$$

where the f^{abc} are called the structure constants of the Lie group. In studying the structure of gauge field theories, nearly all the information of interest can be ascertained by focusing on infinitesimal gauge transformations. In a given representation R , one can expand about the $d(R) \times d(R)$ identity matrix any matrix representation of a group element $U(g(x))$ that is continuously connected to the identity element:

$$U(g(x)) \simeq I_{d(R)} - ig_a \Lambda^a(x) T^a, \quad (4.2.4)$$

where the T^a comprise a $d(R)$ -dimensional representation of the Lie group generators. Then, the infinitesimal gauge transformation corresponding to eq. (4.2.1) is given by $\phi_i(x) \rightarrow \phi_i(x) + \delta\phi_i(x)$, where

$$\delta\phi_i(x) = -ig_a \Lambda^a(x) (T^a)_i^j \phi_j(x). \quad (4.2.5)$$

From the generators T^a , one can reconstruct the group elements $U(g(x))$, so it is sufficient to focus on the infinitesimal group transformations. The group generators T^a span a real vector space, whose general element is $c^a T^a$, where the c^a are real numbers.³ One can formally define a “vector product” of any two elements of the vector space as the commutator of the two vectors. For example, using eq. (4.2.3), it is clear that the vector product of any two vectors $[c^a T^a, d^b T^b]$ is a real linear combination of the generators, which is also an element of the vector space. Consequently, this vector space is also an algebra, called a Lie algebra. Henceforth, the Lie algebra corresponding to the Lie group G will be designated by \mathfrak{g} . A Lie algebra has one additional important property:

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0. \quad (4.2.6)$$

This is called the Jacobi identity, and it is clearly satisfied by any three elements of the Lie algebra.

If the symmetry group G is a direct product of simple Lie group and $U(1)$ factors, then its Lie algebra \mathfrak{g} is a direct sum of a finite number of simple Lie algebras and $\mathfrak{u}(1)$.⁴ A direct sum of simple Lie algebras [*i.e.*, with no $\mathfrak{u}(1)$ factors] is called a semi-simple Lie algebra. If T^a and T^b belong to different classes (*i.e.*, different factors of the direct sum), then $[T^a, T^b] = 0$. Equivalently, $f^{abc} = 0$ if T^a , T^b and T^c do not all belong to the same class.

In the next section, we will see that the gauge fields transform under the adjoint representation of the (global) gauge group. The explicit matrix elements of the

³ With the T^a hermitian, we require the c^a to be real in order that $U(g)$ be unitary. Then, the T^a span a *real* Lie algebra. Mathematicians consider the elements of the real Lie algebra to be $ic^a T^a$, with anti-hermitian generators iT^a . Note that for real Lie algebras, the representation matrices for T^a (or iT^a) may be complex or quaternionic.

⁴ The Lie algebra $\mathfrak{u}(1)$ is equivalent to the vector space of real numbers. The vector product is the commutator, which vanishes for any pair of $\mathfrak{u}(1)$ elements. Thus, $\mathfrak{u}(1)$ is an *abelian* Lie algebra.

adjoint representation generators are given by

$$(T^a)_{bc} = -if_{abc} \quad a, b, c = 1, 2, \dots, d_G. \quad (4.2.7)$$

Thus, the dimension of the adjoint representation matrices coincides with the number of generators, d_G . Hence, we shall often refer to the indices a, b, c as adjoint indices. For real Lie algebras, the f_{abc} are real numbers. Note that $f_{abc} = -f_{acb}$ [as a consequence of eq. (4.2.3)], so that in the adjoint representation the iT^a are real antisymmetric matrices. The representation matrices of the corresponding Lie group elements [eq. (4.2.2)] are therefore real and orthogonal. Thus, the adjoint representation provides a *real* representation of the Lie group and Lie algebra.

The choice of basis vectors (or generators) T^a is arbitrary. Moreover, the values of the structure constants f^{abc} also depend on this choice of basis. Nevertheless, there is a canonical choice which we now adopt. The generators are chosen such that:

$$\text{Tr}(T^a T^b) = T_R \delta^{ab}, \quad (4.2.8)$$

where T_R depends on the irreducible representation of the T^a . Having chosen this basis, there is no distinction between upper and lower adjoint indices.⁵ Moreover, in this basis the f^{abc} are completely antisymmetric under the interchange of a, b and c .⁶ One can show that once T_R is chosen for any one non-trivial irreducible representation R , then the value of T_R for any other irreducible representation is fixed. Corresponding to each simple real (compact) Lie algebra, one can identify one particular irreducible representation, called the *defining* representation (sometimes, but less accurately, called the fundamental representation); the most useful examples are listed in Table 4.1. For the defining (or fundamental) representation (which is indicated by $R = F$), the conventional value for T_R is taken to be:

$$T_F = \frac{1}{2}. \quad (4.2.9)$$

As noted above, the basis choice of eq. (4.2.8) with the normalization convention given by eq. (4.2.9) determines the value of T_R for an arbitrary irreducible representation R . The quantity T_R/T_F is called the *index* of the representation R .

For the record, we mention two other properties of Lie algebras that will be useful in this book. First, given any semi-simple Lie algebra \mathfrak{g} and a corresponding irreducible representation of anti-hermitian generators iT^a , one can always find an equivalent representation $V^{-1}T^aV$ for some unitary matrix V . There exists some choice of V (not necessarily unique) that maximizes the number of simultaneous diagonal generators, $V^{-1}T^aV$. This maximal number r_G , called the *rank* of \mathfrak{g} , is independent of the choice of representation, and is a property of the abstract Lie algebra. The ranks of the classical Lie algebras are given in Table 4.1.

Second, a Casimir operator is defined to be an operator that commutes with

⁵ More generally, in an arbitrary basis, $\text{Tr}(T^a T^b) = T_R g^{ab}$, where g^{ab} is the Cartan-Killing form (which can be used to raise and lower adjoint indices).

⁶ By definition, f^{abc} is antisymmetric under the interchange of a and b . But the complete antisymmetry under the interchange of all three indices requires eq. (4.2.8) to be satisfied.

Table 4.1 Simple real compact Lie algebras, \mathfrak{g} , of dimension d_G and rank r_G . Note that $[\frac{1}{2}n]$ indicates the greatest integer less than $\frac{1}{2}n$. The defining representation refers to arbitrary linear combinations $i c^a T^a$, where the c^a are real and the T^a are the generators of G .

| \mathfrak{g} | d_G | r_G | defining representation for iT^a |
|--------------------|---------------------|------------------|---|
| $\mathfrak{so}(n)$ | $\frac{1}{2}n(n-1)$ | $[\frac{1}{2}n]$ | $n \times n$ real antisymmetric |
| $\mathfrak{su}(n)$ | $n^2 - 1$ | $n - 1$ | $n \times n$ traceless complex anti-hermitian |
| $\mathfrak{sp}(n)$ | $n(2n+1)$ | n | $n \times n$ quaternionic anti-hermitian |

all the generators T^a of G . One can prove that a simple Lie algebra of rank r_G possesses r_G independent Casimir operators. The most important of these is the quadratic Casimir operator, which is defined by:

$$(\mathbf{T}^2)_{i^j} \equiv (T^a)_{i^k} (T^a)_{k^j} = C_R \delta_i^j. \quad (4.2.10)$$

Any operator that commutes with all the generators must be a multiple of the identity (by Schur's lemma). The coefficient of δ_i^j depends on the representation R and is denoted by C_R . By multiplying eq. (4.2.10) by δ_j^i , one derives an important theorem:

$$T_R d_G = C_R d(R), \quad (4.2.11)$$

where $d(R)$ is the dimension of the representation R . Note that for the adjoint representation ($R = A$), $d(A) = d_G$, so that $C_A = T_A$. As an example, for $SU(n)$, Table 4.1 yields $d_G = n^2 - 1$ and $d(F) = n$. Using eq. (4.2.9), one obtains $C_F = (n^2 - 1)/(2n)$. From an explicit representation of the f^{abc} for $SU(n)$, one can also derive $C_A = T_A = n$.

4.3 Non-abelian gauge field theory

In order to construct a non-abelian gauge theory, we follow the recipe presented in the case of the abelian gauge theory. Namely, we introduce a gauge field A_μ and a covariant derivative D_μ . By replacing ∂_μ with D_μ in the kinetic energy terms of the matter fields and introducing an appropriate transformation law for A_μ , the resulting matter kinetic energy terms are invariant under local gauge transformations.

As an example, consider a scalar field theory with the Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi_i)^\dagger (\partial^\mu \Phi_i) - V(\Phi, \Phi^\dagger), \quad (4.3.1)$$

where the scalar potential V is invariant under gauge transformations, $\Phi_i(x) \rightarrow$

$U_i^j(g)\Phi_j(x)$; that is,

$$V(U\Phi, (U\Phi)^\dagger) = V(\Phi, \Phi^\dagger). \quad (4.3.2)$$

Although \mathcal{L} is invariant under *global* gauge transformations, the kinetic energy term is not invariant under *local* gauge transformations due to the presence of the derivative. In particular, under local gauge transformations $\partial_\mu\Phi \rightarrow \partial_\mu(U\Phi) = U\partial_\mu\Phi + (\partial_\mu U)\Phi$. We therefore introduce the covariant derivative acting on a matter field that transforms according to some representation R of the symmetry group G :

$$(D^\mu)_i^j = \delta_i^j \partial^\mu + ig_a A_\mu^a(x) (T^a)_i^j, \quad a = 1, 2, \dots, d_G, \quad (4.3.3)$$

where the flavor indices $i, j = 1, 2, \dots, d(R)$. If the symmetry group is simple, then $g_a = g$. Otherwise $g_a = g_b$ if and only if T^a and T^b belong to the same simple Lie algebra [or $\mathfrak{u}(1)$ factor] in the direct sum decomposition of the Lie algebra \mathfrak{g} .

By introducing a suitable transformation law for $A_\mu^a(x)$, one can arrange $D_\mu\Phi$ to transform under a local gauge transformation as

$$D_\mu\Phi \rightarrow UD_\mu\Phi, \quad (4.3.4)$$

in which case,

$$\mathcal{L} = (D_\mu\Phi_i)^\dagger (D^\mu\Phi_i) - V(\Phi, \Phi^\dagger) \quad (4.3.5)$$

is invariant under local gauge transformations.

The transformation law for $A_\mu^a(x)$ is most easily expressed for the matrix-valued gauge field⁷

$$A_\mu(x) \equiv g_a A_\mu^a(x) T^a. \quad (4.3.6)$$

Under local gauge transformations, the matrix-valued gauge field transforms as

$$A_\mu \rightarrow UA_\mu U^{-1} - iU(\partial_\mu U^{-1}). \quad (4.3.7)$$

Using eq. (4.3.7), one quickly shows that $D_\mu\Phi$ transforms as expected:

$$\begin{aligned} D_\mu\Phi &\equiv (\partial_\mu + iA_\mu)\Phi \rightarrow [\partial_\mu + iUA_\mu U^{-1} + U(\partial_\mu U^{-1})] U\Phi \\ &= UD_\mu\Phi + [\partial_\mu U + U(\partial_\mu U^{-1})U] \Phi \\ &= UD_\mu\Phi. \end{aligned} \quad (4.3.8)$$

In the last step, we noted that

$$\partial_\mu U + U(\partial_\mu U^{-1})U = [(\partial_\mu U)U^{-1} + U(\partial_\mu U^{-1})] U \quad (4.3.9)$$

$$= [\partial_\mu(UU^{-1})] U = 0, \quad (4.3.10)$$

since $UU^{-1} = I$.

It is also useful to exhibit the infinitesimal version of the transformation law, by

⁷ The matrix-valued gauge field that one employs in the covariant derivative depends on the representation of the matter fields on which the covariant derivative acts.

taking the infinitesimal form for U [eq. (4.2.4)]. This allows us to directly write down the transformation law $A_\mu^a(x) \rightarrow A_\mu^a(x) + \delta A_\mu^a(x)$, where

$$\delta A_\mu^a = g_a f^{abc} \Lambda^b A_\mu^c + \partial_\mu \Lambda^a \equiv D_\mu^{ab} \Lambda^b, \quad (4.3.11)$$

and

$$D_\mu^{ab} \equiv \delta^{ab} \partial_\mu + g_a f^{abc} A_\mu^c \quad (4.3.12)$$

is the covariant derivative operator applied to fields in the adjoint representation [*i.e.*, insert eq. (4.2.7) for the T^a in eq. (4.3.3)]. In particular, under global gauge transformations, the gauge fields $A_\mu^a(x)$ transform under the adjoint representation of the gauge group.

To complete the construction of the non-abelian gauge field theory, we must introduce a gauge invariant kinetic energy term for the gauge fields. To motivate the definition of the gauge field strength tensor, we consider $[D_\mu, D_\nu]$ acting on Φ ,

$$\begin{aligned} [D_\mu, D_\nu] \Phi &= [\partial_\mu + iA_\mu, \partial_\nu + iA_\nu] \Phi \\ &= i \{ \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \} \Phi. \end{aligned} \quad (4.3.13)$$

Thus, we define the matrix-valued gauge field strength tensor $F_{\mu\nu} \equiv g_a F_{\mu\nu}^a T^a$ as follows:

$$[D_\mu, D_\nu] \equiv i F_{\mu\nu}. \quad (4.3.14)$$

Using eq. (4.3.13) and the commutation relations of the Lie group generators, it follows that

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_a f^{abc} A_\mu^b A_\nu^c. \quad (4.3.15)$$

Under a gauge transformation, the transformation law for $F_{\mu\nu}$ is easily obtained. Starting with $\Phi \rightarrow U\Phi$ and $D_\mu \Phi \rightarrow U D_\mu \Phi$ [eq. (4.3.4)], it follows that $[D_\mu, D_\nu] \Phi \rightarrow U [D_\mu, D_\nu] \Phi$. One then easily derives $F_{\mu\nu} \Phi \rightarrow U F_{\mu\nu} \Phi = (U F_{\mu\nu} U^{-1}) U \Phi$. That is,

$$F_{\mu\nu} \rightarrow U F_{\mu\nu} U^{-1}, \quad (4.3.16)$$

which is the transformation law for an adjoint field. The infinitesimal form of eq. (4.3.16) is $F_{\mu\nu} \rightarrow F_{\mu\nu} + \delta F_{\mu\nu}$, where

$$\delta F_{\mu\nu}^a(x) = g_a f^{abc} \Lambda^b F_{\mu\nu}^c(x). \quad (4.3.17)$$

Note that in an abelian gauge theory, $U F_{\mu\nu} U^{-1} = U U^{-1} F_{\mu\nu} = F_{\mu\nu}$ so that $F^{\mu\nu}$ is gauge invariant (*i.e.*, neutral under the gauge group). For a non-abelian gauge group, $F_{\mu\nu}$ transforms non-trivially; *i.e.*, it carries non-trivial gauge charge.

We can now construct a gauge-invariant kinetic energy term for the gauge fields:

$$\mathcal{L}_{\text{gauge}} = \frac{-1}{4T_R} \text{Tr}(F_{\mu\nu} F^{\mu\nu}). \quad (4.3.18)$$

Using eq. (4.3.16), we see that $\mathcal{L}_{\text{gauge}}$ is gauge invariant due to the invariance of the trace under cyclic permutation of its arguments. In light of,

$$\text{Tr}(F_{\mu\nu} F^{\mu\nu}) = F_{\mu\nu}^a F^{\mu\nu b} \text{Tr}(T^a T^b) = T_R F_{\mu\nu}^a F^{\mu\nu a}, \quad (4.3.19)$$

the form for $\mathcal{L}_{\text{gauge}}$ does not depend on the representation R and we end up with:

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} \\ &= -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_a f^{abc} A_\mu^b A_\nu^c)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} - g_a f^{ade} A_d^\mu A_e^\nu).\end{aligned}\tag{4.3.20}$$

Thus, for non-abelian gauge theories (in contrast to abelian gauge theories), the gauge kinetic energy term generates three-point and four-point self-interactions among the gauge fields.

To summarize, if $\mathcal{L}_{\text{matter}}$ is invariant under a group G of global (gauge) transformations, then

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}}(\partial_\mu \rightarrow D_\mu)\tag{4.3.21}$$

is invariant under a group G of local gauge transformations. Above, $\mathcal{L}_{\text{matter}}$ contains a sum of kinetic energy terms of the various scalar and fermion matter multiplets, each of which transforms under some irreducible representation of the gauge group. In these terms, we replace the ordinary derivative with $D_\mu = \partial_\mu + ig_a A_\mu^a T^a$ and use the matrix representation T^a appropriate for each of the matter field multiplets.

Note that there is no mass term for the gauge field, since the term:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}m^2 A_\mu^a A^{\mu a}\tag{4.3.22}$$

would violate the local gauge invariance. This is a tree-level result; in the next section we will discuss whether this result persists to all orders in perturbation theory.

4.4 Feynman rules for Gauge theories

The Feynman rules for the self-interactions of the gauge fields and for the interactions of matter are simple to obtain. The triple and quartic gauge boson self-couplings follow from the form of the gauge kinetic energy term [eq. (4.3.20)]. The interactions of the gauge bosons with matter are derived from the matter kinetic energy terms. For example, after replacing the ordinary derivatives with covariant derivatives in the scalar field kinetic energy terms, the gauge field dependence of D_μ generates cubic and quartic terms that are linear and quadratic in A_μ . A similar replacement in the fermion field kinetic energy terms yields interactions between the fermions and gauge bosons that are linear in A_μ , as exhibited by the Feynman rules previously given in Figs. 2.6—2.8 [see also Figs. 3.1 and 3.2].

However, an apparent problem is encountered when one tries to obtain the Feynman rule for the gauge boson propagator. In general the rule for the tree-level propagator is obtained by inverting the operator that appears in the part of the Lagrangian that is quadratic in the fields. In the case of gauge fields, this is the

kinetic energy term, which we can rewrite as follows

$$\begin{aligned} & -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \\ & = \frac{1}{2}A_\mu^a(g^{\mu\nu}\square - \partial^\mu\partial^\nu)A_\nu^a + \text{total divergence}. \end{aligned} \quad (4.4.1)$$

Note that the total divergence does not contribute to the action. However, a more significant observation is that $(g^{\mu\nu}\square - \partial^\mu\partial^\nu)\partial_\nu = 0$, which implies that $g^{\mu\nu}\square - \partial^\mu\partial^\nu$ has a zero eigenvalue and therefore is not an invertible operator.

The solution to this problem is to add the so-called gauge-fixing term and the Faddeev-Popov ghost fields (anticommuting adjoint fields ω^a and ω^{*a}). The justification of this procedure can be found in the standard quantum field theory textbooks (and is most easily explained using path integral techniques). Here, we take a more practical view and simply note that the following Yang-Mills (YM) Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - \frac{1}{2\xi}(\partial_\mu A_a^\mu)^2 + \partial^\mu \omega_a^* D_\mu^{ab} \omega_b \quad (4.4.2)$$

is invariant under a Becchi-Rouet-Stora (BRS) extended gauge symmetry, whose infinitesimal transformation laws are given by:

$$\delta A_\mu^a = \epsilon D_\mu^{ab} \omega_b, \quad (4.4.3)$$

$$\delta \omega_a = \frac{1}{2}\epsilon g_a f^{abc} \omega_b \omega_c, \quad (4.4.4)$$

$$\delta \omega_a^* = -\frac{1}{\xi}\epsilon \partial_\mu A_a^\mu, \quad (4.4.5)$$

where ϵ is an infinitesimal anticommuting parameter. Note that the gauge transformation function $\Lambda^a(x)$ in eq. (4.3.11) has now been promoted to a field $\omega^a(x)$, whose transformation law is given above. Eq. (4.4.2) generates new interaction terms involving the gauge fields and Faddeev-Popov ghosts. The Faddeev-Popov ghosts can therefore appear inside loops of Feynman diagrams. One can show that any scattering amplitudes that involve only physical particles as external states satisfy unitarity. Hence the theory based on eq. (4.4.2) is consistent.

In particular, due to the gauge-fixing term (the term that involves the gauge-fixing parameter ξ), one observes that the part of the Lagrangian quadratic in the gauge fields has changed, and the propagator can now be defined. Converting to momentum space, the Feynman rule for a non-abelian gauge boson propagator is:

$$\begin{array}{c} \xrightarrow{q} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ a, \mu \qquad \qquad \qquad b, \nu \end{array} \quad \frac{-i\delta^{ab}}{q^2 + i\epsilon} \left[-g_{\mu\nu} + (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right]$$

The Feynman gauge ($\xi = 1$) and the Landau gauge ($\xi = 0$) are two of the more common gauge choices made in practical computations. Of course, any physical quantity must be independent of ξ . This provides a good check of Feynman diagram computations of graphs in which internal gauge bosons propagate. Note that the above considerations also apply to abelian gauge theories such as QED. In this case,

one can introduce ghost fields to exhibit the extended BRS symmetry. However, within the class of gauge fixing terms considered here, the ghost fields are non-interacting (since the photon does not carry any U(1)-charge) and hence the ghosts can be dropped. The photon propagator still takes the form above (but with the factor of δ^{ab} removed).

Finally, let us examine the question of the gauge boson mass. We know that the gauge boson is massless at tree-level. But, can one generate mass via radiative corrections? One must compute the gauge boson two-point function (which corresponds to the radiatively-corrected inverse propagator) and check to see whether the zero at $q^2 = 0$ (corresponding to a zero mass gauge boson) is shifted. Summing up the geometric series yields an implicit equation for the fully radiatively-corrected propagator $\mathcal{D}^{\mu\nu}(q)$:

$$\mathcal{D}^{\mu\nu}(q) = D^{\mu\nu}(q) + D^{\mu\lambda}(q)i\Pi_{\lambda\rho}(q)\mathcal{D}^{\rho\nu}(q), \quad (4.4.6)$$

where $D^{\mu\nu}(q)$ is the tree-level gauge field propagator and $i\Pi^{\mu\nu}(q)$ is the sum over all one-particle irreducible (1PI) diagrams (these are the graphs that cannot be split into two separate graphs by cutting through one internal line). The Ward identity of the theory (a consequence of gauge invariance) implies that $q_\mu\Pi^{\mu\nu} = q_\nu\Pi^{\mu\nu} = 0$. It follows that one can write:

$$i\Pi_{ab}^{\mu\nu}(q) = -i(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)\delta^{ab}. \quad (4.4.7)$$

After Multiplying on the left of eq. (4.4.6) by D^{-1} and on the right by \mathcal{D}^{-1} (where, e.g., $D_{\mu\lambda}^{-1}D^{\lambda\nu} = g_\mu^\nu$), one obtains:

$$\mathcal{D}_{\mu\nu}^{-1}(q) = D_{\mu\nu}^{-1}(q) - i\Pi_{\mu\nu}(q). \quad (4.4.8)$$

It is convenient to decompose $D_{\mu\nu}(q)$ and $\mathcal{D}_{\mu\nu}(q)$ into transverse and longitudinal pieces. For example,

$$D_{\mu\nu}(q) = D(q^2)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + D^{(\ell)}(q^2)\frac{q_\mu q_\nu}{q^2}. \quad (4.4.9)$$

Similar decompositions of the inverses $D_{\mu\nu}^{-1}(q)$ and $\mathcal{D}_{\mu\nu}^{-1}(q)$ are easily obtained; for example,

$$D_{\mu\nu}^{-1}(q) = \frac{1}{D(q^2)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{1}{D^{(\ell)}(q^2)}\frac{q_\mu q_\nu}{q^2}. \quad (4.4.10)$$

Since $\Pi_{\mu\nu}(q)$ is transverse [eq. (4.4.7)], one easily concludes from eq. (4.4.8) that

$$\mathcal{D}^{(\ell)}(q^2) = D^{(\ell)}(q^2) = \frac{-i\xi}{q^2 + i\epsilon} \quad (4.4.11)$$

$$\mathcal{D}^{-1}(q^2) = D^{-1}(q^2) + iq^2\Pi(q^2). \quad (4.4.12)$$

Using $D^{-1}(q^2) = iq^2$, we conclude that⁸

$$\mathcal{D}^{\mu\nu}(q) = \frac{-i}{q^2[1 + \Pi(q^2)]}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) - \frac{i\xi q^\mu q^\nu}{q^4}. \quad (4.4.13)$$

⁸ We have dropped the explicit $+i\epsilon$ that is associated with the pole of the propagator.

Thus, the pole at $q^2 = 0$ is not shifted. That is, the gauge boson mass remains zero to all orders in perturbation theory.

This elegant argument has a loophole. Namely, if $\Pi(q^2)$ develops a pole at $q^2 = 0$, then the pole of $\mathcal{D}^{\mu\nu}(q)$ shifts away from zero: $\Pi(q^2) \simeq -m_v^2/q^2$ as $q^2 \rightarrow 0$ implies that $\mathcal{D}(q^2) \simeq -i/(q^2 - m_v^2)$. This requires some non-trivial dynamics to generate a massless intermediate state in $\Pi_{\mu\nu}(q)$. Such a massless state is called a Goldstone boson. The Standard Model employs the dynamics of elementary scalar fields in order to generate the Goldstone mode. We thus turn our attention to the vector boson mass generation mechanism of the Standard Model.

4.5 Spontaneously broken gauge theories

Start with a non-abelian gauge theory with scalar (and fermion) matter, given by eq. (4.3.21). The corresponding scalar potential function must be gauge invariant [eq. (4.3.2)]. In order to identify the physical scalar degrees of freedom of this model, one must minimize the scalar potential and determine the corresponding values of the scalar fields at the potential minimum. These are the scalar vacuum expectation values. Expanding the scalar fields about their vacuum expectation values yields the tree-level scalar masses and self-couplings. However, in general the scalar fields are charged under the global symmetry group G , in which case a non-zero vacuum expectation value would be incompatible with the global symmetry.

4.5.1 Goldstone's theorem

In the absence of the gauge fields, Goldstone proved the following theorem:

Theorem: If the Lagrangian is invariant under a continuous global symmetry group G (of dimension d_G), but the vacuum state of the theory is not invariant under all G -transformations, then the theory exhibits spontaneous symmetry breaking. In this case, if the vacuum is invariant under all H -transformations, where H is a subgroup of G (of dimension d_H), then we say that the gauge group G is spontaneously broken down to H . The physical spectrum will then contain n massless scalar excitations (called Goldstone bosons) where $n \equiv d_G - d_H$.

Goldstone's theorem can be proved independently of perturbation theory [1, 2]. However, it can be demonstrated rather easily with a tree-level computation. Let $\phi_i(x)$ be a set of $n \equiv d(R)$ hermitian scalar fields.⁹ The scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - V(\phi_i) \quad (4.5.1)$$

is assumed to be invariant under a compact symmetry group G , under which the scalar fields transform as $\phi_i \rightarrow \mathcal{Q}_i^j \phi_j$, where \mathcal{Q} is a real representation R of G . Using a well-known theorem, all real representations of a compact group are equivalent

⁹ Classical real scalar fields are promoted to hermitian scalar field operators in quantum field theory.

(via a similarity transformation) to an orthogonal representation. Thus, without loss of generality, we may take \mathcal{Q} to be an orthogonal $n \times n$ matrix. The corresponding infinitesimal transformation law is

$$\delta\phi_i(x) = -ig_a\Lambda^a(T^a)_i^j\phi_j(x), \quad (4.5.2)$$

where g_a and Λ^a are real and the T^a are imaginary antisymmetric matrices. One can check that the scalar kinetic term is automatically invariant under $O(n)$ transformations. The scalar potential, which is not invariant in general under the full $O(n)$ group, is invariant under G [which is a subgroup of $O(n)$] if the following condition is satisfied:

$$V(\phi + \delta\phi) \simeq V(\phi) + \frac{\partial V}{\partial\phi_i}\delta\phi_i = V(\phi). \quad (4.5.3)$$

The first (approximate) equality above is simply a Taylor expansion to first order in the field variation, while the second equality imposes the invariance assumption. Inserting the result for $\delta\phi_i(x)$ from eq. (4.5.2), it follows that:

$$\frac{\partial V}{\partial\phi_i}(T^a)_i^j\phi_j = 0. \quad (4.5.4)$$

Suppose $V(\phi)$ has a minimum at $\langle\phi_i\rangle = v_i$, which is not invariant under the global symmetry group. That is,

$$\mathcal{Q}_i^j v_j \simeq (\delta_i^j - ig_a\Lambda^a(T^a)_i^j)v_j \neq v_i. \quad (4.5.5)$$

It follows that there exists at least one a such that $(T^a v)_j \neq 0$, and we conclude that the global symmetry is spontaneously broken. In general, there is a residual symmetry group H whose Lie algebra \mathfrak{h} is spanned by the maximal number of linearly independent elements of the Lie algebra \mathfrak{g} that annihilate the vacuum expectation value v . That is, we choose a new basis of generators for the Lie algebra \mathfrak{g} (which we denote by \tilde{T}^a), such that

$$(\tilde{T}^a v)_j = 0, \quad a = 1, 2, \dots, d_H, \quad (4.5.6)$$

$$(\tilde{T}^a v)_j \neq 0, \quad a = d_H + 1, d_H + 2, \dots, d_G, \quad (4.5.7)$$

where d_H is the dimension of H , which is identified as the maximal unbroken subgroup of G . Eqs. (4.5.6) and (4.5.7) define the unbroken and broken generators, respectively. We then say that the symmetry group G is spontaneously broken down to the group H .

Now, shift the field by its vacuum expectation value:

$$\phi_i \equiv v_i + \varphi_i, \quad (4.5.8)$$

and express the scalar Lagrangian in terms of the φ_i

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_i)(\partial^\mu\varphi_i) - \frac{1}{2}\mathcal{M}_{ij}^2\varphi_i\varphi_j + \mathcal{O}(\varphi^3), \quad (4.5.9)$$

where we have made use of the scalar potential minimum condition, $(\partial V/\partial\phi_i)_{\phi_i=v_i} =$

0, and

$$\mathcal{M}_{ij}^2 \equiv \left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \right)_{\phi_i=v_i}. \quad (4.5.10)$$

The terms cubic and higher in the φ_i do not concern us here. Since V must satisfy eq. (4.5.4), we can differentiate eq. (4.5.4) with respect to ϕ_k and then set all $\phi_i = v_i$. Invoking the scalar potential minimum condition, the end result is given by

$$\mathcal{M}_{ki}^2 (\tilde{T}^a v)_i = 0. \quad (4.5.11)$$

Hence, for each broken generator [see eq. (4.5.7)], $(\tilde{T}^a v)_i$ is an eigenvector of \mathcal{M}^2 with zero eigenvalue. The corresponding eigenstates can be used to identify the linear combinations of scalar fields φ_i that are massless. These are the Goldstone bosons, $G_a \propto i\varphi_i \tilde{T}_{ij}^a v_j$. Clearly, there are $d_G - d_H$ independent Goldstone modes, corresponding to the number of broken generators.

4.5.2 Massive gauge bosons

If a spontaneously broken global symmetry is promoted to a local symmetry, then a remarkable mechanism called the Higgs mechanism takes place. The Goldstone bosons disappear from the spectrum, and the formerly massless gauge bosons become massive. Roughly speaking, the Goldstone bosons become the longitudinal degrees of freedom of the massive gauge bosons. This is easily demonstrated in a generalization of the tree-level analysis given previously. If the scalar sector of eq. (4.5.1) is coupled to gauge fields, then one must replace the ordinary derivative with a covariant derivative in the scalar kinetic energy term:

$$\mathcal{L}_{\text{KE}} = \frac{1}{2} [\partial_\mu \phi_i + ig_a A_\mu^a (T^a)_i^j \phi_j] [\partial^\mu \phi_i + ig_b A^{\mu b} (T^b)_i^k \phi_k]. \quad (4.5.12)$$

As above, the $\phi_i(x)$ are real fields and the T^a are pure imaginary antisymmetric matrix generators corresponding to the representation of the scalar multiplet. It is convenient to define real antisymmetric matrices

$$L_a \equiv ig_a T^a. \quad (4.5.13)$$

If we expand the ϕ_i around their vacuum expectation values [eq. (4.5.8)], then eq. (4.5.12) yields a term quadratic in the gauge fields:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} M_{ab}^2 A_\mu^a A^{\mu b}, \quad (4.5.14)$$

where

$$M_{ab}^2 = (L_a v, L_b v) \quad (4.5.15)$$

is the gauge boson squared-mass matrix. Here, we have employed a convenient notation where:

$$(x, y) \equiv \sum_i x_i y_i. \quad (4.5.16)$$

If $L_a v \neq 0$ for at least one a , then the gauge symmetry is broken and at least one of the gauge bosons acquires mass. The gauge boson squared-mass matrix is real symmetric, so it can be diagonalized with an orthogonal similarity transformation:

$$\mathcal{O}M^2\mathcal{O}^T = \text{diag}(0, 0, \dots, 0, m_1^2, m_2^2, \dots). \quad (4.5.17)$$

The corresponding gauge boson mass-eigenstates are:

$$\tilde{A}_\mu^a \equiv \mathcal{O}_{ab}A_\mu^b. \quad (4.5.18)$$

Indeed, one can easily check that $M_{ab}^2 A_\mu^a A^{\mu b} = \sum_a m_a^2 \tilde{A}_\mu^a \tilde{A}^{\mu a}$. Likewise, we may define a new basis for the Lie algebra:

$$\tilde{L}_a \equiv \mathcal{O}_{ab}L_b. \quad (4.5.19)$$

It then follows that:

$$(\mathcal{O}M^2\mathcal{O}^T)_{ab} = (\tilde{L}_a v, \tilde{L}_b v) = m_a^2 \delta_{ab}. \quad (4.5.20)$$

In particular $m_a^2 = 0$ when $\tilde{L}_a v = 0$ for $a = 1, \dots, d_H$ (corresponding to the unbroken generators) and $m_a^2 \neq 0$ when $\tilde{L}_a v \neq 0$ for $a = d_H + 1, \dots, d_G$ (corresponding to the broken generators). That is, there are $d_G - d_H$ massive gauge bosons.¹⁰

It is instructive to revisit the scalar Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i + (L_a)_i^j A_\mu^a \phi_j)(\partial^\mu \phi_i + (L_b)_i^k A_\mu^b \phi_k) - V(\phi). \quad (4.5.21)$$

Note that the covariant derivative $D_\mu \equiv \partial_\mu + L_a A_\mu^a$ can also be written in terms of gauge boson mass eigenstates \tilde{A}_μ^a if the new generators \tilde{L}_a are also employed, due to the identity:

$$\tilde{L}_a \tilde{A}_\mu^a = \mathcal{O}_{ac} \mathcal{O}_{ab} L_c A_\mu^b = L_c A_\mu^c. \quad (4.5.22)$$

Expanding the scalar fields in eq. (4.5.21) around their vacuum expectation values [eq. (4.5.8)], and identifying the gauge boson mass eigenstates (while not displaying terms cubic or higher in the fields), we find:¹¹

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \varphi_i)(\partial^\mu \varphi_i) + \frac{1}{2}m_a^2 \tilde{A}_\mu^a \tilde{A}^{\mu a} \\ &\quad + \frac{1}{2}(\tilde{L}_a \tilde{A}_\mu^a v, \partial^\mu \varphi) + \frac{1}{2}(\partial_\mu \varphi, \tilde{L}_a \tilde{A}_\mu^a v) + \dots \\ &= \frac{1}{2}(\partial_\mu G^b)(\partial^\mu G^b) + \frac{1}{2}m_a^2 \tilde{A}_\mu^a \tilde{A}^{\mu a} + m_a \tilde{A}_\mu^a \partial^\mu G_a + \dots, \end{aligned} \quad (4.5.23)$$

where there is an implicit sum over the thrice repeated index a and

$$G_a = \frac{1}{m_a}(\tilde{L}_a v, \varphi), \quad [\text{no sum over } a]. \quad (4.5.24)$$

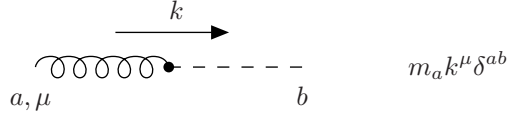
We recognize G_a as the Goldstone bosons that appeared in the scalar theory with a spontaneously broken global symmetry [see eq. (4.5.11) and the discussion that

¹⁰ The number of massive gauge bosons, or equivalently the number of broken generators, is equal to the dimension of the coset space G/H .

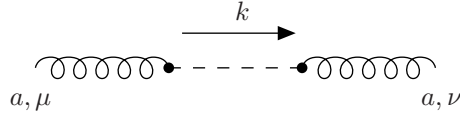
¹¹ Terms involving the physical scalars are also omitted here. These terms will be addressed in Section 4.5.4.

follows]. Here, the normalization of the Goldstone field has been chosen so that G^a possesses a canonically normalized kinetic energy term.

The coupling $m_a A_\mu^a \partial^\mu G_a$ provides an explanation for the vector boson mass generation mechanism. Namely, this coupling yields a new interaction vertex in the theory:



We can then evaluate the contribution of an intermediate Goldstone propagator to $i\Pi^{\mu\nu}(q)$:



$$i\Pi^{\mu\nu}(k) = m_a^2 k^\mu (-k^\nu) \frac{i}{k^2} + \dots = -im_a^2 \frac{k^\mu k^\nu}{k^2} + \dots, \quad (4.5.25)$$

where terms that are finite as $k^2 \rightarrow 0$ are indicated by the ellipsis. That is, the Feynman diagram above is the only source for a pole at $k^2 = 0$ at this order in perturbation theory. But, gauge invariance requires:

$$i\Pi^{\mu\nu}(k) = i(k^\mu k^\nu - k^2 g^{\mu\nu})\Pi(k^2). \quad (4.5.26)$$

It therefore follows that

$$\Pi(k^2) \simeq \frac{-m_a^2}{k^2}, \quad (4.5.27)$$

and (up to an overall wave function renormalization that we absorb into the definition of the renormalized \mathcal{D}),

$$\mathcal{D}(k^2) = \frac{i}{k^2[1 + \Pi(k^2)]} = \frac{i}{k^2 - m_a^2}. \quad (4.5.28)$$

We say that the gauge boson “eats” or absorbs the corresponding Goldstone boson and thereby acquires mass via the Higgs mechanism.

4.5.3 The unitary and R_ξ gauges

The spontaneously broken non-abelian gauge theory Lagrangian contains Goldstone boson fields. However, as we shall now demonstrate, the Goldstone bosons are gauge artifacts that can be removed by a gauge transformation. Consider the transformation law for the shifted scalar field, $\varphi_i(x) \equiv \phi_i(x) - v_i$. Promoting eq. (4.5.2) to a local gauge transformation, where $\Lambda^a = \Lambda^a(x)$, and noting that $\delta v_i = 0$,

$$\delta\varphi_i = -\Lambda(\varphi_i + v_i), \quad (4.5.29)$$

where $\Lambda \equiv igT^a \Lambda^a \equiv L_a \Lambda^a$. If we define $\tilde{\Lambda}^a = \mathcal{O}_{ab} \Lambda^b$, then we can also write $\Lambda = \tilde{L}_a \tilde{\Lambda}^a$. Then, under a gauge transformation, the Goldstone field [eq. (4.5.24)] transforms as $G_a \rightarrow G_a + \delta G_a$, where

$$\begin{aligned} \delta G_a &= \frac{1}{m_a} (\tilde{L}_a v, \delta \varphi) = -\frac{1}{m_a} (\tilde{L}_a v, \Lambda \varphi + \Lambda v) \\ &= -\frac{1}{m_a} (\tilde{L}_a v, \Lambda \varphi) - \frac{1}{m_a} (\tilde{L}_a v, \tilde{L}_b v) \tilde{\Lambda}^b \\ &= -\frac{1}{m_a} (\tilde{L}_a v, \Lambda \varphi) - m_a \tilde{\Lambda}_a, \end{aligned} \quad (4.5.30)$$

after using eq. (4.5.20) for the gauge boson masses. Note that the second term in the last line above, $m_a \tilde{\Lambda}_a$, is an inhomogeneous term independent of φ . As a result, one can always find a $\Lambda_a(x)$ such that $G_a + \delta G_a = 0$. That is, a gauge transformation exists¹² in which the Goldstone field is completely gauged away to zero. The resulting gauge is called the unitary gauge.

So far, we have not mentioned the gauge fixing term and Faddeev-Popov ghosts. In spontaneously-broken non-abelian gauge theories, the R_ξ -gauge turns out to be particularly useful.¹³ The R_ξ -gauge is defined by the following gauge-fixing term:

$$\begin{aligned} \mathcal{L}_{\text{GF}} &= -\frac{1}{2\xi} \left[\partial^\mu \tilde{A}_\mu^a - \xi (\tilde{L}_a v, \varphi) \right]^2 \\ &= -\frac{1}{2\xi} (\partial^\mu \tilde{A}_\mu^a)^2 + m_a G_a (\partial^\mu \tilde{A}_\mu^a) - \frac{\xi m_a^2}{2} G_a G_a, \end{aligned} \quad (4.5.31)$$

after using eq. (4.5.24). At this point, we notice that the term $m_a G_a (\partial^\mu \tilde{A}_\mu^a)$ of eq. (4.5.31) combines with $m_a \tilde{A}_\mu^a \partial^\mu G_a$ of eq. (4.5.23) to yield

$$m_a [G_a \partial^\mu \tilde{A}_\mu^a + \tilde{A}_\mu^a \partial^\mu G_a] = m_a \partial^\mu (G_a \tilde{A}_\mu^a), \quad (4.5.32)$$

which is a total divergence that can be dropped from the Lagrangian. One must also include Faddeev-Popov ghosts [5]:

$$\mathcal{L}_{\text{FP}} = \partial^\mu \omega_a^* D_\mu^{ab} \omega_b - \xi \omega_a^* M_{ab}^2 \omega_b - \xi g_a g_b \omega_a^* \omega_b (\varphi, T^b T^a v), \quad (4.5.33)$$

D_μ^{ab} is defined in eq. (4.3.12) and M_{ab}^2 is the gauge boson squared-mass matrix [eq. (4.5.15)].

In the R_ξ gauges, the Feynman rules for the massless and massive gauge boson propagators take on the following form, respectively:

¹² In a general non-abelian gauge theory, the gauge transformation that eliminates the Goldstone fields $G_a(x)$ [for *all* values of x] cannot be explicitly exhibited [3]. Nevertheless, one can prove that such a gauge transformation must exist [4].

¹³ The R stands for renormalizable, and ξ is the gauge fixing parameter. In the R_ξ -gauges, the theory is manifestly renormalizable although unitarity is not manifest and must be separately proved. The opposite is true for the unitary gauge.

$$\begin{array}{cc}
 \begin{array}{c} \xrightarrow{q} \\ \text{wavy line} \\ a,\mu \qquad b,\nu \end{array} & \frac{i\delta^{ab}}{q^2 + i\epsilon} \left[-g_{\mu\nu} + (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right] \\
 \\
 \begin{array}{c} \xrightarrow{q} \\ \text{curly line} \\ a,\mu \qquad b,\nu \end{array} & \frac{-i\delta^{ab}}{q^2 - m_a^2 + i\epsilon} \left[-g_{\mu\nu} + (1 - \xi) \frac{q_\mu q_\nu}{q^2 - \xi m_a^2} \right]
 \end{array}$$

Above, the massless gauge bosons are indicated by wavy lines, and the massive gauge bosons are indicated by curly lines.¹⁴

In addition, we note from eq. (4.5.31) that the Goldstone bosons have acquired a squared-mass equal to ξm_a^2 . This is an artifact of the gauge choice. Nevertheless, for a consistent computation in the R_ξ -gauge, both the Goldstone bosons and the Faddeev-Popov ghosts must be included as possible internal lines in Feynman graphs. As noted previously, any physical quantity must ultimately be independent of ξ .

As in the unbroken non-abelian gauge theory, the two most useful gauges are $\xi = 1$ (now called the 't Hooft-Feynman gauge) and $\xi = 0$ (still called the Landau gauge). As an additional benefit, the Goldstone bosons are massless in the Landau gauge. Finally, we note that one can attempt to take the limit of $\xi \rightarrow \infty$. This corresponds to the unitary gauge, since the Goldstone boson masses become infinite and thus decouple from all Feynman graphs. Moreover, one can check that the massive gauge boson propagator reduces to

$$\begin{array}{cc}
 \begin{array}{c} \xrightarrow{q} \\ \text{curly line} \\ a,\mu \qquad b,\nu \end{array} & \frac{-i\delta^{ab}}{q^2 - m_a^2 + i\epsilon} \left[-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_a^2} \right]
 \end{array}$$

which resembles the gauge boson propagator of massive QED. The fact that the unitary gauge is a limiting case of the R_ξ -gauge played an essential role in the proof that spontaneously-broken non-abelian gauge theories are unitary and renormalizable [7].

4.5.4 The physical Higgs bosons

An important check of the formalism is the counting of bosonic degrees of freedom. Assume that the multiplet of scalar fields transforms according to some $d(R)$ -dimensional real representation R under the transformation group G (which has dimension d_G). Prior to spontaneous symmetry breaking, the theory contains $d(R)$

¹⁴ In the so-called generalized R_ξ -gauge [6], the ξ that appears in the two Feynman rules above may be different.

scalar degrees of freedom and $2d_G$ vector-boson degrees of freedom. The latter consists of d_G massless gauge bosons (one for each possible value of the adjoint index a), with each massless gauge boson contributing two degrees of freedom corresponding to the two possible transverse helicities. After spontaneous symmetry breaking of G to a subgroup H (which has dimension d_H), there are $d_G - d_H$ Goldstone bosons which are unphysical (and can be removed from the spectrum by going to the unitary gauge). This leaves

$$d(R) - d_G + d_H \text{ physical scalar degrees of freedom,} \quad (4.5.34)$$

which correspond to the physical Higgs bosons of the theory. We also found that there are d_H massless gauge bosons (one for each unbroken generator) and $d_G - d_H$ massive gauge bosons (one for each broken generator). But, for massive gauge bosons which possess a longitudinal helicity state, we must count three degrees of freedom. Thus, we end up with

$$3d_G - d_H \text{ vector boson degrees of freedom.} \quad (4.5.35)$$

Adding the two yields a total of $d(R) + 2d_G$ bosonic degrees of freedom, in agreement with our previous counting.

It is instructive to check that the physical Higgs bosons cannot be removed by a gauge transformation. We divide the scalars into two classes: (i) the Goldstone bosons G_a , $a = d_H + 1, d_H + 2, \dots, d_G$ [see eq. (4.5.24)] and (ii) the scalar states orthogonal to G_a . These are the Higgs bosons:

$$\tilde{H}_k = c_j^{(k)} \varphi_j, \quad (4.5.36)$$

where $k = 1, 2, \dots, d(R) - d_G + d_H$. The $c_j^{(k)}$ are real numbers that satisfy the orthogonality conditions:

$$\sum_j c_j^{(k)} (\tilde{L}_a v)_j = 0, \quad \sum_j c_j^{(k)} c_j^{(\ell)} = \delta_{k\ell}. \quad (4.5.37)$$

Under a gauge transformation [eq. (4.5.29)],

$$\delta(c_j^{(k)} \varphi_j) = c_j^{(k)} \delta \varphi_j = -c_j^{(k)} (\Lambda \varphi + \Lambda v)_j = -c_j^{(k)} (\Lambda \varphi)_j, \quad (4.5.38)$$

where we have noted that $c_j^{(k)} (\Lambda v)_j = c_j^{(k)} \tilde{\Lambda}^a (\tilde{L}_a v)_j = 0$ [after invoking eq. (4.5.37)]. Thus the transformation law for \tilde{H}_k is homogeneous in the scalar fields, and one cannot remove the Higgs boson field by a gauge transformation.

The states \tilde{H}_k are in general not mass-eigenstates. We may write down the physical Higgs mass matrix by employing the $\{G_a, \tilde{H}_k\}$ basis for the scalar fields. This is accomplished by noting that the orthogonality relations of eq. (4.5.37) and eq. (4.5.20) yield:

$$\varphi_j = \frac{(\tilde{L}_a v)_j}{m_a} G_a + c_j^{(k)} \tilde{H}_k, \quad (4.5.39)$$

where the sum over the repeated indices a and k respectively is implied. Finally, we note that the scalar boson squared-mass matrix, given in eq. (4.5.10), is real

symmetric and satisfies $(\mathcal{M}^2)_{ij}(\tilde{L}_a v)_j = 0$ [eq. (4.5.11)]. Thus, the non-derivative quadratic terms in the scalar part of the Lagrangian are given by

$$\mathcal{L}_{\text{scalar mass}} = -\frac{1}{2}(\mathcal{M}_H^2)_{k\ell}\tilde{H}_k\tilde{H}_\ell, \quad (4.5.40)$$

where the physical Higgs squared-mass matrix is given by:

$$(\mathcal{M}_H^2)_{k\ell} = c_i^{(k)}c_j^{(\ell)}\mathcal{M}_{ij}^2. \quad (4.5.41)$$

Diagonalizing \mathcal{M}_H^2 yields the Higgs boson mass-eigenstates, H_k , and the corresponding eigenvalues are the physical Higgs boson squared-masses.

4.6 Complex representations of scalar fields

We now have nearly all the ingredients necessary to construct the Standard Model. However, in our treatment of spontaneously broken non-abelian gauge fields, we considered hermitian scalar fields that transform under a real representation of the symmetry group. In contrast, the Standard Model employs scalars that transform under a complex representation of the symmetry group. In this section, we provide the necessary formalism that will allow us to directly treat the complex case.

Let $\Phi_i(x)$ be a set of $n \equiv d(R)$ complex scalar fields. The scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\Phi_i)^\dagger(\partial^\mu\Phi_i) - V(\Phi_i, \Phi_i^\dagger) \quad (4.6.1)$$

is assumed to be invariant under a compact symmetry group G , under which the scalar fields transform as:

$$\Phi_i \rightarrow \mathcal{U}_i^j\Phi_j, \quad \Phi_i^\dagger \rightarrow \Phi_i^{\dagger j}(\mathcal{U}^\dagger)_j^i, \quad (4.6.2)$$

where \mathcal{U} is a complex representation¹⁵ of G . Using a well-known theorem, all complex representations of a compact group are equivalent (via a similarity transformation) to a unitary representation. Thus, without loss of generality, we may take \mathcal{U} to be a unitary $n \times n$ matrix. Explicitly,

$$\mathcal{U} = \exp[-ig_a\Lambda^a\mathcal{T}^a], \quad (4.6.3)$$

where the generators \mathcal{T}^a are $n \times n$ hermitian matrices. The corresponding infinitesimal transformation law is

$$\delta\Phi_i(x) = -ig_a\Lambda^a(\mathcal{T}^a)_i^j\Phi_j(x), \quad (4.6.4)$$

$$\delta\Phi_i^\dagger(x) = +ig_a\Phi_i^{\dagger j}(x)\Lambda^a(\mathcal{T}^a)_j^i, \quad (4.6.5)$$

where the g_a and Λ^a are real. One can check that the scalar kinetic energy term is invariant under $U(n)$ transformations. The scalar potential, which is not invariant

¹⁵ In this context, we call a representation complex if and only if it is not equivalent (by similarity transformation) to some real representation. This is somewhat broader than the conventional group theoretical definition.

in general under the full $U(n)$ group, is invariant under G [which is a subgroup of $U(n)$] if eq. (??) is satisfied.

There are $2n$ independent scalar degrees of freedom, corresponding to the fields Φ_i and $\Phi^{\dagger i}$. We can also express these degrees of freedom in terms of $2n$ hermitian scalar fields consisting of ϕ_{Aj} and ϕ_{Bj} ($j = 1, 2, \dots, n$) defined by:

$$\Phi_j = \frac{1}{\sqrt{2}}(\phi_{Aj} + i\phi_{Bj}), \quad \Phi^{\dagger j} = \frac{1}{\sqrt{2}}(\phi_{Aj} - i\phi_{Bj}). \quad (4.6.6)$$

It is straightforward to compute the group transformation laws for the hermitian fields ϕ_{Aj} and ϕ_{Bj} . These are conveniently expressed by introducing a $2n$ -dimensional scalar multiplet:

$$\phi(x) = \begin{pmatrix} \phi_A(x) \\ \phi_B(x) \end{pmatrix}. \quad (4.6.7)$$

That is, $\phi_{Aj}(x) = \phi_j(x)$ and $\phi_{Bj}(x) = \phi_{j+n}(x)$. Then the infinitesimal form of the group transformation law for $\phi(x)$ is given by $\phi_k(x) \rightarrow \phi_k(x) + \delta\phi_k(x)$ for $k = 1, 2, \dots, 2n$, where

$$\delta\phi_k(x) = -ig\Lambda^a (T^a)_k{}^\ell \phi_\ell(x), \quad (4.6.8)$$

and [8]

$$iT^a = \begin{pmatrix} -\text{Im } \mathcal{T}^a & -\text{Re } \mathcal{T}^a \\ \text{Re } \mathcal{T}^a & -\text{Im } \mathcal{T}^a \end{pmatrix}. \quad (4.6.9)$$

Note that $\text{Re } \mathcal{T}^a$ is symmetric and $\text{Im } \mathcal{T}^a$ is antisymmetric (which follow from the hermiticity of \mathcal{T}^a). Thus, iT^a is a real antisymmetric $2n \times 2n$ matrix, which when exponentiated yields a real orthogonal $2n$ -dimensional representation of G . Consequently, using the real representation for iT^a [eq. (4.6.9)], we may immediately apply the formalism of Section 4.5 that was established for the case of a real representation, and obtain the corresponding results for the case of a complex representation.

We can also apply the above analysis to a non-abelian gauge theory based on the compact group G coupled to a multiplet of scalar fields. We assume that the scalars transform according to a (possibly reducible) $d(R)$ -dimensional complex unitary representation R of G . We proceed to transcribe some of the results of Section 4.5 to the present case. The vacuum expectation value of the complex scalar field is assumed to be $\langle \Phi_i(x) \rangle \equiv \nu_i$. Consequently, in the real representation,

$$v \equiv \langle \phi(x) \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu + \nu^* \\ -i(\nu - \nu^*) \end{pmatrix}. \quad (4.6.10)$$

We now shift the complex fields by their vacuum expectation values:

$$\Phi_i \equiv \nu_i + \bar{\Phi}_i, \quad \Phi_i^\dagger \equiv \nu_i^* + \bar{\Phi}_i^\dagger. \quad (4.6.11)$$

Using the real scalar field basis, the gauge boson mass matrix is given by eq. (4.5.15),

which can be rewritten as:

$$M_{ab}^2 = g_a g_b (T^a T^b)_{jk} v_j v_k, \quad (4.6.12)$$

after noting that the T^a are antisymmetric. Plugging in eqs. (4.6.9) and (4.6.10), we obtain the corresponding result with respect to the complex scalar field basis [2]

$$M_{ab}^2 = 2g_a g_b \operatorname{Re}(\nu^\dagger \mathcal{T}^a \mathcal{T}^b \nu) = g_a g_b \nu^\dagger (\mathcal{T}^a \mathcal{T}^b + \mathcal{T}^b \mathcal{T}^a) \nu, \quad (4.6.13)$$

where the last step above follows from the hermiticity of the \mathcal{T}^a . As before, M^2 is a real symmetric matrix that can be diagonalized by an orthogonal transformation $\mathcal{O} M^2 \mathcal{O}^T$ [eq. (4.5.17)]. The corresponding gauge boson mass-eigenstates are given by eq. (4.5.18).

It is again convenient to introduce the anti-hermitian generators:

$$\mathcal{L}_a \equiv i g_a \mathcal{T}^a, \quad (4.6.14)$$

and then define a new basis for the Lie algebra:

$$\tilde{\mathcal{L}}_a \equiv \mathcal{O}_{ab} \mathcal{L}_b. \quad (4.6.15)$$

It follows that

$$(\mathcal{O} M^2 \mathcal{O}^T)_{ab} = (\tilde{\mathcal{L}}_a \nu)^\dagger (\tilde{\mathcal{L}}_b \nu) + (\tilde{\mathcal{L}}_b \nu)^\dagger (\tilde{\mathcal{L}}_a \nu) = m_a^2 \delta_{ab}. \quad (4.6.16)$$

Hence, one can easily identify the unbroken and broken generators:

$$(\tilde{\mathcal{T}}^a \nu)_j = 0, \quad a = 1, 2, \dots, d_H, \quad (4.6.17)$$

$$(\tilde{\mathcal{T}}^a \nu)_j \neq 0, \quad a = d_H + 1, d_H + 2, \dots, d_G. \quad (4.6.18)$$

Consider next the scalar squared-mass matrix and the identification of the Goldstone bosons. Although these quantities can be obtained from first principles (see problem 1), our derivations here are based on results already obtained in Section 4.5.1, where we employ the connection between the real and complex basis outlined above. To identify the Goldstone bosons, we insert eqs. (4.6.7), (4.6.9) and (4.6.10) into eq. (4.5.24). Rewriting ϕ_A and ϕ_B in terms of the shifted complex fields $\bar{\Phi}$ and $\bar{\Phi}^\dagger$, we obtain $d_G - d_H$ Goldstone boson fields:

$$G_a = \frac{1}{m_a} \left[\bar{\Phi}^\dagger \tilde{\mathcal{L}}_a \nu + (\tilde{\mathcal{L}}_a \nu)^\dagger \bar{\Phi} \right], \quad (4.6.19)$$

where $a = d_H + 1, d_H + 2, \dots, d_G$. The scalar squared-mass matrix is obtained from eq. (4.5.10):

$$\mathcal{L}_{\text{scalar mass}} = -\frac{1}{2} \mathcal{M}_{ij}^2 \phi_i \phi_j = -\frac{1}{2} \begin{pmatrix} \Phi_k & \Phi^{\dagger \ell} \end{pmatrix} \mathcal{M}^2 \begin{pmatrix} \Phi^{\dagger m} \\ \Phi_n \end{pmatrix}. \quad (4.6.20)$$

To determine the matrix \mathcal{M}^2 , it is convenient to introduce the unitary matrix W

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}, \quad W\phi = \begin{pmatrix} \Phi^\dagger \\ \Phi \end{pmatrix}, \quad (4.6.21)$$

and $\phi^T W^{-1} = (\Phi \quad \Phi^\dagger)$, where I is the $d(R) \times d(R)$ identity matrix. Finally, we use the chain rule to obtain:

$$\mathcal{M}_{ij}^2 = \left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \right)_{\phi_i = v_i} = W^{-1} \mathcal{M}^2 W, \quad (4.6.22)$$

and

$$\mathcal{M}^2 = \begin{pmatrix} \frac{\partial^2 V}{\partial \Phi_k \partial \Phi^\dagger m} & \frac{\partial^2 V}{\partial \Phi_k \partial \Phi_n} \\ \frac{\partial^2 V}{\partial \Phi^\dagger \ell \partial \Phi^\dagger m} & \frac{\partial^2 V}{\partial \Phi_i^\dagger \ell \partial \Phi_n} \end{pmatrix}_{\Phi_i = v_i}, \quad (4.6.23)$$

with $k, \ell, m, n = 1, 2, \dots, d(R)$. Indeed, one can check (see problem 1) that:

$$\mathcal{M}^2 \begin{pmatrix} (\tilde{\mathcal{L}}_a \nu)^{*m} \\ (\tilde{\mathcal{L}}_a \nu)_n \end{pmatrix} = 0, \quad (4.6.24)$$

which confirms the identification of G^a [eq. (4.6.19)] as the Goldstone boson.

Finally, we identify the physical Higgs bosons. In this case, the counting of bosonic degrees of freedom of Section 4.5.4 applies if we interpret $d(R)$ as the number of *complex* degrees of freedom, which is equivalent to $2d(R)$ real degrees of freedom. Following the results of Section 4.5.4, we define the hermitian Higgs fields:

$$\tilde{H}_k \equiv c_j^{(k)} \bar{\Phi}^{\dagger j} + [c^{(k)*}]^j \bar{\Phi}_j, \quad (4.6.25)$$

where $k = 1, 2, \dots, 2d(R) - d_G + d_H$ and the $c_j^{(k)}$ are complex numbers that satisfy orthogonality relations:

$$\sum_j \{ (\tilde{\mathcal{L}}_a \nu)_j [c^{(k)*}]^j + [\tilde{\mathcal{L}}_a \nu]^*{}^j c_j^{(k)} \} = 0, \quad (4.6.26)$$

$$\sum_j [c^{(k)*}]^j c_j^{(\ell)} + [c^{(\ell)*}]^j c_j^{(k)} = \delta_{k\ell}. \quad (4.6.27)$$

The orthonormality of the scalar states $\{G_a, \tilde{H}_k\}$ follows from eqs. (4.6.16), (4.6.25) and (4.6.26). We may therefore solve for the shifted complex fields $\bar{\Phi}$ and $\bar{\Phi}^\dagger$ in terms of G_a and \tilde{H}_k :

$$\bar{\Phi}_j = \frac{(\tilde{\mathcal{L}}_a \nu)_j}{m_a} G_a + c_j^{(k)} \tilde{H}_k. \quad (4.6.28)$$

Inserting this result into eq. (4.6.20) and using eq. (4.6.24), we end up with:

$$\mathcal{L}_{\text{scalar mass}} = -\frac{1}{2} (\mathcal{M}_H^2)_{pq} \tilde{H}_p \tilde{H}_q, \quad (4.6.29)$$

where the physical Higgs squared-mass matrix is given by:

$$(\mathcal{M}_H^2)_{pq} = \left\{ c_k^{(p)} [c^{(q)*}]^m \frac{\partial^2 V}{\partial \Phi_k \partial \Phi^{\dagger m}} + c_k^{(p)} c_n^{(q)} \frac{\partial^2 V}{\partial \Phi_k \partial \Phi_n} + [c^{(p)*}]^\ell [c^{(q)*}]^m \frac{\partial^2 V}{\partial \Phi^{\dagger \ell} \partial \Phi^{\dagger m}} + [c^{(p)*}]^\ell c_n^{(q)} \frac{\partial^2 V}{\partial \Phi^{\dagger \ell} \partial \Phi_n} \right\}_{\Phi_i = \nu_i}. \quad (4.6.30)$$

Note that \mathcal{M}_H^2 is a real symmetric matrix. Diagonalizing \mathcal{M}_H^2 yields the Higgs boson mass-eigenstates, H_k , and the corresponding eigenvalues are the physical Higgs boson squared-masses.

4.7 The Standard Model of particle physics

The Standard Model is a spontaneously broken non-abelian gauge theory based on the symmetry group $SU(3) \times SU(2) \times U(1)$. The color $SU(3)$ group is unbroken, so we put it aside and focus on the spontaneous breaking of $SU(2) \times U(1)_Y$ to $U(1)_{EM}$. Here, we have distinguished between the hypercharge- $U(1)$ gauge group (denoted by Y) which is broken and the electromagnetic $U(1)$ gauge group which is unbroken. The breaking is accomplished by introducing a hypercharge one, $SU(2)$ complex doublet of scalar fields:

$$\Phi = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix}. \quad (4.7.1)$$

The $SU(2) \times U(1)_Y$ covariant derivative acting on Φ is given by:

$$D_\mu \Phi_i = \partial_\mu \Phi_i + W_\mu^k (\mathcal{L}^k)_i^j \Phi_j + B_\mu (\mathcal{L}^4)_i^j \Phi_j, \quad (4.7.2)$$

where $i, j = 1, 2$; $k = 1, 2, 3$ and

$$(\mathcal{L}^k)_i^j = \frac{1}{2} ig (\tau^k)_i^j, \quad (\mathcal{L}^4)_i^j = ig' Y \delta_i^j. \quad (4.7.3)$$

Here, we have introduced the $SU(2)$ gauge fields, W_μ^k , the respective gauge couplings g and g' , and the hypercharge- $U(1)$ gauge field B_μ , and the $SU(2) \times U(1)_Y$ group generators ($\mathcal{L}^a = ig_a \mathcal{T}^a$, where $a = 1, \dots, 4$ with $g_{1,2,3} = g$ and $g_4 = g'$) acting on the hypercharge-one, $SU(2)$ doublet of scalar fields. The τ^k are the usual Pauli matrices¹⁶ and the hypercharge operator Y is normalized such that $Y\Phi = +\frac{1}{2}\Phi$.¹⁷

We begin by focusing on the bosonic sector of the Standard Model. The dynamics of the scalar field are governed by the scalar potential, whose form is constrained by renormalizability and $SU(2) \times U(1)_Y$ gauge invariance,

$$V(\Phi, \Phi^\dagger) = -m^2 (\Phi^\dagger \Phi) + \lambda (\Phi^\dagger \Phi)^2, \quad (4.7.4)$$

¹⁶ Here, $\vec{\tau} = \vec{\sigma}$. We use a different symbol here to distinguish the τ^a from the σ^a that appear in the formalism of two-component spin-1/2 fermions.

¹⁷ Another common normalization in the literature is $(\mathcal{L}^4)_i^j = \frac{1}{2} ig' Y \delta_i^j$, in which case the doublet of scalar fields possesses hypercharge one.

where m^2 and λ are real positive parameters. Minimizing the scalar potential yields a local minimum at $\Phi^\dagger\Phi = m^2/(2\lambda)$. Thus, the scalar doublet acquires a vacuum expectation value. It is conventional to choose:

$$\nu \equiv \langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad (4.7.5)$$

where $v = m/\sqrt{\lambda}$ is real.¹⁸

Using eq. (4.6.13), the gauge boson mass matrix is easily computed. After noting that $\tau^i\tau^j + \tau^j\tau^i = 2\delta^{ij}$ and $\nu^\dagger\nu = v^2/2$, we end up with two mass degenerate states: W^1 and W^2 with $m_W^2 = \frac{1}{4}g^2v^2$, and a non-diagonal matrix which in the W^3 - B basis is given by

$$M^2 = \frac{v^2}{4} \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}. \quad (4.7.6)$$

This matrix is easily diagonalized by

$$\mathcal{O} = \begin{pmatrix} \cos \theta_W & -\sin \theta_W \\ \sin \theta_W & \cos \theta_W \end{pmatrix}, \quad (4.7.7)$$

where the Weinberg angle is defined by

$$\sin \theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}. \quad (4.7.8)$$

From eq. (4.6.15), we deduce that $\tilde{\mathcal{L}}_k = \mathcal{L}_k$ for $k = 1, 2$ and

$$\tilde{\mathcal{L}}_3 = \mathcal{L}_3 \cos \theta_W - \mathcal{L}_4 \sin \theta_W = \frac{ig}{\cos \theta_W} [\mathcal{T}^3 - Q \sin^2 \theta_W], \quad (4.7.9)$$

$$\tilde{\mathcal{L}}_4 = \mathcal{L}_3 \sin \theta_W + \mathcal{L}_4 \cos \theta_W = ieQ, \quad (4.7.10)$$

where

$$e = g \sin \theta_W = g' \cos \theta_W = \frac{gg'}{\sqrt{g^2 + g'^2}}, \quad (4.7.11)$$

and

$$Q = \mathcal{T}^3 + Y. \quad (4.7.12)$$

Note that eqs. (4.7.9), (4.7.10) and (4.7.12) are representation independent, and thus can be applied in any representation.

These results imply that $\tilde{\mathcal{L}}^a\nu \neq 0$ for $a = 1, 2, 3$, while $\tilde{\mathcal{L}}^4\nu = 0$. That is, $\tilde{\mathcal{L}}^4$ is the unbroken generator, which we have identified as ieQ , where Q is the $U(1)_{\text{EM}}$ generator. Indeed, $SU(2) \times U(1)_Y$ is spontaneously broken down to $U(1)_{\text{EM}}$.

¹⁸ Using the $SU(2)$ gauge freedom, it is always possible to perform a gauge transformation to bring $\langle \Phi \rangle$ into the form given by eq. (4.7.5).

The gauge boson mass-eigenstates are given by eq. (4.5.18). Explicitly, these are denoted by W_μ^1 , W_μ^2 , Z_μ and A_μ , where

$$Z_\mu = W_\mu^3 \cos \theta_W - B_\mu \sin \theta_W, \quad (4.7.13)$$

$$A_\mu = W_\mu^3 \sin \theta_W + B_\mu \cos \theta_W. \quad (4.7.14)$$

Note that from eq. (4.7.6), it follows that $\det M^2 = 0$ and $\text{Tr } M^2 = \frac{1}{4}v^2(g^2 + g'^2)$. Thus, the field A_μ is massless and is identified with the photon, while the Z -boson is massive, with $m_Z = \frac{1}{4}v^2(g^2 + g'^2)$. This can be used to set the scale of the vacuum expectation value v . Plugging in the observed values of e , $\sin \theta_W$ and m_Z yields $v = 246$ GeV.

The electric charge of the gauge bosons can be determined by determining the eigenvalues of the charge operator when applied to the gauge boson field. Let \widehat{Q} be the charge operator that acts on the Hilbert space of quantum fields. Then, for a multiplet of gauge fields A_μ^a ,

$$(\widehat{Q}A_\mu)^a = Q_{ab}A_\mu^b, \quad (4.7.15)$$

where Q_{ab} is the representation of \widehat{Q} in the adjoint representation of the gauge group. The adjoint representation of the $\text{SU}(2) \times \text{U}(1)$ Lie algebra consists of a direct sum of the three-dimensional adjoint representation of $\text{SU}(2)$, given by $(T^k)_{ij} = -i\epsilon_{ijk}$ and the trivial one-dimensional representation of $\text{U}(1)$, given by $T^4 = 0$. Thus, in the adjoint representation, Q is a 4×4 matrix. Using eq. (4.7.12), the explicit form for the adjoint representation matrix Q is

$$Q = \begin{pmatrix} T^3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.7.16)$$

In eq. (4.7.16), Q is defined relative to the W^1, W^2, Z, A basis of vector fields. But Q as exhibited in eq. (4.7.16) is not diagonal. The form of Q implies that Z and A are neutral under Q , whereas W^1 and W^2 are not eigenstates of \widehat{Q} . However, it is simple to diagonalize Q by a simple basis change. We introduce:

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}} (W_\mu^1 \mp iW_\mu^2). \quad (4.7.17)$$

With respect to the new W^+, W^-, Z, A basis,

$$Q = \begin{pmatrix} ST^3S^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.7.18)$$

where $W_\mu^\pm = S_{jk}W_\mu^k$ ($k = 1, 2$) with

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \quad (4.7.19)$$

Therefore, $\widehat{Q}W_\mu^\pm = \pm W_\mu^\pm$. That is, W^\pm are the positively and negatively charged W -bosons, respectively, with $m_W^2 = \frac{1}{4}g^2v^2$ [as noted above eq. (4.7.6)]. Having fixed the Z mass by experiment, the value of the W mass is now constrained by the model. At tree-level, the results above imply that $m_W = m_Z \cos \theta_W$. This is often rewritten as:

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1. \quad (4.7.20)$$

The result $\rho = 1$ is a consequence of a “custodial” $SU(2)$ symmetry that is implicit in the choice of the Higgs sector.¹⁹ In theories with more complicated Higgs sectors (that involve different multiplets), the value of ρ is generally a free parameter of the model. The fact that the precision electroweak data finds that $\rho \simeq 1$ is a strong clue as to the fundamental nature of the electroweak symmetry breaking dynamics.

Having identified the gauge boson mass-eigenstates, it is useful to re-express the covariant derivative operator $D_\mu = \partial_\mu + \mathcal{L}_a A_\mu^a = \partial_\mu + \widetilde{\mathcal{L}}_a \widetilde{A}_\mu^a$ in terms of these fields:

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}} (\mathcal{T}^+ W_\mu^+ + \mathcal{T}^- W_\mu^-) + \frac{ig}{\cos \theta_W} (\mathcal{T}^3 - Q \sin^2 \theta_W) Z_\mu + ieQA_\mu, \quad (4.7.21)$$

where $\mathcal{T}^\pm \equiv \mathcal{T}^1 \pm i\mathcal{T}^2$.

We turn next to the scalar sector. First, we note that Φ is a complex $Y = +\frac{1}{2}$, $SU(2)$ -doublet of fields, so we will need to introduce the corresponding antiparticle multiplet which is a complex $Y = -\frac{1}{2}$, $SU(2)$ -doublet of fields. However, the correct anti-particle multiplet is *not* Φ^\dagger , which does not possess the correct transformation properties under $SU(2)$. The correct procedure is to define the antiparticle multiplet by:

$$\widetilde{\Phi}_i = \epsilon^{ij} \Phi_j^\dagger = \begin{pmatrix} [\Phi^0]^\dagger \\ -\Phi^- \end{pmatrix}. \quad (4.7.22)$$

where $\Phi^- \equiv [\Phi^+]^\dagger$ and

$$\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (4.7.23)$$

The proof that $\widetilde{\Phi}$ has the correct $SU(2)$ transformation law, $\widetilde{\Phi} \rightarrow \widetilde{\Phi} + \delta\widetilde{\Phi}$, where

$$\delta\widetilde{\Phi} = -\frac{i}{2} \vec{\theta} \cdot \vec{\tau} \widetilde{\Phi}, \quad (4.7.24)$$

relies on the identity among Pauli matrices given by eq. (1.3.7). The mathematical

¹⁹ In the Standard Model, the Higgs scalar potential [eq. (4.7.4)] possesses a global $SO(4) \cong SU(2) \times SU(2)$ [see problem 2(a)]. One of the $SU(2)$ factors is identified with the gauged $SU(2)$ of electroweak theory. The second $SU(2)$ is the so-called custodial $SU(2)$, of which only a hypercharge $U(1)_Y$ is gauged. The custodial $SU(2)$ is responsible for the tree-level relation $\rho = 1$. Since the custodial $SU(2)$ is not an exact global symmetry of the gauge and fermion sectors of the Standard Model, there exist finite one-loop corrections to the quantity $\rho - 1$.

implication of this result is that the two-dimensional representation of $SU(2)$ is equivalent to its complex conjugated representation.²⁰ The electric charge of the scalar bosons can be determined by determining the eigenvalues of the charge operator when applied to the scalar field. For scalar fields in the doublet representation of $SU(2)$

$$(\widehat{Q}\Phi)_i = Q_i^j \Phi_j \quad (4.7.25)$$

where Q_i^j is the representation of \widehat{Q} in the fundamental representation of the gauge group. In this case,

$$Q = \frac{1}{2}(\tau^3 + I_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.7.26)$$

Note that $\widehat{Q}\Phi^+ = +\Phi^+$ and $\widehat{Q}\Phi^0 = 0$, as expected. For scalar fields in the $Y = -\frac{1}{2}$ complex-conjugated doublet representation of $SU(2) \times U(1)$, we have $\mathcal{T}^a = \{\frac{1}{2}\vec{\tau}^*, -\frac{1}{2}I_2\}$. Consequently, if we denote the charge operator in this representation by Q^* , then

$$(\widehat{Q}\tilde{\Phi})_i = (Q^*)_i^j \tilde{\Phi}_j, \quad \text{where } Q^* = \frac{1}{2}(\tau^{3*} - I_2) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.7.27)$$

Thus, $\widehat{Q}\Phi^- = -\Phi^-$ and $\widehat{Q}\Phi^{*0} = 0$, as expected.

Eq. (4.6.19) provides an explicit formula for the Goldstone bosons. Applying this formula to the electroweak theory is straightforward, and we find:

$$G_1 = \sqrt{2} \operatorname{Im} \Phi^+, \quad G_2 = \sqrt{2} \operatorname{Re} \Phi^+, \quad G_3 = -\sqrt{2} \operatorname{Im} \Phi^0. \quad (4.7.28)$$

The physical Higgs state must be orthonormal to the G_a , so in this case it is trivial to deduce that:

$$H = \sqrt{2} \operatorname{Re} \Phi^0 - v. \quad (4.7.29)$$

Thus, the complex scalar doublet takes the form

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} G_2 + iG_1 \\ v + H - iG_3 \end{pmatrix} \equiv \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}} [v + H + iG^0] \end{pmatrix}, \quad (4.7.30)$$

which defines the Goldstone states of definite charge: G^\pm and G^0 (where $G^- = [G^+]^\dagger$). Finally, the Higgs mass is determined from eq. (4.6.30). In this case, there is only one physical Higgs state, so that $c_1 = 0$ and $c_2 = 1/\sqrt{2}$. Thus,

$$m_H^2 = \left\{ \frac{\partial^2 V}{\partial \Phi^0 \partial \Phi^{0\dagger}} + \frac{1}{2} \frac{\partial^2 V}{\partial \Phi^0 \partial \Phi^0} + \frac{1}{2} \frac{\partial^2 V}{\partial \Phi^{0\dagger} \partial \Phi^{0\dagger}} \right\}_{\Phi_i = \nu_i}. \quad (4.7.31)$$

²⁰ However, neither of these representations is equivalent to a matrix representation of $SU(2)$ in which all the group element matrices are real. Such a representation is called a pseudo-real representation, although such representations were treated as “complex” in Section 4.6.

Plugging in $\nu_1 = 0$ and $\nu_2 = v/\sqrt{2}$ and using the potential minimum condition $m^2 = \lambda v^2$ [see text following eq. (4.7.5)], we end up with:

$$m_H^2 = 2\lambda v^2. \quad (4.7.32)$$

The interactions of the gauge bosons and the Higgs bosons originate from the scalar kinetic energy term

$$\mathcal{L}_{\text{kinetic}} = (D^\mu \Phi)^\dagger (D_\mu \Phi), \quad (4.7.33)$$

where the covariant derivative is given by eq. (4.7.21) with generators $\mathcal{T}^k = \frac{1}{2}\tau^k$ and Φ is given by eq. (4.7.30). In the unitary gauge, we simply set $G^\pm = G^0 = 0$ and obtain:

$$\begin{aligned} \mathcal{L}_{\text{kinetic}} = & \frac{1}{2}(\partial^\mu H)(\partial_\mu H) + \frac{1}{4}g^2(v^2 + 2vH + H^2)W^{\mu+}W_\mu^- \\ & + \frac{g^2}{8\cos^2\theta_W}(v^2 + 2vH + H^2)Z^\mu Z_\mu. \end{aligned} \quad (4.7.34)$$

As expected, eq. (4.7.34) yields $m_W^2 = m_Z^2 \cos^2\theta_W = \frac{1}{4}g^2v^2$, as well as trilinear and quadrilinear interactions of the Higgs boson with a pair of gauge bosons.

Table 4.2 Fermions of the Standard Model and their $SU(3) \times SU(2)_L \times U(1)_Y$ quantum numbers. The generation indices run over $i = 1, 2, 3$. Color indices for the quarks are suppressed. The bars on the two-component antifermion fields are part of their names, and do not denote some form of complex conjugation.

| Two-component fermion fields | SU(3) | SU(2) _L | Y | T ₃ | Q = T ₃ + Y |
|--|--------------|--------------------|----------------|----------------|------------------------|
| $Q_i \equiv \begin{pmatrix} u_i \\ d_i \end{pmatrix}$ | triplet | doublet | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{2}{3}$ |
| | triplet | | $\frac{1}{6}$ | $-\frac{1}{2}$ | $-\frac{1}{3}$ |
| \bar{u}^i | anti-triplet | singlet | $-\frac{2}{3}$ | 0 | $-\frac{2}{3}$ |
| \bar{d}^i | anti-triplet | singlet | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ |
| $L_i \equiv \begin{pmatrix} \nu_i \\ \ell_i \end{pmatrix}$ | singlet | doublet | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 |
| | singlet | | $-\frac{1}{2}$ | $-\frac{1}{2}$ | -1 |
| $\bar{\ell}^i$ | singlet | singlet | 1 | 0 | 1 |

The fermion sector of the Standard Model consists of three generations of quarks and leptons, which are represented by the two-component fermion fields listed in Table 4.2, where Y is the weak hypercharge, T_3 is the third component of the weak isospin, and $Q = T_3 + Y$ is the electric charge. After $SU(2)_L \times U(1)_Y$ breaking, the

quark and lepton fields gain mass in such a way that the above two-component fields combine to make up four-component Dirac fermions:

$$U_i = \begin{pmatrix} u_i \\ \bar{u}^{\dagger i} \end{pmatrix}, \quad D_i = \begin{pmatrix} d_i \\ \bar{d}^{\dagger i} \end{pmatrix}, \quad L_i = \begin{pmatrix} \ell_i \\ \bar{\ell}^{\dagger i} \end{pmatrix}, \quad (4.7.35)$$

while the neutrinos ν_i remain massless. The extension of the Standard Model to include neutrino masses will be treated in Chapter 5.

Note that $u, \bar{u}, d, \bar{d}, \ell$ and $\bar{\ell}$ are two-component fields, whereas the usual four-component quark and charged lepton fields are denoted by capital letters U, D and E . Consider a generic four-component field expressed in terms of the corresponding two-component fields:

$$F = \begin{pmatrix} f \\ \bar{f}^{\dagger} \end{pmatrix}. \quad (4.7.36)$$

The electroweak quantum numbers of f are denoted by T_3^f, Y_f and Q_f , whereas the corresponding quantum numbers for \bar{f} are $T_3^{\bar{f}} = 0$ and $Q_{\bar{f}} = Y_{\bar{f}} = -Q_f$. Thus we have the correspondence to our general notation [eq. (3.1.5)]

$$f \longleftrightarrow \chi, \quad \bar{f} \longleftrightarrow \eta. \quad (4.7.37)$$

The QCD color interactions of the quarks are governed by the following interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = -g_s A_a^\mu q^{\dagger mi} \bar{\sigma}_\mu(\mathbf{T}^a)_m{}^n q_{ni} + g_s A_a^\mu \bar{q}_{ni}^\dagger \bar{\sigma}_\mu(\mathbf{T}^a)_m{}^n \bar{q}^{mi}, \quad (4.7.38)$$

summed over the generations i , where q is a (mass eigenstate) quark field, m and n are SU(3) color triplet indices, A_a^μ is the gluon field (with the corresponding gluons denoted by g_a), and \mathbf{T}^a are the color generators in the triplet representation of SU(3).

The electroweak interactions of the quarks and leptons are governed by the following interaction Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{int}} = & -\frac{g}{\sqrt{2}} \left[(\hat{u}^{\dagger i} \bar{\sigma}^\mu \hat{d}_i + \hat{\nu}^{\dagger i} \bar{\sigma}^\mu \hat{\ell}_i) W_\mu^+ + (\hat{d}^{\dagger i} \bar{\sigma}^\mu \hat{u}_i + \hat{\ell}^{\dagger i} \bar{\sigma}^\mu \hat{\nu}_i) W_\mu^- \right] \\ & - \frac{g}{c_W} \sum_{f=u,d,\nu,\ell} \left\{ (T_3^f - s_W^2 Q_f) \hat{f}^{\dagger i} \bar{\sigma}^\mu \hat{f}_i + s_W^2 Q_f \hat{f}^{\dagger i} \bar{\sigma}^\mu \hat{f}_i \right\} Z_\mu \\ & - e \sum_{f=u,d,\ell} Q^f (\hat{f}^{\dagger i} \bar{\sigma}^\mu \hat{f}_i - \hat{f}^{\dagger i} \bar{\sigma}^\mu \hat{f}_i) A_\mu, \end{aligned} \quad (4.7.39)$$

where $s_W \equiv \sin \theta_W$, $c_W \equiv \cos \theta_W$, the hatted symbols indicate fermion interaction eigenstates and i labels the generations. Before we perform any practical computations, we must convert from fermion interaction eigenstates to mass eigenstates. In order to accomplish this step, we must first identify the quark and lepton mass matrices.

In the electroweak theory, the fermion mass matrices originate from the fermion-Higgs Yukawa interactions. The Higgs field of the Standard Model is a complex

$SU(2)_L$ doublet of hypercharge $Y = \frac{1}{2}$, denoted by Φ_a , where the $SU(2)_L$ index $a = 1, 2$ is defined such that $\Phi_1 \equiv \Phi^+$ and $\Phi_2 \equiv \Phi^0$ [cf. eq. (4.7.1)]. Here, the superscripts $+$ and 0 refer to the electric charge of the Higgs field, $Q = T_3 + Y$, with $Y = \frac{1}{2}$ and $T_3 = \pm\frac{1}{2}$. Since Φ_a is a complex field, its complex conjugate $\Phi^{\dagger a} \equiv (\Phi^-, (\Phi^0)^\dagger)$ is an $SU(2)_L$ doublet scalar field with hypercharge $Y = -\frac{1}{2}$, where $\Phi^- \equiv (\Phi^+)^\dagger$. The $SU(2)_L \times U(1)_Y$ gauge invariant Yukawa interactions of the quarks and leptons with the Higgs field are then given by:

$$\mathcal{L}_Y = \epsilon^{ab} (\mathbf{Y}_u)^i_j \Phi_a \hat{Q}_{bi} \bar{u}^j - (\mathbf{Y}_d)^i_j \Phi^{\dagger a} \hat{Q}_{ai} \hat{d}^j - (\mathbf{Y}_\ell)^i_j \Phi^{\dagger a} \hat{L}_{ai} \hat{\ell}^j + \text{h.c.}, \quad (4.7.40)$$

where ϵ^{ab} is the antisymmetric invariant tensor of $SU(2)_L$, defined such that $\epsilon^{12} = -\epsilon^{21} = +1$. Using the definitions of the $SU(2)_L$ doublet quark and lepton fields given in Table 4.2, one can rewrite eq. (4.7.40) more explicitly as:

$$\begin{aligned} -\mathcal{L}_Y = & (\mathbf{Y}_u)^i_j \left[\Phi^0 \hat{u}_i \hat{u}^j - \Phi^+ \hat{d}_i \hat{u}^j \right] + (\mathbf{Y}_d)^i_j \left[\Phi^- \hat{u}_i \hat{d}^j + \Phi^{0*} \hat{d}_i \hat{d}^j \right] \\ & + (\mathbf{Y}_\ell)^i_j \left[\Phi^- \hat{\nu}_i \hat{\ell}^j + \Phi^{0*} \hat{\ell}_i \hat{\ell}^j \right] + \text{h.c.} \end{aligned} \quad (4.7.41)$$

The Higgs fields can be written in terms of the physical Higgs scalar h_{SM} and Nambu-Goldstone bosons G^0, G^\pm as

$$\Phi^0 = \frac{1}{\sqrt{2}}(v + h_{\text{SM}} + iG^0), \quad (4.7.42)$$

$$\Phi^\pm = G^\pm = (\Phi^\mp)^\dagger = (G^\mp)^\dagger. \quad (4.7.43)$$

where $v = 2m_W/g \simeq 246$ GeV. In the unitary gauge appropriate for tree-level calculations, the Nambu-Goldstone bosons become infinitely heavy and decouple. We identify the quark and lepton mass matrices by setting $\Phi^0 = v/\sqrt{2}$ and $\Phi^\pm = \Phi^\mp = 0$ in eq. (4.7.41):

$$(\mathbf{M}_u)^i_j = \frac{v}{\sqrt{2}}(\mathbf{Y}_u)^i_j, \quad (\mathbf{M}_d)^i_j = \frac{v}{\sqrt{2}}(\mathbf{Y}_d)^i_j, \quad (\mathbf{M}_\ell)^i_j = \frac{v}{\sqrt{2}}(\mathbf{Y}_\ell)^i_j \quad (4.7.44)$$

The neutrinos remain massless as previously indicated.

To diagonalize the quark and lepton mass matrices, we introduce four unitary matrices for the quark mass diagonalization, L_u, L_d, R_u and R_d , and two unitary matrices for the lepton mass diagonalization, L_ℓ and R_ℓ [cf. eq. (1.6.8)] such that

$$\begin{aligned} \hat{u}_i &= (L_u)^i_j u_j, & \hat{d}_i &= (L_d)^i_j d_j, & \hat{u}^i &= (R_u)^i_j \bar{u}^j, & \hat{d}^i &= (R_d)^i_j \bar{d}^j, \\ \hat{\ell}_i &= (L_\ell)^i_j \ell_j, & \hat{\ell}^i &= (R_\ell)^i_j \bar{\ell}^j, \end{aligned} \quad (4.7.45)$$

where the unhatted fields u, d, \bar{u} and \bar{d} are the corresponding quark mass eigenstates and ν, ℓ and $\bar{\ell}$ are the corresponding lepton mass eigenstates. The fermion mass diagonalization procedure consists of the singular value decomposition of the quark and lepton mass matrices:

$$L_u^\dagger \mathbf{M}_u R_u = \text{diag}(m_u, m_c, m_t), \quad (4.7.47)$$

$$L_d^\dagger \mathbf{M}_d R_d = \text{diag}(m_d, m_s, m_b), \quad (4.7.48)$$

$$L_\ell^\dagger \mathbf{M}_\ell R_\ell = \text{diag}(m_e, m_\mu, m_\tau), \quad (4.7.49)$$

where the diagonalized masses are real and non-negative. Since the neutrinos are massless, we are free to define the physical neutrino fields, ν_i , as the weak SU(2) partners of the corresponding charged lepton mass eigenstate fields. That is,

$$\hat{\nu}_i = (L_\ell)_i^j \nu_j. \quad (4.7.50)$$

We can now write out the couplings of the mass eigenstate quarks and leptons to the gauge bosons and Higgs bosons. Consider first the charged current interactions of the quarks and leptons. Using eq. (5.2.49), it follows that $\hat{u}^\dagger \bar{\sigma}^\mu \hat{d}_i = \mathbf{K}_i^j u^\dagger \bar{\sigma}^\mu d_j$, where

$$\mathbf{K} = L_u^\dagger L_d \quad (4.7.51)$$

is the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix [?].²¹ Due to eq. (4.7.50), the corresponding leptonic CKM matrix is the unit matrix. Hence, the charged current interactions take the form

$$\mathcal{L}_{\text{int}} = -\frac{g}{\sqrt{2}} \left[\mathbf{K}_i^j u^\dagger \bar{\sigma}^\mu d_j W_\mu^+ + (\mathbf{K}^\dagger)_i^j d^\dagger \bar{\sigma}^\mu u_j W_\mu^- + \nu^\dagger \bar{\sigma}^\mu \ell_i W_\mu^+ + \ell_i^\dagger \bar{\sigma}^\mu \nu_i W_\mu^- \right], \quad (4.7.52)$$

where $[\mathbf{K}^\dagger]_i^j \equiv [\mathbf{K}_j^i]^*$. Note that in the Standard Model, \bar{u} , \bar{d} and $\bar{\ell}$ do not couple to the W^\pm .

To obtain the neutral current interactions, we insert eqs. (5.2.49)–(4.7.50) into eq. (4.7.39). All factors of the *unitary* matrices L_f and R_f ($f = u, d, \ell$) cancel out, and the resulting interactions are flavor-diagonal. That is, we may simply remove the hats from the quark and lepton fields that couple to the Z and photon fields in eq. (4.7.39). This is the well-known Glashow-Iliopoulos-Maiani (GIM) mechanism for the flavor-conserving neutral currents [?].

The diagonalization of the fermion mass matrices is equivalent to the diagonalization of the Yukawa couplings [cf. eqs. (4.7.44) and (4.7.47)–(4.7.49)]. Thus, we define²²

$$Y_{fi} = \sqrt{2} m_{fi} / v, \quad f = u, d, \ell, \quad (4.7.53)$$

where i labels the fermion generation. It is convenient to rewrite eqs. (4.7.47)–(4.7.49) as follows:

$$(L_f)_{k^j} (\mathbf{Y}_f)^k_m (R_f)^m_i = Y_{fi} \delta_i^j, \quad f = u, d, \ell, \quad (4.7.54)$$

with no sum over the repeated index i . Using the unitarity of L_f ($f = u, d$), eq. (4.7.54) is equivalent to the following convenient form:

$$(\mathbf{Y}_f R_f)^k_i = Y_{fi} (L_f^\dagger)_i^k. \quad (4.7.55)$$

²¹ The CKM matrix elements V_{ij} as defined in ref. [?] are related by, for example, $V_{tb} = \mathbf{K}_3^3$ and $V_{us} = \mathbf{K}_1^2$.

²² Boldfaced symbols are used for the non-diagonal Yukawa matrices, while non-boldfaced symbols are used for the diagonalized Yukawa couplings.

Inserting eqs. (5.2.49), (4.7.50) and (4.7.54) into eq. (4.7.41), the resulting Higgs-fermion Lagrangian is flavor-diagonal:

$$\mathcal{L}_{\text{int}} = -\frac{1}{\sqrt{2}}h_{\text{SM}} [Y_{ui}u_i\bar{u}^i + Y_{di}d_i\bar{d}^i + Y_{\ell i}\ell_i\bar{\ell}^i] + \text{h.c.} \quad (4.7.56)$$

4.8 Parameter count of the Standard Model

The Standard Model Lagrangian appears to contain many parameters. However, not all these parameters can be physical. One is always free to redefine the Standard Model fields in an arbitrary manner. By suitable redefinitions, one can remove some of the apparent parameter freedom and identify the true physical independent parametric degrees of freedom.

To illustrate the procedure, let us first make a list of the parameters of the Standard Model. First, the gauge sector consists of three real gauge couplings (g_3 , g_2 and g_1) and the QCD vacuum angle (θ_{QCD}). The Higgs sector consists of one Higgs squared-mass parameter and one Higgs self-coupling (m^2 and λ). Traditionally, one trades in the latter two real parameters for the vacuum expectation value ($v = 246$ GeV) and the physical Higgs mass. The fermion sector consists of three Higgs-Yukawa coupling matrices \mathbf{y}_u , \mathbf{y}_d , and \mathbf{y}_e . Initially, \mathbf{y}_u , \mathbf{y}_d , and \mathbf{y}_e are arbitrary complex 3×3 matrices, which in total depend on 27 real and 27 imaginary degrees of freedom.

But, most of these degrees of freedom are unphysical. In particular, in the limit where $\mathbf{y}_u = \mathbf{y}_d = \mathbf{y}_e = 0$, the Standard Model possesses a global $U(3)^5$ symmetry corresponding to three generations of the five $SU(3) \times SU(2) \times U(1)$ multiplets: $(\nu_m, e_m^-)_L$, $(e_m^c)_L$, $(u_m, d_m)_L$, $(u_m^c)_L$, $(d_m^c)_L$, where m is the generation label. Thus, one can make global $U(3)^5$ rotations on the fermion fields of the Standard Model to absorb the unphysical degrees of freedom of \mathbf{y}_u , \mathbf{y}_d , and \mathbf{y}_e . A $U(3)$ matrix can be parameterized by three real angles and six phases, so that with the most general $U(3)^5$ rotation, we can apparently remove 15 real angles and 30 phases from \mathbf{y}_u , \mathbf{y}_d , and \mathbf{y}_e . However, the $U(3)^5$ rotations include four exact $U(1)$ global symmetries of the Standard Model, namely B and the three separate lepton numbers L_e , L_μ and L_τ . Thus, one can only remove 26 phases from \mathbf{y}_u , \mathbf{y}_d , and \mathbf{y}_e . This leaves 12 real parameters (corresponding to six quark masses, three lepton masses,²³ and three CKM mixing angles) and one imaginary degree of freedom (the phase of the CKM matrix). Adding up to get the final result, one finds that the Standard Model possesses 19 independent parameters (of which 13 are associated with the flavor sector).

²³ The neutrinos in the Standard Model are automatically massless and are not counted as independent degrees of freedom in the parameter count.

Exercises

4.1 Consider a scalar field theory consisting of a multiplet of n identical complex scalars, $\Phi_i(x)$. Suppose that the scalar potential, $V(\Phi, \Phi^\dagger)$ is invariant under $U(n)$ transformations, $\Phi \rightarrow U\Phi$, where $U = \exp[-ig\Lambda^a \mathcal{T}^a]$.

(a) Show that V must satisfy:

$$(\mathcal{T}^a)_i{}^j \Phi_j \frac{\partial V}{\partial \Phi_i} - (\mathcal{T}^a)_j{}^i \Phi^\dagger{}^j \frac{\partial V}{\partial \Phi^\dagger{}^i} = 0. \quad (4.\text{Ex.1})$$

(b) By taking the derivative of eq. (4.Ex.1) with respect to Φ_k and $\Phi^\dagger{}^\ell$ respectively, show that the resulting expression when evaluated at $\Phi_k = \langle \Phi_k \rangle \equiv \nu_k/\sqrt{2}$ coincides with eq. (4.6.24). Conclude that the Goldstone boson G_a is given by eq. (4.6.19). In particular, check that the normalization of the Goldstone boson state given in eq. (4.6.19) is required for a properly normalized kinetic energy term for G_a .

(c) Expand $V(\Phi, \Phi^\dagger)$ around the scalar field vacuum expectation values up to and including terms quadratic in the fields. Show that the quadratic terms are given by eqs. (4.6.20) and (4.6.23).

(d) Couple this scalar field theory to an $SU(n)$ Yang Mills theory. After replacing ordinary derivatives with covariant derivatives in the scalar kinetic energy term, write out the full scalar Lagrangian in the complex scalar field basis. Writing $\Phi_k = \bar{\Phi}_k + \nu_k/\sqrt{2}$, evaluate all terms up to and including terms quadratic in the fields. Show that one directly obtains the vector boson squared mass matrix given in eq. (4.6.13).

(e) Write down the R_ξ -gauge fixing term in the complex scalar field basis. Show that the resulting couplings of the gauge field and Goldstone boson fields coincide with those previously obtained in eqs. (4.5.23) and (4.5.31).

4.2 Consider a scalar field theory consisting of n identical complex fields Φ_i , with a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi_i)^\dagger (\partial^\mu \Phi_i) - V(\Phi^\dagger \Phi), \quad (4.\text{Ex.2})$$

where the potential function V is a function of $\Phi^\dagger{}^i \Phi_i$. Such a theory is invariant under the $U(n)$ transformation $\Phi \rightarrow U\Phi$, where U is an $n \times n$ unitary matrix.

(a) Rewrite the Lagrangian in terms of hermitian fields ϕ_{A_i} and ϕ_{B_i} defined by:

$$\Phi_j = \frac{1}{\sqrt{2}}(\phi_{A_j} + i\phi_{B_j}), \quad \Phi^\dagger{}^j = \frac{1}{\sqrt{2}}(\phi_{A_j} - i\phi_{B_j}), \quad (4.\text{Ex.3})$$

and introduce the $2n$ -dimensional hermitian scalar field:

$$\phi(x) = \begin{pmatrix} \phi_A(x) \\ \phi_B(x) \end{pmatrix}. \quad (4.\text{Ex.4})$$

Show that the Lagrangian is actually invariant under a larger symmetry group $O(2n)$, corresponding to the transformation $\phi \rightarrow \mathcal{O}\phi$ where \mathcal{O} is a $2n \times 2n$ orthogonal matrix.

(b) Working in the complex basis, show that the Lagrangian [eq. (4.Ex.2)] is invariant under the transformation:

$$\Phi_i \rightarrow U_i^j \Phi_j + \Phi^{\dagger j} (V^\dagger)_j^i, \quad (4.\text{Ex.5})$$

where U and V are complex $n \times n$ matrices, provided that the following two conditions are satisfied:

$$(i) \quad (U^\dagger U + V^\dagger V)_i^j = \delta_i^j, \quad (4.\text{Ex.6})$$

$$(ii) \quad V^T U \text{ is an antisymmetric matrix.} \quad (4.\text{Ex.7})$$

(c) Show that the $2n \times 2n$ matrix

$$\mathcal{Q} = \begin{pmatrix} \text{Re}(U + V) & -\text{Im}(U + V) \\ \text{Im}(U - V) & \text{Re}(U - V) \end{pmatrix} \quad (4.\text{Ex.8})$$

is an orthogonal matrix if U and V satisfy eqs. (4.Ex.6) and (4.Ex.7). Prove that any $2n \times 2n$ orthogonal matrix can be written in the form of eq. (4.Ex.8) by verifying that \mathcal{Q} is determined by $n(2n - 1)$ independent parameters.

(d) Use the results of parts (b) and (c) to conclude that if U is a unitary $n \times n$ matrix, then the $2n \times 2n$ matrix

$$\mathcal{Q}_U = \begin{pmatrix} \text{Re } U & -\text{Im } U \\ \text{Im } U & \text{Re } U \end{pmatrix} \quad (4.\text{Ex.9})$$

provides an explicit embedding of the subgroup $U(n)$ inside $O(2n)$. By writing $\mathcal{Q}_U = \exp[-ig\Lambda^a T^a]$ and $U = \exp[-ig\Lambda^a \mathcal{T}^a]$, show that T^a is given by eq. (4.6.9) in terms of the \mathcal{T}^a .

Likewise, define \mathcal{Q}_V by taking $U = 0$ in \mathcal{Q} [eq. (4.Ex.8)]. Show that \mathcal{Q}_V also provides an embedding of the subgroup $U(n)$ inside $O(2n)$. Prove that \mathcal{Q}_U and \mathcal{Q}_V are in fact equivalent representations of $U(n)$ by explicitly constructing the similarity transformation such that $S^{-1}\mathcal{Q}_U S$ is of the same form as \mathcal{Q}_V (with U substituted for V). Note that \mathcal{Q}_U and \mathcal{Q}_V depend implicitly on the group element $g \in U(n)$, whereas S is a fixed invertible matrix.

(e) Define the unitary matrix

$$A = \begin{pmatrix} I_n & -iI_n \\ iI_n & I_n \end{pmatrix}, \quad (4.\text{Ex.10})$$

where I_n is the $n \times n$ identity matrix. Consider a real orthogonal $2n \times 2n$ matrix R that satisfies:

$$R^T A R = A. \quad (4.\text{Ex.11})$$

Using an infinitesimal analysis (where $R \simeq I + Z$ where Z is an infinitesimal real antisymmetric $2n \times 2n$ matrix), prove that R provides an $\mathfrak{n} \oplus \mathfrak{n}^*$ reducible representation of $U(n)$. Verify that both Q_U and Q_V satisfy the constraint given by eq. (4.Ex.11)

(f) The natural embedding of the $SO(2n)$ subgroup inside $U(2n)$ can be constructed by considering the matrix

$$\mathcal{U} = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix}, \quad (4.\text{Ex.12})$$

where U and V satisfy eqs. (4.Ex.6) and (4.Ex.7). Verify that \mathcal{U} is a unitary matrix. Then, prove that \mathcal{Q} and \mathcal{U} provide equivalent representations of $O(2n)$ by explicitly constructing the similarity transformation $\mathcal{U} = P^{-1} \mathcal{Q} P$ for some fixed invertible matrix P (which is independent of the group element).

4.3 Repeat parts (a)–(d) of problem 2, under the assumption that $\Phi(x)$ is a quaternionic-valued field.

(a) Show that an analogous analysis yields an explicit embedding of $\text{Sp}(n)$ as a subgroup of $U(2n)$ [or equivalently as a subgroup of $O(4n)$].

(b) How does the fundamental representation of $O(4n)$ decompose with respect to the $\text{Sp}(n)$ subgroup? How does the fundamental representation of $U(2n)$ decompose with respect to the $\text{Sp}(n)$ subgroup?

4.4 Consider the spontaneous breaking of a gauge group G down to $U(1)$. The unbroken generator $Q = c_a T^a$ is some real linear combination of the generators of G .

(a) Prove that $x_b \equiv c_b/g_b$ is an (unnormalized) eigenvector of the vector boson squared-mass matrix, M_{ab}^2 , with zero eigenvalue.

(b) Suppose that A_μ is the massless gauge field that corresponds to the generator Q . Show that the covariant derivative can be expressed in the following form:

$$D_\mu = \partial_\mu + ieQA_\mu + \dots, \quad (4.\text{Ex.13})$$

where we have omitted terms in eq. (4.Ex.13) corresponding to all the other

gauge bosons and

$$e = \left[\sum_a \left(\frac{c_a}{g_a} \right)^2 \right]^{-1/2}. \quad (4.\text{Ex.14})$$

HINT: The vector boson mass matrix is diagonalized by an orthogonal transformation $\mathcal{O}M^2\mathcal{O}^T$ according to eq. (4.5.17). The rows of the matrix \mathcal{O} are constructed from the *orthonormal* eigenvectors of M^2 .

(c) Evaluate Q in the adjoint representation (*i.e.*, $Q = c_a T^a$, where the $(T^a)_{bc} = -if_{abc}$ are the generators of G in the adjoint representation). Show that $Q_{bc}x_c = 0$, where x_c is defined in part (a). What is the physical interpretation of this result?

(d) Prove that the commutator $[Q, M^2] = 0$, where Q is the unbroken $U(1)$ generator in the adjoint representation and M^2 is the gauge boson squared-mass matrix. Conclude that one can always choose the eigenstates of the gauge boson squared-mass matrix to be states of definite unbroken $U(1)$ -charge.

(e) Apply the results of part (b) to the spontaneous breaking of $SU(2) \times U(1)_Y$ to $U(1)_{\text{EM}}$ in the Standard Model of electroweak interactions. Show that eq. (4.Ex.14) reproduces eq. (4.7.11).

4.5 The generators of $SU(2) \times U(1)$ in the hypercharge-one $SU(2)$ doublet representation can be expressed as 4×4 matrices by using the real representation of eq. (4.6.9).

(a) Using the explicit form for this real representation, derive the gauge boson squared-mass matrix. Diagonalize the mass matrix and explicitly identify the real antisymmetric matrix generators \tilde{L}^a .

(b) Use eq. (4.5.24) to obtain the Goldstone bosons and identify the physical Higgs boson.

(c) Using the real representation, write down an explicit form for $D_\mu \Phi$ as a four-dimensional column vector. Then, evaluate eq. (4.7.33) in the unitary gauge and verify the results of eq. (4.7.34).

4.6 Derive the interaction Lagrangian for the charged and neutral Nambu-Goldstone bosons and the fermions of the Standard Model and verify that

$$\begin{aligned} \mathcal{L}_{\text{int}} = & Y_{ui}[\mathbf{K}]_i^j d_j \bar{u}^i G^+ - Y_{di}[\mathbf{K}^\dagger]_i^j u_j \bar{d}^i G^- - Y_{\ell i} \nu_i \bar{\ell}^i G^- \\ & + \frac{i}{\sqrt{2}} [Y_{di} d_i \bar{d}^i - Y_{ui} u_i \bar{u}^i + Y_{\ell i} \ell_i \bar{\ell}^i] G^0 + \text{h.c.}, \end{aligned} \quad (4.\text{Ex.15})$$

where \mathbf{K} is the CKM matrix and Y_{fi} (for $f = u, d, \ell$) is given by eq. (4.7.53).

4.7 In Section 4.7, we examined in detail the structure of a spontaneously broken $SU(2) \times U(1)_Y$ gauge theory, in which the symmetry breaking was due to the vacuum expectation value of a $Y = \frac{1}{2}$, $SU(2)$ doublet of complex scalar fields. In this problem, we will replace this multiplet of scalar fields with a different representation.

(a) Consider a spontaneously broken $SU(2) \times U(1)_Y$ gauge theory with a $Y = 0$, $SU(2)$ triplet of *real* scalar fields. Assume that the electrically neutral member of the scalar triplet acquires a vacuum expectation value. After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model. Deduce the Feynman rules for the three-point interactions among the Higgs and vector bosons.

HINT: Since the triplet of scalar fields corresponds to the adjoint representation of $SU(2)$, the corresponding $SU(2)$ generators that act on the triplet of scalar fields can be chosen to be $(T^a)_{bc} = -i\epsilon_{abc}$. The hypercharge operator annihilates the $Y = 0$ fields. Define $L^a = ig_a T^a$, and follow the methods outlined in Section 4.5.4.

(b) Consider a spontaneously broken $SU(2) \times U(1)_Y$ gauge theory with a $Y = 1$, $SU(2)$ triplet of *complex* scalar fields. Again, assume that the electrically neutral member of the scalar triplet acquires a vacuum expectation value. After symmetry breaking, identify the subgroup that remains unbroken. Compute the vector boson masses and the physical Higgs scalar masses in this model.

(c) If both doublet and triplet Higgs fields exist in nature, what does this exercise imply about the parameters of the Higgs Lagrangian?

References

- [1] John C. Taylor, *Gauge Theories of Weak Interactions*, (Cambridge University Press, Cambridge, England, 1976).
- [2] Steven Weinberg, *The Quantum Theory of Fields: Volume II—Modern Applications*, (Cambridge University Press, Cambridge, England, 1996).
- [3] L. O'Raifeartaigh, *Group Structure of Gauge Theories* (Cambridge University Press, Cambridge, England, 1986).
- [4] S. Weinberg, *Phys. Rev.*, **D7** (1973) 1068.
- [5] David Bailin and Alexander Love, *Introduction to Gauge Field Theory* (Institute of Physics Publishing, London, 1993).
- [6] Dima Bardin and Giampiero Passarino, *The Standard Model in the Making: Precision Study of the Electroweak Interactions* (Oxford University Press, Oxford, England, 1999).
- [7] For a review, see E.S. Abers and B.W. Lee, *Physics Reports*, **9C** (1973) 1.
- [8] Kerson Huang, *Quarks, Leptons and Gauge Fields* (World Scientific Publishing, Singapore, 1992).